Problem 1 (20 points)
Let $S_\Omega$ denote the smallest uncountable well-ordered set.

(1) Show that $S_\Omega$ contains a subset with the order type of the positive integers $\mathbb{Z}_+$. [Hint: $S_\Omega$ does not have a largest element.]

(2) Find an element of $S_\Omega$ that does not have an immediate predecessor. [Hint: Use $\mathbb{Z}_+ \subset S_\Omega$.]

Problem 2 (10 points)
If $Z$ is a topological space and $C \subset Z$ a subset, we define the boundary of $C$ by the equation

$$\partial C = \overline{C} \cap (\overline{Z} - C)$$

Let $X$ and $Y$ be topological spaces, $A$ a subset of $X$ and $B$ a subset of $Y$. Then $A \times B$ is a subset of $X \times Y$. Show that

$$\partial (A \times B) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$$

[Hint: $(X \times Y) - (A \times B) = (X - A) \times Y \cup X \times (Y - B)$]

Problem 3 (20 points)
Let $f : S^1 \to \mathbb{R}$ be a continuous map of the circle to the real line.

(1) Show that there is a point $x$ on the circle so that $f(x) = f(-x)$. [Hint: The odd map $g(x) = f(x) - f(-x)$ must take the value 0 at some point.]

(2) Is it possible to imbed the circle $S^1$ in the real line $\mathbb{R}$?

Problem 4 (40 points)
Let $X = \mathbb{Z}_+$ be the set of positive integers with the discrete topology and $\beta(X)$ its Stone–Čech compactification. We consider $X$ as a subset of $\beta(X)$. Let $A$ be any subset of $X \subset \beta(X)$. Let $U$ be an open subset of $\beta(X)$.

(1) Show that there is a continuous function $F : \beta(X) \to \{0, 1\}$ defined on the compactification such that $F(A) = 0$ and $F(X - A) = 1$. Deduce that $\overline{A}$ and $\overline{X - A}$ are disjoint where closures are taken in $\beta(X)$.

(2) Show that $\beta(X) - \overline{A} = \overline{X - A}$ and that $\overline{A}$ is open and closed in $\beta(X)$. [Hint: You may use without proof the general fact that $\overline{C - D} \subset \overline{C} - \overline{D}$.]

(3) Show that $\overline{U} = \overline{U} \cap X$ and that $\overline{U}$ is open and closed in $\beta(X)$.

(4) Show that the connected components of $\beta(X)$ are one-point sets.

(The End)