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Normalizers of maximal tori

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Abstract. Normalizers and p-normalizers of maximal tori in p-compact groups can be characterized by the Euler characteristic of the associated homogeneous spaces. Applied to centralizers of elementary abelian p-groups these criteria show that the normalizer of a maximal torus of the centralizer is given by the centralizer of a preferred homomorphism to the normalizer of the maximal torus; i.e. that "normalizer" commutes with "centralizer".

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1. Introduction

The purpose of this paper is to formulate recognition criteria for the normalizer and the p-normalizer of a maximal torus of a p-compact group.

Fix a prime p and consider a p-compact group X. Let $N(i)\colon N(T) \to X$ denote the normalizer and $N_p(i)\colon N_p(T) \to X$ the p-normalizer [6, 9.8] of a maximal torus [6, 8.9] $i\colon T \to X$. $N(T)$ is the middle term of a short exact sequence [6, 3.2] of loop spaces

$$T \to N(T) \to W_T(X)$$

where $W_T(X)$ is the Weyl group. $N_p(T)$ is the middle term of a short exact sequence of p-compact groups

$$T \to N_p(T) \to W_T(X)_p$$

where $W_T(X)_p$ is a Sylow p-subgroup of $W_T(X)$. Thus $N(T)$ is an extended p-compact torus [7, 3.12] and $N_p(T)$ is a p-compact toral group

[6, 6.3]. The morphism $N_p(i)\colon N_p(T) \to X$ is a monomorphism [6, 3.2] of p-compact groups. Before we can say that $N(i)\colon N(T) \to X$ is a monomorphism, too, we need to define what it means for a morphism of an extended p-compact torus to a p-compact group to be a monomorphism.

Let G be any extended p-compact torus and $f\colon G \to X$ a morphism, i.e. a based map $Bf\colon BG \to BX$. G fits into a short exact sequence

$$S \to G \to \pi$$

where the identity component $S = G_0$ is a p-compact torus and the component group $\pi = \pi_0(G)$ is finite. Define $\tau_0(G)$ to be the kernel of the conjugation action $\pi \to \mathrm{Aut}(\pi_1(S))$ of the component group on the identity component.

Call $G \to X$ a *p-monomorphism* if for some (hence (2.1) any) Sylow p-subgroup $G_p \to G$, the restriction $G_p \to G \to X$ is a monomorphism of p-compact groups. A *Sylow p-subgroup* of G is a morphism $G_p \to G$ of a p-compact toral group G_p to G which restricts to an isomorphism on the identity components and induces a monomorphism on component groups taking $\pi_0(G_p)$ isomorphically onto a Sylow p-subgroup of $\pi_0(G)$. $N_p(T)$ is a Sylow p-subgroup of $N(T)$ and $N(i)\colon N(T) \to X$ is a p-monomorphism.

Definition 1.1. *The morphism $f\colon G \to X$ is a* monomorphism *if it is a p-monomorphism and $\tau_0(G)$ is a p-group.*

With this definition, the p-monomorphism $N(i)\colon N(T) \to X$ is a monomorphism for [13, 3.4.2] $\tau_0(N(T)) = \pi_0(C_X(T))$ is a p-group.

Let X/G denote the homotopy fibre of $Bf\colon BG \to BX$. If f is a monomorphism, X/G is (3.1) $\mathbb{F}_p$-finite so the $\mathbb{F}_p$-Euler characteristic $\chi(X/G)$ is defined. For instance [6, 8.10, 9.5], $\chi(X/T) = |W_T(X)|$, $\chi(X/N_p(T)) = |W_T(X) : W_T(X)_p|$, and (3.3, 3.10) $\chi(X/N(T)) = 1$.

Theorem 1.2. *(3.4, 3.5, 3.6) Let $f\colon G \to X$ be a monomorphism from an extended p-compact torus G to X. Then f is conjugate to*

1. *the maximal torus i if and only if $\chi(X/G) \neq 0$ and $\pi_0(G)$ is trivial.*
2. *the p-normalizer $N_p(i)$ if and only if $\chi(X/G) \neq 0 \mod p$ and $\pi_0(G)$ is a p-group.*
3. *the normalizer $N(i)$ if and only if $\chi(X/G) = 1$.*

We say that two morphisms $G \to X$ and $H \to X$ of extended p-compact tori to X are *conjugate* if there exists an isomorphism $G \to H$, i.e. a homotopy equivalence $BG \to BH$, making

$$\begin{tikzcd} BG \arrow[rr, "\simeq"] \arrow[dr] & & BH \arrow[dl] \\ & BX & \end{tikzcd}$$

homotopy commutative.

Most work goes into proving (1.2.3). Indeed, (1.2.1) is already known [7, 2.15] and (1.2.2) quickly follows from [6, 7].

An application of this recognition principle, shows that, in a certain sense, centralizers commute with normalizers. To be more precise, let $\nu\colon V \to X$ a monomorphism of an elementary abelian p-group $V = (\mathbb{Z}/p)^d$ to X. Suppose that $\mu\colon V \to N(T)$ is a lift of ν over $N(i)$ such that composition with $N(i)$ induces a morphism $C_{N(i)}(V)\colon C_{N(T)}(\mu) \to C_X(\nu)$ of centralizers. Is $C_{N(T)}(\mu)$ the normalizer of a maximal torus of $C_X(\nu)$? With a suitably chosen lift μ, it is.

Theorem 1.3. *Let $\nu\colon V \to X$ be a monomorphism of an elementary abelian p-group into X. There exists a lift $\mu\colon V \to N(T)$ of ν such that*

$$C_{N(i)}(V)\colon C_{N(T)}(\mu) \to C_X(\nu)$$

is conjugate to the normalizer of a maximal torus of $C_X(\nu)$. If V has rank one (4.6), but not in general (4.14.3), the lift μ is unique up to conjugacy

A lift μ as in Theorem 1.3 will be called a *preferred lift* of ν.

This theorem is particularly useful (4.12) in connection with the centralizer decomposition [7, 8.1] of BX and the N-conjecture for p-compact groups. The N-conjecture, see e.g. [10, 5.2] [17, 5.20], asserts that p-compact groups are determined up to isomorphism by their maximal torus normalizers. In [11], preferred lifts are used in a general approach to the N-conjecture leading in [15, 14] to actual verifications of the conjecture in a number of special cases.

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2. Monomorphisms

Suppose that G and H are extended p-compact tori and X a p-compact group. Note first of all, as tacitly required in the definition of a monomorphism, that Sylow p-subgroups of extended p-compact tori are essentially uniquely determined.

Lemma 2.1. *Suppose that $G_p \to G$ and $G'_p \to G$ are Sylow p-subgroups of G. Then there exists an isomorphism $G_p \to G'_p$ such that*

$$\begin{tikzcd} BG_p \arrow[rr, "\simeq"] \arrow[dr] & & BG'_p \arrow[dl] \\ & BG & \end{tikzcd}$$

commutes up to homotopy.

Proof. By covering space theory, there exists a map $BG_p \to BG'_p$, necessarily a homotopy equivalence, that makes the diagram homotopy commutative. □

Let $g\colon G \to H$ and $h\colon H \to X$ be morphisms. Choose Sylow p-subgroups $G_p \to G$ and $H_p \to H$ such that $\pi_0(g)(\pi_0(G_p)) < \pi_0(H_p)$. Then g restricts to a morphism $g_p\colon G_p \to H_p$ and further to a morphism $g_0\colon S \to T$ between the identity components $S = G_0$ and $T = H_0$.

Call $g\colon G \to H$ a *p-monomorphism* if $g_p\colon G_p \to H_p$ is a monomorphism of p-compact toral groups and a *0-monomorphism* if $g_0\colon S \to T$ is a monomorphism of p-compact tori. Any p-monomorphism is a 0-monomorphism.

Define the *rank* of G, $\mathrm{rk}(G)$, to be the rank [6, 6.3] of S. If g is a 0-monomorphism, $\mathrm{rk}(G) \le \mathrm{rk}(H)$ [6, 8.11] and equality holds if and only if g_0 is an isomorphism.

Mapping BS into the fibration sequence $BS \to BG \to B\pi_0(G)$ results in another fibration sequence

$$\mathrm{map}(BS, BS) \to \mathrm{map}(BS, BG) \to \mathrm{map}(BS, B\pi_0(G))$$

which leads to the short exact sequence

$$S \to C_G(S) \to \tau_0(G) \tag{2.2}$$

of extended p-compact tori. Note in particular that $\pi_0(C_G(S)) = \tau_0(G)$. The short exact sequence (2.2) is not quite natural but composition with $g\colon G \to H$ and $g_0\colon S \to T$ induces a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, "g_0"'] & C_G(S) \arrow[r] \arrow[d, "C_g(S)"'] & \tau_0(G) \arrow[d] \\ T \arrow[r] \arrow[d, equal] & C_H(S) \arrow[r] & \pi_0(C_H(S)) \\ T \arrow[r] & C_H(T) \arrow[r] \arrow[u, "C_H(g_0)"] & \tau_0(H) \arrow[u] \end{tikzcd}$$

with exact rows where the two lower short exact sequences are isomorphic provided g_0 is an isomorphism.

Lemma 2.3. *Suppose that g is a 0-monomorphism. Then:*

1. $\ker \pi_0(g) < \tau_0(G)$.
2. *If* $\mathrm{rk}(G) = \mathrm{rk}(H)$, *then* $\tau_0(G) = \pi_0(g)^{-1}(\tau_0(H))$.
3. *If* $\mathrm{rk}(G) = \mathrm{rk}(H)$ *and g is a p-monomorphism, the order of* $\ker \pi_0(g)$ *is prime to p.*

Proof. The first two statements follow from the commutative diagram

$$\begin{tikzcd} \pi_0(G) \times \pi_1(S) \arrow[r] \arrow[d, "\pi_0(g)\times\pi_1(g_0)"'] & \pi_1(S) \arrow[d, "\pi_1(g_0)"'] \\ \pi_0(H) \times \pi_1(T) \arrow[r] & \pi_1(T) \end{tikzcd}$$

expressing naturality of conjugation actions. Here, $\pi_1(g) = \pi_1(g_0)$ is a monomorphism if g is a 0-monomorphism [16, 3.4] and an isomorphism if also G and H have the same rank.

Under the assumptions of point (3), $\pi_0(g)_p$ is a monomorphism. $\square$

It follows that there aren't any monomorphisms of extended p-compact tori to p-compact toral groups besides the already known ones.

Proposition 2.4. *Suppose that $g\colon G \to H$ is a p-monomorphism into a p-compact toral group H. Then g is a monomorphism if and only if G is a p-compact toral group.*

Proof. Suppose g is a monomorphism. Then $\pi_0(G)$ is a p-group since (2.3) $\ker \pi_0(g) < \tau_0(G)$ and $\mathrm{im}\, \pi_0(g) < \pi_0(H)$ are p-groups. $\square$

The p-divisible group $\check{S} = (\pi_1(S) \otimes \mathbb{Q})/\pi_1(S) \cong (\mathbb{Z}/p^\infty)^{\mathrm{rk}(G)}$ is a *discrete approximation* [6, §6] to S.

Also G itself has [7, §3] a discrete approximation: Note that $H^3(\pi_0(G); \pi_1(S)) \cong H^2(\pi_0(G); \check{S})$ for any action $\pi_0(G) \to \mathrm{Aut}(\pi_1(S)) \cong \mathrm{Aut}(\check{S})$. Thus there is a bijection between fibrations

$$BS \to BG \to B\pi_0(G),$$

classified by elements of $H^3(\pi_0(G); \pi_1(S))$, and group extensions

$$\check{S} \to \check{G} \to \pi_0(G),$$

classified by elements of $H^2(\pi_0(G); \check{S})$).

By naturality of this correspondence, the fibre map $Bg\colon BG \to BH$ induces [7, 3.13] a morphism

$$\begin{tikzcd} \check{S} \arrow[r] \arrow[d, "\check{g}_0"'] & \check{G} \arrow[r] \arrow[d, "\check{g}"'] & \pi_0(G) \arrow[d, "\pi_0(g)"] \\ \check{T} \arrow[r] & \check{H} \arrow[r] & \pi_0(H) \end{tikzcd}$$

of group extensions.

Lemma 2.5. *Suppose that $g\colon G \to H$ is a p-monomorphism and $h : H \to X$ a monomorphism where $\mathrm{rk}(G) = \mathrm{rk}(H)$. Then the following conditions are equivalent:*

1. *$h \circ g\colon G \to X$ is a monomorphism.*
2. *$\check{g}\colon \check{G} \to \check{H}$ is a monomorphism.*
3. *$\pi_0(g)\colon \pi_0(G) \to \pi_0(H)$ is a monomorphism.*
4. *$\pi_0(C_g(S))\colon \pi_0(C_G(S)) \to \pi_0(C_H(S))$ is a monomorphism.*

Proof. $(1) \Rightarrow (2)$: Note that the discrete approximation $\check{g}_p\colon \check{G}_p \to \check{H}_p$ to the monomorphism $g_p\colon G_p \to H_p$ is [16, 3.4] a monomorphism and (2.3) that $\ker \pi_0(g) < \tau_0(G)$. Since $\tau_0(G)$ is a p-group, so is $\ker \pi_0(g)$ and we may then assume that $\ker \pi_0(g) < \pi_0(G_p)$. It follows that $\ker \check{g} = \check{G}_p \cap \ker \check{g} = \ker \check{g}_p$ is trivial.

$(2) \Rightarrow (3)$: Since G and H have the same rank, any monomorphism $\check{G} \to \check{H}$ induces a monomorphism $\pi_0(G) \to \pi_0(H)$.

$(3) \Rightarrow (4)$: Obvious, since $\pi_0(C_G(S)) < \pi_0(G)$ and $\pi_0(C_H(S)) < \pi_0(H)$.

$(4) \Rightarrow (1)$: Being isomorphic to a subgroup of the p-group $\tau_0(H) = \pi_0(C_H(S))$, $\tau_0(G) = \pi_0(C_G(S))$ is a p-group. $\square$

Let now $f\colon G \to X$ be a 0-monomorphism. Then $\mathrm{rk}(G) \le \mathrm{rk}(X)$ [6, 8.11] and we say that f is of *maximal rank* if $\mathrm{rk}(G) = \mathrm{rk}(X)$.

Lemma 2.6. *Let $f\colon G \to X$ be a 0-monomorphism of maximal rank. Then there exist 0-monomorphisms, $N(f)|S$ and $N(f)$, unique up to conjugation, such that the diagrams*

$$\begin{tikzcd} & N(T) \arrow[d, "N(i)"] \\ S \arrow[ur, "N(f)|S"] \arrow[r, "f|S"'] & X \end{tikzcd} \qquad \begin{tikzcd} & N(T) \arrow[d, "N(i)"] \\ G \arrow[ur, "N(f)"] \arrow[r, "f"'] & X \end{tikzcd}$$

commute up to conjugacy. Moreover, $N(i)$ induces isomorphisms

$$C_{N(i)}(S)\colon C_{N(T)}(S) \to C_X(S) \quad C_{N(i)}(G)\colon C_{N(T)}(G) \to C_X(G)$$

of extended p-compact tori. If f is a p-monomorphism, so are $N(f)$ and $N(f)|S$.

Proof. Let $B(N(f)|S)$ denote the unique conjugacy class corresponding [13, 3.4] to $B(f|S)$ under the bijection $[BS, BN(T)] \to [BS, BX]$ induced by $BN(i)$ so that $C_{N(i)}(S)\colon C_{N(T)}(S) \to C_X(S)$ is the normalizer for a maximal torus of $C_X(S)$. By maximality, $f|S$ is a maximal torus for X and $C_X(f|S)$ is a p-compact toral group, so $C_{N(i)}(S)$ is an isomorphism.

Equivalently, the space of lifts, $(X/N(T))^{hS}$, of $B(f|S)$ over $BN(i)$ is contractible.

The space of lifts of Bf over $BN(i)$,

$$(X/N(T))^{hG} \simeq ((X/N(T))^{hS})^{h\pi_0(G)},$$

is therefore also contractible. In particular, there exists a based lift $BN(f)$ of Bf over $BN(i)$ and any two such lifts are vertically homotopic. $\square$

Let now, in the maximal rank case, $W(f) = \pi_0(N(f)) : \pi_0(G) \to W_T(X)$ denote the homomorphism (defined up to conjugacy) between the component groups induced by $N(f)$. We now combine (2.5) and (2.6).

Corollary 2.7. *Let $f\colon G \to X$ be a p-monomorphism of maximal rank. Then the following conditions are equivalent:*

1. *$f\colon G \to X$ is a monomorphism.*
2. *$\check{N}(f)\colon \check{G} \to \check{N}(T)$ is a monomorphism.*
3. *$W(f)\colon \pi_0(G) \to W_T(X)$ is a monomorphism.*
4. *$\pi_0(C_f(S))\colon \pi_0(C_G(S)) \to \pi_0(C_X(S))$ is a monomorphism.*

Thus all maximal rank monomorphisms $G \to X$ are obtained by restricting $N(i)\colon N(T) \to X$ to maximal rank subgroups $\check{G} < \check{N}(T)$ where $\check{N}(T)$ is a discrete approximation to $N(T)$.

Corollary 2.8. *For any maximal rank p-monomorphism $f\colon G \to X$,* $\ker$ *$W(f)$ is a subgroup of $\tau_0(G)$ of order prime to p. If X is connected,* $\ker W(f) = \tau_0(G)$.

Proof. Use (2.3) and recall that $\tau_0(N(T)) = \pi_0(C_T(N(T))) = \pi_0(C_X(T))$ is trivial for a connected X. $\square$

3. Euler characteristic criteria

Recall that a space Y is $\mathbb{F}_p$*-finite* if $H^*(Y;\mathbb{F}_p)$ is finite dimensional over $\mathbb{F}_p$. Then also $H^*(Y;\mathbb{Q}_p)$, which stands for $H^*(Y;\mathbb{Z}_p) \otimes \mathbb{Q}$,, is finite dimensional over $\mathbb{Q}_p$ [6, 4.3] and the Euler characteristic

$$\chi(Y) = \sum_{i=0}^{\infty}(-1)^i \dim_{\mathbb{F}_p} H^i(Y;\mathbb{F}_p) = \sum_{i=0}^{\infty}(-1)^i \dim_{\mathbb{Q}_p} H^i(Y;\mathbb{Q}_p)$$

is defined. These notions generalize to pairs.

Consider a p-monomorphism $f\colon G \to X$ of an extended p-compact torus G to a p-compact group X. Let X/G denote the fibre of $Bf : BG \to BX$.

Lemma 3.1. *X/G is $\mathbb{F}_p$-finite.*

The proof of (3.1) depends on a little lemma which we for later reference formulate in greater generality than needed now.

Lemma 3.2. *Let $p_1 \colon E_1 \to B_1$ be an m_1-fold covering and $p_2 \colon E_2 \to B_2$ an m_2-fold covering where m_1 and m_2 are prime to p and $m_1 = m_2 \bmod p$. Suppose that there exists a map*

$$\begin{tikzcd} E_1 \arrow[r, "v"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "u"'] & B_2 \end{tikzcd}$$

of coverings and that (E_2, E_1) is $\mathbb{F}_p$-finite. Then (B_2, B_1) is $\mathbb{F}_p$-finite.

Proof. Consider the diagram, commutative except for the upper square,

$$\begin{tikzcd} H^*(B_1;\mathbb{F}_p) & H^*(B_2;\mathbb{F}_p) \arrow[l, "u^*"'] \\ H^*(E_1;\mathbb{F}_p) \arrow[u, "\mathrm{tr}_1^*"] & H^*(E_2;\mathbb{F}_p) \arrow[l, "v^*"'] \arrow[u, "\mathrm{tr}_2^*"'] \\ H^*(B_1;\mathbb{F}_p) \arrow[u, "p_1^*"] \arrow[uu, bend left=60, "\cdot m_1"] & H^*(B_2;\mathbb{F}_p) \arrow[l, "u^*"] \arrow[u, "p_2^*"'] \arrow[uu, bend right=60, "\cdot m_2"'] \end{tikzcd}$$

where tr_1^* and tr_2^* are transfer homomorphisms. A diagram chase reveals that multiplication by m_1 on $\operatorname{coker} u^*$ factors through a subspace of $\operatorname{coker} v^*$ and that multiplication by m_2 on $\ker u^*$ factors through a subspace of $\ker v^*$. Thus $\operatorname{coker} u^*$ and $\ker u^*$ are finite dimensional vector spaces if $\operatorname{coker} v^*$ and $\ker v^*$ are. □

Proof of 3.1. Since g is a p-monomorphism, the total space of the $|\pi_0(G) : \pi_0(G_p)|$-fold covering map $X/G_p \to X/G$ is $\mathbb{F}_p$-finite so by (3.2) also $(X/G, *)$ is $\mathbb{F}_p$-finite. □

The Euler characteristic $\chi(X/G)$ is, in the maximal rank case, determined by the homomorphism $W(f) \colon \pi_0(G) \to W_T(X)$.

Lemma 3.3. *Let $G \to X$ be a p-monomorphism. The Euler characteristic of X/G is*

$$\chi(X/G) = \begin{cases} |W_T(X) : \operatorname{im} W(f)| & \textit{if } \operatorname{rk}(G) = \operatorname{rk}(X) \\ 0 & \textit{if } \operatorname{rk}(G) < \operatorname{rk}(X) \end{cases}$$

Consequently, $\chi(X/G) = 1$ if and only if f has maximal rank and $W(f)$ is an epimorphism, and $\chi(X/G) = 0$ if and only if $\operatorname{rk}(G) < \operatorname{rk}(X)$.

Proof. Put $s = \mathrm{rk}(G)$ and $r = \mathrm{rk}(X)$.

Assume first that X is connected. Then [6, 9.7] $H^*(BX;\mathbb{Q}_p) = R_T^W$ and $H^*(BG;\mathbb{Q}_p) = R_S^\pi$ where $R_T = H^*(BT;\mathbb{Q}_p)$, $R_S = H^*(BS;\mathbb{Q}_p)$, $W = W_T(X)$, and $\pi \cong \pi_0(G)/\tau_0(G)$ is the image of the action $\pi_0(G) \to \mathrm{Aut}(\pi_1(S))$. Since R_T^W is noetherian (indeed a polynomial ring) and $H^*(X/G;\mathbb{Q}_p)$ finite dimensional over $\mathbb{Q}_p$, the Serre spectral sequence for the fibration $X/G \to BG \to BX$ shows that R_S^π is a finitely generated R_T^W-module. Also, the homological dimension $\mathrm{hdim}_{R_T^W}(R_S^\pi) \leq r < \infty$ by [1, 4.4.4] and hence

$$E_2 = \mathrm{Tor}_{R_T^W}(\mathbb{Q}_p, R_S^\pi)$$

is finite dimensional over $\mathbb{Q}_p$. Moreover, as this is the E_2-term for the Eilenberg-Moore spectral sequence converging to $H^*(X/G;\mathbb{Q}_p)$, its Euler characteristic $\chi(E_2) = \chi(X/G)$.

Change of rings [2, Proposition 7, p. 108],

$$R_T^W \otimes_{\mathbb{Q}_p} E_2 \cong R_T^W \otimes_{\mathbb{Q}_p} \mathrm{Tor}_{R_T^W}(\mathbb{Q}_p, R_S^\pi) \cong \mathrm{Tor}_{R_T^W}(R_T^W, R_S^\pi) \cong R_S^\pi,$$

and the multiplicative property of Poincaré series yield

$$P(E_2,t) = \frac{P(R_S^\pi,t)}{P(R_T^W,t)} = (1-t^2)^{r-s}\frac{|\pi|^{-1} + (1-t^2)O(1-t^2)}{|W|^{-1} + (1-t^2)O(1-t^2)}$$

where the second identity exploits information [1, 2.4.3] about the first term in the Poincaré series of an invariant ring. Evaluation at $t = -1$ provides the formulas for the Euler characteristics. (Recall (2.8) that $\pi \cong \mathrm{im}\, W(f)$ when $s = r$ and X is connected.)

In the general case of a possibly non-connected X, define $B(X_0 \cap G)$ to be the fibre of the composite $BG \to B\pi_0(G) \to B(\mathrm{im}\,\pi_0(f))$ where the last map is induced by the surjection of $\pi_0(G)$ onto its image $\mathrm{im}\,\pi_0(f)$ in $\pi_0(X)$. Then $X_0 \cap G$ is an extended p-compact torus with a p-monomorphism $X_0 \cap G \to X_0$. An inspection of the fibre map

$$\begin{tikzcd} B(X_0 \cap G) \arrow[r] \arrow[d] & BG \arrow[r] \arrow[d] & B(\mathrm{im}\,\pi_0(f)) \arrow[d] \\ BX_0 \arrow[r] & BX \arrow[r] & B\pi_0(X) \end{tikzcd}$$

reveals that the homogeneous space X/G is homotopy equivalent to a disjoint union of $|\pi_0(X) : \mathrm{im}\,\pi_0(f)|$ copies of $X_0/X_0 \cap G$. Hence

$$\chi(X/G) = |\pi_0(X) : \mathrm{im}\,\pi_0(f)| \cdot \chi(X_0/X_0 \cap G)$$

where $\chi(X_0/X_0\cap G)$ is given by the above computations in the connected case. In particular, $\chi(X_0/X_0\cap G)=0=\chi(X/G)$ if $\mathrm{rk}(G)=\mathrm{rk}(X_0\cap G)<\mathrm{rk}(X_0)=\mathrm{rk}(X)$, and in the maximal rank case,

$$\begin{aligned}|W_T(X):\operatorname{im}W(f)| &= |\pi_0(X):\operatorname{im}\pi_0(f)| \\ &\quad\cdot |W_T(X_0):\operatorname{im}W(f|(X_0\cap G))| \\ &= |\pi_0(X):\operatorname{im}\pi_0(f)|\cdot\chi(X_0/X_0\cap G) \\ &= \chi(X/G)\end{aligned}$$

where the first equality is a consequence of the commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \pi_0(X_0\cap G) \arrow[r] \arrow[d, "W(f|(X_0\cap G))"'] & \pi_0(G) \arrow[r] \arrow[d, "W(f)"'] & \operatorname{im}\pi_0(f) \arrow[r] \arrow[d, tail] & 1 \\ 1 \arrow[r] & W_T(X_0) \arrow[r] & W_T(X) \arrow[r] & \pi_0(X) \arrow[r] & 1 \end{tikzcd}$$

with exact rows. □

Proposition 3.4. *The following conditions are equivalent for any p-monomorphism $f\colon G\to X$:*

1. *f has maximal rank and $N(f)\colon G\to N(T)$ is an isomorphism.*
2. *f has maximal rank and $W(f)\colon \pi_0(G)\to W_T(X)$ is an isomorphism.*
3. *f is a monomorphism and $\chi(X/G)=1$.*
4. *$\pi_0(C_f(S))\colon \pi_0(C_G(S))\to\pi_0(C_X(S))$ is injective and $\chi(X/G)=1$.*

Proof. (1) and (2) are evidently equivalent, (2) and (3) are equivalent by (3.3) while (3) and (4) are equivalent by (2.7). □

Suppose in particular that G is a p-compact toral group and $f\colon G\to X$ a monomorphism of maximal rank. Then $W(f)\colon\pi_0(G)\to W_T(X)$ is injective and the image is a Sylow p-subgroup if and only if $\chi(X/G)=|W_T(X):\operatorname{im}W(f)|$ is prime to p.

Proposition 3.5. *Let $f\colon G\to X$ be a monomorphism of a p-compact toral group G to X. Then $\mathrm{rk}(G)\le\mathrm{rk}(X)$ and*

1. *if $\mathrm{rk}(G)=\mathrm{rk}(X)$, then $\chi(X/G)>0$. Moreover, $f\colon G\to X$ and the p-normalizer $N_p(i)\colon N_p(T)\to X$ are conjugate if and only if $\chi(X/G)\neq 0 \bmod p$.*
2. *if $\mathrm{rk}(G)<\mathrm{rk}(X)$, then $\chi(X/G)=0$.*

Consequently,

$$\begin{aligned}&\textit{The morphisms } f \textit{ and } N_p(i) \textit{ are conjugate}\\ &\Leftrightarrow \chi(X/G)\neq 0 \bmod p\\ &\Leftrightarrow p \nmid \chi(X/G)>0\\ &\Leftrightarrow \chi(X/G)=|W_T(X):W_T(X)_p|\end{aligned}$$

And even more special case arises when G is a p-compact torus. Then, in the maximal rank case, $\chi(X/G) = |W_T(X)|$ since $\pi_0(G)$ is trivial.

Proposition 3.6. *(Cf. [7, 2.15].) Let $f\colon G \to X$ be a monomorphism of a nontrivial p-compact torus G to X. Then $\mathrm{rk}(G) \le \mathrm{rk}(X)$ and*

1. *if $\mathrm{rk}(G) = \mathrm{rk}(X)$, then $\chi(X/G) > 0$ and $f\colon G \to X$ is conjugate to the maximal torus $i\colon T \to X$.*
2. *if $\mathrm{cd}(G) < \mathrm{rk}(X)$, then $\chi(X/G) = 0$.*

Consequently,

$$\begin{aligned}\text{The morphisms } f \text{ and } i \text{ are conjugate} &\Leftrightarrow \chi(X/G) \neq 0 \Leftrightarrow \chi(X/G) > 0\\ &\Leftrightarrow \chi(X/G) = |W_T(X)|\end{aligned}$$

Example 3.7. Let $f\colon G \to H$ be a monomorphism between p-compact toral groups. Then $\mathrm{rk}(G) \le \mathrm{rk}(H)$. If $\mathrm{rk}(G) = \mathrm{rk}(H)$, the restriction $f_0\colon G_0 \to H_0$ to the identity components is an isomorphism, $\pi_0(f) : \pi_0(G) \to \pi_0(H)$ a monomorphism, and the Euler characteristic $\chi(H/G) = |\pi_0(H) : \mathrm{im}\, \pi_0(f)|$. If $\mathrm{rk}(G) < \mathrm{rk}(H)$, then the Euler characteristic $\chi(H/G) = 0$. Consequently,

$$\begin{aligned}&f\colon G \to H \text{ is an isomorphism}\\ &\Leftrightarrow \chi(H/G) \neq 0 \bmod p \Leftrightarrow p \not| \chi(H/G) > 0\\ &\Leftrightarrow \chi(H/G) = 1.\end{aligned}$$

Example 3.8. Let Z be a p-compact toral group and $z\colon Z \to X$ a central morphism. Then there exists [6, §7] a lift $N(z)$, unique up to conjugacy, such that

$$\begin{tikzcd} & N(T) \arrow[d, "N(i)"] \\ Z \arrow[ur, "N(z)"] \arrow[r, "z"'] & X \end{tikzcd}$$

commutes up to conjugacy. The induced map $W(z) := \pi_0(N(z)) : \pi_0(Z) \to W_T(X)$ is central since the discrete approximation to $N(z)$ is central.

Define $T \cap Z$ to be the p-compact toral group that fits into the commutative diagram

$$\begin{tikzcd} T \cap Z \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \mathrm{im}\, W(z) \arrow[d, tail] \\ T \arrow[r] \arrow[d, equal] & N_p(T) \arrow[r] \arrow[d] & W_T(X)_p \arrow[d, tail] \\ T \arrow[r] & N(T) \arrow[r] & W_T(X) \end{tikzcd}$$

with exact rows. Consider the induced morphisms

$$T/T \cap Z \to N_p(T)/Z \to N(T)/Z \to X/Z$$

where $N(T)/Z$ is an extended p-compact torus with identity component $T/T \cap Z$ and Sylow p-subgroup $N_p(T)/Z$.

Since $\frac{X/Z}{N_p(T)/Z} \simeq X/N_p(T)$ is $\mathbb{F}_p$-finite with Euler characteristic prime to p, the monomorphism $N_p(i)/Z \colon N_p(T)/Z \to X/Z$ is (3.5) the p-normalizer of the maximal torus $T/T \cap Z \to X/Z$.

Since the $\mathbb{F}_p$-finite space $\frac{X/Z}{N(T)/Z} \simeq X/N(T)$ associated to the p-monomorphism $N(i)/Z \colon N(T)/Z \to X/Z$ has Euler characteristic 1, the induced group homomorphism $W(N(i)/Z) \colon \pi_0(N(T)/Z) \to W_{T/Z}(X/Z)$ is (3.3) surjective. But the regular covering map $X/T \to \frac{X/Z}{T/T\cap Z}$ with the p-group $\operatorname{im} W(z)$ as group of covering transformations shows [6, 4.14] that (3.6) the order $\chi(\frac{X/Z}{T/T\cap Z})$ of the Weyl group $W_{T/T\cap Z}(X/Z)$ equals the order of the component group $\pi_0(N(T)/Z) \cong W_T(X)/\operatorname{im} W(z)$. Hence $W(N(i)/Z)$ is in fact an isomorphism and $N(i)/Z$ is (3.4) the normalizer of the maximal torus of X/Z.

Example 3.9. Let G be a compact Lie group whose component group $\pi_0(G)$ is a finite p-group. Suppose that $T \to G$ is a Lie theoretic maximal torus with normalizer $N(T) \to G$ and p-normalizer $N_p(T) \to G$. Define $B\hat{G} = (BG)_p$, $B\hat{N}_p(T) = (BN_p(T))_p$, and $B\hat{N}(T) = (BN(T))_p$ where K_p is the partial p-completion [3, VII,§6] that preserves the fundamental group and p-completes the universal covering space of the pointed, connected space K. (If the fundamental group of K happens to be a finite p-group, the partial p-completion is the p-completion.) Then $\hat{G}$ is a p-compact group, $\hat{T}$ is a p-compact torus, and $\hat{N}(T)$ is an extended p-compact torus. Since also [3, II.5.3]

$$(G/T)_p = \hat{G}/\hat{T}, \quad (G/N_p(T))_p = \hat{G}/\hat{N}_p(T),$$
$$(G/N(T))_p = \hat{G}/\hat{N}(T)$$

we see [3, VII.6.3] that $\hat{T} \to \hat{G}$, $\hat{N}_p(T) \to \hat{G}$, and $\hat{N}(T) \to \hat{G}$ are p-monomorphisms with $\chi(\hat{G}/\hat{T}) = \chi(G/T) \neq 0$, $\chi(\hat{G}/\hat{N}_p(T)) = \chi(G/N_p(T)) \neq 0 \mod p$, and $\chi(\hat{G}/\hat{N}(T)) = \chi(G/N(T)) = 1$. It now easily follows from (3.4, 3.5, 3.6) that $\hat{N}(T) \to \hat{G}$ is the normalizer and $\hat{N}_p(T) \to \hat{G}$ the p-normalizer of the maximal torus $\hat{T} \to \hat{G}$.

We shall later need a little information on the special case where $G = N(T)$ is the normalizer of the maximal torus.

The maximal torus $i \colon T \to X$ for X factors through the identity component $T \to X_0$. The normalizer $N_0(T)$ of this maximal torus for X_0 is related, cf. [16, 3.8], to the normalizer $N(T)$ of the maximal torus for X by a short exact sequence

$$N_0(T) \to N(T) \to \pi_0(X)$$

of extended p-compact tori. The fibre map

$$\begin{tikzcd} BN_0(T) \arrow[r] \arrow[d] & BN(T) \arrow[r] \arrow[d] & B\pi_0(X) \arrow[d, equal] \\ BX_0 \arrow[r] & BX \arrow[r] & B\pi_0(X) \end{tikzcd}$$

shows that $X/N(T) \simeq X_0/N_0(T)$ and thus that $X/N(T)$ is a connected homogeneous space with fundamental group $\pi_1(X/N(T)) \cong W_T(X_0)$. (Use the fact that $\pi_1(N(T)) \to \pi_1(X)$ is [16, 5.6] surjective to get the expression for the fundamental group.)

Lemma 3.10. $H^*(X/N(T); \mathbb{Q}_p) \cong \mathbb{Q}_p$.

Proof. As noted above, we may assume that X is connected. Then the induced map $H^*(BN(i); \mathbb{Q}_p)\colon H^*(BX; \mathbb{Q}_p) \to H^*(BN(T); \mathbb{Q}_p)$ is an isomorphism [6, 9.7] and the lemma follows from the Serre spectral sequence. □

Finally, we turn to a somewhat different situation.

Let $g\colon Y \to X$ be a monomorphism of some p-compact group Y to X. Then $\mathrm{rk}(Y) \leq \mathrm{rk}(X)$. In the maximal rank case, g induces (2.6) a *normalizer morphism*, $N(g)$, unique up to conjugacy, such that the diagram

$$\begin{tikzcd} N(S) \arrow[r, "N(g)"] \arrow[d] & N(T) \arrow[d] \\ Y \arrow[r, "g"'] & X \end{tikzcd}$$

commutes where $N(S) \to Y$ is the normalizer in Y of a maximal torus $S \to Y$. Define the *Weyl homomorphism* to be the group homomorphism

$$W(g) = \pi_0(N(g))\colon W_S(Y) \to W_T(X)$$

induced by $N(g)$.

Corollary 3.11. *(Cf. [16, 3.12].) Let $g\colon Y \to X$ be a monomorphism of p-compact groups inducing an epimorphism $\pi_0(g)\colon \pi_0(Y) \to \pi_0(X)$ of component groups. Then $\mathrm{rk}(Y) \leq \mathrm{rk}(X)$ and*

1. *if $\mathrm{rk}(Y) = \mathrm{rk}(X)$, then $W(g)\colon W_S(Y) \to W_T(X)$ is injective and the Euler characteristic $\chi(X/Y) = |W_T(X) : \mathrm{im}\, W(g)|$.*

2. *if* $rk(Y) < \mathrm{rk}(X)$*, then* $\chi(X/Y) = 0$.

Consequently,

$$\begin{aligned} &g\colon Y \to X \textit{ is an isomorphism} \\ &\Rightarrow N(g)\colon N(S) \to N(T) \textit{ is an isomorphism} \\ &\Leftrightarrow \chi(X/Y) \neq 0 \textit{ and } W(g) \textit{ is an epimorphism} \\ &\Leftrightarrow \chi(X/Y) = 1 \end{aligned}$$

The first implication can be reversed provided $\pi_0(g)$ *is an isomorphism.*

Proof. The homogeneous space X/Y is connected and the action of the fundamental group $\pi_1(X/Y)$ on $H_*(Y/N(S); \mathbb{F}_p)$ associated to the fibration of $\mathbb{F}_p$-finite spaces

$$Y/N(S) \to X/N(S) \to X/Y$$

is nilpotent because [6, 11.6] it factors through the finite p-group $\pi_0(Y)$. Hence [7, 11.6]

$$\chi(X/N(S)) = \chi(Y/N(S)) \cdot \chi(X/Y) = 1 \cdot \chi(X/Y) = \chi(X/Y)$$

by (3.10).

If $\mathrm{rk}(Y) = \mathrm{rk}(X)$, then $W(g)$ is (2.5) injective and (3.3) the Euler characteristic $\chi(X/N(S)) = |W_T(X) : \mathrm{im}\, W(g)|$; otherwise (3.3) $\chi(X/N(S)) = 0$. This proves (1) and (2).

Suppose that $\chi(X/Y) = 1$. Then Y and X have the same rank and $W(g)$ is bijective. Assuming $\pi_0(g)$ is bijective, the Weyl homomorphism $W(g_0)\colon W_S(Y_0) \to W_T(X_0)$, induced by the restriction $g_0\colon Y_0 \to X_0$ to the identity components, is [16, 3.8] bijective too. Thus g_0 is both a rational isomorphism [6, 9.7] and a monomorphism, hence [16, 3.7] [7, 4.7] an isomorphism.

The remaining statements are easily proved. □

The Euler characteristic conditions formulated in this section have seemingly not played any significant role in classical Lie group theory. In the next section, they will be applied to analyze the centralizer of an elementary abelian p-group.

4. Centralizers of elementary abelian p-groups

The content of this section constitutes a proof of (1.3).

Let $\nu\colon V \to X$ be a monomorphism of an elementary abelian p-group V to the p-compact group X. The centralizer $C_X(\nu) = C_X(\nu V)$ is again a p-compact group and $C_X(\nu) \to X$ is a monomorphism [6, 5.1, 5.2]. The aim here is to identify the normalizer of a maximal torus of $C_X(\nu)$ using the recognition principle of (3.4).

Consider lifts

$$\begin{tikzcd} & BN_p(T) \arrow[d, "Bj_p"] \\ & BN(T) \arrow[d, "BN(i)"] \\ BV \arrow[uur, "B\mu_p"] \arrow[ur, "B\mu"] \arrow[r, "B\nu"'] & BX \end{tikzcd}$$

of $B\nu$ to $BN_p(T)$ and $BN(T)$. According to [7, 2.14] such lifts always exist. Note that V acts on T and, by conjugation, on $W_T(X)$ through the homomorphism $W(\mu) := \pi_0(\mu)\colon V \to W_T(X)$ induced by μ.

The fibration

$$BT^{hV} \to \operatorname{map}(BV, BN(T)) \to \operatorname{map}(BV, BW_T(X))$$

shows that the homotopy groups of $BC_{N(T)}(\mu)$ are concentrated in degrees ≤ 2. The fundamental group,

$$\pi_1(C_{N(T)}(\mu)) \cong \pi_2(BT^{hV}, B\mu) \cong \pi_1(T)^V, \tag{4.1}$$

is a free, finitely generated module over $\mathbb{Z}_p$. For the component group there is a short exact sequence

$$\begin{aligned} 0 &\to H^1(V, \pi_1(T)) \to \pi_0(C_{N(T)}(\mu)) \\ &\to C_{W_T(X)}(\operatorname{im} W(\mu))^{\mu} \to 1 \end{aligned} \tag{4.2}$$

where the group to the right is the isotropy subgroup at $B\mu$ for the action of the fundamental group $\pi_1(\operatorname{map}(BV, BW_T(X)), BW(\mu)) \cong C_{W_T(X)}(\operatorname{im} W(\mu))$ on the set $\pi_0(BT^{hV}) \cong H^2(V, \pi_1(T))$.

Corollary 4.3. *Let $\mu\colon V \to N(T)$ be a lift of $\nu\colon V \to X$. Then the centralizer $C_{N(T)}(\mu)$ is an extended p-compact torus and $C_{N(T)}(\mu) \to C_X(\nu)$ is a p-monomorphism.*

Proof. The above computation of the homotopy groups shows that $C_{N(T)}(\mu)$ is an extended p-compact torus. Moreover, $C_{N_p(T)}(\mu_p)$ is a Sylow p-subgroup of $C_{N(T)}(\mu)$ for a suitable lift $\mu_p\colon V \to N_p(T)$ of μ (4.4) and the composite $C_{N_p(T)}(\mu_p) \to C_{N(T)}(\mu) \to C_X(\nu)$ is [7, 2.5] a monomorphism. $\square$

Lemma 4.4. *Any morphism $\mu\colon V \to N(T)$ admits a lift $\mu_p\colon V \to N_p(T)$ such that $C_{N_p(T)}(\mu_p) \to C_{N(T)}(\mu)$ is a Sylow p-subgroup.*

Proof. Consider the covering map

$$(W_T(X)/W_T(X)_p)^V \to \coprod_{j_p\circ\mu_p\simeq\mu} BC_{N_p(T)}(\mu_p) \to BC_{N(T)}(\mu)$$

obtained by applying $\mathrm{map}(BV,-)$ to the covering map $BN_p(T) \to BN(T)$. The components of the total space are indexed by conjugacy classes of homomorphisms $\mu_p\colon V \to N_p(T)$ with $j_p \circ \mu_p$ conjugate to μ. Since the cardinality of the fibre is congruent mod p to $|W_T(X) : W_T(X)_p|$, there exists at least one μ_p for which the number of sheets of the covering $BC_{N_p(T)}(\mu_p) \to BC_{N(T)}(\mu)$ is prime to p. $\square$

In particular, the homogeneous space $C_X(\nu)/C_{N(T)}(\mu)$ is (3.1) $\mathbb{F}_p$-finite for any lift $\mu\colon V \to N(T)$ of $\nu\colon V \to X$.

In order to establish the base for an inductional proof of (1.3), we first consider the case where V has rank one.

The next lemma, which is the key observation, deals with the space $(X/N(T))^{hV}$ of all lifts to $BN(T)$ of $B\nu\colon BV \to BX$.

Lemma 4.5. *Assume that $\nu\colon V \to X$ is a monomorphism of a rank one elementary abelian p-group V to X. Then the homotopy fixed point space $(X/N(T))^{hV}$ is $\mathbb{F}_p$-finite with Euler characteristic $\chi\left((X/N(T))^{hV}\right) = 1$.*

Proof. Pull back along $B\nu\colon BV \to BX$ of the diagram

$$\begin{tikzcd} BN_p(T) \arrow[rr] \arrow[dr] & & BN(T) \arrow[dl] \\ & BX & \end{tikzcd}$$

produces a similar diagram of homotopy orbit spaces

$$\begin{tikzcd} (X/N_p(T))_{hV} \arrow[rr] \arrow[dr] & & (X/N(T))_{hV} \arrow[dl] \\ & BV & \end{tikzcd}$$

where the horizontal map is a map over BV, i.e. a V-map between the V-spaces $X/N_p(T)$ and $X/N(T)$, and an $|W_T(X) : W_T(X)_p|$-fold covering map. This horizontal map induces [6, 10.6] another covering map

$$(W_T(X)/W_T(X)_p)^V \to (X/N_p(T))^{hV} \to (X/N(T))^{hV}$$

of section, or homotopy fixed point, spaces. Using this map, build a commutative diagram

$$\begin{tikzcd} BV \times (X/N_p(T))^{hV} \arrow[r] \arrow[d] & (X/N_p(T))_{hV} \arrow[d] \\ BV \times (X/N(T))^{hV} \arrow[r] & (X/N(T))_{hV} \end{tikzcd}$$

where both vertical maps are covering maps and the horizontal ones are evaluation maps. From [6, 4.11], using [6, 5.7] to verify the $\mathbb{F}_p$-completeness hypothesis, we infer that the space $(X/N_p(T))^{hV}$ and the pair $((X/N_p(T))_{hV}, BV \times (X/N_p(T))^{hV})$ are $\mathbb{F}_p$-finite. By (3.2), the homotopy fixed point space $(X/N(T))^{hV}$ and the pair $((X/N(T))_{hV}, BV \times (X/N(T))^{hV})$ are $\mathbb{F}_p$-finite, too. In this situation, the Lefschetz number $\Lambda(X/N(T), (X/N(T))^{hV}; V) = 0$ [6, 4.17]. Hence the Euler characteristic of the trivial V-space $(X/N(T))^{hV}$ is given by

$$\begin{aligned} \chi((X/N(T))^{hV}) &= \Lambda((X/N(T))^{hV}; V) = \Lambda(X/N(T); V) \\ &= \Lambda(\mathbb{Q}_p; V) = 1 \end{aligned}$$

using the additive property [6, 4.12] of Lefschetz numbers and (3.10). □

We can now prove (1.3) for elementary abelian p-groups of rank one.

Proposition 4.6. *Assume that $\nu\colon V \to X$ is a monomorphism of a rank one elementary abelian p-group V to X. Then there exists up to conjugacy exactly one lift $\mu\colon V \to N(T)$ of ν such that*

$$C_{N(T)}(\mu)_0 \to C_{N(T)}(\mu) \to C_X(\nu)$$

is a maximal torus for the centralizer of ν. The normalizer of this maximal torus is conjugate to $C_{N(T)}(\mu) \to C_X(\nu)$.

Proof. In the fibration

$$(X/N(T))^{hV} \to \coprod_{N(i)\circ\mu\simeq\nu} \mathrm{map}(BV, BN(T))_{B\mu} \to \mathrm{map}(BV, BX)_{B\nu}$$

the components of the total space are indexed by conjugacy classes of homomorphisms $\mu\colon V \to N(T)$ with $N(i) \circ \mu$ conjugate to ν. The fibre can also be described as a finite disjoint union

$$(X/N(T))^{hV} \simeq \coprod_{N(i)\circ\mu\simeq\nu} C_X(\nu)/C_{N(T)}(\mu)$$

of $\mathbb{F}_p$-finite spaces. Hence

$$1 = \sum_{N(i)\circ\mu\simeq\nu} \chi\left(C_X(\nu)/C_{N(T)}(\mu)\right)$$

by the Euler characteristic computation of (4.5). Since all terms of this sum are nonnegative (3.3), exactly one of them must be equal to 1 and the rest equal to 0.

Let now $\mu\colon V \to N(T)$ denote the uniquely determined preferred lift of ν for which the homogeneous space $C_X(\nu)/C_{N(T)}(\mu)$ has Euler characteristic equal to 1. This lift is characterized by the property that the identity component $S = C_{N(T)}(\mu)_0$ is (3.4, 3.6) a maximal torus for $C_X(\nu)$. It remains to show that the normalizer of the maximal torus S is conjugate to $C_{N(T)}(\mu) = C_{N(T)}(V)$.

Consider the following list of statements

1. $C_{N(T)}(S \times V) \to C_X(S \times V)$ is an isomorphism.
2. $C_{N(T)}(S) \to C_X(S)$ is an isomorphism.
3. $C_X(S)$ is a p-compact toral group.
4. $C_X(S)/S$ is a p-compact toral group.
5. $C_{C_X(S)/S}(V)$ is homotopically discrete.

pertaining to the homomorphism $S \times V \to C_{N(T)}(V) \times V \to N(T) \to X$. According to (3.3, 3.4), the first statement implies the proposition. Moreover, each statement implies the one above it: The implication $(3) \Rightarrow (2)$ holds because, since S is a p-compact torus, $C_{N(T)}(S)$ is known [13, 3.4.(3)] to be conjugate to the maximal torus of $C_X(S)$ and $(5) \Rightarrow (4)$ holds because only in a p-compact toral group can the centralizer of a cyclic p-group be homotopically discrete [5, 1.4]. Thus the proof has been reduced to the verification of (5).

Let

$$(4.7) \qquad BS^{hV} \to \mathrm{map}(BV, BC_X(S)) \to \mathrm{map}(BV, B(C_X(S)/S))$$

be the fibration sequence obtained by mapping BV into the fibation sequence corresponding to the central extension [6, 8.2, 8.3]

$$S \to C_X(S) \to C_X(S)/S$$

of p-compact groups.

Each component of the fibre of (4.7) is homotopy equivalent to BS for $BS_{hV} = BS \times BV$ and $BS^{hV} = \mathrm{map}(BV, BS)$ by centrality. Moreover, $S \to C_{C_X(S)}(V) = C_{C_X(V)}(S)$ is the identity component of the centralizer of the maximal torus $S \to C_X(V)$ and hence the exact homotopy sequence of fibration (4.7) shows that the base space component $BC_{C_X(S)/S}(V)$ is aspherical, i.e. that $C_{C_X(S)/S}(V)$ is homotopically discrete. $\square$

Remark 4.8. Note that (4.5) and (4.6) are valid whenever $V = \mathbb{Z}/p^r$ is a cyclic p-group.

The scene is now set for an inductive proof of Theorem 1.3.

Proof of Theorem 1.3. Let V be an elementary abelian p-group of rank at least two. Write $V = V_1 \times V_2$ where V_1 and V_2 are nontrivial elementary abelian p-groups. Let $\nu_2 = \nu|V_2 \colon V_2 \to X$ be the restriction of and $\nu_1 \colon V_1 \to C_X(\nu_2)$ the adjoint of ν.

We may inductively, by (4.6), assume that there exist a lift $\mu_2 : V_2 \to N(T)$ of $\nu_2 \colon V_2 \to X$ such that $C_{N(T)}(\mu_2) \to C_X(\nu_2)$ is conjugate to the normalizer of a maximal torus and also a lift $\mu_1 \colon V_1 \to C_{N(T)}(\mu_2)$ of $\nu_1 \colon V_1 \to C_X(\nu_2)$ such that

$$C_{C_{N(T)}(\mu_2)}(\mu_1) \to C_{C_X(\nu_2)}(\nu_1)$$

is conjugate to the normalizer of a maximal torus. Equivalently, $C_{N(T)}(\mu) \to C_X(\nu)$, where μ is adjoint to μ_1, is conjugate to the normalizer of a maximal torus. $\square$

Remark 4.9. The proof of Theorem 1.3 shows that any preferred lift $\mu_2 : V_2 \to N$ of the restriction $\nu|V_2$ *extends* to a preferred lift of ν. It is not true (4.14), conversely, that any preferred lift of ν *restricts* to a preferred lift of $\nu|V_2$.

Preferred lifts are easily recognized and constructed in case the monomorphism $C_X(\nu) \to X$ is of maximal rank [7, 4.1] (which happens if and only if $\nu \colon V \to X$ factors through $C_X(T) \to X$).

For any morphism $\mu \colon V \to N(T)$, let $A(\mu) \colon V \to \mathrm{Aut}(\pi_1(T))$ denote the homomorphism $W(\mu) \colon V \to \pi_0(N)$ followed by the monodromy action of $\pi_0(N)$ on $\pi_1(N) = \pi_1(T)$.

Proposition 4.10. *Let $\nu \colon V \to X$ be a monomorphism of a nontrivial elementary abelian p-group into X and assume that $C_X(\nu) \to X$ is a monomorphism of maximal rank. Then*

1. *All preferred lifts of ν are conjugate in $N(T)$.*
2. *A lift $\mu \colon V \to N(T)$ of $\nu \colon V \to X$ is a preferred lift if and only if $A(\mu)$ is trivial.*

Proof. Clearly, if μ is a preferred lift of ν and $C_X(\nu)$ has maximal rank, then (4.1) $A(\mu)(V)$ fixes $\pi_1(T) \otimes \mathbb{Q}$ pointwise so the action of V on $\pi_1(T) \otimes \mathbb{Q}$, and hence on $\pi_1(T)$, is trivial.

The remaining assertions are proved by induction over the rank of V with the induction start provided by (4.6).

Suppose now that V has rank greater than one and write $V = U \oplus U^{\perp}$ as a direct sum of two nontrivial elementary abelian p-groups.

Assume that $A(\mu)$ is the trivial morphism. Let $\nu^\perp\colon U^\perp \to C_X(\nu|U)$ and $\mu^\perp\colon U^\perp \to C_{N(T)}(\mu|U)$ be the adjoints of ν and μ relative to the above splitting of V. Note that the centralizer of $\nu^\perp$ is the centralizer of ν and that all three p-compact groups $C_X(\nu) \to C_X(\nu|U) \to X$ have the same rank. Note also that both $A(\mu|U) = A(\mu)|U$ and $A(\mu^\perp) = A(\mu)|U^\perp \to \mathrm{Aut}(\pi_1(T)^{A(\mu)(V)}) = \mathrm{Aut}(\pi_1(T))$ are trivial homomorphisms. Now apply the induction hypothesis twice to conclude, first, that $\mu|U$ is a preferred lift of $\nu|U$ and, next, that $\mu^\perp$ is a preferred lift of $\nu^\perp$, or, equivalently, that μ is a preferred lift of ν.

Finally, let μ_1 and μ_2 be two preferred lifts of ν. Then the associated homomorphisms $A(\mu_1)$ and $A(\mu_2)$ are trivial. Applying the induction hypothesis twice, we infer that the restrictions $\mu_1|U$ and $\mu_2|U$ are conjugate in $N(T)$ and that the adjoints $\mu_1^\perp$ and $\mu_2^\perp$ are conjugate in $C_{N(T)}(\mu_1|U) = C_{N(T)}(\mu_2|U)$. By adjointness, this shows that μ_1 and μ_2 are conjugate in $N(T)$. $\square$

If the centralizer of the monomorphism $\nu\colon V \to X$ has maximal rank, there is a factorization $\nu\colon V \to C_X(T)$ of ν through the centralizer of the maximal torus. And since the centralizer of T in $N(T)$ is isomorphic (2.6) to the centralizer of T in X we obtain a lift of μ' of ν' as in the diagram

$$\begin{tikzcd} & C_{N(T)}(T) \arrow[r] \arrow[d, "\cong"] & N(T) \arrow[d] \\ V \arrow[ur, "\mu'"] \arrow[r, "\nu'"'] & C_X(T) \arrow[r] & X \end{tikzcd}$$

and hence also a lift of ν. This is the preferred lift of ν.

This applies in particular when X is connected and $V \cong \mathbb{Z}/p$ has rank one.

Corollary 4.11. *Let X be a connected p-compact group and $V \cong \mathbb{Z}/p$ a rank one elementary abelian p-group. Then $\mu\colon V \to N(T)$ is the preferred lift of $\nu = N(i) \circ \mu\colon V \to X$ if and only if $W(\mu)$ is trivial.*

From (4.11) and (4.9) we see that for a *connected* p-compact group any ν admits a preferred lift μ such that $W(\mu)$ has a nontrivial kernel.

Example 4.12. Let $f\colon X \to X$ be an automorphism of X whose normalizer morphism $N(f)\colon N(T) \to N(T)$ is conjugate to the identity and let $\nu\colon V \to X$ be a monomorphism of an elementary abelian p-group to X. Choose a preferred lift $\mu\colon V \to N(T)$ of ν. Note that f restricts to an automorphism $C_f(V)$ of $C_X(\nu)$ for $f \circ \nu = f \circ N(i) \circ \mu = N(i) \circ N(f) \circ \mu =$

$N(i) \circ \mu = \nu$. Moreover, the diagram

$$\begin{tikzcd} C_{N(T)}(\mu) \arrow[r, "C_{N(f)}(V)"] \arrow[d, "C_{N(i)}(V)"'] & C_{N(T)}(\mu) \arrow[d, "C_{N(i)}(V)"] \\ C_X(\nu) \arrow[r, "C_f(V)"'] & C_X(\nu) \end{tikzcd}$$

commutes up to conjugacy. Thus also the normalizer morphism $N(C_f(V)) = C_{N(f)}(V)$ of $C_f(V)$ is conjugate to the identity. See [11] for an application of this example.

Remark 4.13. Let $\check{T} \to \check{N}(T) \to W_T(X)$ be a discrete approximation [7, 3.12] to $N(T)$ and $\check{\mu}\colon V \to \check{N}(T)$ a discrete approximation to a preferred lift $\mu\colon V \to N(T)$ of $\nu\colon V \to X$. The component group homomorphism $W(\mu)\colon V \to W_T(X)$ determines an action of V on $\check{T}$ (and on BT and $\pi_1(T)$). Let $C_{W_T(X)}(\operatorname{im} W(\mu))^{\mu}$ denote the subgroup of those elements w in the centralizer of $\operatorname{im} W(\mu)$ for which $\check{w}\check{\mu}\check{w}^{-1}$, where $\check{w} \in \check{N}(T)$ is a lift of w, is $\check{T}$-conjugate to $\check{\mu}$. Then the short exact sequence (4.2) has the form

$$1 \to H^1(V;\pi_1(T)) \to W(\nu) \to C_{W_T(X)}(\operatorname{im} W(\mu))^{\mu} \to 1$$

where $W(\nu)$ denotes the Weyl group of $C_X(\nu)$.

Example 4.14. (1) For any monomorphism $V \to T$ into the maximal torus, $V \to T \to N(T)$ is the preferred lift of $V \to T \to X$, i.e. $C_{N(T)}(V) \to C_X(V)$ is the normalizer of the maximal torus $T \cong C_T(V) \to C_X(V)$ [13, 3.4(3)].
(2) If $p \not| |W_T(X)|$, X is connected and it follows from [6, 9.5] [7, 2.14] [13, 3.4(1)] that $[BV, BN(T)] \cong [BV, BX]$, so any monomorphism $\nu\colon V \to X$ lifts uniquely to a monomorphism $\mu\colon V \to N(T)$ which therefore must be the preferred lift of ν.
(3) Let $\Delta \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ denote the diagonal subgroup of the normalizer $N(T) = \mathrm{O}(2)$ of the maximal torus $T = \mathrm{SO}(2)$ of $\mathrm{SO}(3)$. Then the inclusion $\mu\colon \Delta \to N(T)$ is a preferred lift of the inclusion $\nu\colon \Delta \to \mathrm{SO}(3)$ for $C_{N(T)}(\Delta^n) = \Delta^n = C_{\mathrm{SO}(3)}(\Delta^n)$. Let P be the permutation matrix of the cycle $(1\,2\,3)$. Then $P\mu P^{-1}$ is another preferred lift of (the conjugacy class of) ν which is *not* conjugate in $N(T)$ to μ.
(4) Write, in the above situation, $\Delta = V \oplus V^{\perp}$ where $V = \Delta \cap T$. Then the restriction $\mu \mid V$ is the preferred lift of $\nu \mid V$ but $\mu \mid V^{\perp}$ is not the preferred lift of $\nu \mid V^{\perp}$ (4.11).

The third of these examples, which shows that (4.11) and the uniqueness part of (4.6) do not hold in general, is understood to take place in the category of 2-compact groups by means of (3.9) and (4.15).

Lemma 4.15. *Let G be a compact Lie group whose component group $\pi_0(G)$ is a p-group and let $C_G(V)$ denote the centralizer of a homomorphism $f\colon V \to G$ of an elementary abelian p-group V into G. Then the component group of $C_G(V)$ is [9, A.4] a p-group and the adjoint to the p-completion of $BV \times BC_G(V) \to BG$,*

$$B\widehat{C_G(V)} \to BC_{\hat{G}}(V),$$

is [8, 1.1] [4, 2.5] a homotopy equivalence.

The above discussion focused on the normalizer of the maximal torus. I close this note with a few words on the analogous, but much easier, problem for the p-normalizer.

With μ denoting a preferred lift of ν as in (1.3) and μ_p a lift of μ as in (4.4), $C_{N_p(T)}(\mu_p) \to C_X(\nu)$ is conjugate to the p-normalizer of a maximal torus of the centralizer $C_X(\nu)$. The existence of such a lift to the p-normalizer is, however, more easily proved directly (even in a more general situation).

Proposition 4.16. *Let $\nu\colon G \to X$ be a monomorphism of a p-compact toral group G into X. There exists a lift $\mu_p\colon G \to N_p(T)$ of ν such that*

$$C_{N_p(i)}(G)\colon C_{N_p(T)}(\mu_p) \to C_X(\nu)$$

is conjugate to the p-normalizer of a maximal torus of $C_X(\nu)$.

Proof. Let $N_p(\nu) \to C_X(\nu)$ denote the p-normalizer of a maximal torus of the centralizer $C_X(\nu)$. The adjoint, $G \times N_p(\nu) \to X$, of this homomorphism admits [7, 2.14] a lift $G \times N_p(\nu) \to N_p(T)$ over $N_p(i)$. The restriction $\mu_p\colon G \to N_p(T)$ to G is a lift of ν with the property that the p-normalizer $N_p(\nu) \to C_X(\nu)$ factors through $C_{N_p(T)}(\mu_p) \to C_X(\nu)$. Conversely, $C_{N_p(T)}(\mu_p) \to C_X(\nu)$ factors through $N_p(\nu) \to C_X(\nu)$ for general reasons [7, 2.14] and hence (3.7) these two p-compact toral groups are isomorphic. $\square$

5. Preferred actions

In this section we look at actions of elementary abelian p-groups or cyclic p-groups on p-compact groups and preferred actions on the maximal torus normalizer.

Let V be an elementary abelian p-group (or (4.8) a cyclic p-group) and X a p-compact group. Assume that BX is a V-space, i.e. that there exists a fibration

$$BX \to BX_{hV} \to BV \tag{5.1}$$

over BV with fibre BX and homotopy orbit space BX_{hV}.

We shall first relate the maximal torus normalizers of X and X_{hV}. Choose a maximal torus normalizer $N_{hV} \to X_{hV}$ for the p-compact group [12, 3.1] X_{hV}. Let BN denote the fibre of $BN_{hV} \to BX_{hV}$ and $BN \to BX$ the factorization of $BN \to BN_{hV} \to BX_{hV}$ through the fibre BX.

Lemma 5.2. *$N \to X$ is conjugate to the normalizer of a maximal torus of X.*

Proof. It is evident from the commutative diagram

$$\begin{tikzcd} N \arrow[r] \arrow[d] & N_{hV} \arrow[r] \arrow[d] & V \arrow[d, equal] \\ X \arrow[r] & X_{hV} \arrow[r] & V \end{tikzcd}$$

with exact rows that $N \to X$ is a monomorphism and that $X/N \simeq X_{hV}/N_{hV}$ has Euler characteristic 1. Now apply 3.4.(3). □

A V-action on X is [5, 1.2] a section $B\nu$ of the fibration (5.1). Define $B(X^{h\nu}) = (BX^{hV}, B\nu)$ to be the component containing $B\nu$ of the section space. Then $X^{h\nu}$ is again a p-compact group [6, 5.8], the homotopy fixed point p-compact group. The following proposition shows the existence of a preferred V-action on N.

Proposition 5.3. *The V-action $B\nu\colon BV \to BX_{hV}$ on X restricts to a V-action $B\mu\colon BV \to BN_{hV}$ on N such that the induced morphism $N^{h\mu} \to X^{h\nu}$ is conjugate to a maximal torus normalizer of the homotopy fixed point p-compact group $X^{h\nu}$.*

Proof. Choose $\mu\colon V \to N_{hV}$ to be a preferred lift of the monomorphism $\nu\colon V \to X_{hV}$. Then $\pi_0 C_{N_{hV}}(\mu)$ maps onto $\pi_0 C_{X_{hV}}(\nu)$ and it follows that there is a commutative diagram with exact rows

$$\begin{tikzcd} N^{h\mu} \arrow[r] \arrow[d] & C_{N_{hV}}(\mu) \arrow[r] \arrow[d] & V' \arrow[d, equal] \\ X^{h\nu} \arrow[r] & C_{X_{hV}}(\nu) \arrow[r] & V' \end{tikzcd}$$

where V' is the image of $\pi_0(C_{X_{hV}}(\nu))$ in V. As in the proof Lemma 5.2 we infer that since the middle vertical monomorphism is conjugate to the normalizer of a maximal torus, so is the left vertical monomorphism. □

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