1st December 2004

Munkres §34

Ex. 34.1. We are looking for a non-regular Hausdorff space. By Example 1 p. 197, $\mathbf{R}_K$ [p. 82] is such a space. Indeed, $\mathbf{R}_K$ is Hausdorff for the topology is finer than the standard topology [Lemma 13.4]. $\mathbf{R}_K$ is 2nd countable for the sets (a,b) and $(a,b)-K$, where the intervals have rational end-points, constitute a countable basis. $\mathbf{R}_K$ is not metrizable for it is not even regular [Example 1, p. 197].
Conclusion: The regularity axiom can not be replaced by the Hausdorff axiom in the Urysohn metrization theorem [Thm 34.1].

Ex. 34.2. We are looking for 1st but not 2nd countable space. By Example 3 p. 192, $\mathbf{R}_\ell$ [p. 82] is such a space. Indeed, the Sorgenfrey right half-open interval topology $\mathbf{R}_\ell$ [p. 82] is completely normal [Ex 32.4], 1st countable, Lindelöf, has a countable dense subset [Example 3, p. 192], but is not metrizable [Ex 30.6].

Ex. 34.3. We characterize the metrizable spaces among the compact Hausdorff spaces.

Theorem 1. *Let X be a compact Hausdorff space. Then*

$$X \text{ is metrizable} \Leftrightarrow X \text{ is 2nd countable}$$

Proof. $\Rightarrow$: Every compact metrizable space is 2nd countable [Ex 30.4].
$\Leftarrow$: Every compact Hausdorff space is normal [Thm 32.3]. Every 2nd countable normal space is metrizable by the Urysohn metrization theorem [Thm 34.1]. □

We may also characterize the metrizable spaces among 2nd countable spaces.

Theorem 2. *Let X be a 2nd countable topological space. Then*

$$X \text{ is metrizable} \overset{\text{Thm 34.1, 32.2}}{\Leftrightarrow} X \text{ is (completely) normal} \overset{\text{Thm 32.1}}{\Leftrightarrow} X \text{ is regular}$$

Ex. 34.4. Let X be a locally compact Hausdorff space. Then

$$X \text{ is metrizable} \Leftarrow X \text{ is 2nd countable}$$

$\not\Rightarrow$: Any discrete uncountable space is metrizable and not 2nd countable.
$\Leftarrow$: Every locally compact Hausdorff space is regular [Ex 32.3] (even completely regular [Ex 33.7]). Every 2nd countable regular space is metrizable by the Urysohn metrization theorem [Thm 34.1].

Ex. 34.5.

Theorem 3. *Let X be a locally compact Hausdorff space and X^+ its one-point-compactification. Then*

$$X^+ \text{ is metrizable} \Leftrightarrow X \text{ is 2nd countable}$$

Proof. $\Rightarrow$: : Every compact metrizable space is 2nd countable [Ex 30.4]. Every subspace of a 2nd countable space is 2nd countable [Thm 30.2].
$\Leftarrow$: Suppose that X has the countable basis $\mathcal{B}$. It suffices to show that also X^+ has a countable basis [Ex 34.3]. Any open subset of X is a union of elements from $\mathcal{B}$. The remaining open sets in X^+ are neighborhoods of ∞. Any neighborhood of ∞ is of the form $X^+ - C$ where C is a compact subspace of X. For each point $x \in C$ there is a basis neighborhood $U_x \in \mathcal{B}$ such that $\overline{U}$ is compact [Thm 29.3]. By compactness, C is covered by finitely many basis open sets $C \subset U_1 \cup \cdots \cup U_k$. Now

$$\infty \in X^+ - (\overline{U}_1 \cup \cdots \overline{U}_k) \subset X^+ - C$$

where $X^+ - (\overline{U}_1 \cup \cdots \overline{U}_k)$ is open in X^+ since $\overline{U}_1 \cup \cdots \overline{U}_k$ is compact in X [Ex 26.3]. This shows that if we supplement $\mathcal{B}$ with all sets of the form $X^+ - (\overline{U}_1 \cup \cdots \overline{U}_k)$, $k \in \mathbf{Z}_+$, $U_i \in \mathcal{B}$, and call the union $\mathcal{B}^+$, then $\mathcal{B}^+$ is a basis for the topology on X^+. Since there are only countable many finite subsets of $\mathcal{B}$ [Ex 7.5.(j)], the enlarged basis $\mathcal{B}^+$ is still countable [Thm 7.5]. □

References