Problem 1

Let $\mathbb{R}$ be the real line (with standard topology) and let $I = [0, 1]$ be the unit interval (considered as a set).

For each natural number $k \in \mathbb{Z}^+$, let $D_k$ be the set of all finite sequences

$$(I_1, \ldots, I_k, x_1, \ldots, x_k)$$

where $I_1, \ldots, I_k \subset I$ are disjoint closed subintervals of $I$ with rational endpoints and $x_1, \ldots, x_k \in \mathbb{Q}$ are rational numbers.

(1) Show that the set $D = \bigcup_{k=1}^{\infty} D_k$ is countable.

For each element $(I_1, \ldots, I_k, x_1, \ldots, x_k) \in D_k$, let $x(\cdot) \in \mathbb{R}$ be the element given by

$$\pi_t x(I_1, \ldots, I_k, x_1, \ldots, x_k) = \begin{cases} x_j & t \in I_j \text{ for some } j \in \{1, \ldots, k\} \\ 0 & t \notin I_1 \cup \cdots \cup I_k \end{cases}$$

where $\pi_t : \mathbb{R} \to \mathbb{R}, t \in I$, is the projection map.

(2) Show that $x(D)$ is dense in $\mathbb{R}$ (with the product topology).

(3) Show that $\mathbb{R}^J$ (with the product topology) does not contain any dense countable subsets when the cardinality of $J$ is strictly bigger than that of the power set $\mathcal{P}(\mathbb{Z}^+)$. 

Hint for (3): Let $D \subset \mathbb{R}^J$ be a dense subset. Show that the map $J \to \mathcal{P}(D) : j \mapsto D \cap \pi_j^{-1}((2003, 2004))$ is injective. You may use without proof that $\overline{D \cap U} = \overline{U}$ for any open set $U \subset \mathbb{R}^J$.

Problem 2

Suppose that $X$ is a locally compact Hausdorff space and $A$ a nonempty closed subset of $X$. Let $\omega(X - A) = (X - A) \cup \{\omega\}$ denote the one-point (Alexandroff) compactification of $X - A$ and $X/A$ the quotient space of $X$ obtained by identifying all points of $A$ to one point and making no further identifications. Are these two spaces homeomorphic?

(1) Explain why $X - A$ is locally compact Hausdorff.

(2) Let $f : X \to \omega(X - A)$ be the surjective map that is the identity on $X - A$ and sends $A$ to $\omega$. Show that $f$ is continuous.

(3) Show that $f$ induces a continuous bijective map $\overline{f} : X/A \to \omega(X - A)$.

(4) Show that $\overline{f}$ is a homeomorphism if and only if $X/A$ is compact.

Let now $X = \mathbb{R}$ and $A = \bigcup_{n \in \mathbb{Z}} [2n, 2n + 1]$.

(5) Is $X/A$ compact? Is $\overline{f} : X/A \to \omega(X - A)$ a homeomorphism?

(6) Show that $X/A$ is not first countable at the point corresponding to $A$.

(7) Describe a subspace of $\mathbb{R}^2$ that is homeomorphic to $\omega(X - A)$. (A proof is not required.) Can you find a subspace of $\mathbb{R}^2$ that is homeomorphic to $X/A$?

(The End)