POLYNOMIAL COMPLEMENTS

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The complement of a polynomial covering is shown to be, up to homotopy, a fibre bundle with fibre a wedge of circles and the braid group as structure group.

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Weierstrass polynomial configuration space
polynomial covering complement fibration
homotopy classification braid group
principal $B(n)$-bundle fundamental group

1. Introduction

A Weierstrass polynomial of degree $n$ over a topological space $X$ is a continuous family, parametrized by $X$, of simple, normed complex polynomials of degree $n$. As shown by Hansen [6, 7, 8], the zero set for such a family traces out a (polynomial) covering space embedded in the trivial complex line bundle over $X$. Here we shall study the associated nonzero sets; i.e. the complements of the polynomial coverings. These polynomial complements turn out to be (total spaces of) sectioned fibrations over $X$.

As normed complex polynomials are determined by their roots, the configuration space, $B^n(C)$, of $n$ unordered distinct points in the complex plane is bound to appear in almost any exposition on polynomial covering spaces. Indeed, $B^n(C)$ is the base space for the canonical $n$-fold polynomial covering [6] from which any other can be obtained by pull back. However, $B^n(C)$ is not a classifying space in the usual sense since nonhomotopic maps may very well induce equivalent polynomial coverings [7, Example 4.3]. This phenomenon is not due to any defect of $B^n(C)$ for polynomial coverings simply do not admit a classifying space [10]. The reason for this seems to be that, ignoring the ambient trivial complex line bundle, we are using a badly adapted notion of equivalence. This point of view is supported by the main result (Theorem 3.3) of this note asserting that $B^n(C)$ does classify polynomial complement fibrations under a restricted class of fibre homotopy equivalences.

Besides this main result, this note contains a computation (Theorem 2.6) of the fundamental group of a polynomial complement. The homology groups of a polynomial complement were computed in [11].

2. Polynomial complement fibrations

Let $X$ denote a 0-connected topological space and $n > 1$ an integer.

Recall the following facts from [6] and [7]. A simple Weierstrass polynomial of degree $n$ over $X$ is a complex function $P : X \times \mathbb{C} \to \mathbb{C}$ of the form

$$P(x, z) = z^n + \sum_{i=1}^{n} a_i(x)z^{n-i}, \quad (x, z) \in X \times \mathbb{C},$$

where $a_1, \ldots, a_n : X \to \mathbb{C}$ are continuous complex functions such that, for any fixed $x \in X$, the complex polynomial $P(x, z)$ has no multiple roots. Then

$$X \times \mathbb{C} \ni V_p := \{(x, z) | P(x, z) = 0\} \xrightarrow{\pi_p} X, \quad \pi_p(x, z) = x,$$

is an $n$-fold (polynomial) covering of $X$ with root map

$$z_p : X \to B^n(\mathbb{C}) := \{b \in \mathbb{C} | \# b = n\}$$

given by $z_p(x) = \{z \in \mathbb{C} | P(x, z) = 0\}$. The canonical polynomial covering,

$$B^n(\mathbb{C}) \times \mathbb{C} \ni V^n(\mathbb{C}) := \{(b, \tau) | \tau \in h\} \xrightarrow{\pi^n(\mathbb{C})} B^n(\mathbb{C}),$$

has the identity as its root map, and $\pi_p \approx z_p^* \pi^n(\mathbb{C})$.

Further, let $B(n)$ denote the group of isotopy classes of geometric $n$-braids in $\mathbb{R}^3$ [3]. The fundamental group $\pi_1(B^n(\mathbb{C}), b_0)$ is canonically isomorphic to $B(n)$ in the following way: If $\alpha : (I, I) \to (B^n(\mathbb{C}), b_0)$ is a loop, representing an element of $\pi_1(B^n(\mathbb{C}), b_0)$, then

$$\alpha^* V^n(\mathbb{C}) \subset I \times \mathbb{C} \subset \mathbb{R}^3$$

is a geometric $n$-braid representing an element of $B(n)$. (Actually, the higher homotopy groups of $B^n(\mathbb{C})$ vanish, so $B^n(\mathbb{C}) = K(B(n), 1)$.)

Abstractly, $B(n)$ is the group on $n-1$ generators, $\sigma_1, \ldots, \sigma_{n-1}$, with relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2,$$

$$\sigma_i \sigma_{i+1} \sigma_{i} = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad 1 \leq i \leq n - 2.$$  

There is a faithful representation, [3, Corollary 1.8.3]

$$\theta(n) : B(n) \to \text{Aut} \mathbb{F}(n),$$
of $B(n)$ as a group of automorphisms of the free group $F(n)$ on $n$ generators $x_1, \ldots, x_n$, given by
\[
\sigma_i(x_j) = \begin{cases} 
  x_{i+1}, & j = i, \\
  x_i^{-1}x_jx_i, & j = i+1, \\
  x_j, & j \neq i, i+1.
\end{cases}
\]
Also this action has a geometric realization as we shall see shortly.

Consider now the nonzero set $(X \times \mathbb{C}) - V_P$ of $P$. Let
\[
(X \times \mathbb{C}) - V_P \xrightarrow{c_P} X
\]
be the maps given by
\[
c_P(x, z) = x, \quad s_P(x) = \left(x, 1 + \sum_{z \in z_P(x)} |z|\right).
\]
Then $c_P$ is a fibration, see [11, Lemma 2.1] or Lemma 2 below, and $s_P$ a section of $c_P$.

In the following we always use $s_P(x)$ as the base point for the fibre $c_P^{-1}(x) = \mathbb{C} - z_P(x)$.

**Definition 2.1.** The sectioned fibration
\[
(X \times \mathbb{C}) - V_P \xrightarrow{c_P} X
\]
is denoted by $C_P$ and is called the *polynomial complement fibration* of degree $n$ associated to the simple Weierstrass polynomial $P$.

Observe that the constructions of $\pi_P$ and $C_P$ are natural: If $g : X \rightarrow Z$ is some map and $P = R \circ (g \times 1)$ for some simple Weierstrass polynomial $R$ over $Z$, then $\pi_P = g^* \pi_R$ and $C_P = g^* C_R$. In particular, $C_P = z_P^* C^n(\mathbb{C})$, where
\[
C^n(\mathbb{C}) = \left((B^n(\mathbb{C}) \times \mathbb{C}) - V^n(\mathbb{C}) \xrightarrow{c^n(\mathbb{C})} B^n(\mathbb{C})\right)
\]
is the canonical polynomial complement fibration formed from the canonical $n$-fold polynomial covering.

For any two configurations $b_0, b_1 \in B^n(\mathbb{C})$ of $n$ points in the complex plane, let
\[
B(n, b_0, b_1) \subset \text{Hom}(\pi_1(\mathbb{C} - b_0, s^n(\mathbb{C})(b_0)), \pi_1(\mathbb{C} - b_1, s^n(\mathbb{C})(b_1)))
\]
be the set (or affine group) of isomorphisms $h$ induced by based homeomorphisms $h : (\mathbb{C} - b_0, s^n(\mathbb{C})(b_0)) \rightarrow (\mathbb{C} - b_1, s^n(\mathbb{C})(b_1))$ that equal the identity outside some compact set (which may depend on $h$). Note that $B(n; n, n)$, where $n = \{1, 2, \ldots, n\} \in B^n(\mathbb{C})$, is a copy [3, Theorem 1.10; 2] of the braid group $B(n)$. 
Let $R$ be a simple Weierstrass polynomial over some space $Z$ and form the complement fibration $C_R$ over and under $Z$.

**Definition 2.2.** A braid map of $C_R$ into $C_R$ is a pair of maps $(h, g)$ such that the diagram

\[
\begin{array}{ccc}
(X \times \mathbb{C}) - V_p & \xrightarrow{h} & (Z \times \mathbb{C}) - V_R \\
\downarrow \scriptstyle{\psi_p} & & \downarrow \scriptstyle{\psi_R} \\
X & \xrightarrow{g} & Z
\end{array}
\]

commutes and such that the induced maps

\[
(h|\mathbb{C} - z_p(x))_*: \pi_1(\mathbb{C} - z_p(x)) \rightarrow \pi_1(\mathbb{C} - z_R(g(x)))
\]

belong to $B(n; z_p(x), z_R(g(x)))$ for all $x \in X$. If $Z = X$ and $g = 1$ is the identity, then $h = (h, 1)$ is called a braid equivalence.

By Dold’s theorem [4], any braid equivalence is indeed a fibre homotopy equivalence and any fibre homotopy inverse is again a braid map.

The category of polynomial complement fibrations of degree $n$ has as objects the sectioned fibrations $C_p$ associated to Weierstrass polynomials $P$ of degree $n$, and as morphisms, braid maps. Any object in this category is in fact locally trivial:

**Lemma 2.3.** Let $P$ be a simple Weierstrass polynomial of degree $n$ over $X$. There exists a numerable [9] open covering $\{U_\alpha\}_{\alpha \in A}$ of $X$ together with homeomorphic braid equivalences

\[
h_\alpha: U_\alpha \times (\mathbb{C} - \mathbb{n}) \rightarrow c_p^{-1}(U_\alpha) = (U_\alpha \times \mathbb{C}) - V_p|U_\alpha
\]

for all $\alpha \in A$.

**Proof.** It suffices, by naturality, to consider the canonical complement $C^n(\mathbb{C})$ over $B^n(\mathbb{C})$. Since $B^n(\mathbb{C})$ is [7] (homeomorphic to) an open set in $\mathbb{C}^n$, $B^n(\mathbb{C})$ can be covered by open contractible sets $U_\alpha$ with compact closures. This covering is numerable as is any open covering of the paracompact Hausdorff space $B^n(\mathbb{C})$.

Let $U = U_\alpha$ be one of the open sets of the covering. Choose a compact disc $D^2 \subset \mathbb{C}$ such that $U \hookrightarrow B^n(\text{int } D^2) \subset B^n(\mathbb{C})$, $\text{pr}_2 \circ s^n(\mathbb{C})(U) \subset \text{int } D^2$, $\mathbb{n} \subset \text{int } D^2$, and $s^n(\mathbb{C})(\mathbb{n}) \subset \text{int } D^2$. Let $\text{TOP}(D^2, S^1)$ be the topological group of all homeomorphisms of $D^2$ that fix the boundary $\partial D^2 = S^1$ pointwise. Consider the fibration [3, Theorem 4.1]

\[
\text{TOP}(D^2, S^1) \rightarrow B^n(D^2) \times D^2
\]

defined by evaluation at $\mathbb{n} \subset D$ and at $s^n(\mathbb{C})(\mathbb{n}) \subset D^2$. Since $U$ is contractible, the map $(i, \text{pr}_2 \circ s^n(\mathbb{C})): U \rightarrow B^n(D^2) \times D^2$ has a lift $\varphi: U \rightarrow \text{TOP}(D^2, S^1)$. Then

\[
h: U \times (\mathbb{C} - \mathbb{n}) \rightarrow (U \times \mathbb{C}) - E^n(\mathbb{C})|U,
\]
given by $h(x, z) = (x, \varphi(x)(z))$ for $z \in D^2 - \mathbb{n}$ and $h(x, z) = z$ for $z \in \mathbb{C} - D^2$, $x \in U$, is a homeomorphic braid equivalence. □
For any two points, $x_0$ and $x_1$, of $X$, let $\pi_1(X; x_0, x_1)$ be the set of homotopy classes (rel. endpoints) of paths from $x_0$ to $x_1$. Define a map

$$\theta_p: \pi_1(X; x_0, x_1) \to \text{Hom}(\pi_1(C - z_p(x_1)), \pi_1(C - z_p(x_0)))$$

by dragging the based fibres of $c_p$ along paths from $x_0$ to $x_1$; i.e. if $u$ is such a path and $H$ is a solution to the homotopy lifting extension problem

$$1 \times (C - z_p(x_1)) \times I \times s_p(x_1) \xrightarrow{\iota \times \iota \times \mu} (X \times C) - V_p$$

$$I \times (C - z_p(x_1)) \xrightarrow{pr_1} I \xrightarrow{u} X$$

then $\theta_p[u] = (H_0)_x$; see [12, IV.8].

Lemma 2.4. Let $H$ be as above. Then

$$(H_t)_x \in B(n; z_p(x_1), z_p(u(t)))$$

for all $t \in I$. In particular, $\theta_p[u] \in B(n; z_p(x_1), z_p(x_0))$.

The proof of Lemma 2.4 is very similar to that of Lemma 2.3 and is therefore omitted.

According to Lemma 2.4, $\theta_p$ can be viewed as a functor, or groupoid morphism,

$$\theta_p: \pi_1(X, X) \to b^n(C)^{\text{op}}$$

of the fundamental groupoid of $X$ into the opposite of $b^n(C)$; here $b^n(C)$ denotes the groupoid with $B^n(C)$ as object set and morphisms

$$b^n(C)(b_0, b_1) = B(n; b_0, b_1).$$

For $\alpha \in b(C)^{\text{op}}(b_0, b_1)$, $\beta \in b^n(C)^{\text{op}}(b_1, b_2)$, the category composition $\alpha \cdot \beta = \alpha \circ \beta \in b^n(C)^{\text{op}}(b_0, b_2)$, so $\theta_p([uv]) = \theta_p[u] \cdot \theta_p[v]$ if $u(1) = v(0)$.

In the canonical situation we obtain a groupoid morphism

$$\theta^n(C): \pi_1(B^n(C), B^n(C)) \to b^n(C)^{\text{op}}$$

of the fundamental groupoid of $B(C)$ into $b^n(C)^{\text{op}}$.

Lemma 2.5. $\theta^n(C)$ is an isomorphism of groupoids.

Proof. Since $\theta^n(C)$ preserves the objects and both groupoids in question are connected, it suffices to show that $\theta^n(C)$ is a group isomorphism on a vertex group. Consider

$$\theta^n(C): \pi_1(B^n(C), n) \to b^n(C)^{\text{op}}(n, n) = B(n; n, n).$$

This action is defined by dragging a disc with $n$ holes up along a geometric $n$-braid. So is $\theta(n)$ [2; 3, Theorem 1.1] and hence $\theta^n(C) = \theta(n)$ under some obvious identifications. But $\theta(n): B(n) \to \text{Aut} F(n)$ is faithful, so $\theta^n(C) = \theta(n): B(n) \to \theta(n)(B(n)) = B(n; n, n)$ is an isomorphism. \qed
Since $\theta_P = \theta^n(\mathbb{C}) \circ (z_P)_*\psi$, by naturality, and $\theta^n(\mathbb{C}) = \theta(n)$, we arrive at the following result, which also (almost) appeared in [2; and 3, Theorem 2.2].

**Theorem 2.6.** Let $P$ be a simple Weierstrass polynomial of degree $n$ over $X$ and $x_0 \in X$ a base point. Then

$$\pi_1((X \times \mathbb{C}) - V_P, s_P(x_0)) \cong \mathbb{F}(n) \times \pi_1(X, x_0)$$

where the semi-direct product is w.r.t. the action

$$\pi_1(X, x_0) \xrightarrow{(z_P)_*} \pi_1(B^n(\mathbb{C}), z_P(x_0)) \cong B(n) \xrightarrow{\theta(n)} \text{Aut} \mathbb{F}(n)$$

induced by the root map $z_P : X \to B^n(\mathbb{C})$.

In the canonical situation, Theorem 2.6 implies

$$(B^n(\mathbb{C}) \times \mathbb{C}) - V^n(\mathbb{C}) = K(\mathbb{F}(n) \times B(n), 1).$$

For $n = 2$, the group $\mathbb{F}(2) \times B(2)$ has two generators, $x, y$, and one relation $[x, y^2] = 1$. I do not know any nice presentation of the semi-direct product when $n > 2$.

### 3. Classification of polynomial complements

The purpose of this section is to verify that $B^n(\mathbb{C})$ is, in some sense, a classifying space for polynomial complements, or, to put in another way, that a polynomial complement fibration is essentially the same thing as a principal $B(n)$-bundle. We do this by constructing explicitly a functor from polynomial complement fibrations to principal $B(n)$-bundles.

In the following, let $P, Q : X \times \mathbb{C} \to \mathbb{C}$ be two simple Weierstrass polynomials of degree $n$ over $X$.

Define the set

$$E_P := \bigsqcup_{x \in X} B(n; \underline{n}, z_P(x))$$

as the disjoint union of the (discrete) affine groups $B(n; \underline{n}, z_p(x))$, $x \in X$. Let $\omega_P : E_P \to X$ be the obvious map. There is a unique topology on $E_P$ such that for any open set $U \subset X$ and any braid equivalence

$$h : \bigcup (U \times \mathbb{C} - \mathcal{N}) \to \omega_P^{-1}(U) = (U \times \mathbb{C}) - V_P|U$$

the induced bijection

$$E(h) : U \times B(n) \to \omega_P^{-1}(U) = \bigsqcup_{x \in U} B(n; \underline{n}, z_P(x))$$

is a homeomorphism. Equip $E_P$ with this topology and with the right action

$$E_P \times B(n) \to E_P$$

obtained by pre-composing with the isomorphisms in $B(n) = B(n; \underline{n}, \underline{n})$. 
Lemma 3.1. \( \omega_p : E_p \to X \) is a numerable principal \( B(n) \)-bundle.

Proof. Use Lemma 2.3. \( \square \)

Thus \( C_p \leadsto \omega_p \) is a functor, \( \omega \), from the category of polynomial complement fibrations of degree \( n \) to the category of numerable principal \( B(n) \)-bundles. Let

\[
\omega^n(C) : E^n(C) \to B^n(C)
\]

be the result of applying this functor in the canonical situation. Then \( \omega_p = z_P^* \omega^n(C) \).

Lemma 3.2. \( \omega^n(C) : E^n(C) \to B^n(C) \) is a universal principal \( B(n) \)-bundle.

Proof. It suffices to show that \( E^n(C) \) is weakly contractible, since \( B^n(C) \), and hence \( E^n(C) \), has the homotopy type of a CW-complex. The only possibly non-zero homotopy group is the fundamental group. Consider the boundary map of \( \omega^n(C) \),

\[
\partial : \pi_1(B^n(C), n) \to B(n; n, n),
\]

which according to Lemma 2.4, equals \( \theta^n(C) \). But \( \theta^n(C) \) is an isomorphism by Lemma 2.5 and hence also \( \pi_1(E^n(C), 1) = 0 \) by exactness. \( \square \)

The main result of this note is the following theorem.

Theorem 3.3. The following are equivalent:

(a) \( z_p \) is homotopic to \( z_Q \),
(b) \( \omega_p \) is equivalent to \( \omega_Q \),
(c) \( C_p \) is equivalent to \( C_Q \).

Proof. The bi-implication between (a) and (b) follows from Lemma 2.3 since \( z_p \) is a classifying map for \( \omega_p \).

Suppose \( C_p \) is braid equivalent to \( C_Q \). Then \( z_P^* \omega^n(C) = \omega_p \cong \omega_Q = z_P^* \omega^n(C) \), since \( \omega \) is a functor, and hence \( z_p \cong z_Q \), since \( \omega^n(C) \) is universal.

Finally, suppose \( z_p \) is homotopic to \( z_Q \). The task is to construct a braid equivalence of \( C_p \) into \( C_Q \). Let \( H : I \times X \to B^n(C) \) be a homotopy of \( H_0 = z_p \) to \( H_1 = z_Q \). Consider \( H \) as the root map of a simple Weierstrass polynomial of degree \( n \) over \( I \times X \) and let

\[
(I \times X \times \mathbb{C}) - V \quad \xrightarrow{\simeq} \quad I \times X
\]

\[
\theta : \pi_1(I \times X, I \times X) \to b^n(C)^{\text{op}}
\]

be the associated polynomial complement fibration and groupoid morphism, respectively. By the homotopy lifting extension property for the fibration \( c \), we can find a homotopy

\[
G : I \times (X \times \mathbb{C} - V_p) \to (I \times X \times \mathbb{C}) - V
\]
such that \( G_0 \) is the inclusion, \( G(t, s_p(x)) = s(t, x) \), and \( cG(t, x, z) = (t, x) \) for all \((t, x, z) \in I \times (X \times \mathbb{C} - V_p)\). Then

\[
h := G_1: (X \times \mathbb{C} - V_p) \to (I \times X \times \mathbb{C} - V_Q)
\]

is a map over and under \( X \). Moreover, \( h \) is a braid map, since, for any \( x \in X \), the induced map

\[
(h|_{\mathbb{C} - z_p(x)})_* = \omega[t \to (1 - t, x)]
\]

and \( \omega \pi_1(I \times X; (1, x), (0, x)) \subset \mathcal{B}(n; z_p(x), z_Q(x)) \) by Lemma 2.4. □

Let \( PCF_n(X) \) and \( k_{B(n)} \) be the sets of equivalence classes of, respectively, polynomial complement fibrations of degree \( n \) over \( X \) and numerable principal \( B(n) \)-bundles over \( X \). Then there is a commutative diagram

\[
\begin{array}{ccc}
[X, B^n(\mathbb{C})] & \leftrightarrow & PCF_n(X) \\
\uparrow & & \uparrow \omega \\
z^* \omega^n(\mathbb{C}) & & C_P \\
\downarrow & & \downarrow \omega_P \\
k_{B(n)}(X) & \leftrightarrow & C_P
\end{array}
\]

of bijections between sets of equivalence classes, so \( B^n(\mathbb{C}) \) is indeed a classifying space for polynomial complement fibrations of degree \( n \). The construction of the principal \( B(n) \)-bundle \( \omega_P = z^* \omega^n(\mathbb{C}) \) from the polynomial complement \( C_P \) is similar to, e.g., the construction of a principal \( GL(n; \mathbb{R}) \)-bundle from a real vector bundle. To regain the vector bundle one forms the associated \( \mathbb{R}^n \) fibre bundle and to regain the polynomial complement one constructs the associated \( K(F(n), 1) \)-bundle.

**Remark 3.4.** The action \( \theta(n): B(n) \to \text{Aut} \mathbb{F}(n) \) can be realized geometrically as an action

\[
B(n) \to \text{Aut}_0 K(\mathbb{F}(n), 1)
\]

of \( B(n) \) on the simplicial set \( K(\mathbb{F}(n), 1) \) by based simplicial automorphisms.

For any simple Weierstrass polynomial \( P \), the associated sectioned fibre bundle

\[
\omega_P[K(\mathbb{F}(n), 1)]: E_P \times_{B(n)} K(\mathbb{F}(n), 1) \Rightarrow X
\]

is homotopy equivalent, over and under \( X \), to the polynomial complement fibration \( C_P \).

Consequently, up to homotopy over and under \( X \), the polynomial complement fibrations of degree \( n \) are precisely the sectioned fibre bundles with \( K(\mathbb{F}(n), 1) \) as fibre and structure group \( B(n) \).
We finish this note by returning to the polynomial coverings. By continuity, any braid equivalence $h$ of $C_p$ into $C_q$ extends in a unique way to a map of triples

$$H: (X \times \mathbb{C}; (X \times \mathbb{C}) - V_p, V_p) \rightarrow (X \times \mathbb{C}; (X \times \mathbb{C}) - V_q, V_q)$$

such that $h|V_p : V_p \rightarrow V_q$ is an equivalence of coverings. Using this observation, we arrive at the following corollary.

**Corollary 3.5.** $P$ and $Q$ have homotopic root maps iff there exists a map of triples

$$H: (X \times \mathbb{C}; (X \times \mathbb{C}) - V_p, V_p) \rightarrow (X \times \mathbb{C}; (X \times \mathbb{C}) - V_q, V_q)$$

such that $h|V_p : V_p \rightarrow V_q$ is an equivalence of coverings and $h|(X \times \mathbb{C}) - V_p : (X \times \mathbb{C}) - V_p \rightarrow (X \times \mathbb{C}) - V_q$ is a braid equivalence.

It is perhaps this formulation of Theorem 3.3 which most clearly expresses exactly what $B^n(C)$ does classify in relation to polynomial coverings.

**Example 3.6.** (a) The polynomial coverings of the polynomials $z^2 - x$ and $z^2 - x^3$ over $S^1$ are equivalent but the corresponding polynomial complement fibrations, classified by $\sigma_1, \sigma_3^2, \sigma_2 \in B(2)$, are inequivalent.

(b) The polynomial covering $\pi : \beta \rightarrow S^1$ obtained by closing the geometric 3-braid $B = \sigma_3 \sigma_2^2 \sigma_1^{-1}$ is trivial, but the complement is non-trivial for so is the conjugacy class of $\beta \in B(3)$.

The first of the above examples was taken from Arnol'd [1, p. 31]. The complement fibration is a geometric realization of what Arnol'd calls the braid group of an algebraic function. The second example shows that a trivial polynomial covering may have a nontrivial complement. This phenomenon can, however, only occur in the presence of other nontrivial polynomial coverings.

**Corollary 3.6.** Suppose that $H_1(X; \mathbb{Z})$ is a finitely generated abelian group. Then all $n$-fold polynomial coverings over $X$ are trivial if and only if all polynomial complement fibrations of degree $n$ over $X$ are trivial.

**Proof.** According to [5, Theorem 1.4] and Theorem 3.5, both statements are equivalent to $\text{Hom}(\pi_1(X), B(n)) = \{1\}$.

**References**


