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POLYNOMIAL COMPLEMENTS

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The complement of a polynomial covering is shown to be, up to homotopy, a fibre bundle with fibre a wedge of circles and the braid group as structure group.

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polynomial covering	complement fibration
homotopy classification	braid group
principal $B(n)$-bundle	fundamental group

1. Introduction

A Weierstrass polynomial of degree n over a topological space X is a continuous family, parametrized by X, of simple, normed complex polynomials of degree n. As shown by Hansen [6, 7, 8], the zero set for such a family traces out a (polynomial) covering space embedded in the trivial complex line bundle over X. Here we shall study the associated nonzero sets; i.e. the complements of the polynomial coverings. These polynomial complements turn out to be (total spaces of) sectioned fibrations over X.

As normed complex polynomials are determined by their roots, the configuration space, $B^n(\mathbb{C})$, of n unordered distinct points in the complex plane is bound to appear in almost any exposition on polynomial covering spaces. Indeed, $B^n(\mathbb{C})$ is the base space for the canonical n-fold polynomial covering [6] from which any other can be obtained by pull back. However, $B^n(\mathbb{C})$ is not a classifying space in the usual sense since nonhomotopic maps may very well induce equivalent polynomial coverings [7, Example 4.3]. This phenomenen is not due to any defect of $B^n(\mathbb{C})$ for polynomial coverings simply do not admit a classifying space [10]. The reason for this seems to be that, ignoring the ambient trivial complex line bundle, we are using a badly adapted notion of equivalence. This point of view is supported by the main result (Theorem 3.3) of this note asserting that $B^n(\mathbb{C})$ does classify polynomial complement fibrations under a restricted class of fibre homotopy equivalences.

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Besides this main result, this note contains a computation (Theorem 2.6) of the fundamental group of a polynomial complement. The homology groups of a polynomial complement were computed in [11].

2. Polynomial complement fibrations

Let X denote a 0-connected topological space and $n>1$ an integer.

Recall the following facts from [6] and [7]. A simple Weierstrass polynomial of degree n over X is a complex function $P: X\times\mathbb{C}\to\mathbb{C}$ of the form

$$P(x,z)=z^n+\sum_{1=1}^{n} a_i(x)z^{n-i},\quad (x,z)\in X\times\mathbb{C},$$

where $a_1,\ldots,a_n: X\to\mathbb{C}$ are continuous complex functions such that, for any fixed $x\in X$, the complex polynomial $P(x,z)$ has no multiple roots. Then

$$X\times\mathbb{C}\supset V_P:=\{(x,z)\mid P(x,z)=0\}\xrightarrow{\pi_P} X,\quad \pi_P(x,z)=x,$$

is an n-fold (polynomial) covering of X with root map

$$z_P: X\to B^n(\mathbb{C}):=\{b\subset\mathbb{C}\mid \# b=n\}$$

given by $z_P(x)=\{z\in\mathbb{C}\mid P(x,z)=0\}$. The canonical polynomial covering,

$$B^n(\mathbb{C})\times\mathbb{C}\supset V^n(\mathbb{C})=\{(b,z)\mid z\in b\}\xrightarrow{\pi^n(\mathbb{C})} B^n(\mathbb{C}),$$

has the identity as its root map, and $\pi_P\cong z_P^*\pi^n(\mathbb{C})$.

Further, let $B(n)$ denote the group of isotopy classes of geometric n-braids in $\mathbb{R}^3$ [3]. The fundamental group $\pi_1(B^n(\mathbb{C}),b_0)$ is canonically isomorphic to $B(n)$ in the following way: If $\alpha:(I,\dot{I})\to(B^n(\mathbb{C}),b_0)$ is a loop, representing an element of $\pi_1(B^n(\mathbb{C}),b_0)$, then

$$\alpha^* V^n(\mathbb{C})\subset I\times\mathbb{C}\subset\mathbb{R}^3$$

is a geometric n-braid representing an element of $B(n)$. (Actually, the higher homotopy groups of $B^n(\mathbb{C})$ vanish, so $B^n(\mathbb{C})=K(B(n),1)$.)

Abstractly, $B(n)$ is the group on $n-1$ generators, $\sigma_1,\ldots,\sigma_{n-1}$, with relations

$$\sigma_i\sigma_j=\sigma_j\sigma_i,\qquad |i-j|\geqslant 2,$$

$$\sigma_i\sigma_{i+1}\sigma_i=\sigma_{i+1}\sigma_i\sigma_{i+1},\quad 1\leqslant i\leqslant n-2.$$

There is a faithful representation, [3, Corollary 1.8.3]

$$\theta(n):\ B(n)\to \operatorname{Aut}\mathbb{F}(n),$$

of $B(n)$ as a group of automorphisms of the free group $\mathbb{F}(n)$ on n generators $x_1, \ldots, x_n$, given by

$$\sigma_i(x_j) = \begin{cases} x_{i+1}, & j = i, \\ x_{i+1}^{-1} x_i x_{i+1}, & j = i+1, \\ x_j, & j \neq i, i+1. \end{cases}$$

Also this action has a geometric realization as we shall see shortly.

Consider now the nonzero set $(X \times \mathbb{C}) - V_P$ of P. Let

$$(X \times \mathbb{C}) - V_P \underset{s_P}{\overset{c_P}{\rightleftarrows}} X$$

be the maps given by

$$c_P(x, z) = x, \qquad s_P(x) = \left(x, 1 + \sum_{z \in z_P(x)} |z| \right).$$

Then c_P is a fibration, see [11, Lemma 2.1] or Lemma 2 below, and s_P a section of c_P.

In the following we always use $s_P(x)$ as the base point for the fibre $c_P^{-1}(x) = \mathbb{C} - z_P(x)$.

Definition 2.1. The sectioned fibration

$$(X \times \mathbb{C}) - V_P \underset{s_P}{\overset{c_P}{\rightleftarrows}} X$$

is denoted by C_P and is called the *polynomial complement fibration* of degree n associated to the simple Weierstrass polynomial P.

Observe that the constructions of π_P and C_P are natural: If $g: X \to Z$ is some map and $P = R \circ (g \times 1)$ for some simple Weierstrass polynomial R over Z, then $\pi_P = g^* \pi_R$ and $C_P = g^* C_R$. In particular, $C_P = z_P^* C^n(\mathbb{C})$, where

$$C^n(\mathbb{C}) = \left((B^n(\mathbb{C}) \times \mathbb{C}) - V^n(\mathbb{C}) \underset{s^n(\mathbb{C})}{\overset{c^n(\mathbb{C})}{\rightleftarrows}} B^n(\mathbb{C}) \right)$$

is the canonical polynomial complement fibration formed from the canonical n-fold polynomial covering.

For any two configurations $b_0, b_1 \in B^n(\mathbb{C})$ of n points in the complex plane, let

$$B(n, b_0, b_1) \subset \operatorname{Hom}(\pi_1(\mathbb{C} - b_0, s^n(\mathbb{C})(b_0)), \pi_1(\mathbb{C} - b_1, s^n(\mathbb{C})(b_1)))$$

be the set (or affine group) of isomorphisms h_* induced by based homeomorphisms

$$h: (\mathbb{C} - b_0, s^n(\mathbb{C})(b_0)) \to (\mathbb{C} - b_1, s^n(\mathbb{C})(b_1))$$

that equal the identity outside some compact set (which may depend on h). Note that $B(n; \underline{n}, \underline{n})$, where $\underline{n} = \{1, 2, \ldots, n\} \in B^n(\mathbb{C})$, is a copy [3, Theorem 1.10; 2] of the braid group $B(n)$.

Let R be a simple Weierstrass polynomial over some space Z and form the complement fibration C_R over and under Z.

Definition 2.2. A *braid map* of C_P into C_R is a pair of maps (h, g) such that the diagram

$$\begin{array}{ccc}
(X\times\mathbb{C})-V_P & \xrightarrow{\;h\;} & (Z\times\mathbb{C})-V_R. \\
{\scriptstyle c_P}\big\downarrow\big\uparrow{\scriptstyle s_P} & & {\scriptstyle c_R}\big\downarrow\big\uparrow{\scriptstyle s_R} \\
X & \xrightarrow{\;g\;} & Z
\end{array}$$

commutes and such that the induced maps

$$(h|\mathbb{C}-z_P(x))_*:\ \pi_1(\mathbb{C}-z_P(x))\to\pi_1(\mathbb{C}-z_R(g(x)))$$

belong to $B(n;\, z_P(x), z_R(g(x)))$ for all $x\in X$. If $Z=X$ and $g=1$ is the identity, then $h=(h, 1)$ is called a *braid equivalence.*

By Dold's theorem [4], any braid equivalence is indeed a fibre homotopy equivalence and any fibre homotopy inverse is again a braid map.

The category of polynomial complement fibrations of degree n has as objects the sectioned fibrations C_P associated to Weierstrass polynomials P of degree n, and as morphisms, braid maps. Any object in this category is in fact locally trivial:

Lemma 2.3. *Let P be a simple Weierstrass polynomial of degree n over X. There exists a numerable* [9] *open covering $\{U_\alpha\}_{\alpha\in A}$ of X together with homeomorphic braid equivalences*

$$h_\alpha:\ U_\alpha\times(\mathbb{C}-\underline{n})\to c_P^{-1}(U_\alpha)=(U_\alpha\times\mathbb{C})-V_P|U_\alpha$$

for all $\alpha\in A$.

Proof. It suffices, by naturality, to consider the canonical complement $C^n(\mathbb{C})$ over $B^n(\mathbb{C})$. Since $B^n(\mathbb{C})$ is [7] (homeomorphic to) an open set in $\mathbb{C}^n$, $B^n(\mathbb{C})$ can be covered by open contractible sets U_α with compact closures. This covering is numerable as is any open covering of the paracompact Hausdorff space $B^n(\mathbb{C})$.

Let $U=U_\alpha$ be one of the open sets of the covering. Choose a compact disc $D^2\subset\mathbb{C}$ such that $U\overset{i}{\hookrightarrow}B^n(\operatorname{int}D^2)\subset B^n(\mathbb{C})$, $\mathrm{pr}_2\circ s^n(\mathbb{C})(U)\subset\operatorname{int}D^2$, $\underline{n}\subset\operatorname{int}D^2$, and $s^n(\mathbb{C})(\underline{n})\in\operatorname{int}D^2$. Let $\mathrm{TOP}(D^2, S^1)$ be the topological group of all homeomorphisms of D^2 that fix the boundary $\partial D^2=S^1$ pointwise. Consider the fibration [3, Theorem 4.1]

$$\mathrm{TOP}(D^2, S^1)\to B^n(D^2)\times D^2$$

defined by evaluation at $\underline{n}\subset D$ and at $s^n(\mathbb{C})(\underline{n})\in D^2$. Since U is contractible, the map $(i, \mathrm{pr}_2\circ s^n(\mathbb{C})): U\to B^n(D^2)\times D^2$ has a lift $\varphi: U\to\mathrm{TOP}(D^2, S^1)$. Then

$$h:\ U\times(\mathbb{C}-\underline{n})\to(U\times\mathbb{C})-E^n(\mathbb{C})|U,$$

given by $h(x, z)=(x, \varphi(x)(z))$ for $z\in D^2-\underline{n}$ and $h(x, z)=z$ for $z\in\mathbb{C}-D^2$, $x\in U$, is a homeomorphic braid equivalence. □

For any two points, x_0 and x_1, of X, let $\pi_1(X; x_0, x_1)$ be the set of homotopy classes (rel. endpoints) of paths from x_0 to x_1. Define a map

$$\theta_P:\ \pi_1(X; x_0, x_1) \to \mathrm{Hom}(\pi_1(\mathbb{C} - z_P(x_1)), \pi_1(\mathbb{C} - z_P(x_0)))$$

by dragging the based fibres of c_P along paths from x_0 to x_1; i.e. if u is such a path and H is a solution to the homotopy lifting extension problem

$$\begin{array}{ccccc} 1\times(\mathbb{C}-z_P(x_1))\cup I\times s_P(x_1) & & \xrightarrow{i\cup s_P u} & & (X\times\mathbb{C})-V_P \\ \downarrow & & \overset{H}{\nearrow} & & \downarrow c_P \\ I\times(\mathbb{C}-z_P(x_1)) & \xrightarrow{pr_1} & I & \xrightarrow{u} & X \end{array}$$

then $\theta_P[u] = (H_0)_*$; see [12, IV.8].

Lemma 2.4. *Let H be as above. Then*

$$(H_t)_* \in B(n; z_P(x_1), z_P(u(t)))$$

for all $t \in I$. In particular, $\theta_P[u] \in B(n; z_P(x_1), z_P(x_0))$.

The proof of Lemma 2.4 is very similar to that of Lemma 2.3 and is therefore omitted.

According to Lemma 2.4, θ_P can be viewed as a functor, or groupoid morphism,

$$\theta_P:\ \pi_1(X, X) \to b^n(\mathbb{C})^{\mathrm{op}}$$

of the fundamental groupoid of X into the opposite of $b^n(\mathbb{C})$; here $b^n(\mathbb{C})$ denotes the groupoid with $B^n(\mathbb{C})$ as object set and morphisms

$$b^n(\mathbb{C})(b_0, b_1) = B(n; b_0, b_1).$$

For $\alpha \in b(\mathbb{C})^{\mathrm{op}}(b_0, b_1)$, $\beta \in b^n(\mathbb{C})^{\mathrm{op}}(b_1, b_2)$, the category composition $\alpha \cdot \beta = \alpha \circ \beta \in b^n(\mathbb{C})^{\mathrm{op}}(b_0, b_2)$, so $\theta_P([uv]) = \theta_P[u] \cdot \theta_P[v]$ if $u(1) = v(0)$.

In the canonical situation we obtain a groupoid morphism

$$\theta^n(\mathbb{C}):\ \pi_1(B^n(\mathbb{C}), B^n(\mathbb{C})) \to b^n(\mathbb{C})^{\mathrm{op}}$$

of the fundamental groupoid of $B(\mathbb{C})$ into $b^n(\mathbb{C})^{\mathrm{op}}$.

Lemma 2.5. *$\theta^n(\mathbb{C})$ is an isomorphism of groupoids.*

Proof. Since $\theta^n(\mathbb{C})$ preserves the objects and both groupoids in question are connected, it suffices to show that $\theta^n(\mathbb{C})$ is a group isomorphism on a vertex group. Consider

$$\theta^n(\mathbb{C}):\ \pi_1(B^n(\mathbb{C}), \underline{n}) \to b^n(\mathbb{C})^{\mathrm{op}}(\underline{n}, \underline{n}) = B(n; \underline{n}, \underline{n}).$$

This action is defined by dragging a disc with n holes up along a geometric n-braid. So is $\theta(n)$ [2; 3, Theorem 1.1] and hence $\theta^n(\mathbb{C}) = \theta(n)$ under some obvious identifications. But $\theta(n): B(n) \to \mathrm{Aut}\,\mathbb{F}(n)$ is faithful, so $\theta^n(\mathbb{C}) = \theta(n): B(n) \to \theta(n)(B(n)) = B(n; \underline{n}, \underline{n})$ is an isomorphism. □

Since $\theta_P = \theta^n(\mathbb{C}) \circ (z_P)_*$, by naturality, and $\theta^n(\mathbb{C}) = \theta(n)$, we arrive at the following result, which also (almost) appeared in [2; and 3, Theorem 2.2].

Theorem 2.6. *Let P be a simple Weierstrass polynomial of degree n over X and $x_0 \in X$ a base point. Then*

$$\pi_1((X \times \mathbb{C}) - V_P, s_P(x_0)) \cong \mathbb{F}(n) \rtimes \pi_1(X, x_0)$$

where the semi-direct product is w.r.t. the action

$$\pi_1(X, x_0) \xrightarrow{(z_P)_*} \pi_1(B^n(\mathbb{C}), z_P(x_0)) \cong B(n) \xrightarrow{\theta(n)} \operatorname{Aut} \mathbb{F}(n)$$

induced by the root map $z_P : X \to B^n(\mathbb{C})$.

In the canonical situation, Theorem 2.6 implies

$$(B^n(\mathbb{C}) \times \mathbb{C}) - V^n(\mathbb{C}) = K(\mathbb{F}(n) \rtimes B(n), 1).$$

For $n = 2$, the group $\mathbb{F}(2) \rtimes B(2)$ has two generators, x, y, and one relation $[x, y^2] = 1$. I do not know any nice presentation of the semi-direct product when $n > 2$.

3. Classification of polynomial complements

The purpose of this section is to verify that $B^n(\mathbb{C})$ is, in some sense, a classifying space for polynomial complements, or, to put in another way, that a polynomial complement fibration is essentially the same thing as a principal $B(n)$-bundle. We do this by constructing explicitly a functor from polynomial complement fibrations to principal $B(n)$-bundles.

In the following, let $P, Q : X \times \mathbb{C} \to \mathbb{C}$ be two simple Weierstrass polynomials of degree n over X.

Define the set

$$E_P := \coprod_{x \in X} B(n; \underline{n}, z_P(x))$$

as the disjoint union of the (discrete) affine groups $B(n; \underline{n}, z_P(x))$, $x \in X$. Let $\omega_P : E_P \to X$ be the obvious map. There is a unique topology on E_P such that for any open set $U \subset X$ and any braid equivalence

$$h:\ U \times (\mathbb{C} - \underline{n}) \to c_P^{-1}(U) = (U \times \mathbb{C}) - V_P | U$$

the induced bijection

$$E(h):\ U \times B(n) \to \omega_P^{-1}(U) = \coprod_{x \in U} B(n; \underline{n}, z_P(x))$$

is a homeomorphism. Equip E_P with this topology and with the right action

$$E_P \times B(n) \to E_P$$

obtained by pre-composing with the isomorphisms in $B(n) = B(n; \underline{n}, \underline{n})$.

Lemma 3.1. *$\omega_P : E_P \to X$ is a numerable principal $B(n)$-bundle.*

Proof. Use Lemma 2.3. □

Thus $C_P \rightsquigarrow \omega_P$ is a functor, ω, from the category of polynomial complement fibrations of degree n to the category of numerable principal $B(n)$-bundles. Let

$$\omega^n(\mathbb{C}): \ E^n(\mathbb{C}) \to B^n(\mathbb{C})$$

be the result of applying this functor in the canonical situation. Then $\omega_P = z_P^* \omega^n(\mathbb{C})$.

Lemma 3.2. *$\omega^n(\mathbb{C}) : E^n(\mathbb{C}) \to B^n(\mathbb{C})$ is a universal principal $B(n)$-bundle.*

Proof. It suffices to show that $E^n(\mathbb{C})$ is weakly contractible, since $B^n(\mathbb{C})$, and hence $E^n(\mathbb{C})$, has the homotopy type of a CW-complex. The only possibly non-zero homotopy group is the fundamental group. Consider the boundary map of $\omega^n(\mathbb{C})$,

$$\partial: \ \pi_1(B^n(\mathbb{C}), \underline{n}) \to B(n; \underline{n}, \underline{n}),$$

which according to Lemma 2.4, equals $\theta^n(\mathbb{C})$. But $\theta^n(\mathbb{C})$ is an isomorphism by Lemma 2.5 and hence also $\pi_1(E^n(\mathbb{C}), 1) = 0$ by exactness. □

The main result of this note is the following theorem.

Theorem 3.3. *The following are equivalent*:

(a) *z_P is homotopic to z_Q,*

(b) *ω_P is equivalent to ω_Q,*

(c) *C_P is equivalent to C_Q.*

Proof. The bi-implication between (a) and (b) follows from Lemma 2.3 since z_P is a classifying map for ω_P.

Suppose C_P is braid equivalent to C_Q. Then $z_P^* \omega^n(\mathbb{C}) = \omega_P \cong \omega_Q = z_P^* \omega^n(\mathbb{C})$, since ω is a functor, and hence $z_P \simeq z_Q$, since $\omega^n(\mathbb{C})$ is universal.

Finally, suppose z_P is homotopic to z_Q. The task is to construct a braid equivalence of C_P into C_Q. Let $H: I \times X \to B^n(\mathbb{C})$ be a homotopy of $H_0 = z_P$ to $H_1 = z_Q$. Consider H as the root map of a simple Weierstrass polynomial of degree n over $I \times X$ and let

$$(I \times X \times \mathbb{C}) - V \underset{s}{\overset{c}{\rightleftarrows}} I \times X$$

$$\theta: \ \pi_1(I \times X, I \times X) \to b^n(\mathbb{C})^{\mathrm{op}}$$

be the associated polynomial complement fibration and groupoid morphism, respectively. By the homotopy lifting extension property for the fibration c, we can find a homotopy

$$G: \ I \times (X \times \mathbb{C} - V_P) \to (I \times X \times \mathbb{C}) - V$$

such that G_0 is the inclusion, $G(t, s_P(x)) = s(t, x)$, and $cG(t, x, z) = (t, x)$ for all $(t, x, z) \in I \times (X \times \mathbb{C} - V_P)$. Then

$$h := G_1 \colon (X \times \mathbb{C}) - V_P \to (I \times X \times \mathbb{C}) - V_Q$$

is a map over and under X. Moreover, h is a braid map, since, for any $x \in X$, the induced map

$$(h | \mathbb{C} - z_P(x))_* = \omega[t \to (1 - t, x)]$$

and $\omega\pi_1(I \times X; (1, x), (0, x)) \subset B(n; z_P(x), z_Q(x))$ by Lemma 2.4. □

Let $PCF_n(X)$ and $k_{B(n)}$ be the sets of equivalence classes of, respectively, polynomial complement fibrations of degree n over X and numerable principal $B(n)$-bundles over X. Then there is a commutative diagram

$$\begin{array}{ccccc}
 & z_P & \longleftarrow & C_P & \\
 & [X, B^n(\mathbb{C})] & \longleftarrow & PCF_n(X) & \\
z & & \searrow \quad \omega \swarrow & & C_P \\
\searrow & & & & \swarrow \\
z^*\omega^n(\mathbb{C}) & & k_{B(n)}(X) & & \omega_P
\end{array}$$

of bijections between sets of equivalence classes, so $B^n(\mathbb{C})$ is indeed a classifying space for polynomial complement fibrations of degree n. The construction of the principal $B(n)$-bundle $\omega_P = z_P^*\omega^n(\mathbb{C})$ from the polynomial complement C_P is similar to, e.g., the construction of a principal $\mathrm{GL}(n; \mathbb{R})$-bundle from a real vector bundle. To regain the vector bundle one forms the associated $\mathbb{R}^n$ fibre bundle and to regain the polynomial complement one constructs the associated $K(\mathbb{F}(n), 1)$-bundle.

Remark 3.4. The action $\theta(n) \colon B(n) \to \operatorname{Aut} \mathbb{F}(n)$ can be realized geometrically as an action

$$B(n) \to \operatorname{Aut}_0 K(\mathbb{F}(n), 1)$$

of $B(n)$ on the simplicial set $K(\mathbb{F}(n), 1)$ by based simplicial automorphisms.

For any simple Weierstrass polynomial P, the associated sectioned fibre bundle

$$\omega_P[K(\mathbb{F}(n), 1)] \colon E_P \times_{B(n)} K(\mathbb{F}(n), 1) \rightleftarrows X$$

is homotopy equivalent, over and under X, to the polynomial complement fibration C_P.

Consequently, up to homotopy over and under X, the polynomial complement fibrations of degree n are precisely the sectioned fibre bundles with $K(\mathbb{F}(n), 1)$ as fibre and structure group $B(n)$.

We finish this note by returning to the polynomial coverings. By continuity, any braid equivalence h of C_P into C_Q extends in a unique way to a map of triples

$$H:\ (X\times\mathbb{C};(X\times\mathbb{C})-V_P,V_P)\to(X\times\mathbb{C};(X\times\mathbb{C})-V_Q,V_Q)$$

such that $h|V_P\to V_Q$ is an equivalence of coverings. Using this observation, we arrive at the following corollary.

Corollary 3.5. *P and Q have homotopic root maps iff there exists a map of triples*

$$H:\ (X\times\mathbb{C};(X\times\mathbb{C})-V_P,V_P)\to(X\times\mathbb{C};(X\times\mathbb{C})-V_Q,V_Q)$$

such that $h|V_P: V_P\to V_Q$ *is an equivalence of coverings and* $h|(X\times\mathbb{C})-V_P:(X\times\mathbb{C})-V_P\to(X\times\mathbb{C})-V_Q$ *is a braid equivalence.*

It is perhaps this formulation of Theorem 3.3 which most clearly expresses exactly what $B^n(\mathbb{C})$ does classify in relation to polynomial coverings.

Example 3.6. (a) The polynomial coverings of the polynomials z^2-x and z^2-x^3 over S^1 are equivalent but the corresponding polynomial complement fibrations, classified by $\sigma_1, \sigma_1^3\in B(2)$, are inequivalent.

(b) The polynomial covering $\pi:\hat{\beta}\to S^1$ obtained by closing the geometric 3-braid $B=\sigma_2\sigma_1^2\sigma_2^{-1}$ is trivial, but the complement is non-trivial for so is the conjugacy class of $\beta\in B(3)$.

The first of the above examples was taken from Arnol'd [1, p. 31]. The complement fibration is a geometric realization of what Arnol'd calls the braid group of an algebraic function. The second example shows that a trivial polynomial covering may have a nontrivial complement. This phenomenen can, however, only occur in the presence of other nontrivial polynomial coverings.

Corollary 3.6. *Suppose that* $H_1(X;\mathbb{Z})$ *is a finitely generated abelian group. Then all n-fold polynomial coverings over X are trivial if and only if all polynomial complement fibrations of degree n over X are trivial.*

Proof. According to [5, Theorem 1.4] and Theorem 3.5, both statements are equivalent to $\mathrm{Hom}(\pi_1(X), B(n))=\{1\}$. □

References

[1] V.I. Arnol'd, On some topological invariants of algebraic functions, Trudy Moskovsk. Mat. Obsc. 21 (1970) 27-46; and, Trans. Moscow Math. Soc. 21 (1970) 30-52.

[2] E. Artin, Theorie der Zöpfe, Abh. Math. Sem. Univ. Hamburg 4 (1925) 47-72.

[3] J.S. Birman, Braids, Links, and Mapping Class Groups, Annals of Mathematics Studies 82 (Princeton University Press/University of Tokyo Press, Princeton, 1975).

[4] A. Dold, Partitions of unity in the theory of fibrations, Ann. Math. (2) 78 (1963) 223-255.

[5] E.A. Gorin and V.Ja. Lin, Algebraic equations with continuous coefficients and certain questions of the algebraic theory of braids, Mat. Sb. 78 (120) (1969) 579-610; and, Math. USSR Sbornik 7 (1969) 569-596.

[6] V.L. Hansen, Coverings defined by Weierstrass polynomials, J. Reine Angew. Math. 314 (1980) 29-39.
[7] V.L. Hansen, Polynomial covering spaces and homomorphisms into the braid groups, Pacific J. Math. 81 (1979) 399-410.
[8] V.L. Hansen, Algebra and topology of Weierstrass polynomials, Expositiones Mathematicae, to appear.
[9] D. Husemoller, Fibre Bundles, Graduate Texts in Mathematics 20 (Springer, Berlin/Heidelberg/New York, 2nd ed., 1975).
[10] J.M. Møller, On polynomial coverings and their classification, Math. Scand. 47 (1980) 116-122.
[11] J.M. Møller, A generalized Gysin sequence, Preprint, 1987.
[12] G.W. Whitehead, Elements of Homotopy Theory, Graduate Texts in Mathematics 61 (Springer, Berlin/New York, 1978).