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THE NUMBER OF p-ELEMENTS IN FINITE GROUPS OF LIE TYPE OF CHARACTERISTIC p

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ABSTRACT. The combinatorics of the poset of p-radical p-subgroups of a finite group is used to count the number of p-elements.

1. Introduction

Let G be a finite group, p a prime number, and $|G|_p$ the p-part of the group order, $|G|$. An element of G is a *p-element* if its order is a power of p. We write

$$G_p = \bigcup \mathrm{Syl}_p(G)$$

for the set of all p-elements in G, the union of all Sylow p-subgroups of G. Frobenius proved in 1907 the general fact, $|G|_p \mid |G_p|$, that the number of p-elements is a multiple of the the p-part of the group order [4]. In the special case where $G = K$ is a finite group of Lie type in characteristic p we even have that $|K_p| = |K|_p^2$ by a theorem of Steinberg from 1968 [11, 15.2]. The main purpose of this note is to present an alternative and more combinatorial proof of Steinberg's theorem.

In addition to the already introduced symbols, G and p, the following notation will be used in this note:

$[H]$	the conjugacy class of the subgroup H of G
$N_G(H, K)$	the transporter set of all group elements $g \in G$ such that $H^g \leq K$
$\mathcal{S}_G^p$ $(\mathcal{S}_G^{p+*})$	the poset of all (nontrivial) p-subgroups of G ordered by inclusion, $H \leq K \iff H \subseteq K$
$[\mathcal{S}_G^p]$ $([\mathcal{S}_G^{p+\mathrm{rad}}])$	the set of conjugacy classes of p-subgroups (p-radical p-subgroups)
q	a power of p
$\mathbf{F}_q$	the finite field with q elements

2. Counting p-elements using Möbius functions

The basic properties of the Möbius function μ of the poset $\mathcal{S}_G^p \cup \{\infty\}$ consisting of the p-subgroup poset $\mathcal{S}_G^p$ with a top element, ∞, added are [9, §3.7]

(1) the Möbius function μ on $\mathcal{S}_G^p \cup \{\infty\}$ restricts to the Möbius function on $\mathcal{S}_G^p$
(2) $\mu(\infty, \infty) = 1$ and $\mu(\infty, K) = 0$ for all $K \in \mathcal{S}_G^p$
(3) $\sum_{H \leq K \in \mathcal{S}_G^p} \mu(H, K) + \mu(H, \infty) = 0$ for all $H \in \mathcal{S}_G^p$
(4) $\sum_{H \leq K \in \mathcal{S}_G^p} \mu(K, \infty) + 1 = 0$ for all $H \in \mathcal{S}_G^p$

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By P. Hall's theorem [9, Proposition 3.8.5] and Quillen's [7, Proposition 6.1], the integer $\mu(H,\infty)$, $H \in \mathcal{S}_G^p$, is the reduced Euler characteristics of the interval (H,∞) in $\mathcal{S}_G^p$ or $\mathcal{S}_{N_G(H)}^p$ or of $\mathcal{S}_{N_G(H)/H}^{p+*}$.

Lemma 2.1. *The number of p-elements in G is* $|G_p| = \sum_{H\in\mathcal{S}_G^p} -\mu(H,\infty)|H|$.

Proof. For a finite p-group H, write $\varphi(H)$ for the number of elements of H generating H. If H is cyclic, $\varphi(H)=1$ if $|H|=1$ and $\varphi(H)=|H|-|H|/p$ if $|H|>1$. If H is not cyclic, $\varphi(H)=0$.

Declare two p-elements of G to be equivalent if they generate the same cyclic subgroup. Since the set of equivalence classes is the set of cyclic p-subgroups C of G and the number of elements in the equivalence class C is $\varphi(C)$, $|G_p| = \sum_{H\in\mathcal{S}_G^p}\varphi(H)$.

For any p-subgroup K of G, $|K| = |K_p| = \sum_{H\in\mathcal{S}_K^p}\varphi(H)$ and $\varphi(K) = \sum_{H\in\mathcal{S}_K^p}\mu(H,K)|H|$ by Möbius inversion [9, Proposition 3.7.1]. The calculation

$$\sum_{H\in\mathcal{S}_G^p} -\mu(H,\infty)|H| \overset{(3)}{=} \sum_{K\in\mathcal{S}_G^p}\sum_{H\in\mathcal{S}_G^p}\mu(H,K)|H| = \sum_{K\in\mathcal{S}_G^p}\varphi(K) = |G_p|$$

now finishes the proof. □

A p-subgroup of G is said to be p-radical if it is the biggest normal p-subgroup of its normaliser in G. Quillen observed that only p-radical p-subgroups contribute to the sum of Lemma 2.1.

Lemma 2.2. $\mu(H,\infty)=0$ *unless H is a p-radical p-subgroup of G.*

Proof. The poset $\mathcal{S}_{N_G(H)/H}^{p+*}$ of nontrivial p-subgroups of $N_G(H)/H$ is contractible if $N_G(H)/H$ contains a nontrivial normal p-subgroup [7, Proposition 2.4]. Thus $\mu(H,\infty) = \widetilde{\chi}(\mathcal{S}_{N_G(H)/H}^{p+*}) = 0$ if H is not p-radical. □

Define $\mathrm{TOM}_G^{p+\mathrm{rad}}$, the *table of marks* for the p-radical p-subgroups of G [2], and $\underline{\mathrm{TOM}}_G^{p+\mathrm{rad}}$, the *normalised table of marks*, to be the square matrices with entries
(2.3)

$$\mathrm{TOM}_G^{p+\mathrm{rad}}([H],[K]) = \frac{|N_G(H,K)|}{|K|}, \qquad \underline{\mathrm{TOM}}_G^{p+\mathrm{rad}}([H],[K]) = \frac{|N_G(H,K)|}{|N_G(K)|}$$

indexed by conjugacy classes of p-radical p-subgroups. Alternatively, $\mathrm{TOM}_G^{p+\mathrm{rad}}([H],[K]) = |(K\backslash G)^H|$ is the mark of H on the right G-set $K\backslash G$ and $\underline{\mathrm{TOM}}_G^{p+\mathrm{rad}}([H],[K])$ the number of H-supergroups conjugate to K. Relation (4) satisfied by the Möbius function μ can be expressed as either of the two equivalent linear equations

(2.4)

$$\mathrm{TOM}_G^{p+\mathrm{rad}}([H],[K])_{[H],[K]\in[\mathcal{S}_G^{p+\mathrm{rad}}]}\left(\frac{-\mu(K,\infty)}{|N_G(K):K|}\right)_{[K]\in[\mathcal{S}_G^{p+\mathrm{rad}}]} = (1)_{[K]\in[\mathcal{S}_G^{p+\mathrm{rad}}]}$$

(2.5) $$\underline{\mathrm{TOM}}_G^{p+\mathrm{rad}}([H],[K])_{[H],[K]\in[\mathcal{S}_G^{p+\mathrm{rad}}]}(-\mu(K,\infty))_{[K]\in[\mathcal{S}_G^{p+\mathrm{rad}}]} = (1)_{[K]\in[\mathcal{S}_G^{p+\mathrm{rad}}]}$$

where the right hand sides are the column vectors whose entries are all 1. By Lemma 2.1, Lemma 2.2 and equation (2.4), the density of p-elements in G,

$$\frac{|G_p|}{|G|} = \sum_{[K]\in[\mathcal{S}_G^{p+\mathrm{rad}}]}\frac{-\mu(K,\infty)}{|N_G(K):K|} = \sum_{[H],[K]\in[\mathcal{S}_G^{p+\mathrm{rad}}]}(\mathrm{TOM}_G^{p+\mathrm{rad}})^{-1}([H],[K])$$

is the sum of the entries of the inverse table of marks for p-radical p-subgroup classes. (Note that the integer $\mu(K,\infty)$ only depends on the conjugacy class of K.)

3. Radical subgroups at the defining characteristic in finite groups of Lie type

Let Σ be a reduced and crystallographic root system with fundamental and positive roots $\Pi, \Sigma^+ \subseteq \Sigma$ [6, Definition 1.8.1]. Suppose $\overline{K}(\Sigma)$ is a semisimple $\overline{\mathbf{F}}_p$-algebraic group with root system Σ [6, Theorem 1.10.4] equipped with a Steinberg endomorphism σ. We can assume that $\sigma = \gamma_\rho\varphi_q$ or $\sigma = \psi\varphi_q$ (in the notation of [6, Definition 1.15.(b), Remarks 2.2.5.(e)]) is of standard form. Assuming Σ to be also irreducible [6, Definition 1.8.4], let $K = O^{p'}C_{\overline{K}(\Sigma)}(\sigma)$ be the finite group in $Lie(p)$ with σ-setup $(\overline{K}(\Sigma),\sigma)$ [6, Definition 2.2.2].

The surjections $\Sigma \to \widetilde{\Sigma} \to \widehat{\Sigma}$ of [6, (2.3.1)] induce surjections $\Pi \to \widetilde{\Pi} \to \widehat{\Pi}$ of sets. Here, $\widetilde{\Sigma}$ is the twisted root system of K [6, p 41], and $\widehat{\Sigma} = \widetilde{\Sigma}/\!\sim$ the set of equivalence classes of twisted roots pointing in the same direction.

For every subset $J \subseteq \widehat{\Pi}$ we have associated subgroups $P_J, U_J, L_J \subseteq K$ such that $U_J = O_p(P_J)$, $P_J = N_K(U_J)$ and $P_J = U_J \rtimes L_J$ [6, Theorem 2.6.5]. The P_J are parabolic subgroups, the U_J are unipotent p-radical p-subgroups and the L_J are Levi complements [6, Definition 2.6.4, Definition 2.6.6]. It is a consequence of the Borel–Tits theorem that $\{U_J \mid J \subseteq \widehat{\Pi}\}$ is complete set of representatives for the K-conjugacy classes of the p-radical p-subgroups of K [6, Corollary 3.1.5]. In the extreme cases $J = \emptyset, \widehat{\Pi}$, $P_\emptyset = U_\emptyset \rtimes L_\emptyset$ is a Borel subgroup of K, $U_\emptyset$ a Sylow p-subgroup [6, p 41, Theorems 2.3.4, 2.3.7], $L_\emptyset = H$ is a maximal torus or Cartan subgroup [6, Theorem 2.4.7, Definition 2.4.12], and $P_{\widehat{\Pi}} = K = L_{\widehat{\Pi}}$, $U_{\widehat{\Pi}} = 1$. If $\emptyset \subseteq J \subseteq I \subseteq \widehat{\Pi}$ then $U_I \subseteq U_J \subseteq P_J \subseteq P_I$ and $U_I \subseteq U_\emptyset \subseteq P_\emptyset \subseteq P_I$.

The next lemma shows that for $J \subseteq I \subseteq \widehat{\Pi}$, the set $P_J\backslash P_I$ of right P_J-cosets in P_I parametrizes the conjugates of U_J containing U_I. The proof relies on the fact that $N_K(U) \geq N_K(V)$ when $U \leq V$ are p-radical p-subgroups of K. Even though both lemma and fact are probably well-known, a proof of the lemma is included here and [12, Proposition 2.13] proves the fact.

Lemma 3.1. *The entries of the normalised table of marks* (2.3) *for the p-radical p-subgroups of K are*

$$\underline{\mathrm{TOM}}_K^{p+\mathrm{rad}}(U_I,U_J) = \begin{cases} |P_I : P_J| & J \subseteq I \\ 0 & \textit{otherwise} \end{cases}$$

for all subsets $I, J \subseteq \widehat{\Pi}$.

Proof. It suffices to show that the transporter set $N_K(U_I,U_J)$ equals P_I if $I \supseteq J$ and is empty otherwise. Assume that $U_I^g \leq U_J$ for some $I, J \subseteq \widehat{\Pi}$, $g \in K$. Then $P_I^g \geq P_J$ are parabolic subgroups containing $P_\emptyset$. The classification of parabolic subgroups [6, Theorem 2.6.5] implies that $I \supseteq J$, $P_I^g = P_I$, $U_I^g = U_I$, and $g \in P_I$. ∎

Let I and J be subsets of $\widehat{\Pi}$. Then $L_I = HM_I$ where $M_I = \langle X_{\widehat{\alpha}} \mid \pm\alpha \in I\rangle$ [6, Definition 2.6.4]. Note that $M_\emptyset = 1$ is the trivial group, $L_\emptyset = H$, $M_{\widehat{\Pi}} = K = L_{\widehat{\Pi}}$

and $|L_I|_p = |M_I|_p$ as $M_I = O^{p'}(L_I)$ [6, Theorem 2.6.5.(f)]. We make the following observations:

- $|P_I : P_J| = |P_I : P_\emptyset|/|P_J : P_\emptyset|$ when $J \subseteq I$
- $|U_\emptyset : U_I| = |L_I|_p$ since $U_\emptyset/U_I$ is the Sylow p-subgroup of $L_I = P_I/U_I$
- $|P_I : P_\emptyset| = |U_I L_I : U_\emptyset L_\emptyset| = \frac{|L_I : L_\emptyset|}{|U_\emptyset : U_I|} = \frac{|HM_I : H|}{|L_I|_p} = \frac{|M_I|}{|H \cap M_I||M_I|_p} = \frac{|M_I|}{|M_I : P_\emptyset \cap M_I|}$ is the index of the Borel subgroup in M_I [6, Theorem 2.6.2.(e)]

The first observation shows that the entire normalised table of marks is determined by the entries, $|P_I : P_\emptyset|$, of the first column. These entries can, by the third observation, be read off from the Dynkin diagram of K as M_I is a subsystem subgroup [6, Theorem 2.6.5.(f)]. Example 4.1 makes this principle explicit in some concrete cases.

Lemma 3.2. $\sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|}|P_I : P_J| = |U_\emptyset : U_I|$ *and* $\sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|}|P_I : P_J||U_\emptyset : U_J| = 1$ *for any subset* I *of* $\widehat{\Pi}$.

Proof. The two identities of the lemma are equivalent under Möbius inversion. We verify the first identity. For any subset I of $\widehat{\Pi}$, let (W_I, I) denote the reflection group generated by the subset I. From the Bruhat decomposition in K we have $|P_I| = |P_\emptyset| W_I(q)$ [8, p. 387] where $W_I(q)$ is the Poincaré polynomial of W_I. Now

$$\sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|}|P_I : P_J| = \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|} \frac{W_I(q)}{W_J(q)} = |U_\emptyset : U_I|$$

by L. Solomon's [8, Corollary 2.2] (or [1, Corollary 7.1.4]) applied to the reflection group W_I. □

Corollary 3.3. $-\mu(U_I, \infty) = (-1)^{|I|}|U_\emptyset : U_I|$ *for any subset* I *of* $\widehat{\Pi}$.

Proof. This follows immediately from the second identity of Lemma 3.2 and the linear relation (2.5). □

We now arrive at a new proof of a version of Steinberg's theorem [11, 15.2] valid for all parabolic subgroups of K.

Theorem 3.4. $|P_p| = |K|_p^2/|O_p(P)|$ *for any parabolic subgroup* P *of* K.

Proof. There is always a bijection between the p-radical p-subgroups of a finite group G and those of $G/O_p(G)$ [5, Proposition 6.3]. In particular, $\{U_J \mid I \supseteq J\}$ is a complete set of representatives for the p-radical p-subgroup classes of P_I corresponding to the p-radical p-subgroup classes of L_I. Obviously, $N_{P_I}(U_J) = P_I \cap N_K(U_J) = P_I \cap P_J = P_J = N_K(U_J)$ and $|P_I : N_{P_I}(U_J)| = |P_I : P_J|$ is the number of conjugates of U_J in P_I or K. According to Lemma 2.1, the number of p-elements in P_I is

$$\begin{aligned}
|(P_I)_p| &= \sum_{\emptyset \subseteq J \subseteq I} -\mu(U_J, \infty)|U_J||P_I : P_J| \\
&\overset{\text{C 3.3}}{=} |U_\emptyset| \sum_{\emptyset \subseteq J \subseteq I} (-1)^{|J|}|P_I : P_J| \\
&\overset{\text{L 3.2}}{=} |U_\emptyset||U_\emptyset : U_I| = |K|_p^2/|O_p(P_I)|
\end{aligned}$$

where we used Corollary 3.3 and Lemma 3.2. □

Theorem 3.4 is valid at all prime powers q for the groups $K = \Omega_{2m+1}(\mathbf{F}_q)$, $\mathrm{SO}_{2m+1}(\mathbf{F}_q)$, $\mathrm{Spin}_{2m+1}(\mathbf{F}_q)$, $\Omega^{\pm}_{2m}(\mathbf{F}_q)$, $P\Omega^{\pm}_{2m}(\mathbf{F}_q)$, $\mathrm{Spin}^{\pm}_{2m}(\mathbf{F}_q)$ of the D-family [6, §2.7]. Note that the groups $\mathrm{SO}^{\pm}_{2m}(\mathbf{F}_q)$ are not of the type considered in the theorem. However, since $\Omega^{\pm}_{2m}(\mathbf{F}_q)$ has index 2 in $\mathrm{SO}^{\pm}_{2m}(\mathbf{F}_q)$, we still have $|\mathrm{SO}^{\pm}_{2m}(\mathbf{F}_{p^e})_p| = |\mathrm{SO}^{\pm}_{2m}(\mathbf{F}_{p^e})|^2_p$ when p is odd but we can not expect this to hold when $p = 2$. Indeed, $\mathrm{SO}^-_4(\mathbf{F}_2) \cong \Sigma_5$ of order $2^3 \cdot 15$ contains $56 < |\mathrm{SO}^-_4(\mathbf{F}_2)|^2_2 = 2^6$ 2-elements and $\mathrm{SO}^+_6(\mathbf{F}_2) \cong \Sigma_8$ of order $2^7 \cdot 315$ contains $11264 < |\mathrm{SO}^+_6(\mathbf{F}_2)|^2_2 = 2^{14}$ 2-elements by Stanley's formula [10, Example 5.2.10]

$$\sum_{n=1}^{\infty} |(\Sigma_n)_p| \frac{x^n}{n!} = \exp(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \cdots + \frac{x^{p^m}}{p^m} + \cdots)$$

for the number of p-elements in symmetric groups.

By Corollary 3.3 and observations at the beginning of Section 2, the poset $\mathcal{S}^{p+*}_{L_I}$ of nontrivial p-subgroups of the Levi complement L_I has reduced Euler characteristics $-\widetilde{\chi}(\mathcal{S}^{p+*}_{L_I}) = (-1)^{|I|}|L_I|_p$.

4. Examples

In the first example we consider (2.5) in case of a concrete Chevalley group, a Steinberg group, and a Suzuki–Ree group. The q-bracket of the natural number d is the polynomial $[d](q) = q^{d-1} + \cdots + q + 1 \in \mathbf{Z}[q]$ of degree $d-1$ with value $[d](1) = d$ at $q = 1$. In case $K = \mathrm{SL}^{\pm}_m(\mathbf{F}_q)$, the index $|P_{\widehat{\Pi}} : P_{\emptyset}| = [m^{\pm}]!(q)$ with $[m^{\pm}]!(q) = \prod_{1 \le d \le m} [d]((\pm 1)^d q)$.

Example 4.1. For the Chevalley group $G = \mathrm{SL}^+_3(\mathbf{F}_q)$, the linear identity (2.5) has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ [2^+]!(q) & 1 & 0 & 0 \\ [2^+]!(q) & 0 & 1 & 0 \\ [3^+]!(q) & [3](q) & [3](q) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -q \\ -q \\ q^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

To see this, let $\Pi = \{\alpha_1, \alpha_2\}$ be the set of fundamental roots for $\mathrm{SL}^+_3(\mathbf{F}_q)$. The p-radical p-subgroups classes are U_I with $|U_{\emptyset} : U_I| = 1, q^{\binom{2}{2}}, q^{\binom{2}{2}}, q^{\binom{3}{2}}$ and $|P_I : P_{\emptyset}| = 1, [2^+]!(q), [2^+]!(q), [3^+]!(q)$ for $I = \emptyset, \{\alpha_1\}, \{\alpha_2\}, \Pi$.

For the Steinberg group $G = \mathrm{SL}^-_5(\mathbf{F}_q)$, the linear identity (2.5) has the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ [2^+]!(q^2) & 1 & 0 & 0 \\ [3^-]!(q) & 0 & 1 & 0 \\ [5^-]!(q) & 1+q^3+q^5+q^8 & 1+q^2+q^5+q^7 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -q^2 \\ -q^3 \\ q^{10} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

To see this, let $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be the set of simple roots for $\mathrm{SL}^+_5(\mathbf{F}_q)$ and $\widehat{\Pi} = \{\widehat{\alpha}_1, \widehat{\alpha}_2\}$, $\widehat{\alpha}_1 = \{\alpha_1, \alpha_4\}$, $\widehat{\alpha}_2 = \{\alpha_2, \alpha_3\}$, the set of simple roots for the Steinberg group $\mathrm{SL}^-_5(\mathbf{F}_q)$. Then $|U_{\emptyset} : U_I| = 1, q^{2\binom{2}{2}}, q^{\binom{3}{2}}, q^{\binom{5}{2}}$ and $|P_I : P_{\emptyset}| = 1, [2^+]!(q^2), [3^-]!(q), [5^-]!(q)$ for $I = \emptyset, \{\widehat{\alpha}_1\}, \{\widehat{\alpha}_2\}, \widehat{\Pi}$.

For the Ree group ${}^2\mathrm{F}_4(q)$, the linear identity (2.5) becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1+q^2 & 1 & 0 & 0 \\ 1+q^4 & 0 & 1 & 0 \\ (1+q^2)(1+q^4)[3^-]!(q^2)[3^-]!(q^4) & (1+q^4)[3^-]!(q^2)[3^-]!(q^4) & (1+q^2)[3^-]!(q^2)[3^-]!(q^4) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -q^2 \\ -q^4 \\ q^{24} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

where now $q = 2^{a+\frac{1}{2}}$, $a \geq 0$. Entry $(4,1)$ of this matrix is the index, $|P_{\widehat{\Pi}} : P_{\emptyset}| = \frac{(q^2-1)(q^6+1)(q^8-1)(q^{12}+1)}{(q^2-1)^2}$, of the Borel subgroup in ${}^2\mathrm{F}_4(q)$ [6, Theorems 2.2.9, 2.4.7].

We can obtain a long list of polynomial identities in q by stating Lemma 3.2 explicitly. The two identities of the lemma in case $I = \widehat{\Pi}$ are

$$\sum_{J \subseteq \widehat{\Pi}} (-1)^J \frac{W_\Pi(q)}{W_J(q)} = q^{|\widehat{\Sigma}^+|}, \qquad \sum_{J \subseteq \widehat{\Pi}} (-1)^J \frac{W_\Pi(q)}{W_J(q)} q^{|\widehat{\Sigma}^+_J|} = 1, \tag{4.2}$$

where we remember that the Poincaré polynomials are products $W_J(q) = \prod_d [d](q)$ over the degrees d of the basic polynomial invariants [1, Theorem 7.1.5]. We shall now consider two concrete examples. Let $\mathrm{OP}(m) = \{(m_1, \ldots, m_k) \mid k \geq 1, m_i \geq 1, \sum m_i = m\}$ denote the set of all the 2^{m-1} ordered partitions of m [9, p 14].

Example 4.3. Subsystems of the root systems A_{m-1} or B_{m-1} are indexed by $\mathrm{OP}(m)$ via the bijection taking $(m_1, \ldots, m_k) \in \mathrm{OP}(m)$ to $A_{m_1-1} \times \cdots \times A_{m_k-1}$ or $A_{m_1-1} \times \cdots \times A_{m_{k-1}-1} \times B_{m_k-1}$ (where A_0 is the empty root system and $B_0 = A_0$, $B_1 = A_1$). The incarnations of equation (4.2) for the Chevalley groups $\mathrm{SL}^+_m(\mathbf{F}_q)$ and $\mathrm{SO}_{2m-1}(\mathbf{F}_q)$ of rank $m-1$ with root systems $\Sigma = A_{m-1}, B_{m-1}$ are the polynomial identities

$$\sum (-1)^k \binom{[m](q)}{[m_1](q), \cdots, [m_k](q)} = (-1)^m q^{\binom{m}{2}},$$

$$\sum (-1)^k \binom{[m](q)}{[m_1](q), \cdots, [m_k](q)} q^{\sum \binom{m_i}{2}} = (-1)^m,$$

$$\sum \frac{(-1)^k \prod_{d=m_k}^{m-1} [2d](q)}{[m_1]!(q) \cdots [m_{k-1}]!(q)} = (-1)^m q^{(m-1)^2},$$

$$\sum \frac{(-1)^k \prod_{d=m_k}^{m-1} [2d](q)}{[m_1]!(q) \cdots [m_{k-1}]!(q)} q^{\sum_{i=1}^{m_{k-1}} \binom{m_i}{2} + (m_k-1)^2} = (-1)^m.$$

The sums are indexed by all $(m_1, \ldots, m_k) \in \mathrm{OP}(m)$ and the identities for A_{m-1} use Gaussian multinomial coefficients [9, §1.7].

Example 4.4 ($\mathrm{SL}^-_m(\mathbf{F}_q)$)**.** The two identities of (4.2) for the Steinberg group $\mathrm{SL}^-_{2m}(\mathbf{F}_q)$ of rank $2m-1$ and twisted rank m are

$$\sum \frac{(-1)^k \prod_{d=1}^{2m} [d]((-1)^d q)}{[m_1]!(q^2) \cdots [m_k]!(q^2)} - \sum \frac{(-1)^k \prod_{d=2m_k+2}^{2m} [d]((-1)^d q)}{[m_1]!(q^2) \cdots [m_{k-1}]!(q^2)} = (-1)^m q^{\binom{2m}{2}}$$

$$\sum \frac{(-1)^k \prod_{d=1}^{2m} [d]((-1)^d q)}{[m_1]!(q^2) \cdots [m_k]!(q^2)} q^{\sum \binom{m_i}{2}} - \sum \frac{(-1)^k \prod_{d=2m_k+2}^{2m} [d]((-1)^d q)}{[m_1]!(q^2) \cdots [m_{k-1}]!(q^2)} q^{\sum \binom{m_i}{2}} = (-1)^m$$

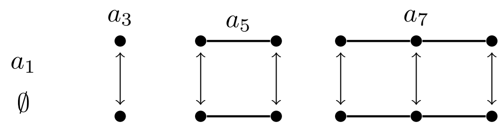
and for the Steinberg group $\mathrm{SL}^-_{2m+1}(\mathbf{F}_q)$ of rank $2m$ and twisted rank m they are

$$\sum \frac{(-1)^k \prod_{d=1}^{2m+1} [d]((-1)^d q)}{[m_1]!(q^2) \cdots [m_k]!(q^2)} - \sum \frac{(-1)^k \prod_{d=2m_k+2}^{2m+1} [d]((-1)^d q)}{[m_1]!(q^2) \cdots [m_{k-1}]!(q^2)} = (-1)^m q^{\binom{2m+1}{2}}$$

$$\sum \frac{(-1)^k \prod_{d=1}^{2m+1} [d]((-1)^d q)}{[m_1]!(q^2) \cdots [m_k]!(q^2)} q^{\sum \binom{m_i}{2}} - \sum \frac{(-1)^k \prod_{d=2m_k+2}^{2m+1} [d]((-1)^d q)}{[m_1]!(q^2) \cdots [m_{k-1}]!(q^2)} q^{\sum \binom{m_i}{2}} = (-1)^m,$$

where the sums run over all $(m_1, \ldots, m_k) \in \mathrm{OP}(m)$. These identities are obtained by analysing the C_2-subsystems of the C_2-root system A_{m-1} [3, 13.3.8]. Write $S(A_{m-1})$ for the multiset of all C_2-subsystems of A_{m-1}. One subsystem of A_{2m-1}

is a_{2m-1} defined to be the C_2-free part of A_{2m-1}, i.e. the subsystem obtained by deleting the middle root α_m. The fundamental roots of the C_2-root systems a_1, a_3, a_5, a_7 are



The first multisets of subsystems are $S(A_1) = \{a_1, A_1\}$, $S(A_2) = \{a_1, A_2\}$, $S(A_3) = \{a_1, A_1, a_3, A_3\} = a_1 \times S(A_1) \cup \{a_3, A_3\}$, $S(A_4) = \{a_1, A_2, a_3, A_4\} = a_1 \times S(A_2) \cup \{a_3, A_4\}$. In general, the 2^m subsystems of A_{2m-1} and A_{2m}, $m \geq 2$, are the multisets

$$\begin{aligned} S(A_{2m-1}) &= a_1 \times S(A_{2m-3}) \cup \cdots \cup a_{2i-1} \times S(A_{2(m-i)-1}) \cup \cdots \cup a_{2m-3} \\ &\quad \times S(A_1) \cup \{a_{2m-1}, A_{2m-1}\} \\ S(A_{2m}) &= a_1 \times S(A_{2m-2}) \cup \cdots \cup a_{2i-1} \times S(A_{2(m-i)}) \cup \cdots \cup a_{2m-3} \\ &\quad \times S(A_2) \cup \{a_{2m-1}, A_{2m}\}. \end{aligned}$$

For each subsystem a of A_m, let $P(a)(q) = |P : B| \in \mathbf{Z}[q]$ be the index of the Borel subgroup B in the parabolic subgroup of $\mathrm{SL}^-_{m+1}(\mathbf{F}_q)$ corresponding to a. In particular, $P(A_m)(q)$ and $P(a_{2m-1})(q)$ are the polynomials

$$P(A_m)(q) = \prod_{1 \leq d \leq m+1} [d]((-1)^d q), \quad P(a_{2m-1}) = \prod_{1 \leq d \leq m} [d](q^2) = [m]!(q^2), \quad m \geq 1$$

of degrees $\binom{m+1}{2}$ and $\binom{m}{2}$. Consider the multiset of signed polynomials associated to all subsystems of A_m

$$P(S(A_m)) = \{(-1)^{|\Pi(a)/C_2|} P(a)(q) \mid a \in S(A_m)\},$$

where $\Pi(a)$ is the set of fundamental roots and $\Pi(a)/C_2$ the orbit set. Then $P(S(A_1)) = \{1, -P(A_1)\}$, $P(S(A_2)) = \{1, -P(A_2)\}$ and one may now determine the multisets of polynomials for all the C_2-root systems A_{2m-1} and A_{2m}, $m \geq 2$. This leads to the above polynomial identities.

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