Solutions to the June 2004 exam

Problem 1

(1) $\Longrightarrow$ (2): Since f is continuous, $f(A) \subset f(\overline{A}) \subset \overline{f(A)}$. Since f is closed, $f(\overline{A})$ is closed and therefore these two inclusions imply $\overline{f(A)} = f(\overline{A})$.
(2) $\Longrightarrow$ (1): Since $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$, f is continuous. If A is closed, then $A = \overline{A}$ so that $f(A) = f(\overline{A}) = \overline{f(A)}$ is also closed. This means that f is closed.

Problem 2

Any continuous map of a compact space to a Hausdorff space is closed. Open subsets of locally path-connected spaces have open path-components.

Problem 3 [2, Ex 38.9]

(1) Suppose that $x_n \in X$ converges to $y \in \beta X - X$. We recursively define a subsequence by x_{n_k} by

$$n_k = \begin{cases} 1 & k = 1 \\ \min\{n > n_{k-1} \mid x_n \notin \{x_{n_1}, \dots, x_{n_{k-1}}\}\} & k > 1 \end{cases}$$

This definition makes sense since the set we are taking the minimal element of a nonempty set. Since x_n converges to y, the subsequence x_{n_k} also converges to y. Clearly, no two points of the subsequence x_{n_k} are identical. We call this subsequence x_n again.

(2) Note that y is in the closure of A as any neighborhood of y contains a point from A. Since therefore $A \subset A \cup \{y\} \subset \overline{A}$, it suffices to show that $A \cup \{y\}$ is closed, ie that the complement of $A \cup \{y\}$ is open: Let z be a point in the complement. Since z is not the limit of the sequence (x_{2n+1}) (there is just one limit point, namely y, in the Hausdorff space βX) there exists a neighborhood, U, of z containing only finitely many points from the sequence. Then $U - (U \cap (A \cup \{y\}))$ is an open neighborhood of z disjoint from $A \cup \{y\}$.

This shows that $\overline{A} = A \cup \{y\}$. Similarly, $\overline{B} = B \cup \{y\}$. Therefore the intersection $\overline{A} \cap \overline{B} = \{y\} \neq \emptyset$.

(3) A and B are disjoint since no two points of the sequence x_n are identical. The set A is closed in X since $\mathrm{Cl}_X A = X \cap \overline{A} = X \cap (A \cup \{y\}) = A$. Similarly for B, of course. By Urysohn's characterization of normal spaces, there exists a continuous function $f\colon X \to [0,1]$ such that $A \subset f^{-1}(0)$ and $B \subset f^{-1}(1)$.

(4) The universal property of the Stone–Čech compactification [1, §27] says that there exists a unique continuous map $\overline{f}$ into the compact Hausdorff space $[0,1]$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & {[0,1]} \\ & \beta X \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

commutes. Since $\overline{A} \subset \overline{f}^{-1}(0)$ and $\overline{B} \subset \overline{f}^{-1}(1)$, $\overline{A}$ and $\overline{B}$ are disjoint.

(5) $\emptyset \overset{(2)}{\neq} \overline{A} \cap \overline{B} \overset{(4)}{=} \emptyset$ $\lightning$

(6) X is a proper subspace of βX since βX is compact which X is not.

(7) First countable spaces satisfy the sequence lemma. If $\beta X - X = \overline{X} - X$ is nonempty then βX does not satisfy the sequence lemma as we have just seen. Thus βX is not first countable, in particular not metrizable.

Here are two applications:

- $\beta\mathbf{Z}_+$ is compact but not sequentially compact: The sequence $\mathbf{Z}_+$ in $\beta\mathbf{Z}_+$ has no convergent subsequence (neither in $\mathbf{Z}_+$ nor in $\beta\mathbf{Z}_+ - \mathbf{Z}_+$).
- $\beta\mathbf{R}$ is connected but not path-connected: $\beta\mathbf{R}$ is connected because it has a dense connected subspace, $\mathbf{R}$. It is not path-connected since there can be no path between a point in $\mathbf{R}$ and a point outside $\mathbf{R}$ as any such path would also give a a sequence of points in $\mathbf{R}$ converging to a point outside $\mathbf{R}$ (the first point of the path that is not in $\mathbf{R}$). To see this note that as $\mathbf{R}$ is locally compact, $\mathbf{R}$ is open in the compact Hausdorff space $\beta\mathbf{R}$ [1, 30.8]. The remainder $\beta\mathbf{R} - \mathbf{R}$ is therefore closed. Let now $p\colon [0,1] \to \beta\mathbf{R}$ be a path from $p(0) \in \mathbf{R}$ to $p(1) \in \beta\mathbf{R} - \mathbf{R}$. Then $p^{-1}(\beta\mathbf{R} - \mathbf{R})$ is a compact subset of $[0,1]$ so it contains its greatest lower bound $t_0 = \inf p^{-1}(\beta\mathbf{R} - \mathbf{R})$. This lower bound is positive for 0 belongs to $p^{-1}(\mathbf{R})$. Hence $t_n = t_0 - \frac{1}{n}$ is a sequence in $p^{-1}(\mathbf{R})$ (for $n \gg 0$) that converges to $t_0 \notin p^{-1}(\mathbf{R})$. The image sequence $p(t_n)$ is then a sequence in $\mathbf{R}$ that converges to a point $p(t_0)$ outside $\mathbf{R}$. But that is impossible!

References

[1] Jesper M. Møller, *General topology*, `http://www.math.ku.dk/ moller/e03/3gt/notes/gtnotes.dvi`.
[2] James R. Munkres, *Topology. Second edition*, Prentice-Hall Inc., Englewood Cliffs, N.J., 2000. MR 57 #4063