Problem 1
Solution 1: Apply the Extreme value theorem [Thm 27.4] to the identity map $X \to X$.
Solution 2: Copy the last two paragraphs on p. 174.

Problem 2
(1) Solution 1: [Ex 30.4] For each $n$ there is by compactness a finite set $A_n \subset X$ such that all points in $X$ are within distance $1/n$ from $A_n$. Then $A = \bigcup A_n$ is a countable dense subset for $d(x, A) \leq d(x, A_n) \leq 1/n$ for all $x \in X$ and all $n \geq 1$.
Solution 2: A careful inspection of pp. 191–192 and [Ex 30.5] reveals (no, that isn’t very well written) the following theorem:

**Theorem 0.1.** Let $X$ be a metrizable space. Then

$$X \text{ is 2nd countable} \iff X \text{ has a countable dense subset} \iff X \text{ is Lindel"of}$$

In our case, $X$ is metrizable and compact, so it is metrizable and Lindelöf, so it has a countable dense subset.

(2) For any metric, there is a bounded metric giving the same topology [Thm 20.1].

(3) If $f(x) = f(y)$, the continuous [Ex 20.3] functions $z \to d(x, z)$ and $z \to d(y, z)$ agree on the dense subspace $A$, so they agree on $X$ [Ex 18.13, Ex 31.5]. In particular they have the same value at the point $x$, so $d(x, y) = d(x, x) = 0$, so $x = y$. We have shown that $f$ is injective.

The map $f$ is continuous because each coordinate, $x \to d(x, a)$, is continuous [Thm 19.6, Ex 20.3].

The continuous bijection $f: X \to f(X)$ is a homeomorphism because $X$ is compact and $[0, 1]^A$ is Hausdorff [Thm 26.6, Thm 19.4, Thm 31.2].

The image $f(X)$ is closed because it is a compact subspace of a Hausdorff space [Thm 26.3, Thm 26.5].

(4) A countable product of metrizable spaces is metrizable [Thm 20.5], any product of compact spaces is compact [Thm 37.3]; in particular, $[0, 1]^{\omega}$ is compact and metrizable. Any subspace of a metrizable space is metrizable [Ex 21.1], any closed subspace of a compact space is compact [Thm 26.2]; in particular, any closed subspace of $[0, 1]^{\omega}$ is metrizable and compact.

Problem 3
(1) The extension $f^*$ is continuous at all points of $X$ (where it agrees with the continuous map $f$). The only problem is continuity at $\infty \in X$. This explains the first step in the following argument:

$f^*$ is continuous $\iff f^*$ is continuous at $\infty \in X$

$\iff$ For any neighborhood $V \subset Y$ of $\infty$ there is a neighborhood $U \subset X$ of $\infty$ such that $U \subset f^{-1}(V)$

$\iff$ For any compact $K \subset Y$ there is a compact $L \subset X$ such that $X - L \subset f^{-1}(Y - K)$

$\iff$ For any compact $K \subset Y$ there is a compact $L \subset X$ such that $f^{-1}(K) \subset L$

$\iff$ For any compact $K \subset Y$, $f^{-1}(K)$ is compact

For the final step, note that if $f^{-1}(K) \subset L$ then $f^{-1}(K)$ is compact [Thm 26.2] as a closed subspace [Thm 26.3] of a compact space.

(2) The map $g^*$ is not continuous for $g^{-1}(S^1) = \mathbb{R}$ is not compact [Example 1,p. 164].

Problem 4
(1) The saturation $p^{-1}(A) = A \cup (-A)$ of any open (resp. closed) subspace $A \subset S^2$ is open (resp. closed) because $x \rightarrow -x$ is a homeomorphism of $S^2$. (To see that $p$ is closed one may also note that any continuous map of a compact space into a Hausdorff space is closed [Ex 26.6].)

(2) Solution 1: The map $p: S^2 \rightarrow P^2$ is perfect, $S^2$ is 2nd countable (in fact a manifold), and perfect maps preserve 2nd countability [Ex 31.7(d)].
Solution 2: Let $\{B_n\}$ be a countable basis for $S^2$. Since $p$ is open, $p(B_n)$ is open for all $n$; indeed $\{p(B_n)\}$ is a countable basis for $P^2$.

(3) Solution 1: The map $p: S^2 \rightarrow P^2$ is perfect, $S^2$ is Hausdorff (in fact a manifold), and perfect maps preserve the Hausdorff property [Ex 31.7(a)].
Solution 2: The map $p: S^2 \rightarrow P^2$ is a closed quotient map, $S^2$ is normal (in fact a manifold), and closed quotient maps preserve normality [Ex 31.6].
Solution 3: It is also possible to give a simple ad hoc argument for this particular map.

(4) Solution 1: The restriction $q = p|U: U \rightarrow p(U)$ is bijective because $U \cap -U = \emptyset$; it is continuous because it is the restriction of a continuous map [Thm 18.2]; it is open because it is the restriction of an open map to an open subspace [Ex 22.5]; so it is a homeomorphism.
Solution 2: The restriction $q = p|U: U \rightarrow p(U)$ is bijective because $U \cap -U = \emptyset$; it is also a quotient map since $U$ is open [Thm 22.1]; a bijective quotient map is a homeomorphism (either directly from the definition or because there exists a continuous map $f: p(U) \rightarrow U$ such that $f \circ q$ is the identity on $U$ [Thm 22.2] so that $q^{-1} = f$ is continuous).

(5) $S^2 - p(U) = S^1$.

**Conclusion:** $P^2$ is a compact manifold [Thm 60.3], $P^2 = S^1 \cup B((0,0),1)$ where $S^1$ is the circle and $B((0,0),1)$ the open unit ball in $\mathbb{R}^2$. 