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ORIGINAL PAPER

Euler Characteristics of p-Coset Posets of Finite Groups

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Abstract

The p-coset poset of a finite group G consists of all cosets of proper p-subgroups of G ordered by set inclusion. We determine the Euler characteristic of the p-coset poset of the finite groups of Lie type in characteristic p.

Keywords Coset posets of finite groups · Euler characteristic · Finite groups of Lie type

Mathematics Subject Classification 06A07 · 20D06

1 Introduction

Let G be a finite group and p a prime number. The coset poset of G consists of all cosets of all proper subgroups, and the p-coset poset of G of all cosets of all proper p-subgroups. By a proper subgroup of G we mean a subgroup H with $H \neq G$. Both $\mathcal{C}_G^*$, the coset poset, and $\mathcal{C}_G^{p+*}$, the p-coset poset, are ordered by set inclusion. In the paper [3] from 2000, K.S. Brown investigates the coset poset of G and remarks in Section 8.4 that it would be interesting to study the p-coset poset as well. This is what we do here.

Thus we shall here focus mainly on the p-coset poset of G,

$$\mathcal{C}_G^{p+*} = \bigcup_{P \lneqq G} G/P = \bigcup_{P \lneqq G} \{gP \mid g \in G\} \tag{1.1}$$

consisting of all cosets of *proper* p-subgroups $P \lneqq G$ ordered by set inclusion. We first observe that the Euler characteristic of $\mathcal{C}_G^{p+*}$ is

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$$\chi(\mathcal{C}_G^{p+*}) \overset{\text{Eq. (2.3)}}{=} \sum_{P \lneqq G} -\mu(P, G)|G : P|$$

where μ is the Möbius function of the p-subgroup poset of G (supplemented by G). As an easy consequence we see that $\chi(\mathcal{C}_G^{p+*}) \neq 1$ and that $\mathcal{C}_G^{p+*}$ is non-contractible for all finite groups G (Corollary 2.7).

Since the index of any p-subgroup is divisible by $|G|_{p'}$, the p'-share of the group order, $|G|_{p'}$ also divides the Euler characteristic of the p-coset poset, and it is sometimes convenient to introduce the *normalised Euler characteristic*

$$\overline{\chi}(\mathcal{C}_G^{p+*}) = \frac{\chi(\mathcal{C}_G^{p+*})}{|G|_{p'}}$$

of the p-coset poset.

In Theorem 3.1 we show that the normalised Euler characteristic of the p-coset poset of a characteristic p finite group of Lie type K is

$$\overline{\chi}(\mathcal{C}_K^{p+*}) = \sum_{\emptyset \subseteq I \subseteq \widehat{\Pi}} (-)^{|I|}|K : P_I||U_\emptyset : U_I|^2$$

where $\widehat{\Pi}$ is the set of fundamental roots, P_I the parabolic subgroup associated to $I \subseteq \widehat{\Pi}$, and $U_I = O_p P_I$ its unipotent radical. The normalised Euler characteristics of the p-coset posets of K and of its parabolic subgroups P_I, described in Corollary 3.4, are closely connected to the combinatorics of K.

Section 4 contains several concrete and detailed computations of normalised Euler characteristics of p-coset posets for finite groups of Lie type in characteristic p. See for instance Example 4.1 which contains the formulas

$$\begin{aligned}
\overline{\chi}(\mathcal{C}^{p+*}_{\mathrm{SL}_2^+(\mathbf{F}_q)}) &= -q^2+q+1 \quad \overline{\chi}(\mathcal{C}^{p+*}_{\mathrm{SL}_2^-(\mathbf{F}_q)}) = -q^2+q+1\\
\overline{\chi}(\mathcal{C}^{p+*}_{\mathrm{SL}_3^+(\mathbf{F}_q)}) &= q^6-2q^4-q^3+2q+1\\
\overline{\chi}(\mathcal{C}^{p+*}_{\mathrm{SL}_4^-(\mathbf{F}_q)}) &= q^{12}-q^8-2q^7+q^6-q^5-q^4+2q^3+q+1
\end{aligned}$$

for the normalised Euler characteristics of the p-coset posets for the groups $\mathrm{SL}_2^+(\mathbf{F}_q)$, $\mathrm{SL}_3^+(\mathbf{F}_q)$, $\mathrm{SL}_2^-(\mathbf{F}_q)$, $\mathrm{SL}_4^-(\mathbf{F}_q)$ where q is a power of p. The relations

$$\begin{aligned}
\frac{q^6}{(q+1)(q^2+q+1)} &= 1 - 2\frac{\overline{\chi}(\mathcal{C}^{p+*}_{\mathrm{SL}_2^+(\mathbf{F}_q)})}{q+1} + \frac{\overline{\chi}(\mathcal{C}^{p+*}_{\mathrm{SL}_3^+(\mathbf{F}_q)})}{(q+1)(q^2+q+1)}\\
\frac{q^{12}}{(q+1)^2(q^2+1)(q^2-q+1)} &= \Big(1 - \frac{\overline{\chi}(\mathcal{C}^{p+*}_{\mathrm{SL}_2^-(\mathbf{F}_q)})}{q+1}\Big) + \Big(\frac{\overline{\chi}(\mathcal{C}^{p+*}_{\mathrm{SL}_2^+(\mathbf{F}_{q^2})})}{q^2+1}\\
&\quad - \frac{\overline{\chi}(\mathcal{C}^{p+*}_{\mathrm{SL}_4^-(\mathbf{F}_q)})}{(q+1)^2(q^2+1)(q^2-q+1)}\Big)
\end{aligned}$$

come from Example 4.9.

1.1 Notation

For quick reference, here is a list of the posets occurring in this note:

$\mathcal{S}_G^*$	The poset of *proper* subgroups of G
$\mathcal{S}_G^{p+*}$	The poset of *proper* p-subgroups of G
$\mathcal{S}_G^{p+\mathrm{rad}+*}$	The poset of *proper* radical p-subgroups of G
$\mathcal{C}_G^*$	The poset of *proper* cosets in G
$\mathcal{C}_G^{p+*}$	The poset of *proper* p-cosets in G
$\mathcal{C}_G^{p+\mathrm{rad}+*}$	The poset of *proper* p-cosets of radical p-subgroups of G

A p-group (p-coset) is a group (coset) of p-power cardinality. A p-subgroup P of G is *radical* if $P = O_p N_G(P)$.

The objects of the (p-)coset poset $\mathcal{C}_G^{(p+)*}$ are cosets xH, $x \in G$, $H \lneqq G$, of proper (p-)subgroups of G. The cosets are ordered by set inclusion: $xH \leq yK$ in $\mathcal{C}_G^{(p+)*}$ if and only if $H \leq K$ and $yK = xK$.

Let $\mathcal{S}$ be a finite poset. If x, y are elements of $\mathcal{S}$, then $x/\mathcal{S} = \{z \in \mathcal{S} \mid x \leq z\}$ is the coslice over x and $x//\mathcal{S} = \{z \in \mathcal{S} \mid x < z\}$ the proper coslice over x. The slice $\mathcal{S}/y$ and the proper slice $\mathcal{S}//y$ have analogous definitions.

The join (or ordinal sum [14, Chp 3.2]) $\mathcal{S} * \{\infty\}$ consists of $\mathcal{S}$ with an adjoined extra element ∞ such that $\infty \geq a$ for all $a \in \mathcal{S}$. The Möbius function, μ, of $\mathcal{S} * \{\infty\}$ restricts to the Möbius function on $\mathcal{S}$, and, for all $a \in \mathcal{S}$,

- $\mu(\infty, \infty) = 1$ and $\mu(\infty, a) = 0$
- $\sum_{b \in a/\mathcal{S}} \mu(a, b) + \mu(a, \infty) = 0$ and $\sum_{b \in a/\mathcal{S}} \mu(b, \infty) + 1 = 0$
- $\chi(a//\mathcal{S}) = \mu(a, \infty)$ [14, 3.8.5, 3.8.6].

In this paper, $\mathcal{S} = \mathcal{S}_G^{(p+)*}$ is the poset of proper (p-)subgroups and μ is the Möbius function of $\mathcal{S} * \{\infty\} = \mathcal{S}_G^{(p+)*} \cup \{G\}$.

2 The Coset Poset and the *p*-Coset Poset of a Finite Group

The p-coset poset $\mathcal{C}_G^{p+*}$ and the coset poset $\mathcal{C}_G^*$, investigated by Brown [3], share several properties.

Proposition 2.1 [3, Proposition 8] *Let N be a normal subgroup of G.*

(a) *If N is contained in the Frattini subgroup, then the projection $q\colon G \to G/N$ induces a homotopy equivalence $\mathcal{C}_G^* \to \mathcal{C}_{G/N}^*$.*
(b) *If N is contained in the Frattini subgroup and is a p-group, then the projection $q\colon G \to G/N$ induces a homotopy equivalence $\mathcal{C}_G^{p+*} \to \mathcal{C}_{G/N}^{p+*}$.*

(c) *If G is not a p-group and N is a p-group, then the projection $q\colon G \to G/N$ induces a homotopy equivalence $\mathcal{C}_G^{p+*} \to \mathcal{C}_{G/N}^{p+*}$.*

Proof (a) and (b) Because N consists of nongenerators [11, 5.2.12], the projection map takes proper subgroups of G to proper subgroups of G/N, and thus induces a poset morphism $q\colon \mathcal{C}_G^{(p+)*} \to \mathcal{C}_{G/N}^{(p+)*}$. There is also a poset morphism $q^{-1}\colon \mathcal{C}_{G/N}^{(p+)*} \to \mathcal{C}_G^{(p+)*}$ in the other direction taking the (p-)subgroup coset $\overline{C} \subsetneq G/N$ to $q^{-1}\overline{C} \subsetneq G$. These poset morphisms are homotopy equivalences because $xP \subseteq xPN$ for all $x \in G$ and $P \lneqq G$, and $\overline{C} = qq^{-1}(\overline{C})$ for all cosets $\overline{C} \subsetneq G/N$ [10, 1.3].
(c) The proof is similar. □

Lemma 2.2 *For any $K \in \mathcal{S}_G^{(p+)*} \cup \{G\}$,*

$$\sum_{H \in \mathcal{S}_K^{(p+)*}} \mu(H,K)|K:H| = -\chi(\mathcal{C}_K^{(p+)*}) = \sum_{H \in \mathcal{S}_K^{(p+)*}} \chi(\mathcal{C}_H^{(p+)*})|K:H|$$

Proof There are isomorphisms of proper coslices and isomorphisms of proper slices

$$\begin{array}{ccccc} H//\mathcal{S}_K^{(p+)*} & \xrightarrow{\cong} & H//\mathcal{C}_K^{(p+)*} & \xrightarrow{\cong} & xH//\mathcal{C}_K^{(p+)*} \\ L & \longrightarrow & L & \longrightarrow & xL \end{array}$$

$$\begin{array}{ccccc} \mathcal{C}_L^{(p+)*} & \xrightarrow{\cong} & \mathcal{C}_K^{(p+)*}//L & \xrightarrow{\cong} & \mathcal{C}_K^{(p+)*}//yL \\ C & \longrightarrow & C & \longrightarrow & yC \end{array}$$

where H, L are proper (p)-subgroups of K and $x, y \in K$. This gives two expressions for the Euler characteristic of $\mathcal{C}_K^{(p+)*}$,

$$\sum_{H \in \mathcal{S}_K^{(p+)*}} \mu(H,K)|K:H| = -\chi(\mathcal{C}_K^{(p+)*}) = \sum_{H \in \mathcal{S}_K^{(p+)*}} \widetilde{\chi}(\mathcal{C}_H^{(p+)*})|K:H| \quad (2.3)$$

as, $\widetilde{\chi}(H//\mathcal{S}_K^{(p+)*}) = \mu(H,K)$, and in general, $\sum_a \widetilde{\chi}(a//\mathcal{S}) = -\chi(\mathcal{S}) = \sum_b \widetilde{\chi}(\mathcal{S}//b)$ for any finite poset $\mathcal{S}$ [5, Corollary 3.8], [8, Example 2.10]. □

Example 2.4 When $G = A_5$ is the alternating group on 5 elements, the table of (2-)subgroups

H	A_4	D_{10}	D_6	C_5	$C_2 \times C_2$	C_3	C_2	C_1
$\|H\|$	12	10	6	5	4	3	2	1
$\|G : N_G(H)\|$	5	6	10	6	5	10	15	1
$\mu(H,G)$	-1	-1	-1	0	0	2	4	-60

H	$C_2 \times C_2$	C_2	C_1
$\|H\|$	4	2	1
$\|G : N_G(H)\|$	5	15	1
$\mu(H,G)$	-1	0	4

and Eq. (2.3) show that $\chi(\mathcal{C}^*_{A_5}) = 1\cdot5\cdot\frac{60}{12}+1\cdot6\cdot\frac{60}{10}+1\cdot10\cdot\frac{60}{6}-2\cdot10\cdot\frac{60}{3}-4\cdot15\cdot\frac{60}{2}+1\cdot60\cdot\frac{60}{1} = 1561$ and $\chi(\mathcal{C}^{2+*}_{A_5}) = 1\cdot5\cdot\frac{60}{4}-4\cdot1\frac{60}{1} = -165 = -11\cdot15 = -11\cdot|A_5|_{2'}$.

Remark 2.5 Let $E(G)$ and $E_p(G)$ be the functions defined recursively by $E(\{1\}) = 1 = E_p(\{1\})$ and

$$E(G) + \sum_{H\in\mathcal{S}^*_G} E(H)|G:H| = 1, \quad E_p(G) + \sum_{H\in\mathcal{S}^{p+*}_G} E_p(H)|G:H| = 1$$

when G is a non-trivial finite group. Then $E(G) = -\widetilde{\chi}(\mathcal{C}^*_G)$ and $E_p(G) = -\widetilde{\chi}(\mathcal{C}^{p+*}_G)$ by the second equality of Lemma 2.2 with $K = G$.

Write $E(n)$, $E_p(n)$ for $E(C_n)$, $E_p(C_n)$ where C_n is cyclic of order $n \geq 1$. Then $E(p^e) = 1 - p = E_p(p^e)$ for all $e > 0$. $E(n) = \prod_{p|n} E(p)$ is a product over the prime divisors of n, and $E_p(p^e m) = 1 - m$ when $e \geq 0$, $m > 1$, $(p, m) = 1$.

The reduced Euler characteristic of the (p-)coset poset is known for p-groups.

Proposition 2.6 [3, Proposition 11] *Let P be a finite p-group and $\Phi(P)$ its Frattini subgroup. Then*

$$\widetilde{\chi}(\mathcal{C}^{p+*}_P) = -\prod_{1\leq j\leq n}(1-p^j)$$

where $n \geq 1$ and $p^n = |P:\Phi(P)|$.

Proof The Frattini quotient $P/\Phi(P)$ is the elementary abelian p-group, $V(n)$, of order p^n [11, 5.3.2], and $-\widetilde{\chi}(\mathcal{C}^*_P) = -\widetilde{\chi}(\mathcal{C}^*_{V(n)})$ by Proposition 2.1. The poset $\mathcal{C}^*_{V(n)}$, which is the poset of proper affine subspaces of the affine space $\mathbf{F}^n_p$, is known to have the homotopy type of a wedge of $\prod_{1\leq j\leq n}(p^i - 1)$ spheres of dimension $n-1$. □

Corollary 2.7 *For any finite group G, $\chi(\mathcal{C}^{p+*}_G)$ is a multiple of $|G|_{p'}$ and $\chi(\mathcal{C}^{p+*}_G) \neq 1$.*

Proof If G is a p-group, $\mathcal{C}^{p+*}_G = \emptyset$ and $\chi(\mathcal{C}^{p+*}_G) = 0$ if $G = \{1\}$ is trivial, while $\chi(\mathcal{C}^{p+*}_P) = 1 - \prod_{1\leq j\leq n}(1-p^j) \neq 1$ if G is nontrivial. If G is not a p-group, $|G|_{p'} > 1$ divides $\chi(\mathcal{C}^{p+*}_G)$ by Eq. (2.3). □

The p-coset poset $\mathcal{C}^{p+*}_G$ is non-contractible as $\chi(\mathcal{C}^{p+*}_G) \neq 1$ for any group G and any prime p. It is unknown if also $\chi(\mathcal{C}^*_G) \neq 1$ for any group G. Nonetheless, Shareshian and Woodroofe, using the classification of the finite simple groups, have shown that $\mathcal{C}^*_G$ is never contractible [12].

The following result shows that only the radical p-subgroups contribute to the Euler characteristic of the p-coset poset of a non-p-group.

Proposition 2.8 [10, Proposition 6.1] *Let P be a proper p-subgroup of G.*

(a) *If P is radical, there are homotopy equivalences*

$$P//\mathcal{S}^{p+*}_{N_G(P)} \underset{r}{\overset{i}{\rightleftarrows}} P//\mathcal{S}^{p+*}_G$$

where i is the inclusion and $r(Q) = N_Q(P) = Q \cap N_G(P)$ for all p-subgroups $Q \gneqq P$.

(b) *If P is not radical and G is not a p-group, $P//\mathcal{S}^{p+*}_G$ is contractible and $\mu(P, G) = 0$.*

Proof Let Q be a p-subgroup of G properly containing P. If $r(Q) = P$ then $N_Q(P) = P$. If $r(Q) = N_G(P)$ and P is radical, then $N_G(P)$ is a p-group and $P = O_pN_G(P) = N_G(P) = N_Q(P)$ again. But $P = N_Q(P)$ for $Q \gneqq P$ is impossible since P satisfies the normalizer condition [11, 5.2.4]. We conclude that $P \lneqq r(Q) \lneqq N_G(P)$, ie that $r(Q) \in P//\mathcal{S}^{p+*}_{N_G(P)}$ for all $Q \in P//\mathcal{S}^{p+*}_G$. The poset morphisms i and r are homotopy equivalences [10, 1.3] since $ir(Q) = N_Q(P) \leq Q$ for $Q \leq G$, and $ri(Q) = Q \cap N_G(P) = Q$ for $Q \leq N_G(P)$.

Now assume that G is not a p-group. Then $P//\mathcal{S}^{p+*}_G$ is the poset $\{Q \mid P \lneqq Q \lneqq G\} = \{Q \mid P \lneqq Q \leq G\}$ of proper p-supergroups of P. By the same argument as above, $\{Q \mid P \lneqq Q \leq G\}$ and $\{Q \mid P \lneqq Q \leq N_G(P)\}$ are homotopy equivalent posets. The latter poset is poset isomorphic to the poset $\{Q \mid 1 \lneqq Q \leq N_G(P)/P\}$ of nontrivial p-subgroups of $N_G(P)/P$ which is contractible if $O_p(N_G(P)/P) \neq 1$ or $P \lneqq O_p(N_G(P))$ [10, Proposition 2.4]. □

Corollary 2.9 *The Euler characteristic of the p-coset poset of a non-p-group G is*

$$\chi(\mathcal{C}^{p+*}_G) = \sum_{P \in \mathcal{S}^{p+\mathrm{rad}+*}_G} -\mu(P, G)|G : P|$$

where the sum runs over all radical p-subgroups of G.

Proof $\chi(\mathcal{C}^{p+*}_G) \overset{\text{Eq. (2.3)}}{=} \sum_{P \in \mathcal{S}^{p+*}_G} -\mu(P, G)|G : P|$ where $\mu(P, G) = 0$ unless P is radical by Proposition 2.8(2.8). □

Corollary 2.10 $\mathcal{C}^{p+\mathrm{rad}+*}_G \hookrightarrow \mathcal{C}^{p+*}_G$ *is a homotopy equivalence when G is not a p-group.*

Proof This follows from Bouc's criterion for homotopy equivalence [2, Proposition 4.(ii)] since

$$xP//\mathcal{C}^{p+*}_G \text{ is non-contractible} \iff P//\mathcal{S}^{p+*}_G \text{ is non-contractible} \overset{\text{Proposition 2.8}}{\Longrightarrow}$$
$$P \text{ is radical} \iff xP \in \mathcal{C}^{p+\mathrm{rad}+*}_G$$

for any proper p-subgroup $P \lneqq G$ and any $x \in G$. □

Define the *normalised Euler characteristic* of the p-coset poset of G,

$$\overline{\chi}(\mathcal{C}^{p+*}_G) = \frac{\chi(\mathcal{C}^{p+*}_G)}{|G|_{p'}} \tag{2.11}$$

as the quotient of the Euler characteristic by the p'-part of the group order.

Proposition 2.12 $\overline{\chi}(\mathcal{C}_G^{p+*}) \equiv 0 \bmod p$ *if* G *is a* p*-group and* $\overline{\chi}(\mathcal{C}_G^{p+*}) \equiv 1 \bmod p$ *otherwise.*

Proof Use Proposition 2.6 if G is a p-group. If G is not a p-group, Eq. (2.3) shows that

$$\chi(\mathcal{C}_G^{p+*}) \equiv \sum_{S \in \mathrm{Syl}_p(G)} -\mu(S, G)|G|_{p'} = |\,\mathrm{Syl}_p(G)||G|_{p'} \equiv |G|_{p'} \bmod p$$

where $\mathrm{Syl}_p(G)$ is the set of Sylow p-subgroups of G. □

Proposition 2.13 *For any abelian group* A *that is not a* p*-group,* $\overline{\chi}(\mathcal{C}_A^{p+*}) = 1$.

Proof $\chi(\mathcal{C}_A^{p+*}) = |A : O_p(A)| = |A|_{p'}$ by Corollary 2.9 since, $O_p(A)$, the only radical p-subgroup, is the Sylow p-subgroup. □

I finish this section with a quick review of Hall's Eulerian functions for finite groups [7]. Let F_s denote the free group of rank $s \geq 0$, $G^s = \mathrm{Hom}(F_s, G)$ the set of homomorphisms of F_s into G, and $\mathrm{Epi}(F_s, G)$ the set of epimorphisms of F_s onto G. For any subgroup K of G, $|\,\mathrm{Hom}(F_s, K)| = \sum_{1 \leq H \leq K} |\,\mathrm{Epi}(F_s, H)|$, and $|\,\mathrm{Epi}(F_s, K)| = \sum_{1 \leq H \leq K} \mu(H, K)|\,\mathrm{Hom}(F_s, H)|$ by Möbius inversion. The density of the generating s-tuples in G^s is

$$\begin{aligned} P(G, s) &= \frac{|\,\mathrm{Epi}(F_s, G)|}{|\,\mathrm{Hom}(F_s, G)|} = \sum_{1 \leq H \leq G} \mu(H, G) \frac{|\,\mathrm{Hom}(F_s, H)|}{|\,\mathrm{Hom}(F_s, G)|} \\ &= \sum_{1 \leq H \leq G} \mu(H, G)|G : H|^{-s}. \end{aligned}$$

The finite Dirichlet series $P(G, s)$ evaluates at $s = -1$ to

$$P(G, -1) = \sum_{1 \leq H \leq G} \mu(H, G)|G : H| \overset{\text{Lemma 2.2}}{=} -\widetilde{\chi}(\mathcal{C}_G^*).$$

This is the connection, pointed out in [3, (1),(4)], and attributed to Bouc, between the function $P(G, s)$ and the Euler characteristic of $\mathcal{C}_G^*$. There seems to be no similar interpretation of $\chi(\mathcal{C}_G^{p+*})$.

3 Coset Posets of Finite Groups of Lie Type

In this section we determine the normalised Euler characteristic of the p-coset poset of a characteristic p finite group of Lie type.

Let Σ be a reduced and crystallographic root system with fundamental and positive roots Π, $\Sigma^+ \subseteq \Sigma$ [6, Definition 1.8.1]. Suppose $\overline{K}(\Sigma)$ is a semisimple $\overline{\mathbf{F}}_p$-algebraic group with root system Σ [6, Theorem 1.10.4] equipped with a Steinberg endomorphism σ. We can assume, for some power q of p, that $\sigma = \gamma_\rho\varphi_q$ or $\sigma = \psi\varphi_q$ (in the notation of [6, Definition 1.15.(b), Remarks 2.2.5.(e)]) is of standard form. Assuming Σ to be also irreducible [6, Definition 1.8.4], let $K = O^{p'}C_{\overline{K}(\Sigma)}(\sigma)$ be the finite group in $Lie(p)$ with σ-setup $(\overline{K}(\Sigma), \sigma)$ [6, Definition 2.2.2].

The surjections $\Sigma \to \widetilde{\Sigma} \to \widehat{\Sigma}$ of [6, (2.3.1)] induce surjections $\Pi \to \widetilde{\Pi} \to \widehat{\Pi}$ of sets. Here, $\widetilde{\Sigma}$ is the twisted root system of K [6, p 41], and $\widehat{\Sigma} = \widetilde{\Sigma}/\sim$ the set of equivalence classes of twisted roots pointing in the same direction.

For every subset $I \subseteq \widehat{\Pi}$ we have associated subgroups $P_I, U_I, L_I \subseteq K$ such that $U_I = O_p(P_I)$, $P_I = N_K(U_I)$ and $P_I = U_I \rtimes L_I$ [6, Theorem 2.6.5]. The P_I are parabolic subgroups, the U_I are unipotent radical p-subgroups and the L_I are Levi complements [6, Definition 2.6.4, Definition 2.6.6]. It is a consequence of the Borel–Tits theorem that $\{U_I \mid I \subseteq \widehat{\Pi}\}$ is complete set of representatives for the K-conjugacy classes of the radical p-subgroups of K [6, Corollary 3.1.5]. In the extreme cases $J = \emptyset, \widehat{\Pi}$, $P_\emptyset = U_\emptyset \rtimes L_\emptyset$ is a Borel subgroup of K, $U_\emptyset$ a Sylow p-subgroup [6, p 41, Theorems 2.3.4, 2.3.7], $L_\emptyset = H$ is a maximal torus or Cartan subgroup [6, Theorem 2.4.7, Definition 2.4.12], and $P_{\widehat{\Pi}} = K = L_{\widehat{\Pi}}$, $U_{\widehat{\Pi}} = 1$. If $\emptyset \subseteq J \subseteq I \subseteq \widehat{\Pi}$ then $U_I \subseteq U_J \subseteq P_J \subseteq P_I$ and $U_I \subseteq U_\emptyset \subseteq P_\emptyset \subseteq P_I$.

Theorem 3.1 *The normalised Euler characteristic (2.11) of the p-coset poset of $K \neq 1$ is*

$$\overline{\chi}(\mathcal{C}_K^{p+*}) = \sum_{\emptyset \subseteq I \subseteq \widehat{\Pi}} (-)^{|I|}|K : P_I||U_\emptyset : U_I|^2.$$

More generally, the normalised Euler characteristic of the p-coset poset of the parabolic subgroup P_I is

$$\overline{\chi}(\mathcal{C}_{P_I}^{p+*}) = \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|}|P_I : P_J||U_\emptyset : U_J|^2$$

for any $I \subseteq \widehat{\Pi}$ such that $L_I \neq 1$.

Proof Since $-\mu(U_I, K) = (-1)^{|I|}|U_\emptyset : U_I|$ [9, Corollary 3.3], Corollary 2.9 shows that the Euler characteristic of the p-coset poset is

$$\begin{aligned}
\chi(\mathcal{C}_K^{p+*}) &= \sum_{\emptyset \subseteq I \subseteq K} (-1)^{|I|}|U_\emptyset : U_I||K : U_I||K : P_I| \\
&= \sum_{\emptyset \subseteq I \subseteq K} (-1)^{|I|}|U_\emptyset : U_I||K : U_\emptyset||U_\emptyset : U_I||K : P_I| \\
&= |K : U_\emptyset| \sum_{\emptyset \subseteq I \subseteq K} (-1)^{|I|}|K : P_I||U_\emptyset : U_I|^2
\end{aligned}$$

where $|K : U_\emptyset| = |K|_{p'}$ as $U_\emptyset$ is a Sylow p-subgroup of K. This is the first formula of the theorem.

Assume that $L_I \neq 1$ for some $I \subseteq \widehat{\Pi}$. Then P_I is not a p-group. As in the proof of [9, Theorem 3.4], we note that the radical subgroups classes in P_I are the p-groups U_J for all subsets J of I. For $J \subseteq I$, there is an inclusion

$$U_J//\mathcal{S}_{P_I}^{p+*} \hookrightarrow U_J//\mathcal{S}_K^{p+*}$$

and a poset morphism in the other direction that takes Q with $U_J < Q < K$ to $N_Q(U_J) = Q \cap N_K(U_J) = Q \cap P_J$. Observe that $U_J < N_Q(U_J)$ [11, 5.2.4.(iii)] and $Q \cap P_J < P_J \leq P_I$ since P_J is not a p-group. Thus $Q \cap P_J$ lies in the coslice $U_J//\mathcal{S}_{P_I}^{p+*}$. Since these two poset morphisms are homotopy equivalences [10, §1.3], the Euler characteristics of the two posets are identical, $-\widetilde{\chi}(U_J//\mathcal{S}_{P_I}^{p+*}) = -\widetilde{\chi}(U_J//\mathcal{S}_K^{p+*}) = (-1)^{|J|}|U_\emptyset : U_J|$ by [9, Corollary 3.3]. An application of Corollary 2.9 now yields

$$\chi(\mathcal{C}_{P_I}^{p+*}) = \sum_{\emptyset \subseteq J \subseteq I} (-1)^J |U_\emptyset : U_J||P_I : U_J||P_I : P_J|.$$

Rewriting this, in much the same way as we just did for $\chi(\mathcal{C}_K^{p+*})$ and noting that $|P_I : U_\emptyset| = |P_I|_{p'}$, gives the second formula of the theorem. □

The p-coset poset $\mathcal{C}_K^{p+*}$ is independent of the version of K since the order of the center of the universal version of K is prime to p [6, Theorems 2.2.7, 1.12.5].

For any $I \subseteq \widehat{\Pi}$, $L_I = HM_I$ where $M_I = \langle X_{\widehat{\alpha}} \mid \pm\alpha \in I\rangle$ [6, Definition 2.6.4]. Note that $M_\emptyset = 1$ is the trivial group, $L_\emptyset = H$, $M_{\widehat{\Pi}} = K = L_{\widehat{\Pi}}$ and $|L_I|_p = |M_I|_p$ as $M_I = O^{p'}(L_I)$ [6, Theorem 2.6.5.(f)]. Let $\mathrm{KB}(M_I) = |P_I : P_\emptyset|$ for any $I \subseteq \widehat{\Pi}$. In particular, the *Borel index* of K,

$$\mathrm{KB}(K) = \mathrm{KB}(M_{\widehat{\Pi}}) = |K : P_\emptyset| = |K : U_\emptyset H| = \frac{|K : U_\emptyset|}{|H|} = \frac{|K|_{p'}}{|H|} = |K|/|B|$$

is the index in K of the parabolic subgroup $B = P_\emptyset$. Furthermore,

$$\begin{aligned}\mathrm{KB}(M_I) = |P_I : P_\emptyset| = |U_I L_I : U_\emptyset L_\emptyset| &= \frac{|L_I : L_\emptyset|}{|U_\emptyset : U_I|} \\ = \frac{|HM_I : H|}{|L_I|_p} &= \frac{|M_I|}{|H \cap M_I||M_I|_p} = \frac{|M_I|_{p'}}{|H \cap M_I|}\end{aligned}$$

and, since $H \cap M_I$ is the Cartan subgroup for M_I [6, Theorem 2.6.2.(e)], this is the product of the Borel indices for the components of M_I [6, Theorem 2.6.5.(f)]. The structure of M_I, the subsystem subgroup associated to I [6, Theorem 2.6.5.(f)], can be determined from the Dynkin diagram of K with its symmetry ρ [6, Propositions 2.2..11, 2.6.2]. With this notation $|K : P_I| = \frac{|K:P_\emptyset|}{|P_I:P_\emptyset|} = \frac{\mathrm{KB}(K)}{\mathrm{KB}(M_I)} = \frac{\mathrm{KB}(M_{\widehat{\Pi}})}{\mathrm{KB}(M_I)}$ and $|U_\emptyset : U_I| = |M_I|_p = q^N$ where N is the degree of the Borel index $\mathrm{KB}(M_I)$ considered as a polynomial in q. The next corollary is a consequence of the more general version of Theorem 3.1.

Corollary 3.2 *For any subset $I \subseteq \widehat{\Pi}$,*

$$\overline{\chi}(\mathcal{C}_{P_I}^{p+*}) = \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|} \frac{\mathrm{KB}(M_I)}{\mathrm{KB}(M_J)} q^{2\deg(\mathrm{KB}(M_J))}$$
$$q^{2\deg(\mathrm{KB}(M_I))} = \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|} \frac{\mathrm{KB}(M_I)}{\mathrm{KB}(M_J)} \overline{\chi}(\mathcal{C}_{P_J}^{p+*})$$

with the understanding that $\overline{\chi}(\mathcal{C}_{P_\emptyset}^{p+*})$ means 1.

Proof The second identity of Theorem 3.1 and its Möbius inverse [14, 3.8.3] are

$$\frac{\overline{\chi}(\mathcal{C}_{P_I}^{p+*})}{|P_I|} = \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|} \frac{|U_\emptyset : U_J|^2}{|P_J|} \quad \frac{|U_\emptyset : U_I|^2}{|P_I|} = \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|} \frac{\overline{\chi}(\mathcal{C}_{P_J}^{p+*})}{|P_J|}.$$

Use that $|P_I : P_J| = \frac{|K:P_J|}{|K:P_I|} = \frac{\mathrm{KB}(M_I)}{\mathrm{KB}(M_J)}$ and $|U_\emptyset : U_J| = q^{\deg \mathrm{KB}(M_J)}$ to bring them to the forms displayed in the corollary. □

Placing the Solomon identities [4, Chapter 14], [9, Lemma 3.2], [13, Corollary 1.1] in the first line and the identities from Theorem 3.1 and Corollary 3.2 in the second line, we obtain a table

$$1 = \sum_{\emptyset \subseteq I \subseteq \widehat{\Pi}} (-1)^{|I|} \frac{\mathrm{KB}(K)}{\mathrm{KB}(M_I)} q^{\deg \mathrm{KB}(M_I)}$$
$$q^{\deg \mathrm{KB}(K)} = \sum_{\emptyset \subseteq I \subseteq \widehat{\Pi}} (-1)^{|I|} \frac{\mathrm{KB}(K)}{\mathrm{KB}(M_I)}$$
$$\overline{\chi}(\mathcal{C}_K^{p+*}) = \sum_{\emptyset \subseteq I \subseteq \widehat{\Pi}} (-1)^{|I|} \frac{\mathrm{KB}(K)}{\mathrm{KB}(M_I)} q^{2\deg \mathrm{KB}(M_I)}$$
$$q^{2\deg \mathrm{KB}(K)} = \sum_{\emptyset \subseteq I \subseteq \widehat{\Pi}} (-1)^{|I|} \frac{\mathrm{KB}(K)}{\mathrm{KB}(M_I)} \overline{\chi}(\mathcal{C}_{P_I}^{p+*})$$

of four polynomial identities associated to K. These identities can also be written as

$$1 = \sum_{\emptyset \subseteq I \subseteq \widehat{\Pi}} (-1)^{|I|} |K : P_I||U_\emptyset : U_I| \quad |K|_p = \sum_{\emptyset \subseteq I \subseteq \widehat{\Pi}} (-1)^{|I|} |K : P_I|$$
$$\overline{\chi}(\mathcal{C}_K^{p+*}) = \sum_{\emptyset \subseteq I \subseteq \widehat{\Pi}} (-1)^{|I|} |K : P_I||U_\emptyset : U_I|^2 \quad |K|_p^2 = \sum_{\emptyset \subseteq I \subseteq \widehat{\Pi}} (-1)^{|I|} |K : P_I| \overline{\chi}(\mathcal{C}_{P_I}^{p+*}).$$

The two identities of each line are equivalent under Möbius inversion. (It is here understood that $\overline{\chi}(\mathcal{C}_{P_\emptyset}^{p+*})$ equals 1 in all cases. This is indeed the correct normalised Euler characteristic of $\mathcal{C}_{P_\emptyset}^{p+*}$ unless $q = 2$ where $P_\emptyset = U_\emptyset$ is a 2-group. For instance, if $K = A_n^+(2) = \mathrm{SL}_{n+1}^+(\mathbf{F}_2)$, the (normalised) Euler characteristic of $\mathcal{C}_{P_\emptyset}^{2+*}$ is not 1 but rather $1 - \prod_{1\le j\le n}(1-2^j)$ by Proposition 2.6.)

Corollary 3.2 can be reformulated as a linear equation. Let $[\mathrm{KB}](K)$ be the matrix with entries

$$[\mathrm{KB}](K)(I,J) = \begin{cases} (-1)^{|J|}\frac{\mathrm{KB}(M_I)}{\mathrm{KB}(M_J)} & I \supseteq J \\ 0 & \text{otherwise} \end{cases}$$

indexed by subsets $I, J \subseteq \widehat{\Pi}$. ($[\mathrm{KB}](K)$ is a lower triangular square matrix if the subsets are ordered with decreasing cardinality.) By Corollary 3.2, the vector $\overline{X} = \left(\overline{\chi}(\mathcal{C}_{P_I}^{p+*})\right)_{I\subseteq\widehat{\Pi}}$ of normalised Euler characteristics is the solution to the linear equation

$$[\mathrm{KB}](K)_{I,J\subseteq\widehat{\Pi}}\overline{X} = \left(q^{2\deg \mathrm{KB}(M_I)}\right)_{I\subseteq\widehat{\Pi}}. \tag{3.3}$$

From this follows a multiplicative principle for Euler characteristics of p-poset cosets of parabolic subgroups.

Corollary 3.4 *For any subset* $I \subseteq \widehat{\Pi}$,

$$\overline{\chi}(\mathcal{C}_{P_I}^{p+*}) = \prod_{L\in\pi_0(M_I)} \overline{\chi}(\mathcal{C}_L^{p+*})$$

where $\pi_0(M_I)$ *is the multiset of components of* M_I [6, Theorem 2.6.5.(f)].

Proof The vector $\overline{X} = \left(\prod_{L\in\pi_0(M_I)}\overline{\chi}(\mathcal{C}_L^{p+*})\right)_{I\subseteq\widehat{\Pi}}$ is also a solution to (3.3). □

The identity for $I = \widehat{\Pi}$ to the right in Corollary 3.2 takes the form

$$\frac{q^{2\deg \mathrm{KB}(K)}}{\mathrm{KB}(K)} = \sum_{\emptyset\subseteq J\subseteq\widehat{\Pi}} (-1)^{|J|} \prod_{L\in\pi_0(M_J)} \frac{\overline{\chi}(\mathcal{C}_L^{p+*})}{\mathrm{KB}(L)} \tag{3.5}$$

thanks to the factorisations $\overline{\chi}(\mathcal{C}_{P_J}^{p+*}) = \prod_{L\in\pi_0(M_J)}\overline{\chi}(\mathcal{C}_L^{p+*})$ and $\mathrm{KB}(M_J) = \prod_{L\in\pi_0(M_J)}\mathrm{KB}(L)$.

4 Calculations of $\overline{\chi}(\mathcal{C}_K^{p+*})$ for K in Characteristic p

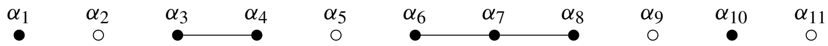
The q-bracket of the natural number d is the polynomial $[d](q) = \frac{q^d-1}{q-1} = q^{d-1} + \cdots + q + 1 \in \mathbf{Z}[q]$ of degree $d-1$ with value $[d](1) = d$ at $q = 1$.

Let $\mathrm{OP}(m) = \{(m_1,\ldots,m_k) \mid k \ge 1, m_i \ge 1, \sum m_i = m\}$ denote the set of all the 2^{m-1} ordered partitions of m [14, p 14]. The map that sends $(m_1,\ldots,m_k) \in \mathrm{OP}(m+1)$ to $\{1,\ldots,m\} - \{m_1, m_1+m_2, \ldots, m_1+m_2+\cdots+m_{k-1}\}$ is a bijection between $\mathrm{OP}(m+1)$ and the set of all subsets of $\{1,\ldots,m\}$, $m \ge 1$.

Example 4.1 ($A_m^\varepsilon(q)$, $\varepsilon = \pm 1$) The Borel index of $A_m^\varepsilon(q)$ is the polynomial

$$\mathrm{KB}(A_m^\varepsilon(q)) = \prod_{2 \le d \le m+1} [d](\varepsilon^d q)$$

of degree $\binom{m+1}{2}$. The root system A_m has m fundamental roots and the subsystem generated by the subset indexed by $(m_1, \dots, m_k) \in \mathrm{OP}(m+1)$ is isomorphic to $A_{m_1-1} \times \cdots \times A_{m_k-1}$. For instance, the ordered partition $(2, 3, 4, 2, 1) \in \mathrm{OP}(12)$ corresponds to the subsystem



of type $A_1 \times A_2 \times A_3 \times A_1 \times A_0$ of A_{11}. The Solomon identities and Theorem 3.1 for $A_m^+(q)$ state that

$$q^{\binom{m+1}{2}} = \sum_{(m_1,\dots,m_k) \in \mathrm{OP(m+1)}} (-1)^{m-k+1} \frac{\mathrm{KB}(A_m^+(q))}{\prod_{1 \le i \le k} \mathrm{KB}(A_{m_i-1}^+(q))}$$

$$1 = \sum_{(m_1,\dots,m_k) \in \mathrm{OP(m+1)}} (-1)^{m-k+1} \frac{\mathrm{KB}(A_m^+(q))}{\prod_{1 \le i \le k} \mathrm{KB}(A_{m_i-1}^+(q))} q^{\sum_{1 \le i \le k} \binom{m_i}{2}}$$

$$\overline{\chi}(\mathcal{C}^{p+*}_{A_m^+(q)}) = \sum_{(m_1,\dots,m_k) \in \mathrm{OP(m+1)}} (-1)^{m-k+1} \frac{\mathrm{KB}(A_m^+(q))}{\prod_{1 \le i \le k} \mathrm{KB}(A_{m_i-1}^+(q))} q^{\sum_{1 \le i \le k} 2\binom{m_i}{2}}.$$

The first terms of the sequence $\left(\overline{\chi}(\mathcal{C}^{2+*}_{A_m^+(2)})\right)_{m \ge 1}$ (where $q = 2$) are $-1, 29, -2561, 814309, -944455609$.

The C_2-subsystems of $A_{2m-\delta}$, $\delta \in \{0, 1\}$, are indexed by two disjoint copies of $\mathrm{OP}(m)$ with $(m_1, \dots, m_k) \in \mathrm{OP}(m)$ corresponding respectively to the C_2-subsystems

- $(A_{m_1-1} \times A_{m_1-1}) \times \cdots \times (A_{m_k-1} \times A_{m_k-1})$
- $(A_{m_1-1} \times A_{m_1-1}) \times \cdots \times (A_{m_{k-1}-1} \times A_{m_{k-1}-1}) \times A_{2m_k-\delta}$.

The Solomon identities and Theorem 3.1 for the Steinberg groups $A_{2m-\delta}^-(q)$ of twisted rank m state that

$$q^{\binom{2m+1-\delta}{2}} = \sum_{(m_1,\dots,m_k) \in \mathrm{OP}(m)} (-1)^{m-k} \Bigg(\frac{\mathrm{KB}(A_{2m-\delta}^-(q))}{\prod_{1 \le i \le k} \mathrm{KB}(A_{m_i-1}^+(q^2))} - \frac{\mathrm{KB}(A_{2m-\delta}^-(q))}{\prod_{1 \le i < k} \mathrm{KB}(A_{m_i-1}^+(q^2)) \cdot \mathrm{KB}(A_{2m_k-\delta}^-(q))} \Bigg)$$

$$1 = \sum_{(m_1,\dots,m_k) \in \mathrm{OP}(m)} (-1)^{m-k} q^{\sum_{1 \le i < k} 2\binom{m_i}{2}} \Bigg(\frac{q^{2\binom{m_k}{2}} \mathrm{KB}(A_{2m-\delta}^-(q))}{\prod_{1 \le i \le k} \mathrm{KB}(A_{m_i-1}^+(q^2))} - \frac{q^{\binom{2m_k+1-\delta}{2}} \mathrm{KB}(A_{2m-\delta}^-(q))}{\prod_{1 \le i < k} \mathrm{KB}(A_{m_i-1}^+(q^2)) \cdot \mathrm{KB}(A_{2m_k-\delta}^-(q))} \Bigg)$$

$$\overline{\chi}(\mathcal{C}^{p+*}_{A^-_{2m-\delta}(q)}) = \sum_{(m_1,\ldots,m_k)\in \mathrm{OP}(m)} (-1)^{m-k} q^{\sum_{1\le i<k} 4\binom{m_i}{2}} \Big(\frac{q^{4\binom{m_k}{2}} \mathrm{KB}(A^-_{2m-\delta}(q))}{\prod_{1\le i\le k} \mathrm{KB}(A^+_{m_i-1}(q^2))} - \frac{q^{2\binom{2m_k+1-\delta}{2}} \mathrm{KB}(A^-_{2m-\delta}(q))}{\prod_{1\le i<k} \mathrm{KB}(A^+_{m_i-1}(q^2)) \cdot \mathrm{KB}(A^-_{2m_k-\delta}(q))} \Big).$$

(The first two of these formulas disagree with [9, Example 4.4] where some exponents of q are incorrect.) The sequences $\big(\overline{\chi}(\mathcal{C}^{2+*}_{A^-_{2m}(2)})\big)_{m\ge 1}$ and $\big(\overline{\chi}(\mathcal{C}^{2+*}_{A^-_{2m-1}(2)})\big)_{m\ge 1}$ start with -55, 1034749, -4395007776655 and -1, 3619, -1067484529. The formulas of this example apply for instance to versions of the groups $A^{\pm}_m(q) = \mathrm{SL}^{\pm}_{m+1}(\mathbf{F}_q)$ and to their extensions $\mathrm{GL}^{\pm}_{m+1}(\mathbf{F}_q)$.

Example 4.2 ($B_m(q)$ or $C_m(q)$) The Borel index of $B_m(q)$ or $C_m(q)$ is the polynomial

$$\mathrm{KB}(B_m(q)) = \prod_{1\le d\le m} [2d](q), \quad m \ge 2$$

of degree m^2. By convention, B_0, $\mathrm{KB}(B_0(q)) = 1$ and $B_1 = A_1$, $\mathrm{KB}(B_1(q)) = \mathrm{KB}(A^+_1(q))$. The subsystems of B_m are indexed by $\mathrm{OP}(m+1)$ with $(m_1,\ldots,m_k) \in \mathrm{OP}(m+1)$ corresponding to the subsystem $A_{m_1-1} \times \cdots \times A_{m_{k-1}-1} \times B_{m_k-1}$ of B_m. The polynomial identities

$$q^{m^2} = \sum_{(m_1,\ldots,m_k)\in \mathrm{OP}(m+1)} (-1)^{m+1-k} \frac{\mathrm{KB}(B_m(q))}{\prod_{1\le i<k} \mathrm{KB}(A^+_{m_i-1}(q)) \cdot \mathrm{KB}(B_{m_k-1}(q))}$$

$$1 = \sum_{(m_1,\ldots,m_k)\in \mathrm{OP}(m+1)} (-1)^{m+1-k} \frac{\mathrm{KB}(B_m(q))}{\prod_{1\le i<k} \mathrm{KB}(A^+_{m_i-1}(q)) \cdot \mathrm{KB}(B_{m_k-1}(q))} \times q^{\sum_{1\le i<k} \binom{m_i}{2} + m_k^2}$$

$$\overline{\chi}(\mathcal{C}^{p+*}_{B^-_m(q)}) = \sum_{(m_1,\ldots,m_k)\in \mathrm{OP}(m+1)} (-1)^{m+1-k} \frac{\mathrm{KB}(B_m(q))}{\prod_{1\le i<k} \mathrm{KB}(A^+_{m_i-1}(q)) \cdot \mathrm{KB}(B_{m_k-1}(q))} \times q^{\sum_{1\le i<k} 2\binom{m_i}{2} + 2m_k^2}$$

apply for instance to the groups $B_m(q) = \mathrm{SO}_{2m+1}(\mathbf{F}_q)$ or $C_m(q) = \mathrm{Sp}_{2m}(\mathbf{F}_q)$. The first terms of the sequence $\big(\overline{\chi}(\mathcal{C}^{2+*}_{B_m(2)})\big)_{m\ge 2} = \big(\overline{\chi}(\mathcal{C}^{2+*}_{C_m(2)})\big)_{m\ge 2}$ are 181, -240841, 4219541101, -1121407861986721.

Example 4.3 ($D^{\varepsilon}_m(q)$, $\varepsilon = \pm 1$) The Borel index of $D^{\varepsilon}_m(q)$ is the polynomial

$$\mathrm{KB}(D^{\varepsilon}_m(q)) = \frac{(q^2-1)\cdots(q^{2m-2}-1)(q^m-\varepsilon)}{(q-1)\cdots(q-1)(q-\varepsilon)} = \begin{cases} [2](q)[4](q)\cdots[2m-2](q)[m](q) & \varepsilon = +1 \\ [2](q^m)[4](q)\cdots[2m-2](q) & \varepsilon = -1 \end{cases}$$

of degree $m(m-1)$.

The subsystems of D_m, with fundamental roots $\{\alpha_1, \ldots, \alpha_{m-1}, \alpha_m\}$, are indexed by the disjoint union of four copies of $\mathrm{OP}(m-1)$ with $(m_1, \ldots, m_k) \in \mathrm{OP}(m-1)$ corresponding respectively to the subsystems

- $A_{m_1-1} \times \cdots A_{m_k-1}$ (not including α_{m-1} nor α_m)
- $A_{m_1-1} \times \cdots \times A_{m_{k-1}-1} \times A_{m_k}$ (including α_{m-1} but not α_m)
- $A_{m_1-1} \times \cdots \times A_{m_{k-1}-1} \times A_{m_k}$ (including α_m but not α_{m-1})
- $A_{m_1-1} \times \cdots A_{m_{k-1}-1} \times D_{m_k+1}$ (including both α_{m-1} and α_m)

and we let

$$D_{00}(m_1, \ldots, m_k) = \frac{\mathrm{KB}(D_m^+(q))}{\prod_{1 \le i < k} \mathrm{KB}(A^+_{m_i-1}(q)) \cdot \mathrm{KB}(A^+_{m_k-1}(q))}$$
$$D_{01}(m_1, \ldots, m_k) = \frac{\mathrm{KB}(D_m^+(q))}{\prod_{1 \le i < k} \mathrm{KB}(A^+_{m_i-1}(q)) \cdot \mathrm{KB}(A^+_{m_k}(q))}$$
$$D_{11}(m_1, \ldots, m_k) = \frac{\mathrm{KB}(D_m^+(q))}{\prod_{1 \le i < k} \mathrm{KB}(A^+_{m_i-1}(q)) \cdot \mathrm{KB}(D^+_{m_k+1}(q))}.$$

The polynomial identities

$$q^{m(m-1)} = \sum_{(m_1 \ldots, m_k) \in \mathrm{OP}(m-1)} (-1)^{m-1-k} \big(D_{00} - 2D_{01} + D_{11}\big)$$
$$1 = \sum_{(m_1 \ldots, m_k) \in \mathrm{OP}(m-1)} (-1)^{m-1-k} q^{\sum_{1 \le i < k} \binom{m_i}{2}} \times \left(D_{00} q^{\binom{m_k}{2}} - 2D_{01} q^{\binom{m_k+1}{2}} + D_{11} q^{m_k(m_k+1)}\right)$$
$$\overline{\chi}(\mathcal{C}^{p+*}_{D_m^+(q)}) = \sum_{(m_1 \ldots, m_k) \in \mathrm{OP}(m-1)} (-1)^{m-1-k} q^{\sum_{1 \le i < k} 2\binom{m_i}{2}} \times \left(D_{00} q^{2\binom{m_k}{2}} - 2D_{01} q^{2\binom{m_k+1}{2}} + D_{11} q^{2m_k(m_k+1)}\right)$$

apply for instance to the groups $D_m^+(q) = D_m^+(q) = \mathrm{Spin}^+_{2m}(\mathbf{F}_q)$, $\Omega^+_{2m}(\mathbf{F}_q)$, $P\Omega^+_{2m}(\mathbf{F}_q)$ [6, 2.7]. It also applies to $\mathrm{SO}^+_{2m}(\mathbf{F}_q)$, containing $\Omega^+_{2m}(\mathbf{F}_q)$ with index 2, when q is odd. The first terms of the sequence $\big(\overline{\chi}(\mathcal{C}^{2+*}_{D_m^+(2)})\big)_{m \ge 2}$ of normalised Euler characteristics are 1, −2561, 15448861, −1086597998849, 1150481040643422181. A computer calculation based on Corollary 2.9 yields $\overline{\chi}(\mathcal{C}^{2+*}_{\mathrm{SO}_6^+(\mathbf{F}_2)}) = -32009 \neq \overline{\chi}(\mathcal{C}^{2+*}_{\Omega_6^+(\mathbf{F}_2)})$.

The C_2-invariant subsystems of the C_2-root system D_m are indexed by the set $\mathrm{OP}(m)$ of ordered partitions of m with $(m_1, \ldots, m_{k-1}, m_k) \in \mathrm{OP}(m)$ corresponding to the C_2-subsystem $A_{m_1-1} \times \cdots A_{m_{k-1}-1} \times D_{m_k}$ of D_m. (By convention, $A_0 = \emptyset$, $D_1 = \emptyset$, $D_2 = A_1 \times A_1$. Also, $\mathrm{KB}(A_0^+(q)) = 1 = \mathrm{KB}(D_1^-(q))$ and, following [6, Propositions 2.2.11, 2.6.2], $\mathrm{KB}(D_2^-(q)) = \mathrm{KB}(A_1^+(q^2))$.) The polynomial identities

$$q^{m(m-1)} = \sum_{(m_1,\ldots,m_k)\in\mathrm{OP}(m)} (-1)^{m-k} \frac{\mathrm{KB}(D_m^-(q))}{\prod_{1\le i<k}\mathrm{KB}(A_{m_i-1}^+(q))\cdot\mathrm{KB}(D_{m_k}^-(q))}$$

$$1 = \sum_{(m_1,\ldots,m_k)\in\mathrm{OP}(m)} (-1)^{m-k} \frac{\mathrm{KB}(D_m^-(q))}{\prod_{1\le i<k}\mathrm{KB}(A_{m_i-1}^+(q))\cdot\mathrm{KB}(D_{m_k}^-(q))} \times q^{\sum_{1\le i<k}\binom{m_i}{2}+m_k(m_k-1)}$$

$$\overline{\chi}(\mathcal{C}^{p+*}_{D_m^-(q)}) = \sum_{(m_1,\ldots,m_k)\in\mathrm{OP}(m)} (-1)^{m-k} \frac{\mathrm{KB}(D_m^-(q))}{\prod_{1\le i<k}\mathrm{KB}(A_{m_i-1}^+(q))\cdot\mathrm{KB}(D_{m_k}^-(q))} \times q^{\sum_{1\le i<k}2\binom{m_i}{2}+2m_k(m_k-1)}$$

apply for instance to the groups $D_m^-(q) = \mathrm{Spin}_{2m}^-(\mathbf{F}_q)$, $\Omega_{2m}^-(\mathbf{F}_q)$, $P\Omega_{2m}^-(\mathbf{F}_q)$ [6, 2.7]. It also applies to $\mathrm{SO}_{2m}^-(\mathbf{F}_q)$, containing $\Omega_{2m}^-(\mathbf{F}_q)$ with index 2, when q is odd. The first terms of the sequence $\left(\overline{\chi}(\mathcal{C}^{2+*}_{D_m^-(2)})\right)_{m\ge 2}$ of normalised Euler characteristics at $q=2$ are $-11, 3619, -16250471, 1091026687411, -115069798619695075 1$. A computer calculation based on Corollary 2.9 with Magma software [1] yields $\overline{\chi}(\mathcal{C}^{2+*}_{\mathrm{SO}_6^-(\mathbf{F}_2)}) = -27331 \neq \overline{\chi}(\mathcal{C}^{2+*}_{\Omega_6^-(\mathbf{F}_2)})$.

Example 4.4 ($E_6^-(q)$) The Borel index of E_6^ε is the product $\prod_d [d](\varepsilon^d q)$ over $d \in \{2,5,6,8,9,12\}$ of degree 36. Let $\alpha_1,\ldots,\alpha_6$ be the fundamental roots of $E_6^+(q)$ numbered as in [4, 13.3.3] and $\widehat{\alpha}_1 = \{\alpha_1,\alpha_6\}$, $\widehat{\alpha}_2 = \{\alpha_2,\alpha_5\}$, α_3, α_4 those of $E_6^-(q)$. Using the data of the tables

I	$\emptyset$	$\{\widehat{\alpha}_1\}$	$\{\widehat{\alpha}_2\}$	$\{\alpha_3\}$	$\{\alpha_4\}$
M_I	1	$A_1^+(q^2)$	$A_1^+(q^2)$	$A_1^+(q)$	$A_1^+(q)$

I	$\{\widehat{\alpha}_1,\widehat{\alpha}_2\}$	$\{\widehat{\alpha}_1,\alpha_3\}$	$\{\widehat{\alpha}_1,\alpha_4\}$	$\{\widehat{\alpha}_2,\alpha_3\}$	$\{\widehat{\alpha}_2,\alpha_4\}$	$\{\alpha_3,\alpha_4\}$
M_I	$A_2^+(q^2)$	$A_1^+(q^2)\times A_1^+(q)$	$A_1^+(q^2)\times A_1^+(q)$	$A_3^-(q)$	$A_1^+(q^2)\times A_1^+(q)$	$A_2^+(q)$

I	$\{\widehat{\alpha}_2,\alpha_3,\alpha_4\}$	$\{\widehat{\alpha}_1,\alpha_3,\alpha_4\}$	$\{\widehat{\alpha}_1,\widehat{\alpha}_2,\alpha_4\}$	$\{\widehat{\alpha}_1,\widehat{\alpha}_2,\alpha_3\}$	$\widehat{\Pi}$
M_I	$D_4^-(q)$	$A_1^+(q^2)\times A_2^+(q)$	$A_2^+(q^2)\times A_1^+(q)$	$A_5^-(q)$	$E_6^-(q)$

one can extract the p-coset poset Euler characteristic $\overline{\chi}(\mathcal{C}^{p+*}_{E_6^-(q)})$ for $E_6^-(q)$. For instance, the normalised Euler characteristic $\overline{\chi}(\mathcal{C}^{2+*}_{E_6^-(2)}) = 472236184021809092886 1$ at $q=2$.

Example 4.5 ($^3D_4(q)$) The Borel index of $^3D_4(q)$ is the polynomial [4, pp. 251–252, Theorem 14.3.2]

$$\mathrm{KB}(^3D_4(q)) = \frac{(q^2-1)(q^6-1)[3](q^4)}{(q-1)(q^3-1)} = [2](q)[2](q^3)[3](q^4)$$

of degree 12. Let $\Pi = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$ be the fundamental roots of $D_4(q)$ and $\widehat{\Pi} = \{\widehat{\alpha}_1,\widehat{\alpha}_2\}$, $\widehat{\alpha}_1 = \{\alpha_1,\alpha_3,\alpha_4\}$, $\widehat{\alpha}_2 = \{\alpha_2\}$ those of $^3D_4(q)$. The data of the table

I	$\emptyset$	$\{\widehat{\alpha}_1\}$	$\{\widehat{\alpha}_2\}$	$\widehat{\Pi}$
M_I	1	$A_1^+(q^3)$	$A_1^+(q)$	$^3D_4(q)$

determine the Solomon identities and the Euler characteristic

$$\begin{aligned}\overline{\chi}(\mathcal{C}^{p+*}_{{}^3D_4(q)}) &= \mathrm{KB}({}^3D_4(q)) - \frac{\mathrm{KB}({}^3D_4(q))}{\mathrm{KB}(A_1^+(q^3))}q^6 - \frac{\mathrm{KB}({}^3D_4(q))}{\mathrm{KB}(A_1^+(q)}q^2 + q^{24} \\ &= q^{24} - q^{15} - q^{14} - q^{13} + q^{12} - 2q^{10} + 2q^8 - 2q^6 + 2q^4 \\ &\quad + q^3 - q^2 + q + 1\end{aligned}$$

of the coset poset of ${}^3D_4(q)$ at the defining characteristic.

Example 4.6 (${}^2B_2(q)$, $q = 2^{a+\frac{1}{2}}$) The Borel index of the Suzuki group ${}^2B_2(q)$ [4, pp. 251–252, Theorem 14.3.2] is

$$\mathrm{KB}({}^2B_2(q)) = \frac{(q^2-1)(q^4+1)}{(q^2-1)} = 1 + q^4.$$

The data of the table

I	$\emptyset$	$\widehat{\Pi}$
M_I	1	${}^2B_2(q)$

determine the Solomon identities $\mathrm{KB}({}^2B_2(q)) - 1 = q^4$, $\mathrm{KB}({}^2B_2(q)) - q^4 = 1$, and the normalised Euler characteristic

$$\overline{\chi}(\mathcal{C}^{2+*}_{{}^2B_2(q)}) = \mathrm{KB}({}^2B_2(q)) - q^8 = 1 + q^4 - q^8.$$

With $q^2 = 2, 2^3, 2^5$ the Euler characteristics are $-11, -4031, -1047551$.

Example 4.7 (${}^2F_4(q)$, $q = 2^{a+\frac{1}{2}}$) The Borel index of the Ree group ${}^2F_4(q)$ is [4, pp. 251–252, Theorem 14.3.2]

$$\mathrm{KB}({}^2F_4(q)) = \frac{(q^2-1)(q^6+1)(q^8-1)(q^{12}+1)}{(q^2-1)(q^2-1)} = [4](q^2)[2](q^6)[2](q^{12})$$

of degree 24. Let $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the fundamental roots of F_4 and $\widehat{\Pi} = \{\widehat{\alpha}_1, \widehat{\alpha}_2\}$ those of ${}^2F_4(q)$. The data of the table

I	$\emptyset$	$\{\widehat{\alpha}_1\}$	$\{\widehat{\alpha}_2\}$	$\widehat{\Pi}$
M_I	1	$A_1^+(q^2)$	${}^2B_2(q)$	${}^2F_4(q)$

determine the Solomon identities and the identity of Theorem 3.1,

$$1 = \mathrm{KB}({}^2F_4(q)) - q^2\frac{\mathrm{KB}({}^2F_4(q))}{\mathrm{KB}(A_1^+(q^2))} - q^4\frac{\mathrm{KB}({}^2F_4(q))}{\mathrm{KB}({}^2B_2(q))} + q^{24}$$

$$q^{24} = \mathrm{KB}({}^2F_4(q)) - \frac{\mathrm{KB}({}^2F_4(q))}{\mathrm{KB}(A_1^+(q^2))} - \frac{\mathrm{KB}({}^2F_4(q))}{\mathrm{KB}({}^2B_2(q))} + 1$$

$$\begin{aligned}\overline{\chi}(\mathcal{C}^{2+*}_{^2F_4(q)}) &= \mathrm{KB}(^2F_4(q)) - q^4\frac{\mathrm{KB}(^2F_4(q))}{\mathrm{KB}(A_1^+(q^2))} - q^8\frac{\mathrm{KB}(^2F_4(q))}{\mathrm{KB}(^2B_2(q))} + q^{48} \\ &= q^{48} - q^{28} - 2q^{26} + q^{24} - q^{22} - q^{20} + 2q^{18} - q^{16} - q^{14} + 2q^{12} \\ &\quad - q^{10} - q^8 + 2q^6 + q^2 + 1.\end{aligned}$$

For example, the 2-coset poset of $^2F_4(2^{\frac{1}{2}})$ has normalised Euler characteristic 16746211.

Example 4.8 ($G_2(q)$ and $^2G_2(q)$ for $q = 3^{a+\frac{1}{2}}$) The Borel indices of the Chevalley group $G_2(q)$ and the Ree group $^2G_2(q)$ are [4, pp. 251–252, Theorem 14.3.2]

$$\mathrm{KB}(G_2(q)) = \frac{(q^2-1)(q^6-1)}{(q-1)(q-1)} = [2](q)[6](q),$$
$$\mathrm{KB}(^2G_2(q)) = \frac{(q^2-1)(q^6+1)}{q^2-1} = q^6 + 1.$$

The data of the tables

I	$\emptyset$	$\{\alpha_1\}$	$\{\alpha_2\}$	Π
M_I	1	$A_1^+(q)$	$A_1^+(q)$	$G_2(q)$

I	$\emptyset$	$\widehat{\Pi}$
M_I	1	$^2G_2(q)$

explain the normalised Euler characteristics

$$\overline{\chi}(\mathcal{C}^{p+*}_{G_2(q)}) = q^{12} - 2q^7 - q^6 + 2q + 1, \qquad \overline{\chi}(\mathcal{C}^{3+*}_{^2G_2(q)}) = -q^{12} + q^6 + 1,$$
$$q = 3^{a+\frac{1}{2}}, \quad a \geq 0.$$

For example, the normalized Euler characteristic of the 3-coset poset for $^2G_2(3^{\frac{1}{2}})$ is -701.

Example 4.9 Here are the explicit versions of Eq. (3.5)

$$\frac{q^{2\deg \mathrm{KB}(A_m^+(q))}}{\mathrm{KB}(A_m^+(q))} = \sum_{(m_1,\dots,m_k)\in \mathrm{OP}(m+1)} (-1)^{m-k+1} \prod_{1\leq i\leq k} \frac{\overline{\chi}(\mathcal{C}^{p+*}_{A^+_{m_i-1}(q)})}{\mathrm{KB}(A^+_{m_i-1}(q))}$$

$$\begin{aligned}\frac{q^{2\deg \mathrm{KB}(A^-_{2m-\delta}(q))}}{\mathrm{KB}(A^-_{2m-\delta}(q))} &= \sum_{(m_1,\dots,m_k)\in \mathrm{OP}(m)} (-1)^{m-k} \prod_{1\leq i<k} \frac{\overline{\chi}(\mathcal{C}^{p+*}_{A^+_{m_i-1}(q^2)})}{\mathrm{KB}(A^+_{m_i-1}(q^2))} \\ &\quad \cdot \left(\frac{\overline{\chi}(\mathcal{C}^{p+*}_{A^+_{m_k-1}(q^2)})}{\mathrm{KB}(A^+_{m_k-1}(q^2))} - \frac{\overline{\chi}(\mathcal{C}^{p+*}_{A^-_{2m_k-\delta}(q)})}{\mathrm{KB}(A^-_{2m_k-\delta}(q))} \right)\end{aligned}$$

$$\frac{q^{2\deg \mathrm{KB}(B_m(q))}}{\mathrm{KB}(B_m(q))} = \sum_{(m_1,\dots,m_k)\in \mathrm{OP}(m+1)} (-1)^{m-k+1} \prod_{1\leq i<k} \frac{\overline{\chi}(\mathcal{C}^{p+*}_{A^+_{m_i-1}(q)})}{\mathrm{KB}(A^+_{m_i-1}(q))}$$

$$\cdot \frac{\overline{\chi}(\mathcal{C}^{p+*}_{B_{m_k-1}(q)})}{\mathrm{KB}(B_{m_k-1}(q))}$$

$$\frac{q^{2\deg \mathrm{KB}(D^+_m(q))}}{\mathrm{KB}(D^+_m(q))} = \sum_{(m_1\dots,m_k)\in \mathrm{OP}(m-1)} (-1)^{m-k+1} \prod_{1\le i<k} \frac{\overline{\chi}(\mathcal{C}^{p+*}_{A^+_{m_i-1}(q)})}{\mathrm{KB}(A^+_{m_i-1}(q))} \cdot \left(\frac{\overline{\chi}(\mathcal{C}^{p+*}_{A^+_{m_k-1}(q)})}{\mathrm{KB}(A^+_{m_k-1}(q))} - 2\frac{\overline{\chi}(\mathcal{C}^{p+*}_{A^+_{m_k}(q)})}{\mathrm{KB}(A^+_{m_k}(q))} + \frac{\overline{\chi}(\mathcal{C}^{p+*}_{D^+_{m_k+1}(q)})}{\mathrm{KB}(D^+_{m_k+1}(q))} \right)$$

$$\frac{q^{2\deg \mathrm{KB}(D^-_m(q))}}{\mathrm{KB}(D^-_m(q))} = \sum_{(m_1\dots,m_k)\in \mathrm{OP}(m-1)} (-1)^{m-k} \prod_{1\le i<k} \frac{\overline{\chi}(\mathcal{C}^{p+*}_{A^+_{m_i-1}(q)})}{\mathrm{KB}(A^+_{m_i-1}(q))} \cdot \frac{\overline{\chi}(\mathcal{C}^{p+*}_{D^-_{m_k}(q)})}{\mathrm{KB}(D^-_{m_k}(q))}$$

for the groups $A^{\pm}_m(q)$, $B_m(q)$, $C_m(q)$, $D^{\pm}_m(q)$.

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Declarations

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