9th June 2005

# Munkres §33

**Ex. 33.1 (Morten Poulsen).** Let $r \in [0,1]$. Recall from the proof of the Urysohn lemma that if $p < q$ then $\overline{U}_p \subset U_q$. Furthermore, recall that $U_q = \emptyset$ if $q < 0$ and $U_p = X$ if $p > 1$.

**Claim 1.** $f^{-1}(\{r\}) = \bigcap_{p>r} U_p - \bigcup_{q<r} U_q$, $p, q \in \mathbf{Q}$.

*Proof.* By the construction of $f\colon X \to [0,1]$,

$$\bigcap_{p>0} U_p - \bigcup_{q<0} U_q = \bigcap_{p>0} U_p = f^{-1}(\{0\})$$

and

$$\bigcap_{p>1} U_p - \bigcup_{q<1} U_q = X - \bigcup_{q<1} U_q = f^{-1}(\{1\}).$$

Now assume $r \in (0,1)$.

"$\subset$": Let $x \in f^{-1}(\{r\})$, i.e. $f(x) = r = \inf\{\, p \,|\, x \in U_p \,\}$. Note that $x \notin \bigcup_{q<r} U_q$, since $f(x) = r$. Suppose there exists $t > r$, $t \in \mathbf{Q}$, such that $x \notin U_t$. Since $f(x) = r$, there exists $s \in \mathbf{Q}$ such that $r \leq s < t$ and $x \in U_s$. Now $x \in U_s \subset \overline{U}_s \subset U_t$, contradiction. It follows that $x \in \bigcap_{p>r} U_p - \bigcup_{q<r} U_q$.

"$\supset$": Let $x \in \bigcap_{p>r} U_p - \bigcup_{q<r} U_q$. Note that $f(x) \leq r$, since $x \in \bigcap_{p>r} U_p$. Suppose $f(x) < r$, i.e. there exists $t < r$ such that $x \in U_t \subset \bigcup_{q<r} U_q$, contradiction. It follows that $x \in f^{-1}(\{r\})$. □

**Ex. 33.4 (Morten Poulsen).**

**Theorem 2.** *Let $X$ be normal. There exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) > 0$ for $x \notin A$, if and only if $A$ is a closed $G_\delta$ set in $X$.*

*Proof.* Suppose $A = f^{-1}(\{0\})$. Since

$$A = f^{-1}(\{0\}) = f^{-1}\Big(\bigcap_{n\in\mathbf{Z}_+} [0, 1/n)\Big) = \bigcap_{n\in\mathbf{Z}_+} f^{-1}([0,1/n))$$

it follows that $A$ is a closed $G_\delta$ set.

Conversely suppose $A$ is a closed $G_\delta$ set, i.e. $A = \bigcap_{n\in\mathbf{Z}_+} U_n$, $U_n$ open. Then $X - U_n$ and $A$ are closed and disjoint for all $n$. By Urysohn's lemma there exists a continuous function $f_n : X \to [0,1]$, such that $f_n(A) = \{0\}$ and $f_n(X - U_n) = \{1\}$.

Now define $f : X \to [0,1]$ by

$$f(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x).$$

Clearly $f$ is well-defined. Furthermore $f$ is continuous, by theorem 21.6, since the sequence of continuous functions $\big(\sum_{i=1}^{n} \frac{1}{2^i} f_i(x)\big)_{n\in\mathbf{Z}_+}$ converges uniformly to $f$, since

$$\left|\sum_{i=1}^{\infty} \frac{1}{2^i} f_i(x) - \sum_{i=1}^{n} \frac{1}{2^i} f_i(x)\right| = \sum_{i=n+1}^{\infty} \frac{1}{2^i} f_i(x) \leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} \to 0$$

for $n \to \infty$.

Clearly $f(x) = 0$ for $x \in A$. Furthermore note that if $x \notin A$ then $x \in X - U_n$ for some $n$, hence $f(x) \geq \frac{1}{2^n} f_n(x) = \frac{1}{2^n} > 0$. □

**Ex. 33.5 (Morten Poulsen).**

**Theorem 3** (Strong form of the Urysohn lemma)**.** *Let $X$ be a normal space. There is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in A$, and $f(x) = 1$ for $x \in B$, and $0 < f(x) < 1$ otherwise, if and only if $A$ and $B$ are disjoint closed $G_\delta$ sets in $X$.*

*Proof.* Suppose $f : X \to [0,1]$ is a continuous function. Then clearly $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$ are disjoint. Since

$$A = f^{-1}(\{0\}) = f^{-1}(\bigcap_{n\in\mathbf{Z}_+}[0,1/n)) = \bigcap_{n\in\mathbf{Z}_+} f^{-1}([0,1/n))$$

and

$$B = f^{-1}(\{1\}) = f^{-1}(\bigcap_{n\in\mathbf{Z}_+}(1-1/n,1]) = \bigcap_{n\in\mathbf{Z}_+} f^{-1}((1-1/n,1])$$

it follows that $A$ and $B$ are disjoint closed $G_\delta$ sets in $X$.

Conversely suppose $A$ and $B$ are disjoint closed $G_\delta$ sets in $X$. By ex. 33.4 there exists continuous functions $f_A : X \to [0,1]$ and $f_B : X \to [0,1]$, such that $f_A^{-1}(\{0\}) = A$ and $f_B^{-1}(\{0\}) = B$. Now the function $f : X \to [0,1]$ defined by

$$f(x) = \frac{f_A(x)}{f_A(x) + f_B(x)}$$

is well-defined and clearly continuous. Furthermore $f^{-1}(\{0\}) = A$ and $f^{-1}(\{1\}) = B$, since

$$f(x) = 0 \Leftrightarrow f_A(x) = 0 \Leftrightarrow x \in A$$

and

$$f(x) = 1 \Leftrightarrow f_A(x) = f_A(x) + f_B(x) \Leftrightarrow f_B(x) = 0 \Leftrightarrow x \in B.$$

□

**Ex. 33.7.** For any topological space $X$ we have the following implications:

$X$ is locally compact Hausdorff

$\overset{\text{Cor 29.4}}{\Rightarrow}$ $X$ is an open subspace of a compact Hausdorff space

$\overset{\text{Thm 32.2}}{\Rightarrow}$ $X$ is a subspace of a normal space

$\overset{\text{Thm 33.1}}{\Rightarrow}$ $X$ is a subspace of a completely regular space

$\overset{\text{Thm 33.2}}{\Rightarrow}$ $X$ is completely regular

**Ex. 33.8.** Using complete regularity of $X$ and compactness of $A$, we see that there is a continuous real-valaued function $g\colon X \to [0,1]$ such that $g(a) < \frac{1}{2}$ for all $a \in A$ and $g(B) = \{1\}$. (There are finitely many continuous functions $g_1, \dots, g_k \colon X \to [0,1]$ such that $A \subset \bigcup\{g_i < \frac{1}{2}\}$ and $g_i(B) = 1$ for all $i$. Put $g = \frac{1}{k}\sum g_i$.) The continuous [Ex 18.8] function $f = 2\max\{0, g - \frac{1}{2}\}$ maps $X$ into the unit interval, $g(A) = \{0\}$, and $g(B) = \{1\}$.

## References