Hatcher §2.1

Ex 2.1.2

Let $S = [012] \cup [123] \subset \Delta^3 = [0123]$ be the union of two faces of the 3-simplex Δ^3. Let $\sim$ be the equivalence relation that identifies $[01] \sim [13]$ and $[02] \sim [23]$. The quotient space $S/\sim$ is the Klein bottle KB; remember [2, p 51] that $KB = (S^1 \vee S^1)_{a^2b^2} D^2$. Let $R\colon I \times \Delta^3 \to S$ be a deformation retraction. Since R_t is the identity on S, there is an induced deformation retraction $\overline{R}\colon I \times \Delta^3/\sim\, \to S/\sim$ of $\Delta^3/\sim$ onto $S/\sim$. The induced map $\overline{R}$ is continuous because the unit interval I is locally compact Hausdorff so that $I \times \Delta^3 \to I \times \Delta^3/\sim$ is a quotient map [3, §29, Ex 11].

Ex 2.1.4

The simplicial chain complex of the triangular parachute

$$0 \longleftarrow \mathbf{Z} \overset{0}{\longleftarrow} \mathbf{Z}\{[12],[02],[01]\} \overset{\begin{pmatrix} 1 & -1 & 1 \end{pmatrix}}{\longleftarrow} \mathbf{Z}\{[012]\} \longleftarrow 0$$

has homology $H_0 = \mathbf{Z}$, $H_1 = \mathbf{Z} \oplus \mathbf{Z}$ and $H_i = 0$ for $i \geq 2$.

Ex 2.1.5

Since $\partial_2 U = b - c + a$ and $\partial_2 L = a - b - c$ we have the simplicial chain complex

$$0 \longleftarrow \mathbf{Z}\{v\} \overset{0}{\longleftarrow} \mathbf{Z}\{a,b,c\} \overset{\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}}{\longleftarrow} \mathbf{Z}\{U,L\} \longleftarrow 0$$

Since the matrix ∂_2 is equivalent (over $\mathbf{Z}$) to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}$$

we conclude that $H_0(K) = \mathbf{Z}$, $H_1(K) = \mathbf{Z} \oplus \mathbf{Z}/2$, $H_2(K) = 0$.

Ex 2.1.7

The boundary of the 3-simplex $\Delta^3 = [0123]$ is the simplicial chain $[123] - [023] + [013] - [012]$ in $\Delta_2(\Delta^3)$. If we want a Δ-complex structure on S^3 it is tempting to identify the the 2-simplices $[123] = [023]$ and $[013] = [012]$ of Δ^3. This gives a Δ-complex S with one 3-simplex, two 2-simplexes,

$$A\colon [012] = [013], \quad B\colon [123] = [023],$$

three 1-simplexes,

$$a\colon [01], \quad b\colon [13] = [03] = [12] = [02], \quad c\colon [23]$$

and two 0-simplexes, $0 = 1$ and $2 = 3$. From the simplicial chain complex $\Delta_*(S)$

$$0 \longleftarrow \mathbf{Z}\{0,1\} \overset{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}}{\longleftarrow} \mathbf{Z}\{a,b,c\} \overset{\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}}{\longleftarrow} \mathbf{Z}\{A,B\} \overset{0}{\longleftarrow} \mathbf{Z} \longleftarrow 0$$

it follows that $H_i(S) = H_i(S^3)$. Since the 2-skeleton of S is contractible, S is homotopy equivalent to S^3.

Ex 2.1.8

The Δ-complex X has

- n three-simplices $T_1, \ldots, T_n$
- $2n$ two-simplices $R_1, \ldots, R_n$ (roof) and $W_1, \ldots, W_n$ (walls)
- $n + 2$ one-simplices $s_1, \ldots, s_n$ (slanted), h (horizontal), and v (vertical)
- 2 zero-simplices x and y

The boundary maps are

$$\partial_3 T_i = W_i - W_{i-1} + R_i - R_{i+1}$$
$$\partial_2 R_i = s_i - s_{i-1} + h, \quad \partial_2 W_i = v - s_i + s_{i+1}$$
$$\partial_1 s_i = y - x, \quad \partial_1 h = 0, \quad \partial_1 v = 0$$

It follows that the one-cycles are

$$\ker \partial_1 = \mathbf{Z}\{s_1 - s_2, \ldots, s_{n-1} - s_n, h, v\}$$

and from the expressions

$$\partial_2 R_1 = h + (s_1 - s_2) + \cdots + (s_{n-1} - s_n), \quad \partial_2 R_2 = h - (s_1 - s_2), \ldots, \partial_2 R_n = h - (s_{n-1} - s_n)$$
$$\partial_2 W_1 = v - (s_1 - s_2), \ldots, \partial_2 W_{n-1} = v - (s_{n-1} - s_n), \quad \partial_2 W_n = v + (s_1 - s_2) + \cdots + (s_{n-1} - s_n)$$

we see that all $n+1$ generators of $\ker \partial_1$ are homologous and that $H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2 = \mathbf{Z}/n$, generated by the homology class represented by for instance the one-cycle h.

It is not difficult to see that $\ker \partial_2 = \operatorname{im} \partial_3$ and $\ker \partial_3 = \mathbf{Z}\{T_1 + \cdots + T_n\}$. This means that $H_2(X) = 0$ and $H_3(X) = \mathbf{Z}$, generated by the homology class of the three-cycle $T_1 + \cdots + T_n$.

Ex 2.1.14

Proposition 0.1. *For an abelian group A the following are equivalent:*
   (1) A is the middle term of a short exact sequence $0 \to \mathbf{Z}/p^m \to A \to \mathbf{Z}/p^n \to 0$
   (2) $A \cong \mathbf{Z}/p^k \times \mathbf{Z}/p^{m+n-k}$ where $0 \leq k \leq \min\{m, n\}$

Proof. $(1) \Rightarrow (2)$: Let $0 \to \mathbf{Z}/p^m \to A \to \mathbf{Z}/p^n \to 0$ be a short exact sequence of abelian groups. Form the pull-back

along the mod p^n reduction homomorphism $\mathbf{Z} \to \mathbf{Z}/p^n$. The Snake Lemma (the associated long exact sequence in homology) shows that the middle vertical arrow is surjective with kernel isomorphic to $\mathbf{Z}$. In fact, it shows that there is an enlarged commutative diagram

with exact rows and columns. The middle column is a short exact sequence of the form

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\binom{r}{p^n}} \mathbf{Z}/p^m \times \mathbf{Z} \longrightarrow A \longrightarrow 0$$

for some natural number r. From this presentation of A we see that there is a short exact sequence

$$0 \longrightarrow \mathbf{Z} \times \mathbf{Z} \xrightarrow{\begin{pmatrix} p^m & r \\ 0 & p^n \end{pmatrix}} \mathbf{Z} \times \mathbf{Z} \longrightarrow A \longrightarrow 0$$

The integer matrix in this short exact sequence is equivalent [1, Chp 7] [4, Chp 10] to the integer matrix

$$\begin{pmatrix} p^k & 0 \\ 0 & p^{m+n-k} \end{pmatrix}$$

where $(p^k) = (p^m, p^n, r)$ as ideals. We conclude that

$$A \cong \mathbf{Z}/p^k \times \mathbf{Z}/p^{m+n-k}, \quad 0 \leq k \leq \min\{m,n\}.$$

$(2) \Rightarrow (1)$: Let m and n be natural numbers and k a nonnegative integer such that $0 \leq k \leq \min\{m,n\}$. Since $0 \leq k \leq min\{m,n\} \leq m$, there is an epimorphism $\mathbf{Z}/p^m \xrightarrow{\alpha} \mathbf{Z}/p^k$; since

$$m+n-k \geq m+n-\min\{m,n\} = (\max\{m,n\} + \min\{m,n\}) - \min\{m,n\} = \max\{m,n\} \geq m$$

there is a monomorphism $\mathbf{Z}/p^m \xrightarrow{\beta} \mathbf{Z}/p^{m+n-k}$. The homomorphism $\mathbf{Z}/p^m \xrightarrow{(\alpha,\beta)} \mathbf{Z}/p^k \times \mathbf{Z}/p^{m+n-k}$ is injective and the cokernel is cyclic (think of a presentation for the cokernel) of order p^n. Thus we have a short exact sequence

$$0 \longrightarrow \mathbf{Z}/p^m \xrightarrow{(\alpha,\beta)} \mathbf{Z}/p^k \times \mathbf{Z}/p^{m+n-k} \longrightarrow \mathbf{Z}/p^n \longrightarrow 0$$

of abelian groups. □

For example, the abelian groups A that can occur in the short exact sequence

$$0 \longrightarrow \mathbf{Z}/4 \longrightarrow A \longrightarrow \mathbf{Z}/4 \longrightarrow 0$$

are $\mathbf{Z}/16$, $\mathbf{Z}/2 \times \mathbf{Z}/8$, and $\mathbf{Z}/4 \times \mathbf{Z}/4$. The two first groups from this list are also middle terms of short exact sequences of the form

$$0 \longrightarrow \mathbf{Z}/2 \longrightarrow A \longrightarrow \mathbf{Z}/8 \longrightarrow 0$$

but the third group is not.

Proposition 0.2. *For an abelian group A the following are equivalent:*
(1) *A is the middle term of a short exact sequence $0 \to \mathbf{Z} \to A \to \mathbf{Z}/n \to 0$*
(2) *$A \cong \mathbf{Z}/d \times \mathbf{Z}$ where $d|n$.*

Proof. $(1) \Rightarrow (2)$: Assume that A is the middle term of the short exact sequence in item (1). As above, we get a short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\binom{r}{n}} \mathbf{Z} \times \mathbf{Z} \longrightarrow A \longrightarrow 0$$

for some integer r. Since the matrix $\binom{r}{n}$ is equivalent to the matrix $\binom{d}{0}$ where $d = (r,n)$ is the greatest common divisor, A is isomorphic to $\mathbf{Z}/d \times \mathbf{Z}$.
$(2) \Rightarrow (1)$: If d divides n,

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\binom{1}{n/d}} \mathbf{Z}/d \times \mathbf{Z} \longrightarrow \mathbf{Z}/n \longrightarrow 0$$

is a short exact sequence of abelian groups. The reason for this is that the matrices

$$\begin{pmatrix} d & 1 \\ 0 & n/d \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$$

are equivalent so that

$$\frac{\mathbf{Z}/d \times \mathbf{Z}}{(1,n/d)\mathbf{Z}} = \frac{\mathbf{Z} \times \mathbf{Z}}{(d,0)\mathbf{Z} + (1,n/d)\mathbf{Z}} = \frac{\mathbf{Z} \times \mathbf{Z}}{(1,0)\mathbf{Z} + (0,n)\mathbf{Z}} = \mathbf{Z}/n$$

□

Ex 2.1.16

Consider first the special case where X is path-connected. For a nonempty path-connected space X with a subspace $A \subset X$, we have

$$H_0(A) \to H_0(X) \text{ is surjective} \Leftrightarrow A \text{ is nonempty}$$
$$H_0(A) \to H_0(X) \text{ is injective} \Leftrightarrow A \text{ is path-connected}$$
$$H_0(A) \to H_0(X) \text{ is bijective} \Leftrightarrow A \text{ is nonempty and path-connected}$$

Indeed, if A is nonempty, the commutative diagram

$$
\begin{array}{ccc}
H_0(\{x\}) & === & H_0(\{x\}) \\
\downarrow & & \downarrow {\scriptstyle \cong} \\
H_0(A) & \longrightarrow & H_0(X)
\end{array}
$$

where x is a point in A, shows that $H_0(X) \to H_0(A)$ is surjective. If A is nonempty and path-connected the left vertical arrow is an isomorphism as well, so that $H_0(A) \to H_0(X)$ is an isomorphism. If $H_0(A) \to H_0(X) \cong \mathbf{Z}$ is injective, then $H_0(A) = 0$, i.e. $A = \emptyset$, or $H_0(A) = \mathbf{Z}$, i.e. A is path-connected.

Proposition 0.3. *Let X be a topological space and $A \subset X$ a subspace. Then $H_0(X, A) = 0$ iff each path-component of X contains at least one path-component of A.*

Proof. Let X_j, $j \in J$, be the path components of X. Since each path-component of A is contained in a path-component X_j of X, $X_j \cap A$ is a union of path-components of A; it is the union of the path-components of A contained in X_j. (Make a mental picture of A in X.) The inclusion maps $X_j \to X$ provide a commutative diagram

(1)
$$
\begin{array}{ccc}
\bigoplus H_0(X_j \cap A) & \longrightarrow & \bigoplus H_0(X_j) \\
{\scriptstyle \cong} \downarrow & & \downarrow {\scriptstyle \cong} \\
H_0(A) & \longrightarrow & H_0(X)
\end{array}
$$

form which we see, by taking cokernels, that

$$H_0(X, A) = \operatorname{coker}\left(H_0(A) \to H_0(X)\right) \cong \bigoplus_{j \in J} \operatorname{coker}\left(H_0(X_j \cap A) \to H_0(X_j)\right) \cong \bigoplus_{j \in J} H_0(X_j, X_j \cap A)$$

We conclude that $H_0(X, A)$ is the free abelian group on the set $\{j \in J \mid X_j \cap A = \emptyset\}$ of path-components of X that do not meet A. $\qquad\square$

Proposition 0.4. *Let X be a topological space and $A \subset X$ a subspace. Then $H_1(X, A) = 0$ iff*
  (1) *Each path-component of X contains at most one path-component of A*
  (2) *$H_1(A) \to H_1(X)$ is surjective*

Proof. The long exact sequence

$$\cdots \to H_1(A) \to H_1(X) \to H_1(X, A) \to H_0(A) \to H_0(X) \to \cdots$$

shows that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and $H_0(A) \to H_0(X)$ is injective. From diagram (1) we see that

$$\ker\left(H_0(A) \to H_0(X)\right) = \bigoplus_{j \in J} \ker\left(H_0(X_j \cap A) \to H_0(X_j)\right)$$

and therefore $H_0(A) \to H_0(X)$ is injective iff $X_j \cap A$ is path-connected for all j, i.e. iff each path-component of X contains at most one path-component of A. $\qquad\square$

Ex 2.1.20

Let CX be the cone on the space X. The suspension $SX = CX \cup CX$ is the union of two cones. Excision implies (see p. 123) that

$$\tilde{H}_{n+1}(SX) = \tilde{H}_{n+1}(CX \cup CX) \cong H_{n+1}(CX, X) \cong \tilde{H}_n(X)$$

where the last isomorphism comes from the long exact sequence for (CX, X) because CX is contractible. More generally,

$$\tilde{H}_{n+1}(\underbrace{CX \cup \cdots \cup CX}_{k}) = \tilde{H}_{n+1}(\underbrace{CX \cup \cdots \cup CX}_{k-1} \cup CX) \cong H_{n+1}(\underbrace{CX \cup \cdots \cup CX}_{k-1}, X)$$

$$\cong \tilde{H}_{n+1}(\underbrace{SX \vee \cdots \vee SX}_{k-1}) \cong \underbrace{\tilde{H}_{n+1}(SX) \oplus \cdots \oplus \tilde{H}_{n+1}(SX)}_{k-1} \cong \underbrace{\tilde{H}_n(X) \oplus \cdots \oplus \tilde{H}_n(X)}_{k-1}$$

where we use that $(CX \cup \cdots \cup CX, X)$ is a good pair with quotient space $SX \vee \cdots \vee SX$.

Ex 2.1.29

We already know (Example 2.3, Corollary 2.25) that $S^1 \times S^1$ and $S^2 \vee S^1 \vee S^1$ have isomorphic homology groups. The universal covering space $\mathbf{R} \times \mathbf{R}$ (Ex 1.3.2) of $S^1 \times S^1$ is contractible; in particular, $H_2(\mathbf{R} \times \mathbf{R}) = 0$. The universal covering space E of $S^2 \vee S^1 \vee S^1$ must have $H_2(E) \neq 0$. To see this, note that the inclusion $i \colon S^2 \hookrightarrow S^2 \vee S^1 \vee S^1$ of the first wedge summand factors through the universal covering space. This gives the commutative diagram

$$
\begin{array}{ccc}
 & & H_2(E) \\
 & \nearrow & \downarrow \\
H_2(S^2) & \xrightarrow{\ H_2(i)\ } & H_2(S^2 \vee S^1 \vee S^1)
\end{array}
$$

where $H_2(S^2) \cong \mathbf{Z}$ and we know from Ex 2.1.11 that $H_2(i)$ is injective. (The universal covering space of $S^1 \vee S^1$ is a tree, and the universal covering space of $S^2 \vee S^1 \vee S^1$ is a tree decorated with balloons. This CW-structure may be used to calculate the homology of E explicitly.)

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