Munkres §38

**Ex. 38.4.** Let $X \to \beta X$ be the Stone–Čech compactification and $X \to cX$ an arbitrary compactification of the completely regular space $X$. By the universal property of the Stone–Čech compactification, the map $X \to cX$ extends uniquely

$$
\begin{array}{c}
X \to cX \\
\beta X
\end{array}
$$

to a continuous map $\beta X \to cX$. Any continuous map of a compact space to a Hausdorff space is closed. In particular, $\beta X \to cX$ is closed. It is also surjective for it has a dense image since $X \to cX$ has a dense image. Thus $\beta X \to cX$ is a closed quotient map.

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**Ex. 38.5.**

(a). For any $\varepsilon > 0$ there exists an $\alpha \in S_\beta = [0, \Omega)$ such that $|f(\beta) - f(\alpha)| < \varepsilon$ for all $\beta > \alpha$. For if no such element existed we could find an increasing sequence of elements $\gamma_n \in (0, \Omega)$ such that $|f(\gamma_n) - f(\gamma_{n-1})| \geq \varepsilon$ for all $n$. But any increasing sequence in $(0, \Omega)$ converges to its least upper bound whereas the image sequence $f(\gamma_n) \in \mathbb{R}$ does not converge; this contradicts continuity of the function $f: (0, \Omega) \to \mathbb{R}$. So in particular, there exist elements $\alpha_n$ such that $|f(\beta) - f(\alpha_n)| < 1/n$ for all $\beta > \alpha_n$. Let $\alpha$ be an upper bound for these elements. Then $f$ is constant on $(\alpha, \Omega)$.

(b). Since any real function on $(0, \Omega)$ is eventually constant, any real function, in particular any bounded real function, on $(0, \Omega)$ extends to the one-point-compactification $(0, \Omega]$. But the Stone–Čech compactification is characterized by this property [Thm 38.5] so $(0, \Omega] = \beta(0, \Omega]$.

(c). Use that any compactification of $(0, \Omega]$ is a quotient of $(0, \Omega]$ [Ex 38.4].

**Ex. 38.6.** ([1, Thm 6.1.14]) Let $X$ be a completely regular space and $\beta(X)$ its Stone–Čech compactification. Then

- $X$ is nonconnected $\iff$ There exists a continuous surjective function $X \to \{0, 1\}$

- $\Rightarrow$ There exists a continuous surjective function $\beta(X) \to \{0, 1\} \iff \beta(X)$ is nonconnected

If $X$ is connected then also $\beta X$ is connected since it has a connected dense subset [3, Thm 23.4]

**Ex. 38.7.** ([Exam June 03, Problem 4] [5, 6, 4]) Let $X$ be a discrete space; $A$ a subset of $X \subset \beta(X)$ and $U$ an open subset of $\beta(X)$.

1. Let $F: \beta(X) \to \{0, 1\}$ be the extension [Thm 38.4] of the continuous function $f: X \to \{0, 1\}$ given by $f(A) = 0$ and $f(X - A) = 1$. Then $\overline{A} \subset F^{-1}(0)$ and $\overline{X - A} \subset F^{-1}(1)$ so these two subsets are disjoint; in other words $X - \overline{A} \subset \beta(X) - \overline{A}$. The inclusions

$$
\beta(X) - \overline{A} \equiv X - \overline{A} \supset \overline{X - A} \subset \beta(X) - \overline{A}
$$

tell us that $\beta(X) - \overline{A} = X - \overline{A}$. In particular, $\overline{A}$ is open (and closed).

2. Since $U \cap X$ is a subset of $U$, it is clear that $U \cap X \subset \overline{U}$ [Ex 17.6.(a)]. Conversely, let $x$ be a point in $\overline{U}$ and $V$ any neighborhood of $x$. Then $V \cap U \neq \emptyset$ is nonempty for $x$ lies in the closure of $U$, and hence $(V \cap U) \cap X = V \cap (U \cap X) \neq \emptyset$ is also nonempty as $X$ is dense. Thus every neighborhood $V$ of $x$ intersects $U \cap X$ nontrivially. This means that $x \in U \cap X$. We conclude that $U \cap X = \overline{U}$. From (1) (with $A = U \cap X$) we see that $\overline{U}$ is open (and closed).

3. Let $Y$ be any subset of $\beta(X)$ containing at least two distinct points, $x$ and $y$. We shall show that $Y$ is not connected. Let $U \subset \beta(X)$ be an open set such that $x \in U$ and $y \notin U$; such an open set $U$ exists because $\beta(X)$ is Hausdorff [Definition, p. 237]. Then $Y = (Y \cap U) \cup (Y - U)$ is a separation of $Y$, so $Y$ is not connected.
A Hausdorff space is said to be extremally disconnected if the closure of every open set is open. A space is totally disconnected if the connected components are one-point sets. Any extremally disconnected space is totally disconnected. We have shown that \( \beta(X) \) is extremally disconnected.

**Ex. 38.8.** The compact Hausdorff space \( I' \) is a compactification of \( \mathbb{Z}_+ \) since [3, Ex 30.16] it has a countable dense subset (and is not finite). Any compactification of \( \mathbb{Z}_+ \) is a quotient of the Stone–Čech compactification \( \beta\mathbb{Z}_+ \) [3, Ex 38.4]. In particular, \( I' \) is a quotient of \( \beta\mathbb{Z}_+ \) so \( \text{card} \beta\mathbb{Z}_+ \geq \text{card} I' \).

**Ex. 38.9.** ([Exam June 04, Problem 3])

(a). Suppose that \( x_n \in X \) converges to \( y \in \beta X - X \). We will show that then \( y \) is actually the limit point of two sequences with no points in common. The first step is to find a subsequence where no two points are identical. We recursively define a subsequence \( x_{n_k} \) by

\[
n_k = \begin{cases} 1 & k = 1 \\ \min\{n > n_{k-1} \mid x_n \notin \{x_{n_1}, \ldots, x_{n_{k-1}}\} \} & k > 1 \end{cases}
\]

This definition makes sense since the set we are taking the minimal element of a nonempty set. Since \( x_n \) converges to \( y \), the subsequence \( x_{n_k} \) also converges to \( y \). Clearly, no two points of the subsequence \( x_{n_k} \) are identical. We call this subsequence \( x_n \) again.

Let now \( A = \{x_1, x_2, \ldots\} \) be the set of odd points and \( B = \{x_2, x_4, \ldots\} \) the set of even points in this sequence. We claim that \( \overline{A} = A \cup \{y\} \) and \( \overline{B} = B \cup \{y\} \).

Any neighborhood of \( y \) contains a point from \( A \), so \( y \) is in the closure of \( A \). Since \( A \subset A \cup \{y\} \subset \overline{A} \), it suffices to show that \( A \subset A \cup \{y\} \) is closed, ie that the complement of \( A \cup \{y\} \) is open: Let \( z \) be a point in the complement. Since \( z \) is not the limit of the sequence \( (x_{2n+1}) \) (there is just one limit point, namely \( y \), in the Hausdorff space \( \beta X \)) there exists a neighborhood of \( z \), even one that doesn’t contain \( y \), containing only finitely many elements from this sequence. Since \( z \) is not in \( A \) we can remove these finitely many points from the neighborhood to get a neighborhood of \( z \) that is disjoint from \( A \cup \{y\} \).

This shows that \( \overline{A} = A \cup \{y\} \). Similarly, \( \overline{B} = B \cup \{y\} \). Therefore the intersection \( \overline{A} \cap \overline{B} = \{y\} \neq \emptyset \).

On the other hand, the sets \( A \) and \( B \) are disjoint since no two points of the sequence \( x_n \) are identical. They are closed subsets of \( X \) for Cl\(_X\) \( A = X \cap \overline{A} = X \cap (A \cup \{y\}) = A \) and similarly for \( B \), of course. By Urysohn’s characterization of normal spaces, there exists a continuous function \( f: X \to [0,1] \) such that \( A \subset f^{-1}(0) \) and \( B \subset f^{-1}(1) \). The universal property of the Stone–Čech compactification [2, §27] says that there exists a unique continuous map \( \beta f \) into the compact Hausdorff space \([0,1] \) such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & [0,1] \\
\downarrow{\beta} & & \downarrow{\beta f} \\
\beta X & & \\
\end{array}
\]

commutes. Since \( \overline{A} \subset \beta f^{-1}(0) \) and \( \overline{B} \subset \beta f^{-1}(1) \), \( \overline{A} \) and \( \overline{B} \) are disjoint.

We have now shown that \( \overline{A} \cap \overline{B} \) is both empty an nonempty. This contradiction means that no point in \( \beta X - X \) can be the limit of a sequence of points in \( X \).

(b). Assume that \( X \) is normal and noncompact. \( X \) is a proper subspace of \( \beta X \) since \( \beta X \) is compact which \( X \) is not. No point in \( \beta X - X = \overline{X} - X \) is the limit of a sequence of points in \( X \). Thus \( \beta X \) does not satisfy the Sequence lemma so \( \beta X \) is not first countable, in particular not metrizable.
References


