HOMOTOPY LIE GROUPS: A SURVEY

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Abstract. With homotopy Lie groups we see the incarnation within homotopy theory of the same concept that shows up in set theory as finite groups, in algebraic geometry as group schemes, and in differential geometry as Lie groups.

1. Introduction

The aim of these notes from a minicourse on Homotopy Lie Groups held at the University of Lille, May 31 – June 02, 2000, is to advertise the discovery by W.G. Dwyer and C.W. Wilkerson of a remarkable class of spaces called homotopy Lie groups. These purely homotopy theoretic objects capture the essence of the idea of a Lie group.

I shall here focus on [16] where Dwyer and Wilkerson introduce and prove the first general structural facts about homotopy Lie groups and in the final sections I discuss general structure theorems for p-compact groups and the classification theorem.

1.1. **Finite loop spaces.** Suppose G is a compact Lie group. Take a free, contractible G-space EG and define $BG = EG/G$ to be the orbit space. Then the associated fibre sequence

$$\Omega EG \to \Omega BG \to G \to EG \to BG$$

contains a homotopy equivalence $\Omega BG \to G$.

This phenomenon is embedded in the general concept of a *finite loop space*.

Definition 1.2. *A finite loop space is a connected, pointed space BX such that $X = \Omega BX$ is homotopy equivalent to a finite CW-complex.*

Note that X — *by definition* — is the loop space of BX. It is customary, though ambiguous, to refer to the finite loop space BX by its underlying space X and then call BX the *classifying space* of X.

We have already seen that compact Lie groups are finite loop spaces. The classifying space of SU(2), for instance, is the infinite quaternionic projective space $B\,\mathrm{SU}(2) = HP^\infty$. However, the class of finite loop spaces is much larger. A striking example was provided almost 25 years ago by Rector [45] who found an uncountable family of homotopically distinct finite loop spaces BX with X homotopy equivalent to SU(2). In other words, the homotopy type SU(2) supports uncountable many distinct loop space structures.

Rector's example destroyed all hopes of a classification theorem in the spirit of compact Lie groups — as long as one sticks to integral spaces, that is. The situation looks brighter in the category of $\mathbb{F}_p$-local spaces.

1.3. **Notation.** In the following, p denotes a fixed prime number, $\mathbb{F}_p$ the field with p elements, $\mathbb{Z}_p$ the ring of p-adic integers, and $\mathbb{Q}_p = \mathbb{Z}_p \otimes \mathbb{Q}$ the field of p-adic numbers.

$H^*(-)$ denotes singular cohomology with $\mathbb{F}_p$-coefficients, $H^*(-;\mathbb{F}_p)$, while $H^*(-;\mathbb{Q}_p)$ denotes $H^*(-;\mathbb{Z}_p) \otimes \mathbb{Q}$ (and *not* singular cohomology with $\mathbb{Q}_p$-coefficients).

A space U is $\mathbb{F}_p$*-finite* if $H^*(U)$ is finite dimensional over $\mathbb{F}_p$. A map $A \to B$ is an $\mathbb{F}_p$*-equivalence* if it induces an isomorphism $H^*(B) \to H^*(A)$ on $H^*(-)$.

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1.4. **$\mathbb{F}_p$-local spaces.** A space U is *$\mathbb{F}_p$-local* if any $\mathbb{F}_p$-equivalence $A \to B$ induces a homotopy equivalence $\mathrm{map}(B, U) \to \mathrm{map}(A, U)$ of mapping spaces.

$\mathbb{F}_p$-local spaces exist — the classifying space $K(\mathbb{F}_p, i)$ for the functor $H^i(-)$ is an obvious example. In fact, any space can be made $\mathbb{F}_p$-local in a minimal and functorial way.

Theorem 1.5. *(Bousfield* [7, 3.2]*) The exists a functor $U \rightsquigarrow U_{\mathbb{F}_p}$, of the (homotopy) category of CW-complexes into itself, with a natural transformation $\eta_U \colon U \to U_{\mathbb{F}_p}$ such that η_U is an $\mathbb{F}_p$-equivalence and $U_{\mathbb{F}_p}$ is $\mathbb{F}_p$-local.*

Note these categorical consequences of the definition and the theorem:

- Any $\mathbb{F}_p$-equivalence between $\mathbb{F}_p$-local spaces is a homotopy equivalence.
- A map is an $\mathbb{F}_p$-equivalence if and only if its $\mathbb{F}_p$-localization is a homotopy equivalence.
- U is $\mathbb{F}_p$-local if and only if $\eta_U \colon U \to U_{\mathbb{F}_p}$ is a homotopy equivalence.

If the pointed space U is nilpotent or is connected and either $H_1(U; \mathbb{F}_p) = 0$ or $\pi_1(U)$ is finite, then [8, VI.5.3, VII.3.2, VII.5.1] [16, §11] the Bousfield localization, $U_{\mathbb{F}_p}$, coincides with the (perhaps more familiar) Bousfield–Kan localization, $(\mathbb{F}_p)_\infty U$. In particular [8, VI.5.2], $\pi_i(U_{\mathbb{F}_p}) \cong \pi_i(U) \otimes \mathbb{Z}_p$ if U is connected, pointed and nilpotent with finitely generated abelian homotopy groups. (This is far from true without the nilpotency hypothesis (2.2).)

1.6. **Homotopy Lie groups.** By design, actually inescapably [23], the uncountably many finite loop spaces BX of Rector's example all $\mathbb{F}_p$-localize to the standard $(B\,\mathrm{SU}(2))_{\mathbb{F}_p}$. This observation indicates that "$\mathbb{F}_p$-local finite loop spaces" are better behaved than integral loop spaces. The problem, however, is that this term is meaningless as finite complexes are unlikely to be $\mathbb{F}_p$-local. The solution proposed in [16] is to replace the topological finiteness criterion in (1.2) by a cohomological one.

Definition 1.7. [16, 2.2] *A p-compact group is an $\mathbb{F}_p$-local space BX such that $X = \Omega BX$ is $\mathbb{F}_p$-finite.*

Again, it is customary to use X, by definition the loop space of BX, when referring to the p-compact group BX, the *classifying space* of X.

The main purpose of [16] is to associate to any p-compact group X a *maximal torus* (7.1) with an action of a *Weyl group* (7.4). For a connected X, the Weyl group is a p-adic reflection group whose invariant ring determines the p-adic rational cohomology of BX (7.8). (This justifies the term "homotopy Lie group" which has been proposed as an alternative to "p-compact group" as used in [16].)

To verify the existence of a maximal torus, Dwyer and Wilkerson first show that any nontrivial p-compact group possesses a nontrivial element (3.8). Next, they use an inductive procedure to build the maximal torus (§7). A prominent feature of their construction is the systematic use of algebraic Smith theory (§6) for homotopy fixed point spaces (§4).

2. Examples of p-compact groups

Let G be any compact Lie group G whose component group $\pi_0(G)$ is a p-group. Define $B\hat{G} = (BG)_{\mathbb{F}_p}$. Then $\hat{G}$ is a p-compact group with $H^*(B\hat{G}) = H^*(BG)$, $\pi_0(\hat{G}) = \pi_0(G)$ and $\pi_i(\hat{G}) = \pi_i(G) \otimes \mathbb{Z}_p$ for all $i \geq 1$ [16, §11].

This example includes all finite p-groups such as the trivial group $\{1\}$ and the cyclic p-groups $\mathbb{Z}/p^n, n \geq 0$.

2.1. **Toral groups.** When applied to an r-torus $S = \mathrm{SO}(2)^r$, the above construction produces a *p-compact r-torus* $T = \hat{S}$. The classifying space $BT = K(\mathbb{Z}_p, 2)^r$ is an Eilenberg-MacLane space with homotopy in dimension two and $H^*(BT) = \mathbb{F}_p[t_1, \ldots, t_r]$ is polynomial on r generators of degree two.

Alternatively, $BT = (B\check{T})_{\mathbb{F}_p}$, where $\check{T} = (\mathbb{Z}/p^\infty)^r$ is a *p-discrete r-torus*.

More generally, a *p-compact toral group* P is a p-compact group with $BP = (B\check{P})_{\mathbb{F}_p}$, where $\check{P}$, a *p-discrete toral group*, is an extension of a p-discrete torus $\check{T}$ by a finite p-group π. Note that the $\mathbb{F}_p$-localized sequence $BT \to BP \to B\pi$ is [8, II.5.1] a fibration sequence as π necessarily acts nilpotently on $H_i(B\check{T}; \mathbb{F}_p)$.

Since $\mathrm{map}(BP, BX) \simeq \mathrm{map}(B\check{P}, BX)$ for any p-compact group X and $\check{P}$ is the union of an ascending chain of finite p-groups, results for finite p-groups can often, by *discrete approximation* [16, §6], be extended to p-compact toral groups.

2.2. **Exotic examples.** Call a connected p-compact group *exotic* if it isn't of the form $\hat{G}$ for any connected compact Lie group G. Sullivan spheres and, more generally, (many) Clark–Ewing p-compact groups are exotic. Indeed, any connected $\mathbb{F}_p$-local space with polynomial mod p cohomology is a p-compact group.

Proposition 2.3. *(Sullivan) Assume that $n > 2$ is an integer dividing $p-1$. Then the $\mathbb{F}_p$-local sphere $(S^{2n-1})_{\mathbb{F}_p}$ is a p-compact group.*

The construction goes as follows: The cyclic group $\mathbb{Z}/n$ acts on $\check{T} = \mathbb{Z}/p^\infty$ as $\mathbb{Z}/n < \mathrm{Aut}(\check{T}) \cong \mathbb{Z}_p^*$ when $n|(p-1)$. Define $BX = (B\check{N})_{\mathbb{F}_p}$ where $\check{N} = \check{T} \rtimes \mathbb{Z}/n$ is the semi-direct product. The computation

$$H^*(BX) = H^*(B\check{N}) = H^*(B\check{T})^{\mathbb{Z}/n} = \mathbb{F}_p[t]^{\mathbb{Z}/n} = \mathbb{F}_p[t^n],$$

which uses (1.4) and the fact that n is prime to p, shows that the mod p cohomology of BX is polynomial on one generator of degree $2n$. Thus the $\mathbb{F}_p$-local space BX is $(2n-1)$-connected [8, I.6.1], and its loop space X is $(2n-2)$-connected with $H^*(X)$ abstractly isomorphic to $H^*(S^{2n-1})$. Now the Hurewicz theorem tells us that this abstract isomorphism is realizable by an $\mathbb{F}_p$-equivalence $S^{2n-1} \to X$, i.e. (1.4) by a homotopy equivalence $(S^{2n-1})_{\mathbb{F}_p} \to X_{\mathbb{F}_p} = X$.

Clark and Ewing [12] observed that applicability of the Sullivan construction isn't restricted to the rank one case. Let $\check{T}$ be a p-discrete r-torus and $W < \mathrm{Aut}(\check{T}) = \mathrm{GL}_r(\mathbb{Z}_p)$ a finite group of order prime to p acting on $\check{T}$. Define $BX = (B\check{N})_{\mathbb{F}_p}$ where $\check{N} = \check{T} \rtimes W$. The invariant ring

$$H^*(BX) = H^*(B\check{N}) = H^*(B\check{T})^W = \mathbb{F}_p[t_1, \dots, t_r]^W$$

is, essentially by the Shephard–Todd theorem [4, 7.2.1], a finitely generated polynomial algebra if and only if W is a reflection group in $\mathrm{GL}_r(\mathbb{Q}_p)$. If this is the case, $H^*(X)$ is an exterior algebra on finitely many odd degree generators. In particular, X is $\mathbb{F}_p$-finite and BX a p-compact group. These Clark–Ewing p-compact groups are fairly well understood [15].

The Clark–Ewing p-compact group associated to any non-Coxeter group from the list [12] of irreducible p-adic reflection groups is (7.9) exotic. This scheme produces many exotics if p is odd but none for $p = 2$ as the only non-Coxeter 2-adic reflection group, number 24 on the list, has even order.

To come up with an exotic 2-compact group, a much more sophisticated approach is required. In their landmark paper [26], Jackowski and McClure showed how to decompose BG, for any compact Lie group G, as a generalized pushout of classifying spaces of subgroups (proper subgroups if the center of G is trivial). Dwyer and Wilkerson realized that a similar decomposition applies in the case of p-compact groups [17, §8] and that this could be used in the construction of an exotic 2-compact group.

Theorem 2.4. [21] *There exists a connected 2-compact group* $\mathrm{DI}(4)$ *such that* $H^*(B\,\mathrm{DI}(4); \mathbb{F}_2)$ *is isomorphic (as an algebra over the Steenrod algebra) to the rank 4 mod 2 Dickson algebra* $H^*(B(\mathbb{F}_2)^4; \mathbb{F}_2)^{\mathrm{GL}_4(\mathbb{F}_2)}$ *and* $H^*(B\,\mathrm{DI}(4)); \mathbb{Q}_2)$ *to the invariant ring of the number 24 reflection group.*

Assuming that such a space exists, it is possible [20] to read off from its cohomology a finite diagram that looks like the cohomological image of a diagram of spaces. Some effort is required to verify that the picture seen in cohomology actually is realizable on the level of spaces. The generalized pushout of this diagram is the exotic 2-compact group.

2.5. **Cohomological invariants.** If the space U is $\mathbb{F}_p$-finite, also $H^*(U; \mathbb{Q}_p)$ is finite dimensional over $\mathbb{Q}_p$, and the *Euler characteristic*

$$\chi(U) = \sum (-1)^i \dim_{\mathbb{F}_p} H^i(U) = \sum (-1)^i \dim_{\mathbb{Q}_p} H^i(U; \mathbb{Q}_p)$$

and the *cohomological dimension*

$$\mathrm{cd}(U) = \max\{i \mid H^i(U) \neq 0\}$$

are defined [16, 4.3, 6.13].

For a connected p-compact group X, in particular, $H^*(X;\mathbb{Q}_p)$ is a connected finite dimensional Hopf algebra so, by Milnor–Moore [31], $H^*(X;\mathbb{Q}_p) = E(x_1,\dots,x_r)$ is an exterior algebra on finitely many generators of odd degree $|x_i| = 2d_i - 1, 1 \le i \le r$. The number, $r = \mathrm{rk}(X)$, of generators is the *rank* of X. The cohomological dimension of X is [18, 3.8] given by $\mathrm{cd}(X) = \max\{i \mid H^i(X;\mathbb{Q}_p) \neq 0\} = \sum_{i=1}^r (2d_i - 1)$. For instance, $\mathrm{cd}(\hat{G}) = \dim G$, $\mathrm{rk}(\hat{G})$ is the rank of G, a p-compact r-torus has rank r, $\mathrm{cd}(P) = \mathrm{rk}(P)$ if (and only if) P is a p-compact toral group, $\mathrm{rk}(S^{2n-1}) = 1$ and $\mathrm{cd}(S^{2n-1}) = 2n-1$, while $\mathrm{rk}(\mathrm{DI}(4)) = 3$ and $\mathrm{cd}(\mathrm{DI}(4)) = 45$.

Exercise 2.6. The trivial p-compact group has Euler characteristic $\chi(\{1\}) = 1$. The empty space has Euler characteristic $\chi(\emptyset) = 0$. For a connected p-compact group X, X is trivial $\Leftrightarrow \chi(X) \neq 0 \Leftrightarrow \mathrm{rk}(X) = 0$.

3. Morphisms

A *p-compact group morphism* $f\colon X \to Y$ is a based map $Bf\colon BX \to BY$ between the classifying spaces. The *trivial morphism* $0\colon X \to Y$ is the constant map $B0\colon BX \to BY$, and the *identity morphism* $1\colon X \to X$ is the identity map $B1\colon BX \to BX$.

Note the fibration sequence

$$X \xrightarrow{f} Y \to Y/f \to BX \xrightarrow{Bf} BY \tag{3.1}$$

where Y/f, or Y/X when f is understood, denotes the homotopy fibre of Bf.

Two morphisms $f, g\colon X \to Y$ are *conjugate* if the maps $Bf, Bg\colon BX \to BY$ are freely homotopic and $\mathrm{Rep}(X,Y) = \pi_0 \operatorname{map}(BX,BY) = [BX,BY]$ denotes the set of conjugacy classes of morphisms of X to Y.

3.2. **Monomorphisms, epimorphisms, and isomorphisms.** The morphism $f\colon X \to Y$ is a *monomorphism* if Y/X is $\mathbb{F}_p$-finite, an *epimorphism* if Y/X is the classifying space of some p-compact group, and an *isomorphism* if Y/X is contractible.

Example 3.3. $\{1\} \to X$ is a monomorphism with $X/\{1\} = X$, $X \to \{1\}$ is an epimorphism with $\{1\}/X = BX$, $1\colon X \to X$ is an isomorphism with $X/X = \{1\}$, and the diagonal $\Delta\colon X \to X^n$ is a monomorphism since X^n/X is homotopy equivalent to X^{n-1}.

These definitions are motivated by

Example 3.4. Let $f\colon G \to H$ be a monomorphism (an epimorphism) of compact Lie groups. The homotopy fibre of the induced map $Bf\colon BG \to BH$ is $H/f(G)$ $(B(\ker f))$ so the corresponding p-compact group morphism $\hat{f}\colon \hat{G} \to \hat{H}$ is a monomorphism (an epimorphism). (Not all morphisms between $\hat{G}$ and $\hat{H}$ are induced from homomorphisms between G and H.)

A diagram $X \to Y \to Z$ of p-compact group morphisms is a *short exact sequence* if $BX \to BY \to BZ$ is a fibration sequence. Any p-compact group sits in a short exact sequence of the form $X_0 \to X \to \pi_0(X)$ where X_0 is the *identity component* of X; the identity component of a p-compact toral group, for instance, is a p-compact torus (2.1).

Exercise 3.5. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms.

(1) If f and g are monomorphisms, then $g \circ f$ is a monomorphism.
(2) If X is a p-compact toral group and $g \circ f$ a monomorphism, then f is a monomorphism.
(3) Assume that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a short exact sequence. Show that f is a monomorphism and g an epimorphism. Show also that if X is a p-compact r-torus and Z a p-compact s-torus, then Y is a p-compact $(r+s)$-torus.

To be fair, part (2) of this exercise, requiring the theory of *kernels* [16, 7.1–7.3], is highly nontrivial. (The condition on X can be removed [16, 9.11].)

An inspection of the Serre spectral sequence for the left segment of (3.1) yields

Proposition 3.6. [16, 6.14] [17, 4.6] *If $f\colon X \to Y$ is a monomorphism, then* $\mathrm{cd}(Y) = \mathrm{cd}(X) + \mathrm{cd}(Y/X)$. *In particular,* $\mathrm{cd}(X) = \mathrm{cd}(Y)$ *if and only if f gives an isomorphism between X and some components of Y.*

3.7. **Nontrivial elements.** The existence of nontrivial elements in nontrivial p-compact groups represents the first and decisive step in constructing the maximal torus.

Theorem 3.8. [16, 5.4, 5.5, 7.2, 7.3] *Let X be a nontrivial p-compact group.*

(1) *There exists a monomorphism $\mathbb{Z}/p \to X$.*
(2) *If X is connected, there exists a monomorphism $S \to X$ from a p-compact 1-torus S to X.*

Note that (2) implies (1): In case the identity component X_0 is nontrivial, use (2) to get (3.5) a monomorphism $\mathbb{Z}/p \to S \to X_0 \to X$. Otherwise, (1) reduces to obstruction theory.

Analogously, any nontrivial, connected compact Lie group contains a copy of SO(2) (in its maximal torus).

Exercise 3.9. Use (3.8) and Lannes theory [28] to show that BX is $\mathbb{F}_p$-finite only if X is trivial. Next show that a p-compact group morphism which is both a monomorphism and an epimorphism, is an isomorphism.

The following sections contain the background material for the proof of Theorem 3.8 to be presented in §6.

4. Homotopy Fixed Point Spaces

Let π be a finite p-group and K a space. A *π-space* with underlying space K is a fibration $K_{h\pi} \to B\pi$ over $B\pi$ with fibre K. A *π-map* is a map $u_{h\pi} \colon K_{h\pi} \to L_{h\pi}$ over $B\pi$.

The *homotopy fixed orbit space* is the total space, $K_{h\pi}$, and the *homotopy fixed point space*, $K^{h\pi}$, is the space of sections (which may very well be empty). These spaces are connected by an evaluation map $B\pi \times K^{h\pi} \to K_{h\pi}$.

For brevity, a π-space will often be denoted by its underlying space and a π-map by its restriction to the underlying spaces.

Example 4.1. The *trivial* π-space with underlying space K is the trivial fibration $K \times B\pi \to B\pi$ with homotopy orbit space $K_{h\pi} = B\pi \times K$ and homotopy fixed point space $K^{h\pi} = \mathrm{map}(B\pi, K)$.

The homotopy fixed point construction $K^{h\pi}$ is functorial in both variables:

- For any π-map $u \colon K \to L$, composition with $u_{h\pi} \colon K_{h\pi} \to L_{h\pi}$ determines a map $u^{h\pi} \colon K^{h\pi} \to L^{h\pi}$.
- For any subgroup $\kappa < \pi$, any π-space, K, is also an κ-space. The inclusion $\iota \colon \kappa \to \pi$ induces a map $K_{h\iota} \colon K_{h\kappa} \to K_{h\pi}$ over $B\iota \colon B\pi \to B\kappa$ and a map $K^{h\iota} \colon K^{h\pi} \to K^{h\kappa}$ of homotopy fixed point spaces.

The homotopy orbit space and the homotopy fixed point space are homotopy invariant constructions in that any π-map $u \colon K \to L$ which is an ordinary (non-equivariant) homotopy equivalence induces homotopy equivalences $u_{h\pi} \colon K_{h\pi} \to L_{h\pi}$ and $u^{h\pi} \colon K^{h\pi} \to L^{h\pi}$.

4.2. **Exactness.** Let U denote the (ordinary, non-equivariant) homotopy fibre of a π-map $u \colon K \to L$ (where L is assumed to be connected), or, equivalently, the homotopy fibre of $u_{h\pi} \colon K_{h\pi} \to L_{h\pi}$. The pull back diagram

$$\begin{tikzcd} U_{h\pi} \arrow[r] \arrow[d] & K_{h\pi} \arrow[d, "u_{h\pi}"] \\ B\pi \arrow[r, "l"'] & L_{h\pi} \end{tikzcd}$$

shows that any homotopy fixed point $l \in L^{h\pi}$ makes U into a π-space such that

Proposition 4.3. [16, 10.6] *$U \to K \to L$ is a fibration sequence of π-maps between π-spaces and $U^{h\pi} \to K^{h\pi} \to L^{h\pi}$ is a fibration sequence of homotopy fixed point spaces (where $l \in L^{h\pi}$ serves as base point).*

4.4. **Exponential Laws.** Let κ be a subgroup of π. In the situations

$$\begin{tikzcd} & K_{h\kappa} \arrow[r] \arrow[d] & K_{h\pi} \arrow[d] \\ \pi/\kappa \arrow[r] & B\kappa \arrow[r, "B\iota"] & B\pi \end{tikzcd} \qquad \begin{tikzcd} & K_{h\pi} \arrow[d] & \\ B\kappa \arrow[r, "B\iota"] & B\pi \arrow[r] & B(\pi/\kappa) \end{tikzcd}$$

where the horizontal sequences are fibrations (and, to the right, κ is normal in π), the fibrewise exponential law reads

$$K^{h\kappa} = (K^{|\pi:\kappa|})^{h\pi} \qquad K^{h\pi} = (K^{h\kappa})^{h(\pi/\kappa)}.$$

On the left, $K^{|\pi:\kappa|} = \mathrm{map}(\pi/\kappa, K)$ is a π-space in the standard way such that the diagonal $\Delta\colon K \to K^{|\pi:\kappa|}$ is a π-map.

Proposition 4.5. [16, 10.7] *The two maps*

$$\Delta^{h\pi}\colon K^{h\pi} \to \left(K^{|\pi:\kappa|}\right)^{h\pi} \quad \text{and} \quad K^{h\iota}\colon K^{h\pi} \to K^{h\kappa}$$

correspond to each other under the identification $K^{h\kappa} = \left(K^{|\pi:\kappa|}\right)^{h\pi}$.

In combination with (4.3) the homotopy fibre of $K^{h\iota}$ can now be determined as in (4.7) below.

4.6. **Applications to classifying spaces.** The case where $K = BX$ and $L = BY$ are classifying spaces of p-compact groups is of special interest to us.

Corollary 4.7. *Suppose that BX and BY are π-spaces, that $Bf\colon BX \to BY$ is a π-map, and $\iota\colon \kappa \to \pi$ a subgroup inclusion.*

(1) *The homotopy fibre over any point of*

$$(Bf)^{h\pi}\colon (BX)^{h\pi} \to (BY)^{h\pi}$$

is $(Y/X)^{h\pi}$.

(2) *The homotopy fibre over any point of the restriction map*

$$(BX)^{h\iota}\colon (BX)^{h\pi} \to (BX)^{h\kappa}$$

is $(X^{|\pi:\kappa|}/X)^{h\pi}$.

(3) *The homotopy fibre of the evaluation map*

$$(BX)^{h\pi} \to BX$$

is $(X^{|\pi|}/X)^{h\pi}$.

Point (3) is just the special case $\kappa = \{1\}$ of (2).

Now specialize to the case where BX is a trivial π-space.

Example 4.8. (Extensions of morphisms.) Let $h\colon \kappa \to X$ be a morphism defined on the subgroup $\kappa < \pi$. The space of extensions of Bh,

$$\begin{tikzcd} B\kappa \arrow[d, "B\iota"'] \arrow[dr, "Bh"] & \\ B\pi \arrow[r, dashed] & BX \end{tikzcd}$$

is homotopy equivalent to the homotopy fibre $(X^{|\pi:\kappa|}/X)^{h\pi}$ over Bh of $\mathrm{map}(B\iota, BX)\colon \mathrm{map}(B\pi, BX) \to \mathrm{map}(B\kappa, BX)$. This homotopy fixed point space is nonempty if and only if h extends (up to conjugacy) to π.

Assuming additionally that $\pi = \mathbb{Z}/p^n, n \geq 0$, is cyclic, we consider the π-space $X^{|\pi:\kappa|}/X$ of Example 4.8 more closely: Note that π acts trivially on BX and permutes cyclically the factors of $BX^{|\pi:\kappa|} = \mathrm{map}(\pi/\kappa, BX)$. This specifies the (monodromy) action of π on the fibration sequence $X^{|\pi:\kappa|}/X \to BX \xrightarrow{\Delta} BX^{|\pi:\kappa|}$ and thus the induced action of π on the p-adic rational cohomology

$$H^*(X^{|\pi:\kappa|}/X; \mathbb{Q}_p) \cong \mathrm{Tor}_{H^*(BX;\mathbb{Q}_p)^{\otimes|\pi:\kappa|}}(\mathbb{Q}_p, H^*(BX;\mathbb{Q}_p))$$

of the fibre.

Lemma 4.9. [16, 5.11] *Suppose that X is connected of rank r. Then the Lefschetz number* $\Lambda(X^{|\pi:\kappa|}/X;\pi) = |\pi:\kappa|^r$.

Recall that the *Lefschetz number* of an $\mathbb{F}_p$-finite $\mathbb{Z}/p^n$-space U is the alternating sum

$$\Lambda(U;\mathbb{Z}/p^n) = \sum_{i=0}^{\infty}(-1)^i \operatorname{trace} H^i(\xi;\mathbb{Q}_p)$$

where $H^i(\xi;\mathbb{Q}_p)$ is the automorphism of $H^i(U;\mathbb{Q}_p)$ induced by a generator $\xi \in \mathbb{Z}/p^n$. (This p-adic number is independent of the choice of ξ.) The Lefschetz number of the trivial π-space U is $\Lambda(U,\{1\}) = \chi(U)$.

The standing assumption made here that π be finite isn't essential. Corollary 4.7, for instance, holds (suitably modified) in more general settings with π replaced by a p-discrete toral group or even a p-compact toral group.

5. Centralizers

Let P be a p-compact toral group, Y any p-compact group, and $g\colon P \to Y$ a morphism of P to Y.

The *centralizer* of g, $C_Y(g)$, or $C_Y(P)$ when g is understood, is the loop space of $BC_Y(g) = \operatorname{map}(BP,BY)_{Bg}$, the mapping space component containing Bg. Note the evaluation map $BC_Y(g)\times BP \to BY$. Base point evaluation, $BC_Y(g)\to BY$, in particular, provides the first nontrivial example of a monomorphism.

Theorem 5.1. [16, 5.1, 5.2, 6.1] *$C_Y(g)$ is a p-compact group and $C_Y(g)\to Y$ is a monomorphism.*

The difficulty here is to show that $C_Y(g)$ and $Y/C_Y(g)$ are $\mathbb{F}_p$-finite spaces. (It is unknown if this remains true with P replaced by a general p-compact group). §7 contains some information on the proof.

5.2. **Central maps.** The morphism $g\colon P\to Y$ is said to be *central* if

(1) $C_Y(g)\to Y$ is an isomorphism, or,
(2) g extends to a morphism $Y\times P\to Y$ which is the identity on Y.

These two conditions are equivalent as the adjoint of a morphism as in (2) is an inverse to the evaluation monomorphism in (1).

These definitions are motivated by

Example 5.3. [22] [46, Theorem 9.6] [17, 12.5] Let $C_G(f)$ be the centralizer of a homomorphism $f\colon \pi\to G$ of a finite p-group π into a connected compact Lie group G. Then $\pi_0(C_G(f))$ is a p-group and there is an isomorphism

$$\widehat{C_G(f)} \to C_{\hat{G}}\hat{f}$$

which is adjoint to the $\mathbb{F}_p$-localization of the map $BC_G(f)\times B\pi \to BG$ induced by the homomorphism $C_G(f)\times\pi\to G$. Thus $\hat{f}\colon \pi\to\hat{G}$ is a central morphism of p-compact groups if $f\colon\pi\to G$ of is a central homomorphism of Lie groups.

Here are some more examples of central morphisms.

Theorem 5.4. [16, 5.3, 6.1] *The constant morphism $0\colon P\to Y$ is central.*

This first example is an immediate consequence of the Sullivan conjecture as proved by H. Miller [30].

Example 5.5. (Liftings and existence of morphisms.) Writing $\operatorname{map}(BP,BY) = \coprod_{g\in\operatorname{Rep}(P,Y)} BC_Y(g)$ leads to alternative expressions for the homotopy fibres of (4.7). Assume in point (2) that the p-compact toral group P is a finite p-group, π.

(1) Let $f\colon X\to Y$ be a morphism. The space of lifts of Bg,

$$\begin{tikzcd} & BX \arrow[d, "Bf"] \\ BP \arrow[ur, dashed] \arrow[r, "Bg"'] & BY \end{tikzcd}$$

is homotopy equivalent to the the homotopy fibre

$$\coprod_{g'} C_Y(g)/C_X(g') = (Y/X)^{hP}$$

over Bg of $\mathrm{map}(BP, Bf)\colon \mathrm{map}(BP, BX) \to \mathrm{map}(BP, BY)$. (The above disjoint union is indexed by all conjugacy classes of morphisms $g'\colon P \to X$ with $f \circ g'$ conjugate to g.) This homotopy fixed point space is nonempty if and only if g lifts over f.

(2) The space $\mathrm{map}_*(B\pi, BY)$ of based maps is homotopy equivalent to the homotopy fibre

$$\coprod_{g \in \mathrm{Rep}(\pi, Y)} Y/C_Y(g) = \left(Y^{|\pi|}/Y\right)^{h\pi}.$$

of the evaluation $\mathrm{map}(B\pi, BY) \to BY$. This homotopy fixed point space is contractible if and only if $\mathrm{Rep}(\pi, Y) = \{0\}$.

In contrast to the very deep Theorem 5.4, nothing more than elementary obstruction theory is needed for the second example of a central morphism.

Lemma 5.6. *The identity map $1\colon S \to S$ of a p-compact torus S is central.*

In other words, S is *abelian*. (Conversely, any connected abelian p-compact group is a p-compact torus [17, 5.1], [32, 3.1]). Version (2) of the definition of centrality shows that any morphism $P \to S$ is central

5.7. **Centralizers of p-compact tori.** When $P = S$ is a p-compact torus, composition of maps,

$$BC_Y(g) \times BS \xleftarrow{\simeq} BC_Y(g) \times BC_S(1) \xrightarrow{\circ} BC_Y(g),$$

is a factorization $C_Y(g) \times S \to C_Y(g)$ of the evaluation morphism through the centralizer. The restriction to S of this morphism is thus a central factorization $g'\colon S \to C_Y(g)$ of $g\colon S \to Y$ through its centralizer.

Lemma 5.8. [16, 8.2, 8.3] *Suppose that $g\colon S \to Y$ is a monomorphism of a p-compact torus S to Y. Then there exist a short exact sequence*

$$S \xrightarrow{g'} C_Y(g) \to C_Y(g)/g'$$

of p-compact groups such that $S \xrightarrow{g'} C_Y(g) \to Y$ is conjugate to g.

Note that (5.8) asserts the existence of a classifying space $B(C_X(g)/g')$ for the homogeneous space $C_X(g)/g'$.

Exercise 5.9. Let S be a p-compact torus. Then:

(1) Any monomorphism $S \to S$ is an isomorphism.
(2) No monomorphisms $S \times \mathbb{Z}/p \to S$ exist.
(3) The only central extension $S \to P \to \mathbb{Z}/p$ is the trivial one.

6. Algebraic Smith theory

Suppose that π is a finite p-group and that the p-compact group classifying spaces BX and BY are π-spaces. Let $f\colon X \to Y$ be a monomorphism such that $Bf\colon BX \to BY$ is a π-map. Choose a base point $y \in (BY)^{h\pi}$ and equip Y/X with the corresponding π-space structure such that (4.3) $Y/X \to BX \to BY$ is a fibration sequence of π-maps and $(Y/X)^{h\pi} \to (BX)^{h\pi} \to (BY)^{h\pi}$ a fibration sequence of homotopy fixed point spaces.

Algebraic Smith theory, based on work by J. Lannes and his collaborators and concerned with the cohomological properties, in particular the Euler characteristic (2.5), of the fibre $(Y/X)^{h\pi}$, can be summarized as follows.

Theorem 6.1. [16, 4.5, 4.6, 5.7] [19] *Under the above assumptions the following holds:*

(1) *$(Y/X)^{h\pi}$ is $\mathbb{F}_p$-finite.*
(2) $\chi((Y/X)^{h\pi}) = \chi(Y/X) \mod p$.
(3) *$\chi((Y/X)^{h\pi}) = \Lambda(Y/X; \pi)$ if π is cyclic.*

The analogous Euler characteristic formulas were known to be true in classical Smith theory dealing with fixed point spaces for (reasonable) group actions on finite complexes [6].

I refrain from commenting on the proof of Theorem 6.1 but refer to [29] for more detailed information.

A particularly advantageous situation arises when the finite p-group π can be replaced by a p-discrete torus (2.1) $\check{T}$.

Corollary 6.2. [16, 4.7, 5.7] [19] *Suppose that* $Bf\colon BX \to BY$ *is a* $\check{T}$*-map. Then:*

(1) $\chi((Y/X)^{hA}) = \chi(Y/X)$ *for any finite subgroup* $A < \check{T}$.
(2) $(Y/X)^{h\check{T}} \neq \emptyset$ *if* $\chi(Y/X) \neq 0$.

It is unknown if (1) also holds for infinite subgroups, such as $\check{T}$ itself.

The proof of the first half of this corollary, obviously true for the trivial group, is by induction on the order of A: Suppose that $B < A < \check{T}$ with A/B cyclic. Then $(Y/X)^{hB}$ is an A/B-space as well as a $\check{T}/B$-space and

$$\chi((Y/X)^{hA}) = \chi(((Y/X)^{hB})^{h(A/B)}) = \Lambda((Y/X)^{hB}; A/B)$$

by the exponential law (4.4) and Theorem 6.1.

The action of $A/B < \check{T}/B \cong \check{T}$ on $H^*((Y/X)^{hB}; \mathbb{Q}_p)$ is trivial since [16, 4.18] p-discrete tori admit no nontrivial finite dimensional representations over $\mathbb{Q}_p$. Thus

$$\Lambda((Y/X)^{hB}; A/B) = \Lambda((Y/X)^{hB}; \{1\}) = \chi((Y/X)^{hB}) = \chi(Y/X)$$

where the last equality is by induction hypothesis.

If $\chi(Y/X) \neq 0$, also $\chi((Y/X)^{hA}) \neq 0$ and hence (2.6) $(Y/X)^{hA} \neq \emptyset$, for all finite subgroups $A < \check{T}$. Then also $(Y/X)^{\check{T}} \neq \emptyset$ by a limit argument.

6.3. **Applications.** It is time to assemble the information presented until now into proofs of the existence of nontrivial elements (3.8) and of centralizer p-compact groups (5.1).

Proof of Theorem 3.8. Let X be a nontrivial, connected p-compact group. Then (2.6) the rank, r, of X is positive.

The Euler characteristic of the $\mathbb{F}_p$-finite homotopy fibre, $(X^p/X)^{h\mathbb{Z}/p^{n+1}}$, of the restriction $\mathrm{map}(B\mathbb{Z}/p^{n+1}, BX) \to \mathrm{map}(B\mathbb{Z}/p^n, BX)$, $n \geq 0$, is

$$\chi((X^p/X)^{h\mathbb{Z}/p^{n+1}}) = \Lambda(X^p/X; \mathbb{Z}/p^{n+1}) = p^r$$

by (3.3, 6.1, 4.9). This homotopy fixed point space is thus (2.6) nonempty and non-contractible.

Since the homotopy fibre of the evaluation $\mathrm{map}(B\mathbb{Z}/p, BX) \to BX$ is non-contractible, there exists (5.5.2) a nontrivial morphism $\mathbb{Z}/p \to X$. Any such morphism is a monomorphism [16, §7].

Suppose, inductively, that a monomorphism $h\colon \mathbb{Z}/p^n \to X$, $n \geq 1$, has been found. Since the homotopy fibre over Bh of the restriction $\mathrm{map}(B\mathbb{Z}/p^{n+1}, BX) \to \mathrm{map}(B\mathbb{Z}/p^n, BX)$ is nonempty, h extends (4.8) to $\mathbb{Z}/p^{n+1}$. Any such extension of a monomorphism is again a monomorphism [16, §7]. This shows the existence of a map $B\mathbb{Z}/p^\infty \to BX$ that restricts to a monomorphism $\mathbb{Z}/p^n \to X$ for all $n \geq 0$. The $\mathbb{F}_p$-localization of this map is [16, §7] a monomorphism $S \to X$ of a p-compact 1-torus to X. □

Proof of Theorem 5.1 *for finite* P. Let $g\colon \pi \to Y$ be a morphism from a finite p-group π to a p-compact group Y. The claim is (essentially) that $C_Y(g) = \Omega\,\mathrm{map}(B\pi, BY)_{Bg}$ and $Y/C_Y(g)$ are $\mathbb{F}_p$-finite spaces.

Consider the π-map $B\pi = B\{1\} \times B\pi \to BY \times B\pi$ that takes any $b \in B\pi$ to $(Bg(b), b)$. The loop space of $BC_Y(g)$ is the homotopy fibre over Bg of the induced map $B\{1\} = (B\{1\})^{h\pi} \to (BY)^{h\pi} = \mathrm{map}(B\pi, BY)$, taking the point $B\{1\}$ to Bg. By (4.7) this homotopy fibre can also be described as the homotopy fixed point space $(Y/\{1\})^{h\pi}$ which is $\mathbb{F}_p$-finite by (3.3, 6.1).

The homogeneous space $Y/C_Y(g)$ is $\mathbb{F}_p$-finite as a direct summand (5.5.2) of the $\mathbb{F}_p$-finite (6.1) space $(Y^{|\pi|}/Y)^{h\pi}$. □

The general case of Theorem 5.1 follows from this special case by discrete approximation (2.1).

7. Maximal tori and Weyl groups

Let X be any p-compact group. The maximal torus of X is constructed by an inductive procedure.

If $X(= C_X(\{1\})/\{1\})$ isn't homotopically discrete, it is (3.8) the target of a monomorphism $S_1 \to X$ defined on a p-compact 1-torus S_1. This monomorphism factors through its own centralizer (5.8) to give a short exact sequence

$$S_1 \to C_X(S_1) \to C_X(S_1)/S_1$$

of p-compact groups.

If $C_X(S_1)/S_1$ isn't homotopically discrete, it is (3.8) the target of a monomorphism $S_2/S_1 \to C_X(S_1)/S_1$ defined on a p-compact 1-torus S_2/S_1. Pull back along this monomorphism induces a commutative diagram of p-compact group morphisms

$$\begin{tikzcd} S_1 \arrow[r] \arrow[d, equal] & S_2 \arrow[r] \arrow[d] & S_2/S_1 \arrow[d] \\ S_1 \arrow[r] & C_X(S_1) \arrow[r] & C_X(S_1)/S_1 \end{tikzcd}$$

where S_2 is (3.5) a p-compact 2-torus and the middle arrow a monomorphism ($C_X(S_1)/S_2 \simeq \frac{C_X(S_1)/S_1}{S_2/S_1}$ is $\mathbb{F}_p$-finite). Thus X is the target of a monomorphism (3.5) $S_2 \to C_X(S_1) \to X$ defined on a p-compact 2-torus.

For dimension reasons (3.6), this inductive procedure eventually stops at a *maximal torus* for X where

Definition 7.1. [16, 8.8, 8.9] *A maximal torus is a monomorphism $i\colon T \to X$ of a p-compact torus T to X such that $C_X(T)/T$ is a homotopically discrete p-compact group.*

We have thus established the existence part of

Theorem 7.2. [16, 8.13, 9.4] *Any p-compact group admits a maximal torus, unique up to conjugacy.*

Next, we introduce the Weyl group of a maximal torus.

Let $i\colon T \to X$ be a maximal torus such that $Bi\colon BT \to BX$ is a fibration. The *Weyl space* $\mathcal{W}_T(X)$ is the topological monoid of self-maps of BT over BX. As a space (5.5.1),

$$(X/T)^{hT} = \mathcal{W}_T(X) = \coprod_w C_X(i)/C_T(w) = \coprod_w C_X(T)/T \tag{7.3}$$

with the disjoint union indexed by all $w \in \mathrm{Rep}(T,T)$, necessarily (3.5, 5.9) central automorphisms, with $i \circ w$ conjugate to i. The right hand side shows that the Weyl space is homotopically discrete.

Definition 7.4. [16, 9.6] *The* Weyl group $W_T(X)$ *is the component group $\pi_0\mathcal{W}_T(X)$ of the Weyl space.*

The left hand side of (7.3), by discrete approximation (2.1) homotopy equivalent [16, 6.1, 6.7] to $(X/T)^{hA}$ for some finite subgroup $A < \check{T}$, is $\mathbb{F}_p$-finite (6.1) and the computation (6.2)

$$\chi(X/T) = \chi((X/T)^{hA}) = \chi((X/T)^{hT}) = |W_T(X)| \tag{7.5}$$

shows that the Euler characteristic of the homogeneous space X/T equals the order of the Weyl group; in particular, $\chi(X/T) > 0$.

Another application of (6.2) now yields the uniqueness part of Theorem 7.2: Suppose that $i_1\colon T_1 \to X$ and $i_2\colon T_2 \to X$ are maximal tori. The fact that $(X/T_2)^{h\check{T}_1} = (X/T_2)^{hT_1} \neq \emptyset \neq (X/T_1)^{h\check{T}_2} = (X/T_1)^{hT_2}$ means that there exist morphisms $u\colon T_1 \to T_2$ and $v\colon T_2 \to T_1$ necessarily isomorphisms (3.5, 5.9), such that i_1 is conjugate to $i_2 \circ u$ and i_2 to $i_1 \circ v$.

We now specialize to connected p-compact groups and head directly for the culmination of [16].

Proposition 7.6. [16, 9.1] *Let X be a connected p-compact group and $T \to X$ a maximal torus. Then $T \to C_X(T)$ is an isomorphism.*

Proof. Note that, by maximality, $C_X(T)$ has T as its identity component so that $T \to C_X(T) \to \pi_0(C_X(T))$ is a central extension. Assume that $\pi_0(C_X(T))$ is nontrivial and produce (5.9.3) a monomorphism $\mathbb{Z}/p \times T \to C_X(T)$. To obtain a contradiction, it suffices (5.9.2) to show that the composite monomorphism $\mathbb{Z}/p \times T \to X$ factors through T, i.e. that the space of lifts (5.5.1)

$$(X/T)^{h(\mathbb{Z}/p\times T)} = ((X/T)^{h\mathbb{Z}/p})^{hT} \simeq ((X/T)^{h\mathbb{Z}/p})^{h\check{T}}$$

is nonempty. But that follows from (6.2) since (6.1) the Euler characteristic

$$\chi((X/T)^{h\mathbb{Z}/p}) = \Lambda(X/T; \mathbb{Z}/p) = \Lambda(X/T; \{1\}) = \chi(X/T)$$

is nonzero (7.5). Note that this computation uses triviality of the monodromy action, factoring through $\pi_1(BX) = \pi_0(X) = \{1\}$, of $\mathbb{Z}/p$ on $H^*(X/T;\mathbb{Q}_p)$. □

With $C_X(T)/T = \{1\}$, (7.3) shows that the monoid morphism $W_T(X) \to \mathrm{Rep}(T,T) = [BT, BT]$ is injective, i.e. that the $W_T(X)$ is faithfully represented in $H_2(BT;\mathbb{Z}_p)\otimes\mathbb{Q}$. Moreover, the $W_T(X)$-invariant map $Bi\colon BT \to BX$ induces an algebra map

$$H^*(BX;\mathbb{Q}_p) \to H^*(BT;\mathbb{Q}_p)^{W_T(X)} \tag{7.7}$$

into the invariant ring of this faithful representation.

Theorem 7.8. [16, 9.7] *Let X be a connected p-compact group with maximal torus $T \to X$. Then:*

1. *T and X have the same rank.*
2. *The Weyl group $W_T(X)$ is faithfully represented as a reflection group in the $\mathbb{Q}_p$-vector space $H_2(BT;\mathbb{Z}_p) \otimes \mathbb{Q}$.*
3. *The homomorphism (7.7) is an isomorphism.*

By the classical Shephard–Todd theorem [4, 7.2.1], the fact (3) that the invariant ring is polynomial (2.5) implies (2) that the Weyl group is represented as a reflection group in the p-adic vector space $H_2(BT;\mathbb{Z}_p)\otimes\mathbb{Q}$. Thus $W_T(X)$ must be isomorphic to a product irreducible p-adic reflection groups from the Clark–Ewing list [12]. This revives old hopes about a classification theorem.

Example 7.9. Suppose that G is a compact Lie group with $\pi_0(G)$ a p-group. Then any Lie-theoretic maximal torus $T \to G$ induces a maximal torus $\hat{T} \to \hat{G}$ of the p-compact group $\hat{G}$. The associated Weyl groups are isomorphic. The Weyl group of a p-compact toral group P is $\pi_0(P)$. The Weyl group of the Sullivan sphere $(S^{2n-1})_{\mathbb{F}_p}$ is $\mathbb{Z}/n$ and the Weyl group of the Clark–Ewing p-compact group $B(\check{T}\rtimes W)_{\mathbb{F}_p}$ is W. The Weyl group of DI(4) is group number 24 on the Clark–Ewing list, abstractly isomorphic to the product of a cyclic group of order two and the simple group of order 168.

Exercise 7.10. Modify the above construction of the maximal torus for a p-compact group to obtain an unconventional construction of the maximal torus for a compact Lie group. (See [29] and [16, 1.2].)

The maximal torus normalizer BN is defined [16, 9.8] as the Borel construction for the action of the topological monoid $\mathcal{W}$ on BT. Up to homotopy, BN sits as the total space in a fibration $BT \to BN \to BW$ over the classifying space of the Weyl group. BN is in general not a p-compact group since $\pi_1(BN) = W$ need not be a p-group. Since the map $BT \to BX$ is invariant under the action of the Weyl monoid, it extends to map $Bj\colon BN \to BX$.

Example 7.11. The semi-direct product $\check{T} \rtimes W$ is a discrete approximation [17, 3.12] to the maximal torus normalizer for the Clark–Ewing p-compact group $B(\check{T}\rtimes W)_{\mathbb{F}_p}$.

Theorem 7.12 (Adams-Mahmud [1] for p-compact groups)**.** [34, 5.1] *Any automorphism $f\colon X \to X$ of the p-compact group X restricts to an automorphism $\mathrm{AM}(f)$ of BN, unique up to the action of the Weyl group $W(X_0) = \pi_1(X/N(X))$ of the identity component X_0 of X, such that the diagram*

$$\begin{tikzcd} BN \arrow[r, "B(\mathrm{AM}(f))"] \arrow[d, "Bj"'] & BN \arrow[d, "Bj"] \\ BX \arrow[r, "Bf"'] & BX \end{tikzcd}$$

commute up to based homotopy.

The *Adams–Mahmud homomorphism* is the resulting homomorphism

$$\mathrm{AM}\colon \operatorname{Aut}(X) \to \operatorname{Aut}(N(X))/W(X_0) \tag{7.13}$$

of automorphism groups.

8. Structure of p-compact groups

For a p-compact group X, let

- $T(X)$ denote the maximal torus of X [16, 8.9],
- $L(X) = \pi_2(BT(X))$ the lattice of X,
- $\check{T}(X) = L(X) \otimes \mathbb{Z}/p^\infty$ the p-discrete maximal torus of X [16, §6],
- $W(X)$ the Weyl group of X [16, 9.6], $r_0W(X)$ the rational and $r_pW(X)$ the mod p Weyl group of X [37, 4.3],
- $N(X)$ the maximal torus normalizer of X [16, 9.8],
- $Z(X)$ the center of X (8.1)[17, 32],
- $\operatorname{Out}(X)$ the group of invertible elements in the monoid $\operatorname{End}(X) = [BX, BX]$ [35, §3], and,

Structure theorems for p-compact groups exhibit a pronounced analogy to Lie theory.

8.1. **Centers.** A p-compact group is *abelian* if its identity map is central, i.e. if the evaluation map $\operatorname{map}(BX, BX)_{B1} \to BX$ is an equivalence or, equivalently, if BX is an H-space. By (5.6), p-compact tori are abelian.

Theorem 8.2. [17, 1.1] [32, 3.1] *A p-compact group is abelian if and only if it is isomorphic to a product of a finite abelian p-group and a p-compact torus.*

If $Z \to X$ is a central monomorphism, then [17, 5.1] [32, 3.5] Z is abelian.

Theorem 8.3. [17, 1.2] [32, 4.4] *For any p-compact group X there exists a central monomorphism $Z(X) \to X$ such that any central monomorphism into X factors through $Z(X)$.*

The terminal central monomorphism, essentially unique, of (8.3) is the *center* of X.

The (discrete approximation to) the center can be defined as the group of elements in (the discrete approximation to) $C_X(T)$ that are central in X. Another candidate to the center title is the centralizer of the identity morphism. Fortunately, there is no discrepancy.

Theorem 8.4. [17, 1.3] [37, 4.7] *The map $BZ(X) \to \operatorname{map}(BX, BX)_{B1}$, corresponding to the isomorphism $C_X(Z(X)) \to X$, is a homotopy equivalence.*

The highly nontrivial proof of (8.4) involves decomposing BX as a generalized push out — a technique also applied in the proof of (2.4).

The discrete approximation to the center can for a connected p-compact group be computed from the action of the Weyl group on the discrete maximal torus.

Theorem 8.5. *If X is a connected p-compact group and p is odd, then $H^0(W(X); \check{T}(X)) = \check{Z}(X)$* [17, §7] *and $H_0(W(X); L(X)) = \pi_1(X)$* [37, 4.7].

Example 8.6. Let G be a connected compact Lie group with Lie theoretic center $Z(G)$. Then the maps

$$B\widehat{Z(G)} \to \operatorname{map}(B\hat{G}, B\hat{G})_{B1} \leftarrow BZ(\hat{G})$$

adjoint to the $\mathbb{F}_p$-localization of $BZ(G) \times BG \to BG$ and $BZ(\hat{G}) \times B\hat{G} \to B\hat{G}$, respectively, are homotopy equivalences [17, 1.4] (8.4). They constitute an isomorphism $\widehat{Z(G)} \cong Z(\hat{G})$ of abelian p-compact groups.

For a connected p-compact group X, $\operatorname{rk}(Z(X)) = \dim_{\mathbb{Q}_p}(\pi_1(X) \otimes \mathbb{Q}_p)$ [32, 5.2], so the center is finite if and only if the fundamental group is. The quotient p-compact group [16, 8.3] $PX = X/Z(X)$, the *adjoint form* of X, has trivial center [17, 6.3] [32, 4.6].

Connected p-compact groups can to some extent be recovered from their universal covering group and their center or from their adjoint form and their fundamental group.

Theorem 8.7. [32, 5.4] [37, 4.7, 4.12] *Let X be a connected p-compact group with adjoint form PX and with universal covering p-compact group SX. Then $SX = SSX = SPX$, $PX = PPX = PSX$, and there are short exact sequences*

$$\pi(X) \to SX \times Z(X)_0 \to X, \quad \check{\pi}(X) \to X \to PX \times K(\pi_1(X)_0, 1)$$

where $Z(X)_0$ is the identity component of the center of X, $\pi_1(X)_0$ is the fundamental group of X modulo torsion, and $\pi(X)$ and $\check{\pi}(X)$ are some finite abelian groups.

8.8. **Semisimplicity.** Call a connected p-compact group *simple* if the faithful representation of the Weyl group $W_T(X)$ in $H_2(BT;\mathbb{Q}_p)$ is irreducible. Sullivan spheres and Clark–Ewing p-compact groups (2.2) are simple by design. $\hat{G}$ is simple for any connected compact simple Lie group G.

Suppose X is connected with maximal torus $T \to X$. The $W_T(X)$-representation $H_2(BT;\mathbb{Q}_p)$ splits as a direct sum

$$H_2(BT;\mathbb{Q}_p) = M_1 \oplus \cdots \oplus M_n$$

of irreducible $\mathbb{Q}_p[W_T(X)]$-modules. Provided the center $Z(X) = 0$ is trivial, all M_i are nontrivial and [18, 1.5] this splitting of $\mathbb{Q}_p[W_T(X)]$-modules descends to a splitting of $\mathbb{Z}_p[W_T(X)]$-modules,

$$H_2(BT;\mathbb{Z}_p) = L_1 \oplus \cdots \oplus L_n$$

where $L_i = H_2(BT;\mathbb{Z}_p) \cap M_i$. The splitting criterion [18, 1.4], guaranteeing the realizability as a splitting of X of any splitting of the $\mathbb{Z}_p[W_T(X)]$-module $H_2(BT;\mathbb{Z}_p)$, now leads to the main result on semisimplicity.

Theorem 8.9. [18, 1.3] [42] *Any connected p-compact group with trivial center is isomorphic to a product of simple p-compact groups.*

For any connected p-compact group X we thus have product decompositions

$$PX = \prod P_i(X), \quad SX = \prod S_i(X) \tag{8.1}$$

where the $P_i(X)$s are simple, center-less p-compact groups, called the *simple factors* of X, and the $S_i(X)$s are simple, simply connected p-compact groups ($S_i = SP_i$).

The decompositions of (8.7) and (8.9) are natural [34] [33] (with respect to *some* morphisms).

9. Classification of p-compact groups for odd p

A (maybe not so often seen) form of the classification theorem for compact, connected Lie groups says that two such Lie groups are isomorphic if and only their maximal torus normalizers are isomorphic [44] [13]. It has been conjectured [14, 5.3] that this is also true for p-compact groups. The following definition is intended to give some meaning to this statement.

Definition 9.1. [35, 7.1] *Let p be an odd prime. The p-compact group BX is N-determined if any diagram of the form*

$$\begin{tikzcd} & BN \arrow[dl, "Bj"'] \arrow[dr, "Bj'"] & \\ BX \arrow[rr, dotted] & & BX' \end{tikzcd} \tag{9.2}$$

where $Bj\colon BN \to BX$ is the maximal torus normalizer for X and $Bj'\colon BN \to BX'$ the maximal torus normalizer for some other p-compact group X', there exists an isomorphism $BX \to BX'$ making the diagram commutative up to homotopy.

The maximal torus normalizer also carries information about p-compact group automorphisms.

Definition 9.3. *The p-compact group BX has N-determined automorphisms if the Adams–Mahmud homomorphism* (7.13) *is injective.*

We say that the p-compact group X is *totally N-determined* if it is N-determined (9.1) and has N-determined automorphisms (9.3).

Theorem 9.4 (Classification of p-compact groups)**.** [37, 2] *All p-compact groups ($p > 2$) are totally N-determined and the Adams–Mahmud homomorphism* (7.13) *is an isomorphism for all p-compact groups.*

For *connected* p-compact groups we can formulate the classification theorem in terms of Weyl groups.

An R-reflection group, $R = \mathbb{Z}_p$, $\mathbb{F}_p$, $\mathbb{Q}_p$, is a pair (W, L) where L is a free R-module of finite rank and W is a finite group of R-linear automorphisms of L which is generated by the reflections that it contains. Two R-reflection groups, (W_1, L_1) and (W_2, L_2), are *similar* if $\theta W_1 \theta^{-1} = W_2$ for some R-linear isomorphism $\theta \colon L_1 \to L_2$. A $\mathbb{Z}_p$-reflection group (W, L) is said to be *simple* if its rationalization $(r_0 W, L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ is simple in the sense that $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a simple $\mathbb{Q}_p[r_0 W]$-module. From the Clark–Ewing list [12] of all simple $\mathbb{Q}_p$-reflection groups it is easy to derive the list of all simple $\mathbb{Z}_p$-reflection groups [37, 10.18].

The pair $(W(X), L(X))$ is a $\mathbb{Z}_p$-reflection group for any connected p-compact group X. By rationalization and mod p reduction we obtain the $\mathbb{Q}_p$-Weyl group $(r_0 W(X), L(X) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ and the $\mathbb{F}_p$-Weyl group $(r_p W(X), L(X) \otimes_{\mathbb{Z}_p} \mathbb{F}_p)$.

The point is that for a connected p-compact group X the maximal torus normalizer $N(X)$ is determined by the the Weyl reflection group $(W(X), L(X))$; the extension class is always zero [3] at all odd primes.

Corollary 9.5 (Classification of connected p-compact groups)**.** *The map*

$$\left\{ \begin{matrix} \textit{Isomorphism classes of} \\ \textit{connected } p\textit{-compact groups} \end{matrix} \right\} \xrightarrow{(W,L)} \left\{ \begin{matrix} \textit{Similarity classes of} \\ \mathbb{Z}_p\textit{-reflection groups} \end{matrix} \right\}$$

is a bijection. The Adams–Mahmud homomorphism is an isomorphism

$$\mathrm{Aut}(X) \cong N_{\mathrm{Aut}_{\mathbb{Z}_p}(L(X))}(W(X))/W(X)$$

between the automorphism group of X and the Weyl group of the Weyl group.

For a connected p-compact group the target of the Adams–Mahmud homomorphism is simply the group $\mathrm{Out}(N(X))$ of outer automorphisms, easily determined in purely algebraic terms using the results of [3].

For most of the connected simple p-compact groups X, the normalizer of the Weyl group is just $\mathbb{Z}_p^\times W(X)$ so that $\mathrm{Aut}(X) \cong \mathbb{Z}_p^\times W(X)/W(X) \cong \mathbb{Z}_p^\times / Z(W(X))$. However, F_4 at the prime $p = 3$, for instance, admits an exotic self-homotopy equivalence [9].

A connected p-compact group X is said to be *determined by its $\mathbb{F}_p$-Weyl group* if any any other connected p-compact group with the same $\mathbb{F}_p$-Weyl group as X is actually isomorphic to X. Any such p-compact group will be *cohomologically unique among p-compact groups*, because, thanks to Lannes theory [28], the $\mathbb{F}_p$-Weyl group can be read off from the mod p cohomology algebra. All connected p-compact groups with Weyl group order prime to p (non-modular p-compact groups) are determined by their $\mathbb{F}_p$-Weyl groups [37, 4.5.2].

Corollary 9.6. *All simple p-compact groups, except those of the form $\mathrm{SU}(n)/Z$ where Z is non-trivial and proper subgroup of the center of the p-compact group $\mathrm{SU}(n)$, are determined by their $\mathbb{F}_p$-Weyl group so are cohomologically unique among p-compact groups.*

The proof of this is purely algebraic since we know from the classification theorem that connected p-compact groups are determined by their Weyl groups. An inspection of the Clark–Ewing list shows that all simple $\mathbb{Z}_p$-reflection groups, except those associated to $SU(n)/Z$, $1 \lneqq Z \lneqq \check{Z}(\mathrm{SU}(n))$, are determined by their mod p reductions [37, 10.25.3]. The mod p cohomology algebras $H^*(BG; \mathbb{F}_p)$ are not known for all simple Lie groups G.

A finite loop space BX (1.2) is said to admit a maximal torus if there exists a map $BT \to BX$ from the classifying space of a torus to BX whose fibre is homotopy equivalent to a finite complex with non-zero Euler characteristic. Among the uncountably many finite loop spaces in Rector's example only the genuine article $B\mathrm{SU}(2)$ admits a maximal torus. It is a conjecture that the only connected finite loop spaces that admit maximal tori are the ones that we already know, namely the classifying spaces of connected Lie groups.

Corollary 9.7 (The maximal torus conjecture)**.** *Let BX be a connected finite loop space that admits a maximal torus. Then the localization away from 2, $BX[\frac{1}{2}]$, is homotopy equivalent to $BG[\frac{1}{2}]$ for some connected Lie group G.*

A connected finite loop space with a maximal torus has a Weyl group which, as an integrally defined reflection group, must be the Weyl group of some compact, connected Lie group G. At each odd prime p, the p-compact groups $(BX)_p$ and $(BG)_p$ are therefore homotopy equivalent by the classification theorem (9.5). Thus Sullivan's Arithmetic Square shows that BX and BG are homotopy equivalent away from the prime 2.

9.8. **Polynomial p-compact groups.** A polynomial p-compact group is a connected $\mathbb{F}_p$-local space with polynomial mod p cohomology algebra. (Any such space is a p-compact group because its loop space will have a mod p cohomology algebra which is exterior and in particular finite.) The polynomial p-compact groups are precisely the connected p-compact groups BX for which the map

$$H^*(BX;\mathbb{F}_p) \to H^*(BT(X);\mathbb{F}_p)^{W(X)} = H^*(\check{T}(X);\mathbb{F}_p)^{W(X)}$$

induced by the (discrete) maximal torus $\check{T}(X) \to X$, is an isomorphism [37, 6.1]. Any connected p-compact group with Weyl group order prime to p is polynomial. Any simple p-compact group is polynomial but

- $\mathrm{SU}(r+1)/Z$ where Z is a non-trivial central p-subgroup,
- F_4, PE_6, E_6, $\mathrm{E}_7, \mathrm{E}_8$ at $p = 3$,
- E_8 at $p = 5$.

In particular, any exotic, simple p-compact group is polynomial [37, 6.9].

The $\mathbb{Z}_p$-reflection group (W, L) is said to be polynomial if its mod p ring of invariants, $H^*(\check{T};\mathbb{F}_p)^W$, where $\check{T} = L \otimes_{\mathbb{Z}_p} \mathbb{Z}/p^\infty$, is polynomial. Any $\mathbb{Z}_p$-reflection group of order prime to p is polynomial by the Shephard–Todd theorem [4, 7.2.1]. Kemper and Malle [27] and also Notbohm [40, 39] show that the non-polynomial simple $\mathbb{Z}_p$-reflection groups are precisely the Weyl groups of the p-compact groups on the above list; $(W, L)(\mathrm{PSU}(3))$ at $p = 3$ *is* polynomial, though, since it has rank two [38, 5.1].

The Weyl group of any polynomial p-compact group is a polynomial $\mathbb{Z}_p$-reflection group, but not any polynomial $\mathbb{Z}_p$-reflection group is the Weyl group of a polynomial p-compact group as evidenced by $(W, L)(\mathrm{PSU}(3))$ at $p = 3$ [37, 6.1, 6.27] [11].

Theorem 9.9 (Steenrod's problem)**.** [37, 6.26] [41] *The map*

$$\left\{\begin{matrix}\text{Isomorphism classes of}\\ \text{polynomial } p\text{-compact groups}\end{matrix}\right\} \xrightarrow{(W,L)} \left\{\begin{matrix}\text{Similarity classes of polynomial}\\ \mathbb{Z}_p\text{-reflection groups with } H_1(W;\check{T}) = 0\end{matrix}\right\}$$

is a bijection when $p > 2$.

The proof of this theorem is partly by inspection of the Clark–Ewing classification table for $\mathbb{Z}_p$-reflection groups [12] [37, 10.17]

Polynomial p-compact groups have several appealing properties. For instance, any center-free polynomial p-compact group, such as BG_2 at $p = 3$ (9.15), can, as a kind of generalized Clark–Ewing construction, be built as the homotopy colimit of a diagram of polynomial p-compact groups of smaller dimension where the set of diagram nodes is in bijection with the set of subgroups of the Sylow p-subgroup of the Weyl group. Also, polynomial p-compact groups are almost determined by their mod p cohomology algebras.

Corollary 9.10. [37, 6.23] *Any polynomial p-compact group is determined up to local isomorphism by its* mod p *cohomology algebra considered as an unstable algebra over the Steenrod algebra.*

The next example indicates that polynomial p-compact groups are rare and that we can not improve the above corollary.

Example 9.11. [37, 6.25] The local isomorphism system of the polynomial p-compact group $\mathrm{U}(p^\nu)$, $\nu \geq 1$, contains $\frac{1}{2}(\nu+1)(\nu+2)$ distinct p-compact groups of which $\nu + 1$ are polynomial, namely $\mathrm{U}(p^\nu)$ and ν other polynomial p-compact groups with $\mathbb{F}_p$-Weyl group similar to that of $\mathrm{SU}(p^\nu) \times \mathrm{U}(1)$.

9.12. **Proof of the classification theorem.** The first step in the proof is a reduction to the simple and center-less case.

Lemma 9.13. [37, 3.3, 3.7] *Let p be an odd prime and X any p-compact group. If the simple factors* (8.1) *of the identity component of X are totally N-determined, X itself is totally N-determined.*

Thus it suffices to show that all center-less simple p-compact groups are totally N-determined. The possible maximal torus normalizers are listed in the Clark–Ewing classification table. In each case we must show that any two center-less simple p-compact groups with this maximal torus normalizer are isomorphic. For this we use Theorems 9.18 and 9.19 and so we need to verify the conditions of these theorems. Proceeding by induction over cohomological dimension, we may assume that all elementary abelian subgroups of X have totally N-determined centralizers. The remaining conditions are harder to verify but this is done in [37] for the polynomial p-compact groups and for the simple p-compact groups whose Weyl group belong to Clark–Ewing family 1 (see also [10]), and in [2] for simple p-compact groups with $\mathbb{Z}_p$-Weyl group similar to the Weyl group of PE_6, E_6, $\mathrm{E}_7, \mathrm{E}_8$ at $p = 3$, and E_8 at $p = 5$. In these cases we use a detailed description of the non-toral elementary abelian subgroups. This covers all simple p-compact groups. (A more ideal proof would be to verify the conditions in a way independent of the classification table.)

Theorems 9.18 and 9.19 providing sufficient criteria for a connected, center-less p-compact group to be totally N-determined are obtained by combining the Homology Decomposition Theorem [17] and the preferred lifts of [36]. In outline, we proceed as follows.

Let $\mathbf{A}(X)$ denote the Quillen category of X. The objects (E, ν) of $\mathbf{A}(X)$ are conjugacy classes of monomorphism $\nu\colon E \to X$ of non-trivial elementary abelian p-groups E into X. The morphisms $(E_0, \nu_0) \to (E_1, \nu_1)$ of $\mathbf{A}(X)$ consists of all group homomorphisms $f\colon E_0 \to E_1$ such that $(E_0, \nu_0) = (E_0, \nu_1 f)$. There are functors

$$BC_X\colon \mathbf{A}(X)^{\mathrm{op}} \to \mathbf{Top}, \quad (V, \nu) \to \mathrm{map}(BV, BX)_{B\nu} = BC_X(V, \nu)$$
$$\pi_i BZC_X\colon \mathbf{A}(X) \to \mathbf{Ab}, \quad (V, \nu) \to \pi_i\big(\mathrm{map}(BC_X(V, \nu), BX), \mathrm{ev}(\nu)\big)$$

from (the opposite of) the Quillen category to the category of topological spaces and to the category of abelian groups, respectively. Note that the evaluation monomorphisms (5.1) $\mathrm{ev}(\nu)\colon C_X(V, \nu) \to X$ respect the morphisms in the Quillen category.

Theorem 9.14. [26, 17] *The map* $\mathrm{hocolim}_{\mathbf{A}(X)} BC_X \to BX$ *is an $\mathbb{F}_p$-equivalence for any connected p-compact group X.*

This theorem is only helpful when X is center-less because otherwise one of the centralizers will equal X and we have just described X in terms of itself in a very complicated way. However, when X has no center, then $\mathrm{cd}\, C_X(V, \nu) < \mathrm{cd}\, X$ for all objects (V, ν) of the Quillen category making inductive arguments possible.

Example 9.15. [37, 6.10] The polynomial 3-compact group BG_2 is the homotopy colimit of the diagram

$$\begin{tikzcd} BSU(3) \arrow[loop left, "Z(W(G_2))^{\mathrm{op}}"] & BT \arrow[l, "W(\mathrm{SU}(3))^{\mathrm{op}}\backslash W(G_2)^{\mathrm{op}}"'] \arrow[loop right, "W(G_2)^{\mathrm{op}}"] \end{tikzcd}$$

where $Z(W(G_2)) \cong \mathbb{Z}/2$ acts on $B\mathrm{SU}(3)$ via the unstable Adams operations $\psi^{\pm 1}$

A *preferred lift* of a monomorphism $\nu\colon V \to X$ of an elementary abelian p-group to V to X is a monomorphism $\nu\colon V \to N(X)$ to the maximal torus normalizer $j\colon N(X) \to X$ such that $j \circ \mu$ is conjugate to ν and the induced morphism $C_{N(X)}(V, \mu) \to C_X(V, \nu)$ is a maximal torus normalizer for the centralizer. In short, preferred lifts make "maximal torus normalizer" commute with "centralizer". As observed by H. Miller in his review of [36], an elaboration of the Borel–Serre theorem [5] [47, 5.16] will provide the classical analogue to preferred lifts.

Theorem 9.16. [36] *Any monomorphism of an elementary abelian p-group to a p-compact group admits a preferred lift.*

A monomorphism $\nu\colon V \to X$ is *toral* if it admits a factorization $\mu\colon V \to T(X)$ through the maximal torus. Toral objects of the Quillen category have uniquely defined preferred lifts, namely the factorization through the maximal torus, but non-toral objects of the Quillen category have several (non-conjugate) preferred lifts.

Example 9.17. All objects of the Quillen category of a polynomial p-compact group are toral. The image in $\mathrm{PU}(p^d)$ of the extra-special subgroup of order p^{1+2d} in $\mathrm{U}(p^d)$ [25, 7.5, 7.6.b] is a non-toral elementary abelian subgroup of order p^{2d}.

Combining Theorems 9.14 and 9.16 we can now formulate inductive criteria for N-determinism of center-less p-compact groups.

Theorem 9.18. [37, 3.4] *Let X be connected, center-less p-compact group. Assume that*

(1) *all objects of $\mathbf{A}(X)$ of rank 1 have totally N-determined centralizers, and,*
(2) $\lim^j(\mathbf{A}(X); \pi_j BZC_X) = 0$ *for* $j = 1, 2$.

Then X has N-determined automorphisms.

To see this, let $f\colon X \to X$ be an automorphism under $N(X)$ and let (V, ν) be an object of $\mathbf{A}(X)$ with preferred lift (V, μ). Then f restricts to an isomorphism of $C_X(V, \nu)$ under $C_{N(X)}(V, \mu)$ which therefore must be conjugate to the identity by the induction hypothesis. The obstructions to a globally defined homotopy to the identity map lie in the higher limits groups which we have assumed to vanish.

Theorem 9.19. [37, 3.8] *In the situation of* (9.2)*, assume that X is a connected, center-less p-compact group and that*

(1) *all objects of $\mathbf{A}(X)$ of rank ≤ 2 have totally N-determined centralizers,*
(2) *for any non-toral rank 2 object (V, ν) of $\mathbf{A}(X)$ there is an object (V, ν') of $\mathbf{A}(X')$ such that $(V, j'\mu) = (V, \nu')$ for all preferred lifts (V, μ) of (V, ν) and such that the isomorphism $f(V, \mu)$ under $C_N(V, \mu)$,*

$$\begin{tikzcd} & C_N(V,\mu) \arrow[dl] \arrow[dr] & \\ C_X(V,\nu) \arrow[rr, "\cong", "f(V,\mu)"'] & & C_{X'}(V,\nu') \end{tikzcd}$$

is independent of the $p + 1$ possible choices of preferred lift (V, μ), and

(3) $\lim^{j+1}(\mathbf{A}(X); \pi_j BZC_X) = 0$ *for* $j = 1, 2$.

Then X is N-determined.

By the inductive assumption (9.19.1), there is for any preferred lift of any non-toral, rank 2 object (V, ν) of $\mathbf{A}(X)$ an isomorphism $f(V, \mu)\colon C_X(V, \nu) \to C_{X'}(V, j'\nu)$ under $C_N(V, \mu)$. In order to verify assumption (9.19.2), we need to show that the object $(V, j'\mu) \in \mathrm{Ob}(\mathbf{A}(X'))$ and the isomorphism $f(V, \mu)$ are in fact independent of the choice of (V, μ). Oliver's cochain complex [43] reduces the problem of the higher limits (9.18.2, 9.19.3) to a question about homomorphisms out of Steinberg modules.

Here is an example of the application of the criteria for total N-determinism.

Theorem 9.20. [37] *The following hold for the* 3*-compact group* F_4*:*

(1) F_4 *is totally N-determined.*
(2) F_4 *is determined by its R-Weyl group for* $R = \mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{F}_p$.
(3) F_4 *is a cohomologically unique p-compact group.*
(4) $\mathrm{End}(\mathrm{F}_4) - \{0\} = \mathrm{Out}(\mathrm{F}_4) = N_{\mathrm{GL}(L(\mathrm{F}_4))}(W(\mathrm{F}_4))/W(\mathrm{F}_4)$ *is an abelian group isomorphic to* $\mathbb{Z}_3^\times/\mathbb{Z}^\times \times C_2$ *where the group C_2 of order* 2 *is generated by an exotic automorphism.*

The case of $(\mathrm{F}_4, p = 3)$ is particularly strightforward as F_4 contains just one non-toral elementary abelian subgroup which has rank 3 [24, 7.4]. In particular, F_4 does not contain any of the problematic non-toral rank 2 elementary abelian subgroups.

10. 2-compact groups

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