Ex. 34.1. We are looking for a non-regular Hausdorff space. By Example 1 p. 197, $R_K$ [p. 82] is such a space. Indeed, $R_K$ is Hausdorff for the topology is finer than the standard topology [Lemma 13.4]. $R_K$ is 2nd countable for the sets $(a, b)$ and $(a, b) - K$, where the intervals have rational end-points, constitute a countable basis. $R_K$ is not metrizable for it is not even regular [Example 1, p. 197].

Conclusion: The regularity axiom can not be replaced by the Hausdorff axiom in the Urysohn metrization theorem [Thm 34.1].

Ex. 34.2. We are looking for 1st but not 2nd countable space. By Example 3 p. 192, $R^\ell$ [p. 82] is such a space. Indeed, the Sorgenfrey right half-open interval topology $R^\ell$ is completely normal [Ex 32.4], 1st countable, Lindelöf, has a countable dense subset [Example 3, p. 192], but is not metrizable [Ex 30.6].

Ex. 34.3. We characterize the metrizable spaces among the compact Hausdorff spaces.

Theorem 1. Let $X$ be a compact Hausdorff space. Then

$X$ is metrizable $\iff X$ is 2nd countable

Proof. $\Rightarrow$: Every compact metrizable space is 2nd countable [Ex 30.4].

$\Leftarrow$: Every compact Hausdorff space is normal [Thm 32.3]. Every 2nd countable normal space is metrizable by the Urysohn metrization theorem [Thm 34.1].

We may also characterize the metrizable spaces among 2nd countable spaces.

Theorem 2. Let $X$ be a 2nd countable topological space. Then

$X$ is metrizable $\iff X$ is (completely) normal $\iff X$ is regular

Ex. 34.4. Let $X$ be a locally compact Hausdorff space. Then

$X$ is metrizable $\iff X$ is 2nd countable

$\not\Rightarrow$: Any discrete uncountable space is metrizable and not 2nd countable.

$\Leftarrow$: Every locally compact Hausdorff space is regular [Ex 32.3] (even completely regular [Ex 33.7]). Every 2nd countable regular space is metrizable by the Urysohn metrization theorem [Thm 34.1].

Ex. 34.5. Theorem 3. Let $X$ be a locally compact Hausdorff space and $X^+$ its one-point-compactification. Then

$X^+$ is metrizable $\iff X$ is 2nd countable

Proof. $\Rightarrow$: Every compact metrizable space is 2nd countable [Ex 30.4]. Every subspace of a 2nd countable space is 2nd countable [Thm 30.2].

$\Leftarrow$: Suppose that $X$ has the countable basis $B$. It suffices to show that also $X^+$ has a countable basis [Ex 34.3]. Any open subset of $X$ is a union of elements from $B$. The remaining open sets in $X^+$ are neighborhoods of $\infty$. Any neighborhood of $\infty$ is of the form $X^+ - C$ where $C$ is a compact subspace of $X$. For each point $x \in C$ there is a basis neighborhood $U_x \in B$ such that $\overline{U}$ is compact [Thm 29.3]. By compactness, $C$ is covered by finitely many basis open sets $C \subset U_1 \cup \cdots \cup U_k$. Now

$\infty \in X^+ - (\overline{U_1} \cup \cdots \cup \overline{U_k}) \subset X^+ - C$

where $X^+ - (\overline{U_1} \cup \cdots \cup \overline{U_k})$ is open in $X^+$ since $\overline{U_1} \cup \cdots \cup \overline{U_k}$ is compact in $X$ [Ex 26.3]. This shows that if we supplement $B$ with all sets of the form $X^+ - (\overline{U_1} \cup \cdots \cup \overline{U_k})$, $k \in \mathbb{Z}_+$, $U_i \in B$, and call the union $B^+$, then $B^+$ is a basis for the topology on $X^+$. Since there are only countable many finite subsets of $B$ [Ex 7.5.(j)], the enlarged basis $B^+$ is still countable [Thm 7.5].

References