Chromatic numbers of simplicial manifolds

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/home/moller/projects/simplicial/version05/presentation/5min.tex

Chromatic numbers

The s-chromatic number of a finite ASC K

$\chi_s(K) = r$ if K admits a vertex coloring in r colors without monochrome s-simplices and r is minimal.

The s-chromatic number of a compact manifold M

$$\chi_s(M) = \sup\{\chi_s(K) \mid |K| = M\} \leq \infty$$

$$\infty \geq \chi_1(M^d) \geq \chi_2(M^d) \geq \cdots \geq \chi_d(M^d) \geq \chi_{d+1}(M^d) = 0$$

The chromatic numbers of S^1 and S^2 (The 4-color theorem)

$\chi_1(S^1) = 3$. $\chi_1(S^2) = 4$ and $\chi_2(S^2) = 2$.

- What are the s-chromatic numbers of the d-sphere S^d?
- What are the s-chromatic numbers of the surfaces?

Chromatic numbers of spheres

Determine the *s*-chromatic numbers S^d!

The 3 chromatic numbers of the 3-sphere

$\chi_1(S^3) = \chi_2(S^3) = \infty$ and $\chi_3(S^3) \geq 3$.

$\underline{\chi_2(S^3)}$: There are 'well-known' triangulations with $\chi_2 = 4$

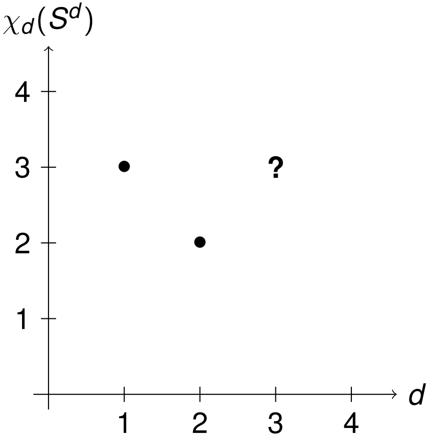
A triangulated 3-sphere with $\chi_2 = 5$

There is a triangulated 3-sphere with 167 vertices and 1412 3-simplices and $\chi_2 = 5$. No explicit examples with $\chi_2 = 6$ are known.

$\underline{\chi_3(S^3)}$:

- We do not know any triangulated 3-sphere with $\chi_3 > 3$
- Is $\chi_3(S^3)$ finite?

The d-chromatic number of the d-sphere



Chromatic numbers of compact surfaces

Determine the 1- and 2-chromatic numbers of surfaces!

- The 1-chromatic numbers are known
- The 2-chromatic numbers are known only in very few cases

The 1-chromatic number of a surface (Map color theorem)

$$\chi_1(M^2) = \left\lfloor \frac{7 + \sqrt{49 - 24E(M)}}{2} \right\rfloor \qquad (M \neq S^2, \mathrm{KB})$$

The 2-chromatic number

$$\chi_2(M^2) \leq \left\lceil \frac{\chi_1(M^2)}{2} \right\rceil$$

is finite.

2-chromatic numbers of compact surfaces

The known 2-chromatic numbers

$\chi_2(M^2) \geq 3$ except for $M = S^2$, and $\chi_2(M^2) = 3$ when M is the torus, the projective plane or the Klein bottle.

Examples of surfaces with $\chi_2 = 4$

$\chi_2(M^2) \geq 4$ if M is orientable of genus ≥ 20 or nonorientable of genus ≥ 26.

There are surfaces with large 2-chromatic numbers

$$\sup\{\chi_2(M) \mid M \text{ compact surface}\} = \infty$$

Find an explicit triangulated surface with $\chi_2 > 4$!

Triangulated surfaces with large 2-chromatic number

We construct

- an orientable surface of genus 620 and a nonorientable surface of genus 1240 with $\chi_2 = 5$ and *f*-vector $(2017, 9765, 6510)$
- an orientable surface of genus 9680 and a nonorientable surface of genus 19360 with $\chi_2 = 6$ and *f*-vector $(29647, 147015, 98010)$
- a nonorientable surface of genus 2542 with $\chi_2 \in \{5, 6\}$ and *f*-vector $(127, 8001, 5334)$