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RATIONAL ISOMORPHISMS OF p-COMPACT GROUPS

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1. INTRODUCTION

THE NOTION of a p-compact group was introduced by Dwyer and Wilkerson as a homotopy theoretic candidate for a replacement of compact Lie groups. They defined [7, 2.2] a p-compact group to be an $\mathbb{F}_p$-complete loop space with finite $\mathbb{F}_p$-cohomology and with a finite p-group as component group. Subsequent investigation in [17, 8] strengthened the candidacy in finding that much of the internal structure of compact Lie groups does seem to be present also in p-compact groups. The process of gathering support for p-compact groups continues here where the outlining idea is to translate Baum's paper [3], describing local isomorphism systems of Lie groups, into the setting of p-compact groups.

The rational isomorphisms of the title form the most prominent concept of this paper. However, for the sake of stressing the similarity with Lie groups, I shall in this introduction restrict myself to the particular rational isomorphisms called finite covering homomorphisms or isogenies: an isogeny between two p-compact groups is an epimorphism whose kernel is a finite p-group (Definition 2.3). It was shown in [17, 5.4] that for any connected p-compact group X there exists an isogeny $q: Y \times S \to X$ where Y is a simply connected p-compact group and S is a p-compact torus; the homomorphism q can even be chosen to be what is here called a *special* isogeny. Locally isomorphic p-compact groups (Definition 2.7) are characterized by having isomorphic finite covering groups of this kind (Proposition 2.8).

As an example of how these concepts behave as our experience with Lie groups tells us they should, the following theorem—containing elements from Theorem 4.3 and Corollary 4.5—is an almost mechanical translation of the basic lifting property for covering homomorphisms exploited by Baum [3, Proposition 5] in computing local isomorphism systems.

THEOREM 1.1. *Suppose that X_1 and X_2 are two locally isomorphic connected p-compact groups and that $X_1 \xleftarrow{q_1} Y \times S \xrightarrow{q_2} X_2$ are special isogenies.*

For any isogeny $f: X_1 \to X_2$ there exist an automorphism g of Y and a self-isogeny h of S such that the diagram of p-compact group homomorphisms

$$\begin{array}{ccc} Y \times S & \xrightarrow{g \times h} & Y \times S \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array}$$

commutes up to conjugacy; g and h are uniquely determined up to conjugacy. Furthermore, f is an isomorphism if and only if both g and h are automorphisms.

The formulation of Theorem 1.1 freely employs the basic definitions from the dictionary of [7] (and shall I continue this habit throughout this paper).

The most notable tool used to derive Theorem 1.1 is an analog of the admissible homomorphisms of Adams and Mahmud [1] or Adams and Wojtkowiak [2].

THEOREM 1.2. *Let X_1 and X_2 be two connected p-compact groups with maximal tori $i_1: T_1 \to X_1$ and $i_2: T_2 \to X_2$. For any homomorphism $f: X_1 \to X_2$ there exists a homomorphism $\varphi: T_1 \to T_2$ such that the diagram of p-compact group homomorphisms*

$$\begin{array}{ccc} T_1 & \xrightarrow{\varphi} & T_2 \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array}$$

commutes up to conjugacy; the conjugacy class of the homomorphism φ is unique up to the action of the Weyl group of X_2.

Assume additionally that X_1 and X_2 are locally isomorphic so that we may take $T_1 = T_2$. Then φ is an isogeny which is an automorphism if and only if f is an isomorphism.

The above theorem is a reformulation of ingredients from Theorem 3.5 and 3.6. As the center [17, 8] of a connected p-compact group is a Weyl group invariant subgroup of the maximal torus this theorem implies (Corollary 4.1) that the center is a functor on the category of connected p-compact groups with isogenies as morphisms.

In particular X_1 and X_2 could be identical so that we are discussing endomorphisms of some p-compact group X. By Theorem 1.2, any endomorphism $f: X \to X$ restricts to an endomorphism $\varphi: T \to T$ of the maximal torus. If X is *simple* in the sense (Definition 5.4) that $\pi_2(BT) \otimes \mathbb{Q}$ is an irreducible representation of the Weyl group, then the induced homomorphism $\pi_2(B\varphi)$ must be multiplication by some p-adic integer λ (and $Bf = \psi^\lambda$ an Adams operation of exponent λ); in particular, f is a rational isomorphism if $\lambda \neq 0$. This observation leads to the following main result of Section 4.

THEOREM 1.3. *Any nontrivial endomorphism $f: X \to X$ of a connected simple p-compact group X is a rational isomorphism, and if p divides the order of the Weyl group, even an automorphism.*

The proof of Theorem 1.3 is based on a triviality criterion valid for any p-compact group homomorphism $f: X \to Y$. By Theorem 6.1, f is trivial if $f \circ x$ is trivial for any homomorphism $x: \mathbb{Z}/p^n \to X$ of a cyclic p-group into X. In case X is connected it follows (Corollary 6.7) that f is trivial if the induced map $\tilde{H}^*_{\mathbb{Q}_p}(Bf)$ in reduced rational cohomology is trivial.

2. RATIONAL ISOMORPHISMS AND FINITE COVERINGS

This section contains the basic definitions and a few auxiliary results to be used later.

Let X, X_1 and X_2 be p-compact groups. A homomorphism $f: X_1 \to X_2$ of p-compact groups is [8, 3.1] a based map $Bf: BX_1 \to BX_2$; the homotopy fiber of Bf over the base point is denoted X_2/fX_1 or just X_2/X_1 when no confusion may arise [8, 3.2]. Let $\mathrm{Rep}(X_1, X_2)$ denote the set of conjugacy classes [7, 3.1] of p-compact group homomorphisms of X_1 to X_2, i.e. $\mathrm{Rep}(X_1, X_2) = [BX_1, BX_2]$ is the set of free homotopy classes of maps of BX_1 to BX_2.

Denoting the identity component of X_j by X_j^0, $j = 1, 2$,

$$X_j^0 \to X_j \to \pi_0(X_j)$$

may serve as our first example of a short exact sequence [7, 3.2] of p-compact groups. Note that any homomorphism $f: X_1 \to X_2$ will respect this short exact sequence in the sense that f restricts to a homomorphism $f^0: X_1^0 \to X_2^0$ between the identity components.

The central concept of this paper is that of a rational isomorphism.

Definition 2.1. The homomorphism $f: X_1 \to X_2$ is a *rational isomorphism* if the homomorphism $H^*_{\mathbb{Q}_p}(Bf^0): H^*_{\mathbb{Q}_p}(BX_1^0) \leftarrow H^*_{\mathbb{Q}_p}(BX_2^0)$, induced by the restriction f^0 of f to the identity components, is an isomorphism.

As in [7, 1.5], the cohomology theory $H^*_{\mathbb{Q}_p}(-)$ is defined as $H^*(-;\mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$.

Let $\varepsilon_{\mathbb{Q}}(X_1, X_2) \subseteq \mathrm{Rep}(X_1, X_2)$ denote the set of conjugacy classes of rational isomorphisms from X_1 to X_2 and $\varepsilon_{\mathbb{Q}}(X) := \varepsilon_{\mathbb{Q}}(X, X)$ the monoid of rational isomorphisms from X to itself.

Assume from now on that X, X_1, and X_2 are *connected* p-compact groups.

LEMMA 2.2. *Let $f: X_1 \to X_2$ be a homomorphism between connected p-compact groups. Then the following are equivalent*:

(1) *$\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism.*

(2) *$H^*_{\mathbb{Q}_p}(f)$ is an isomorphism.*

(3) *$H^*_{\mathbb{Q}_p}(Bf)$ is an isomorphism.*

(4) *$\pi_*(Bf) \otimes \mathbb{Q}$ is an isomorphism.*

Proof. By the Milnor–Moore theorem [14], the Hopf algebra $H_*^{\mathbb{Q}_p}(X_j)$, where $H_*^{\mathbb{Q}_p}(-) := H_*(-;\mathbb{Z}_p^\wedge) \otimes \mathbb{Q}$, is naturally isomorphic to the universal enveloping algebra of the Lie algebra $\pi_*(X_j) \otimes \mathbb{Q}$ with the Samelson product determining the Lie bracket, $j = 1, 2$. Thus (1) $\Rightarrow$ (2). That (2) $\Rightarrow$ (3) follows from the (collapsing) bar construction spectral sequence [12, Corollary 7.18]

$$\mathrm{Tor}^{H_*^{\mathbb{Q}_p}(X_j)}(\mathbb{Q}_p, \mathbb{Q}_p) \Rightarrow H_*^{\mathbb{Q}_p}(BX_j).$$

If (3) holds, the Eilenberg–Moore spectral sequence [12, Theorem 7.1]

$$\mathrm{Tor}_{H^*_{\mathbb{Q}_p}(BX_2)}(\mathbb{Q}_p, H^*_{\mathbb{Q}_p}(BX_1)) \Rightarrow H^*_{\mathbb{Q}_p}(X_2/X_1)$$

shows that the fiber X_2/X_1 of Bf is a torsion space and (4) follows. Finally (4)$\Leftrightarrow$(1) is an immediate consequence of the homotopy exact sequence. □

A special class of rational isomorphisms are provided by the isogenies.

Definition 2.3. A homomorphism $f: X_1 \to X_2$ between connected p-compact groups is *an isogeny* if $X_2/X_1 \simeq B\pi$ for some finite p-group π.

Let $\mathrm{Cov}(X_1, X_2) \subseteq \varepsilon_{\mathbb{Q}}(X_1, X_2)$ denote the set of isogenies, or finite covering homomorphisms, of X_1 to X_2 and $\mathrm{Cov}(X) := \mathrm{Cov}(X, X)$ the monoid of isogenies of X to itself.

Obstruction theory and [16, Theorem 6.3] prove a lifting criterion similar to [19, Proposition 1.2]. Below, and elsewhere in the paper, I use the notation from [7, 3.3] for homotopy fixed point spaces: If the solid arrows

$$\begin{array}{ccc} & & BG \\ & \nearrow & \downarrow \\ BX & \xrightarrow[Bf]{} & BH \end{array}$$

form a diagram of p-compact group classifying spaces and based maps then the homotopy fixed point space $(H/G)^{hfX}$, or just $(H/G)^{hX}$, is the (possibly empty!) space of all free maps $BX \to BG$ that make the diagram commute.

LEMMA 2.4. *Let $f: X_1 \to X_2$ be an isogeny and $g: X_1 \to X_2$ a homomorphism from the connected p-compact group X into the base space X_2. Then*

$$\begin{array}{ccc} & & X_1 \\ & \nearrow & \downarrow f \\ X & \xrightarrow[g]{} & X_2 \end{array}$$

can be completed to a diagram commuting up to conjugacy if and only if $g_*\pi_1(X) \subseteq f_*\pi_1(X_1)$. Moreover, if g factors through X_1, then the space $(X_2/X_1)^{hX}$ of all lifts of Bg is homotopy equivalent (by evaluation) to X_2/X_1; in particular, the factorization $X \to X_1$ is unique up to conjugacy.

Lemma 2.4 shows that if g is conjugate to $f \circ h$ for some homomorphism $h: X \to X_1$, then there exists a fibration

$$X_2/X_1 \to \mathrm{map}(BX, BX_1)_{Bh} \xrightarrow{Bf} \mathrm{map}(BX, BX_2)_{Bg}$$

of the corresponding mapping spaces.

An isogeny is an example of an epimorphism [7, 3.2] of p-compact groups. In fact, if $\mathrm{cd}_{\mathbb{F}_p}(X_1) = \mathrm{cd}_{\mathbb{F}_p}(X_2)$ then $f: X_1 \to X_2$ is an isogeny if and only if f is an epimorphism, [7, Proposition 6.14]. Any isogeny to a simply connected p-compact group is an isomorphism.

The mod p cohomological dimension $\mathrm{cd}_{\mathbb{F}_p}(X)$ occurring above is defined [7, Definition 6.13] to be the largest integer d such that $H^d(X; \mathbb{F}_p) \neq 0$. A homomorphism $X_1 \to X_2$ of p-compact groups is said to be a monomorphism if [7, 3.2] the homogeneous space X_2/X_1 is $\mathbb{F}_p$-finite.

LEMMA 2.5. *Let $f: X_1 \to X_2$ be a rational isomorphism. Then:*
(1) f is an isomorphism if and only if f is a monomorphism.
(2) f is an isogeny if and only if f is an epimorphism.

Proof. The first assertion is [17, Proposition 3.7]. If f is an epimorphic rational isomorphism, $\Omega(X_2/X_1)$ is a p-compact group whose identity component has rational rank [7, 5.9] zero, hence [7, Lemma 5.10] a finite p-group. □

Thus an isogeny is the same thing as an epimorphic rational isomorphism.

The endomorphism $\Omega\psi^3$ of the 3-compact group $SU(2)\hat{_3}$ is an example of a rational isomorphism that is not an epimorphism, in particular not an isogeny; see also Corollary 4.5 for more information about the relation between isogenies and rational isomorphisms.

Recall [17, Lemma 3.3] that the universal cover $X\langle 1\rangle$ of the connected p-compact group X is again a p-compact group.

PROPOSITION 2.6. *The following conditions are equivalent:*
(1) *X_1 and X_2 have identical* mod p *cohomological dimensions and isomorphic universal covering p-compact groups.*
(2) *There exists a simply connected p-compact group Y, a p-compact torus S, and isogenies $X_1 \leftarrow Y \times S \to X_2$.*

Proof. Assuming (1), both X_1 and X_2 are finitely covered [17, Theorem 5.4] by a p-compact group of the form $Y \times S$ where $X_1\langle 1\rangle \cong Y \cong X_2\langle 1\rangle$ and S is p-compact torus of mod p cohomological dimension $\mathrm{cd}_{\mathbb{F}_p}(Y) - \mathrm{cd}_{\mathbb{F}_p}(X_1) = \mathrm{cd}_{\mathbb{F}_p}(Y) - \mathrm{cd}_{\mathbb{F}_p}(X_2)$. □

Definition 2.7. The two connected p-compact groups X_1 and X_2 are *locally isomorphic* if they satisfy either of the conditions in Proposition 2.6.

We may elaborate a little on point (2) in Proposition 2.6. Let Y be a *simply connected* p-compact group with center [17, 8] $Z(Y)$ and let S be a p-compact torus. For any subgroup $K < Z(Y)$ and any homomorphism $\varphi \in \mathrm{Rep}(K, S)$, define (inspired by and borrowing notation form [3]) the p-compact group $Y \times S/(K, \varphi)$ by requiring

$$K \xrightarrow{(\mathrm{incl},\varphi)} Y \times S \xrightarrow{q} Y \times S/(K, \varphi) \tag{1}$$

to be an exact sequence of p-compact groups; see [7, Proposition 8.3] and [17, Proposition 4.6]. An isogeny as q of the short exact sequence (1), arising by factoring out the graph of a homomorphism $\varphi: K \to S$ defined on a central subgroup of Y, will be called a *special* isogeny. Note that the isogeny of [17, Theorem 5.4] occurring in point (2) of Proposition 2.6 is special. The following proposition is therefore an immediate consequence of Proposition 2.6.

PROPOSITION 2.8. *Let X be a connected p-compact group; let $Y = X\langle 1 \rangle$ denote the universal covering p-compact group of X and $S = Z(X)_0$ the identity component of the center of X. Then any p-compact group locally isomorphic to X is isomorphic to*

$$Y \times S/(K, \varphi)$$

for some subgroup $K < Z(Y)$ and some $\varphi \in \mathrm{Rep}(K, S)$.

Since Y is simply connected, its center $Z(Y)$ is a finite abelian p-group [17, Theorem 5.3] and it follows in particular that the local isomorphism type of X contains only finitely many isomorphism classes of p-compact groups.

Endomorphisms of p-compact tori have particularly nice properties.

Let, for any endomorphism $\varphi: T \to T$ of a p-compact torus T, $\breve{\varphi}: \breve{T} \to \breve{T}$ denote the discrete approximation [17, 2.7; 8, Proposition 3.2] to φ.

LEMMA 2.9. *Let T be a p-compact torus and $\varphi: T \to T$ an endomorphism of T. The following conditions are equivalent*:

(1) *φ is a rational isomorphism.*
(2) *$\pi_1(\varphi)$ is a monomorphism.*
(3) *φ is an epimorphism.*
(4) *$\breve{\varphi}$ is an epimorphism.*

If any of these conditions is satisfied, $T/\varphi(T) \simeq B(\operatorname{coker} \pi_1(\varphi)) \simeq B(\ker \breve{\varphi})$ and φ is an isogeny.

Proof. (1) $\Rightarrow$ (2) is clear. If (2) holds, the homotopy exact sequence yields $T/\varphi(T) \simeq B(\operatorname{coker} \pi_1(\varphi))$ with $\operatorname{coker} \pi_1(\varphi)$ a finite p-group; i.e. φ is an epimorphism (even an isogeny). If (3) holds, $\Omega(T/\varphi(T))$ is a p-compact group and $\pi_1(T/\varphi(T)) \cong \operatorname{coker} \pi_1(\varphi)$ a finite p-group. Hence $\pi_1(\varphi) \otimes \mathbb{Q}$ is an automorphism and the induced endomorphism $\breve{\varphi}$ on $\pi_1(T) \otimes \mathbb{Q}/\pi_1(T) = \breve{T}$ an epimorphism. If (4) holds, the Snake lemma tells that the $\mathbb{Q}_p$-vector space $\operatorname{coker}(\pi_1\ (\varphi) \otimes \mathbb{Q})$ is a quotient of the countable abelian $\operatorname{coker} \pi_1(\varphi)$. Hence $\pi_1(\varphi) \otimes \mathbb{Q}$ must be an isomorphism which is the content of (1). □

In other words, Lemma 2.9 says that $\varepsilon_{\mathbf{Q}}(T) = \mathrm{Cov}(T)$.

Finally, let

$$G \xrightarrow{j} U \xrightarrow{q} V$$

be a short exact sequence of p-compact groups. Let Y be any p-compact group and let

$$\overline{Bq}: \mathrm{map}(BV, BY) \to \coprod_{g|G \simeq 0} \mathrm{map}(BU, BY)_{Bg}$$

denote precomposition with Bq; here, the disjoint union is indexed by the set of those $g \in \mathrm{Rep}(U, Y)$ for which $Bg \circ Bj$ is null homotopic. In this situation we have a version of [19, Proposition 1.1].

LEMMA 2.10. *The above map $\overline{Bq}$ is a homotopy equivalence.*

Proof. Inclusion of constant maps provide by the Sullivan conjecture for p-compact groups [8, Theorem 9.3] a homotopy equivalence $BY \to \mathrm{map}(BG, BY)_0$ inducing a homotopy equivalence

$$BY^{hV} \to \mathrm{map}(BG, BY)_0^{hV}$$

which (as $BY^{hV} = \mathrm{map}(BV, BY)$ and $\mathrm{map}(BG, BY)_0^{hV}$ is a collection of components of $\mathrm{map}(BG, BY)^{hV} = \mathrm{map}(BU, BY)$ [7, Lemma 10.5]) can be identified to the map $\overline{Bq}$. □

COROLLARY 2.11. *Let $g: U \to Y$ be a homomorphism from the total space U into some p-compact group Y. Then*

$$\begin{array}{ccc} U & \xrightarrow{g} & Y \\ {\scriptstyle q}\downarrow & \nearrow & \\ V & & \end{array}$$

can be completed to a diagram commuting up to conjugacy if and only if $g|G$ is trivial. Moreover, if g factors through V, then the factorization $V \to Y$ is unique up to conjugacy and $C_Y(V) \cong C_Y(U)$.

The factorization criterion, an analog of a very basic fact in (Lie) group theory, of Corollary 2.11 is obtained from Lemma 2.10 by applying the functor π_0 to $\overline{Bq}$. The centralizer of U in Y, denoted $C_Y(U)$, $C_Y(g)$, or $C_Y(gU)$, is [7, 3.4] the loop space of $\mathrm{B}C_Y(U) = \mathrm{map}(BU, BY)_{Bg}$.

3. CENTRALIZERS OF p-COMPACT TORAL GROUPS

The key technical results of this paper, dealing with centralizers [7, 3.4] of p-compact toral groups, are contained in this section. They lead to the consequence, analogous to the Lie group case [1, 2], that homomorphisms between p-compact groups lift to homomorphisms between the respective maximal tori and that these lifts are unique up to left action by the Weyl group of the target.

Let G be a p-compact toral group, i.e. an extension [7, Definition 6.3] of a p-compact torus by a finite p-group, X a p-compact group with maximal torus $i: T \to X$, and $C_X(fG)$ the centralizer of some homomorphism $f: G \to X$; the centralizer $C_X(fG)$ is [7, Proposition

5.1, Theorem 6.1] again a p-compact group and the natural homomorphism $C_X(fG) \to X$ is a monomorphism, in fact an example of a homomorphism of maximal rank [8, Definition 4.1, Proposition 4.3]. Assume that f lifts to a homomorphism $\varphi: G \to T$ into the maximal torus T; i.e. that the space $(X/T)^{hG}$ of lifts of Bf is nonempty. This always is the case if [17, Lemma 3.13; 8, Proposition 2.14] the Weyl group order $|W_T(X)|$ is not divisible by p or if G is a p-compact torus.

Mapping BG into $Bi: BT \to BX$, which we assume has been turned into a fibration, produces another fibration of mapping spaces

$$(X/T)^{hG} \to \text{map}(BG, BT) \xrightarrow{B_i} \text{map}(BG, BX) \tag{2}$$

where the fiber over Bf is displayed. The Weyl space monoid [8, Definition 9.2] acts on this fibration with trivial action on the base space and the Weyl group $W_T(X)$ acts on the associated exact sequence

$$\pi_0((X/T)^{hG}) \to \text{Rep}(G, T) \xrightarrow{i_*} \text{Rep}(G, X)$$

of sets. Define $W_T(X)^{B\varphi} < W_T(X)^{\varphi} < W_T(X)$ to be the isotropy subgroup at $B\varphi \in \pi_0((X/T)^{hG})$, respectively $\varphi \in \text{Rep}(G, T)$. Note that the set $i_*^{-1}(f) \subseteq \text{Rep}(G, T)$ of conjugacy classes of lifts of f is a $W_T(X)$-set.

The essential idea of the proof of Theorem 3.1 is due to Bill Dwyer during a conversation at the Čech Centennial Homotopy Conference.

THEOREM 3.1. *Let $f: G \to X$ be a homomorphism from a p-compact toral group G into a p-compact group X with maximal torus $i: T \to X$. Let $C_X(fG)_0$ denote the identity component of the centralizer $C_X(fG)$ of f in X. Assume that $(X/T)^{hG} \neq \emptyset$ and let $\varphi: G \to T$ be a lift of f. Then*:

(1) *The homomorphism $C_T(\varphi G) \to C_X(fG)$, induced by i, is a maximal torus for $C_X(fG)$.*
(2) *The Weyl group of $C_X(fG)_0$ is isomorphic to $W_T(X)^{B\varphi}$.*
(3) *$W_T(X)$ acts transitively on the set $\pi_0((X/T)^{hG})$.*

Proof. Put $C := C_X(fG)$ and $C_0 := C_X(fG)_0$.

The first assertion is contained in the proof of [8, Proposition 4.3]. Note that the factorization

$$BT \simeq BC_T(\varphi G) \to BC \to BX$$

of $B_i: BT \to BX$ and the fact [7, Theorem 9.7] that Weyl groups are faithfully represented in maximal tori, allow us to consider $W_T(C)$ as a subgroup of $W_T(X)$.

Computing the cardinality of the set of components of the space $(X/T)^{hG}$ of lifts of Bf is the first step. This space of lifts occurs as the fiber of the fibration

$$(X/T)^{hG} \to \coprod_{\psi \in i_*^{-1}(f)} BC_T(\psi G) \to BC \tag{3}$$

obtained by restricting fibration (2) to the connected component BC of the base space map(BG, BX). Thus we may describe the fiber

$$(X/T)^{hG} \simeq \coprod_{\psi \in i_*^{-1}(f)} C/C_T(\psi G) \tag{4}$$

as a disjoint union of homogeneous spaces.

Replacing G by $G/\ker\varphi$ if necessary, we may assume (as $C_X(G/\ker\varphi) \simeq C_X(G)$ by [7, Lemma 7.5] and $(X/T)^{hG} \simeq ((X/T)^{h\ker\varphi})^{hG/\ker\varphi} \simeq (X/T)^{hG/\ker\varphi}$ by [7, Lemma 10.5] and the Sullivan conjecture [13]) that φ is a monomorphism [7, Section 7] and hence [17, Proposition 3.4] that G has a discrete approximation which is a subgroup of $(\mathbb{Z}/p^\infty)^r$. Using [7, Theorem 4.7, Proposition 5.7, Theorem 6.1, Proposition 6.7] we compute the Euler characteristic $\chi((X/T)^{hG}) = \chi(X/T) = |W_T(X)|$ where the last equality is [7, Proposition 8.10, Proposition 9.5]. As each disjoint summand $C/C_T(\psi G)$ of the right-hand side of (4) has Euler characteristic $|W_T(C)|$, the order of the index set is $|i_*^{-1}(f)| = |W_T(X): W_T(C)|$. Hence

$$|\pi_0((X/T)^{hG})| = |W_T(X): W_T(C)| \cdot |\pi_0(C)| = |W_T(X): W_T(C_0)|$$

because the index of the Weyl group of C_0 in the Weyl group of C equals the number of components $|\pi_0(C)|$ of C by [17, Proposition 3.8] or [8, Remark 2.11].

Observe that $W_T(X)^\varphi < W_T(C)$ and that $W_T(X)^{B\varphi} < W_T(C_0)$: if $w \in W_T(X)^\varphi$, the two maps $w \circ B\varphi, B\varphi: BG \to BT$ are homotopic so composition with w is a self-map $\underline{w}$ of $BC_T(\varphi G) = \mathrm{map}(BG, BT)_{B\varphi}$ over BC. Thus w represents an element of the Weyl group of C. If $w \in W_T(X)^{B\varphi}$, the fiber map $\underline{w}$ preserves the component containing φ of the fiber $C/C_T(\varphi G) \subseteq (X/T)^{hG}$; i.e. is homotopic over BC to a based map, meaning [17, Proposition 3.8] that $w \in W_T(C_0) < W_T(C)$.

The final step is a counting argument.

Consider the orbit $W_T(X)\cdot B\varphi \subseteq \pi_0((X/T)^{hG})$. As

$$\begin{aligned}|\pi_0((X/T)^{hG})| &\geq |W_T(X)\cdot B\varphi| \\ &= |W_T(X): W_T(X)^{B\varphi}| \\ &\geq |W_T(X): W_T(C_0)| \\ &= |\pi_0((X/T)^{hG})|\end{aligned}$$

we have $W_T(X)\cdot B\varphi = \pi_0((X/T)^{hG})$ and $W_T(X)^{B\varphi} = W_T(C_0)$. □

COROLLARY 3.2. *Let $f: G \to X$ and $\varphi: G \to T$ be as in Theorem 3.1. Then:*

(1) *The Weyl group of $C_X(fG)$ is isomorphic to the isotropy subgroup $W_T(X)^\varphi$.*
(2) *$W_T(X)$ acts transitively on the set $i_*^{-1}(f)$.*
(3) *$(X/T)^{hG}$ is homotopy equivalent to a disjoint union of $|W_T(X): W_T(X)^\varphi|$ copies of the homogeneous space $C_X(fG)/C_T(\varphi G)$.*

Proof. According to the proof of Theorem 3.1, $|i_*^{-1}(f)| = |W_T(X): W_T(C)|$ and $W_T(X)^\varphi < W_T(C)$.

Consider the orbit $W_T(X)\cdot\varphi \subseteq i_*^{-1}(f) \subseteq \mathrm{Rep}(G, T)$. A counting argument similar to the one in the proof of Theorem 3.1 shows that $W_T(X)\cdot\varphi = i_*^{-1}(f)$ and $W_T(X)^\varphi = W_T(C)$. The final assertion is the homotopy equivalence (4) from the proof of Theorem 3.1. □

The p-compact toral group G could in particular be a p-compact torus.

COROLLARY 3.3. *Let $f: S \to X$ be a homomorphism from a p-compact torus S into a p-compact group X with maximal torus $i: T \to X$. Then:*

(1) *There exists a homomorphism $\varphi: S \to T$ such that $B_i \circ B\varphi = Bf$.*
(2) *$W_T(X)$ acts transitively on the set $\pi_0((X/T)^{hS})$ of vertical homotopy classes of lifts of Bf.*
(3) *If X is connected, $\pi_0((X/T)^{hS}) = i_*^{-1}(f)$, i.e. two lifts $B\varphi_1, B\varphi_2: BS \to BT$ of Bf are homotopic over Bf if (and only if) they are homotopic.*

Proof. The existence of φ is [7, Proposition 8.11]. If X is connected, the centralizer $C_X(fS)$ of S in X is also connected [17, Proposition 3.11], so the homotopy exact sequence for the fibration (3) shows that $\pi_0((X/T)^{hS}) = i_*^{-1}(f) \subseteq \mathrm{Rep}(S, T)$. □

Thus the maximal torus homomorphism $i: T \to X$ induces a bijection of sets

$$W_T(X)\backslash \mathrm{Rep}(S, T) \xrightarrow{\cong} \mathrm{Rep}(S, X)$$

for any p-compact torus S and any connected p-compact group X.

Example 3.4. (1) Corollary 3.2(2) says that

$$W_T(X)\backslash \mathrm{Rep}(G, T) \to \mathrm{Rep}(G, X)$$

is injective for any p-compact group G as in Theorem 3.1. If $G = S$ is a p-compact torus this map is even a bijection by Corollary 3.3. This was also proved in [18].

(2) By Corollary 3.2(1), the component group $\pi_0(C_X(T))$ is isomorphic to the kernel of the action $W_T(X) \to \mathrm{Aut}(BT)$ of the Weyl group on the maximal torus.

(3) Let $N(T) \to X$ denote the normalizer [7, 9.8] of the maximal torus. Corollary 3.2(1) implies that the centralizer $C_{N(T)}(G)$ of G in the normalizer of the maximal torus is isomorphic to the normalizer of the maximal torus $C_T(G)$ of the centralizer $C_X(G)$.

The material of Theorem 3.1 and Corollary 3.3 establishes the analogs to the main results of the papers [1, 2] by Adams and Mahmud, respectively Adams and Wojtkowiak.

THEOREM 3.5. *Let X_1 and X_2 be p-compact groups with maximal tori $i_1: T_1 \to X_1$, $i_2: T_2 \to X_2$, and Weyl groups W_1, W_2. Let $f: X_1 \to X_2$ be a homomorphism of X_1 into X_2. Then:*

(1) *There exists a homomorphism $\varphi: T_1 \to T_2$ such that the diagram*

$$\begin{array}{ccc} BT_1 & \xrightarrow{B\varphi} & BT_2 \\ {\scriptstyle Bi_1}\downarrow & & \downarrow{\scriptstyle Bi_2} \\ BX_1 & \xrightarrow[Bf]{} & BX_2 \end{array}$$

commutes. (We say that $B\varphi$ covers Bf.)

(2) *The Weyl group W_2 acts transitively on the set $\pi_0((X_2/T_2)^{hT_1})$ of vertical homotopy classes of maps covering Bf.*

(3) *If X_2 is connected, two maps $B\varphi_1$, $B\varphi_2: BT_1 \to BT_2$ that cover Bf are vertically homotopic if they are homotopic.*

Proof. Apply Corollary 3.3 to the homomorphism $f \circ i_1: T_1 \to X_1$. □

In case f is a rational isomorphism a stronger version is possible, cf. [10, Proposition 1.2]. If there exists a rational isomorphism between the p-compact groups X_1 and X_2 then they have the same rational rank [8, 5.9] so [7, Theorem 9.7] there exists a p-compact torus T with homomorphisms $i_1: T \to X_1$ and $i_2: T \to X_2$ making T into a maximal torus for both p-compact groups.

THEOREM 3.6. *Let X_1 and X_2 be p-compact groups with maximal tori $i_1: T \to X_1$, $i_2: T \to X_2$ and Weyl groups W_1, W_2. Let $f: X_1 \to X_2$ be a rational isomorphism of X_1 into X_2. Then:*

(1) *There exists an isogeny $\varphi: T \to T$ such that $B\varphi$ covers Bf, i.e. such that the diagram*

$$\begin{array}{ccc} BT & \xrightarrow{B\varphi} & BT \\ {\scriptstyle Bi_1}\downarrow & & \downarrow{\scriptstyle Bi_2} \\ BX_1 & \xrightarrow[Bf]{} & BX_2 \end{array}$$

commutes.

(2) *The space $(X_2/T)^{hT}$ of all maps covering Bf is homotopically discrete* [7, Remark 6.15] *and W_2 acts simply and transitively on its set of components $\pi_0((X_2/T)^{hT})$.*

(3) *There exists a homomorphism $W_T(\varphi): W_1 \to W_2$, such that $B\varphi \circ w = W_T(\varphi)(w) \circ B\varphi$ holds in $\pi_0((X_2/T)^{hT})$ for all $w \in W_1$; $W_T(\varphi)$ takes the Weyl group $W_1^0 < W_1$ of the identity component of X_1 isomorphically onto the Weyl group $W_2^0 < W_2$ of the identity component of X_2.*

(4) *If X_2 is connected, two maps $B\varphi_1, B\varphi_2: BT \to BT$ that cover Bf are vertically homotopic if they are homotopic.*

Under the additional assumption that $\pi_0(f): \pi_0(X_1) \to \pi_0(X_2)$ is an isomorphism, the homomorphism $W_T(\varphi): W_1 \to W_2$ in point (3) *is an isomorphism, and the homomorphism φ in point* (1) *is an isomorphism if and only if f is an isomorphism.*

Proof. (1) We only need to show that φ is a rational isomorphism for then φ is actually an isogeny by Lemma 2.9. Note that by replacing X_1 and X_2 by their respective identity components it represents no loss of generality to assume that X_1 and X_2 are connected p-compact groups.

Let $w_1 \in W_1$ be represented by a map $w_1: BT \to BT$ over BX. Since $B\varphi \circ w_1$ covers $Bf, B\varphi \circ w_1$ is (Theorem 3.5(2)) (vertically) homotopic to $w_2 \circ B\varphi$ for some $w_2 \in W_2$. Hence the image $\varphi^*(V) \subseteq V$ of $V := H^*_{\mathbb{Q}_p}(BT)$ under $\varphi^* := H^*_{\mathbb{Q}_p}(B\varphi)$ is W_1-invariant. Moreover, as $H^*_{\mathbb{Q}_p}(Bf)$ is an isomorphism, the endomorphism $S(\varphi^*)$ of the symmetric algebra $S(V) = H^*_{\mathbb{Q}_p}(BT)$ restricts to an isomorphism $S(V)^{W_1} \leftarrow S(V)^{W_2}$ of invariant algebras [8, Theorem 9.7]; thus

$$S(V)^{W_1} = S(\varphi^*)(S(V)^{W_2}) = S(\varphi^*(V))^{W_1}$$

which, as any nontrivial W_1-invariant complement to $\varphi^*(V)$ would contribute effectively to $S(V)^{W_1}$, implies that $\varphi^*(V) = V$, i.e. that φ^* is an isomorphism.

(2) Since φ is an epimorphism,

$$C := C_{X_2}(fi_1 T) = C_{X_2}(i_2 \varphi T) = C_{X_2}(i_2 T)$$

by [8, Lemma 7.5]. Thus C is a p-compact toral group with $C_0 = T$ as its identity component [8, Section 8]. By Corollary 3.2,

$$(X_2/T)^{hT} \simeq \coprod C/T$$

is homotopically discrete and the W_2-action on the set of components is transitive and simple by Theorem 3.1.

(3) The order of W_j^0 equals the rank of $H^*_{\mathbb{Q}_p}(BT)$ as an $H^*_{\mathbb{Q}_p}(BX_j)$-module [8, Theorem 9.7; [4, Ch. 5, Section 5, no. 2] so since $H^*_{\mathbb{Q}_p}(B\varphi)$ and $H^*_{\mathbb{Q}_p}(Bf^0)$ are isomorphisms (Definition 2.1), $|W_1^0| = |W_2^0|$.

Since W_2 acts simply and transitively on $\pi_0((X_2/T)^{hT})$, the equation $B\varphi \circ w = W_T(\varphi)(w) \circ B\varphi$ does define a homomorphism between the Weyl groups. Naturality of this construction implies that $W_T(\varphi)$ is in fact a homomorphism

$$\begin{array}{ccccccccc} 1 & \to & W_1^0 & \to & W_1 & \xrightarrow{\lambda} & \pi_0(X_1) & \to & 1 \\ & & \downarrow\cong & & \downarrow W_T(\varphi) & & \downarrow \pi_0(f) & & \\ 1 & \to & W_2^0 & \to & W_2 & \underset{\lambda}{\to} & \pi_0(X_2) & \to & 1 \end{array}$$

of the short exact sequences relating the full Weyl groups to those of the identity components [17, Proposition 3.8]. Here, the restriction $W_T(\varphi)\,|\,W_1^0 : W_1^0 \to W_2^0$ is a monomorphism because $\pi_2(B\varphi) \otimes \mathbb{Q}$ is an isomorphism and the subgroups W_j^0 are faithfully represented [7, Theorem 9.7] in $\pi_2(BT) \otimes \mathbb{Q}$. (Alternatively one may view the Weyl groups W_1^0 and W_2^0 as Galois groups of field extensions connected by isomorphisms induced by φ and the restriction f^0 of f to the identity components.)

(4) This is point (3) of Corollary 3.3.

Now assume additionally that $\pi_0(f) : \pi_0(X_1) \to \pi_0(X_2)$ is an isomorphism. By the above diagram, $W_T(\varphi) : W_1 \to W_2$ is then an isomorphism.

Suppose $\varphi : T \to T$ is an isomorphism and that $B\varphi$ covers Bf as in point (1). Replacing i_2 by $i_2 \circ \varphi$ we may even assume that φ is the identity, i.e. that $i_2 : T \to X_2$ admits the factorization $i_2 = f \circ i_1$. Since also $W_1 = W_2$, this implies that the p-normalizer [7, Definition 9.8] of T in X_1, N_p, equals the p-normalizer of T in X_2 and that the natural monomorphism $N_p \to X_1$ followed by f is the natural monomorphism $N_p \to X_2$. Thus $f\,|\,N_p$ is a monomorphism and so is f by [17, Theorem 2.17]. Consequently, f is a monomorphic rational isomorphism inducing an isomorphism on the groups of components, i.e. an isomorphism by Lemma 2.5. □

In the language of [20, Definition 1.1], the homomorphism φ from point (1) of Theorem 3.6 induces a $W_T(\varphi)$-admissible monomorphism $\pi_1(\varphi) : \pi_1(T) \to \pi_1(T)$ (provided $\pi_0(f)$ is an isomorphism).

4. CLASSIFICATION OF RATIONAL ISOMORPHISMS

The aim of this section is to relate the set of rational isomorphisms between two locally isomorphic p-compact groups to the monoid of rational automorphisms of their universal covering p-compact group. We begin by showing that the center construction is a functor on the category of p-compact groups with rational isomorphisms.

Let X_1 and X_2 be two *connected* p-compact groups with maximal tori $i_1 : T \to X_1$ and $i_2 : T \to X_2$ of the same rank; let W_1 and W_2 denote the associated Weyl groups.

Rational isomorphisms between p-compact groups behave like epimorphisms between Lie groups in that they preserve centrality.

LEMMA 4.1. *Let $f : X_1 \to X_2$ be a rational isomorphism and $z : Z \to X_1$ a central* [7, 3.4] *homomorphism from some p-compact toral group Z into X_1. Then the composition $f \circ z : Z \to X_2$ is central.*

Proof. Choose [17, Lemma 4.1] a homomorphism $y : Z \to T$ such that $z = i_1 \circ y$ and (Theorem 3.6) a finite covering homomorphism $T(f) : T \to T$ such that $Bi_2 \circ T(f) = Bf \circ Bi_1$. The commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{y} & T & \xrightarrow{T(f)} & T \\ & \searrow^{z} & \downarrow i_1 & & \downarrow i_2 \\ & & X_1 & \underset{f}{\to} & X_2 \end{array}$$

induces another commutative diagram

$$\begin{array}{ccccc} C_T(yZ) & \to & C_1 & \xrightarrow{\cong} & X_1 \\ \downarrow & & \downarrow & & \downarrow f \\ C_T(T(f)yZ) & \to & C_2 & \to & X_2 \end{array}$$

where $C_1 := C_{X_1}(zZ)$ and $C_2 := C_{X_2}(fzZ)$ have $C_T(yZ) \cong T$ and $C_T(T(f)yZ) \cong T$ as maximal tori (Theorem 3.1). Note that C_2 here can be replaced by its identity component C_2^0 leading to a factorization of i_2 which allows us to view the Weyl group of C_2^0 as a subgroup of W_2; in the language of [8, Section 4], $C_2^0 \to X_2$ is a subgroup of maximal rank. Applying the functor $H^*_{\mathbb{Q}_p}(-)$ leads to a third commutative diagram

$$\begin{array}{ccccc} H^*_{\mathbb{Q}_p}(BT) & \leftarrow & H^*_{\mathbb{Q}_p}(BC_1) & \xleftarrow{\cong} & H^*_{\mathbb{Q}_p}(BX_1) \\ {\scriptstyle H^*_{\mathbb{Q}_p}(B\varphi)}\uparrow\cong & & \uparrow & & \cong\uparrow {\scriptstyle H^*_{\mathbb{Q}_p}(Bf)} \\ H^*_{\mathbb{Q}_p}(BT) & \leftarrow & H^*_{\mathbb{Q}_p}(BC_2^0) & \leftarrow & H^*_{\mathbb{Q}_p}(BX_2) \end{array}$$

in cohomology; here C_2 has been replaced by C_2^0. This diagram shows that the rank, $|W_T(C_2^0)|$, of $H^*_{\mathbb{Q}_p}(BT)$ as an $H^*_{\mathbb{Q}_p}(BC_2^0)$-module equals the rank, $|W_1|$, of $H^*_{\mathbb{Q}_p}(BT)$ as an $H^*_{\mathbb{Q}_p}(BX_1)$-module. Since $|W_1| = |W_2|$ (Theorem 3.6), we have $W_T(C_2^0) = W_2$ so $C_2^0 \to X_2$ is both a monomorphism and a rational isomorphism [8, Theorem 9.7], hence (Lemma 2.5) an isomorphism. Now $C_2 \to X$ is both an epimorphism and monomorphism, i.e. [7, 3.2] an isomorphism, meaning that $f \circ z : Z \to X_2$ is central. □

Functoriality of Z is now an easy consequence.

COROLLARY 4.2. *For any rational isomorphism $f \in \varepsilon_{\mathbb{Q}}(X_1, X_2)$ there exists a uniquely determined rational isomorphism $Z(f) \in \varepsilon_{\mathbb{Q}}(Z(X_1), Z(X_2))$ such that*

$$\begin{array}{ccc} Z(X_1) & \xrightarrow{Z(f)} & Z(X_2) \\ \downarrow & & \downarrow \\ T & \xrightarrow{T(f)} & T \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_2} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array}$$

commutes up to conjugacy for any finite covering homomorphism $T(f)$ that covers f.

Proof. In order to prove existence, let $\breve{T}(f): \breve{T} \to \breve{T}$ be the discrete approximation to any finite covering homomorphism $T(f): T \to T$ that covers f. By Lemma 4.1, $\breve{T}(f)$ restricts to a homomorphism $\breve{Z}(f): \breve{Z}(X_1) \to \breve{Z}(X_2)$ between the p-discrete centers [17, Definition 4.3] of X_1 and X_2. Define $Z(f): Z(X_1) \to Z(X_2)$ to be the closure [7, Proposition 6.9] of $\breve{Z}(f)$. Then the above diagram commutes and $Z(f)$ is a rational isomorphism because [17, Corollary 5.2] $Z(X_j) \to X_j$ induces an isomorphism on $\pi_1(-) \otimes \mathbb{Q}$.

There is [17, Lemma 4.8] a homotopy equivalence

$$(X_2/Z(X_2))^{hZ(X_1)} \simeq X_2/Z(X_2)$$

between $X_2/Z(X_2)$ and the space of all lifts to $BZ(X_2)$ of the central (Lemma 4.1) homomorphism $Z(X_1) \to X_1 \to X_2$. In particular, the space of lifts is connected meaning that all lifts are vertically homotopic. This proves uniqueness of $Z(f)$. □

The uniqueness clause ensures that $Z(f_2 \circ f_1) = Z(f_2) \circ Z(f_1)$ whenever the rational isomorphisms f_1 and f_2 are composable; in particular, $Z(f)$ is an isomorphism for any isomorphism f. This center functor plays a central role in the computation of the set of rational isomorphisms between locally isomorphic p-compact groups.

Assume from now on that X_1 and X_2 are locally isomorphic, i.e. (Proposition 2.8) that there exist special finite covering homomorphisms (as in the short exact sequence (1) of Section 2)

$$K_j \xrightarrow{(\mathrm{incl}, \varphi_j)} Y \times S \xrightarrow{q_j} X_j = Y \times S/(K_j, \varphi_j)$$

for some simply connected p-compact group Y, some p-compact torus S, some finite subgroups $K_j < Z(Y)$, and some homomorphisms $\varphi_j : K_j \to S$, $j = 1, 2$.

The aim is to relate $\varepsilon_{\mathbf{Q}}(X_1, X_2)$ to $\varepsilon_{\mathbf{Q}}(y)$ and $\varepsilon_{\mathbf{Q}}(S)$. Let $f: X_1 \to X_2$ be any rational isomorphism.

The composite maps

$$BY \xrightarrow{B(\mathrm{incl})} BY \times BS = B(Y \times S) \xrightarrow{Bq_j} BX_j, \quad j = 1, 2$$

induce homotopy equivalences $BY \to BX_j\langle 2\rangle$ of 2-connected covers. Let f_Y be the rational isomorphism corresponding to $Bf\langle 2\rangle$. Then

$$\begin{array}{ccc} Y & \xrightarrow{f_Y} & Y \\ {\scriptstyle q_1|Y}\downarrow & & \downarrow{\scriptstyle q_2|Y} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array} \tag{5}$$

commutes up to conjugacy.

The composite homomorphisms

$$S \xrightarrow{(\mathrm{incl})} Z(Y) \times S = Z(Y \times S) \xrightarrow{Z(q_j)} Z(X_j), \quad j = 1, 2$$

take S isomorphically to the identity component $Z(X_j)_0$ of the center [17, Corollary 5.5]. Let $f_S : S \to S$ be the homomorphism corresponding to the finite covering (Lemma 2.9) homomorphism $Z(f)_0 : Z(X_1)_0 \to Z(X_2)_0$ of identity components. Then

$$\begin{array}{ccc} S & \xrightarrow{f_S} & S \\ {\scriptstyle q_1|S}\downarrow & & \downarrow{\scriptstyle q_2|S} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array} \tag{6}$$

commutes up to conjugacy (Corollary 4.2).

Now put $\Lambda(f) := (f_Y, f_S)$. The following main result is the analog of [3, Corollary 6] for Lie groups and [16, Theorem 0.3] for classifying spaces of Lie groups.

THEOREM 4.3. *Let $X_1 = Y \times S/(K_1, \varphi_1)$ and $X_2 = Y \times S/(K_2, \varphi_2)$ be two locally isomorphic connected p-compact groups.*

(1) *The map*

$$\Lambda : \varepsilon_{\mathbf{Q}}(X_1, X_2) \to \varepsilon_{\mathbf{Q}}(Y) \times \varepsilon_{\mathbf{Q}}(S)$$

is injective.

(2) *For* $f \in \varepsilon_{\mathbb{Q}}(X_1, X_2)$, $g \in \varepsilon_{\mathbb{Q}}(Y)$, *and* $h \in \varepsilon_{\mathbb{Q}}(S)$, $\Lambda(f) = (g, h)$ *if and only if the diagram*

$$\begin{array}{ccc} Y \times S & \xrightarrow{g \times h} & Y \times S \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle q_2} \\ X_1 & \xrightarrow[f]{} & X_2 \end{array} \tag{7}$$

commutes up to conjugacy.

(3) *The image of* Λ *consists of all pairs* $(g, h) \in \varepsilon_{\mathbb{Q}}(Y) \times \varepsilon_{\mathbb{Q}}(S)$ *for which* $Z(g)(K_1) < K_2$ *and the diagram*

$$\begin{array}{ccc} K_1 & \xrightarrow{\varphi_1} & S \\ {\scriptstyle Z(g)|K_1}\downarrow & & \downarrow{\scriptstyle h} \\ K_2 & \xrightarrow[\varphi_2]{} & S \end{array} \tag{8}$$

commutes up to conjugacy.

Proof. The first step is to see that diagram (7) with $g = f_Y$ and $h = f_S$ commutes up to conjugacy: diagram (6) shows that both $Bq_2 \circ (Bf_Y \times Bf_S)$ and $Bf \circ Bq_1$ have adjoints belonging to the mapping space

$$\operatorname{map}(BY, \operatorname{map}(BS, BX_2)_{Bf \circ B(q_1|S)}) \simeq \operatorname{map}(BY, BX_2)$$

where the homotopy equivalence is by centrality (Lemma 4.1) of $f \circ (q_1|S)$; furthermore diagram (5) shows that both adjoints land in the component of $Bf \circ B(q_1|Y)$ of $\operatorname{map}(BY, BX_2)$ under this homotopy equivalence. Thus the adjoints, and with them also the original maps, are homotopic.

In other words, $f_Y \times f_S$ is a lift of $f \circ q_1$ or f is a factorization of $q_2 \circ (f_Y \times f_S)$; as lifts are unique by Lemma 2.4, point (2) follows, and as factorizations are unique by Corollary 2.11, point (1) follows.

Suppose $(g, h) = \Lambda(f)$ for some rational isomorphism f. The functor Z applied to the commutative diagram (7) determines a homomorphism

$$\begin{array}{ccccc} K_1 & \xrightarrow{(\mathrm{incl}, \varphi_1)} & Z(Y) \times S & \xrightarrow{Z(q_1)} & Z(X_1) \\ \downarrow & & \downarrow{\scriptstyle Z(g) \times h} & & \downarrow{\scriptstyle Z(f)} \\ K_2 & \xrightarrow[(\mathrm{incl}, \varphi_2)]{} & Z(Y) \times S & \xrightarrow[Z(q_2)]{} & Z(X_2) \end{array}$$

between short exact sequences of p-compact group homomorphisms; the horizontal short exact sequences come from [17, Corollary 5.5]. Commutativity of the left square in this diagram means that the pair (g, h) satisfies the conditions of point (3).

Suppose, conversely, that $(g, h) \in \varepsilon_{\mathbb{Q}}(Y) \times \varepsilon_{\mathbb{Q}}(S)$ satisfies the conditions in point (3) meaning that the left square in the above diagram exists or, equivalently, that the left square in the commutative diagram

$$\begin{array}{ccccc} BK_1 & \xrightarrow{(B(\mathrm{incl}), B\varphi_1)} & B(Y \times S) & \xrightarrow{Bq_1} & BX_1 \\ \downarrow & & \downarrow{\scriptstyle B(g \times h)} & & \vdots \\ BK_2 & \xrightarrow[(B(\mathrm{incl}), B\varphi_2)]{} & B(Y \times S) & \xrightarrow[Bq_2]{} & BX_2 \end{array}$$

exists. According to Corollary 2.11, this commutative diagram can be completed by some homomorphism $f: X_1 \to X_2$. Obviously, f is a rational isomorphism and $\Lambda(f) = (g, h)$ since, by construction, diagram (7) commutes. □

In the situation of Theorem 4.3 there is a simple relation between the mapping space components containing f, g, and h.

PROPOSITION 4.4. *Suppose that the homomorphism $f: X_1 \to X_2$ is covered by a product homomorphism $g \times h: Y \times S \to Y \times S$. Then there exists a fibration of the form*

$$BK_2 \to \mathrm{map}(BY, BY)_{Bg} \times \mathrm{map}(BS, BS)_{Bh} \to \mathrm{map}(BX_1, BX_2)_{Bf}$$

relating the centralizers of f, g, and h.

Proof. Recall that for any two p-compact groups U and V, the trivial homomorphism is central [7, Proposition 10.1] meaning that the maps

$$\mathrm{map}(BU, BV)_0 \underset{c_{BV}}{\overset{e_{BV}}{\rightleftarrows}} BV$$

given by evaluation and inclusion of constant maps are each other's homotopy inverses. (If U is a p-compact toral group this already follows from Miller's Sullivan conjecture [13; 7, Proposition 5.3, Theorem 6.1] and if V is a p-compact toral group from elementary obstruction theory.)

Suppose that $Bg: BY \to BY$ and $Bh: BS \to BS$ are maps; not necessarily rational isomorphisms. Then

$$\begin{aligned}
&\mathrm{map}(BY \times BS, BY \times BS)_{Bg \times Bh} \\
&\quad = \mathrm{map}(BY \times BS, BY)_{Bg \circ \mathrm{pr}_{BY}} \times \mathrm{map}(BY \times BS, BS)_{Bh \circ \mathrm{pr}_{BS}} \\
&\quad = \mathrm{map}(BY, \mathrm{map}(BS, BY)_0)_{c_{BY} \circ Bg} \times \mathrm{map}(BS, \mathrm{map}(BY, BS)_0)_{c_{BS} \circ Bh} \\
&\quad \simeq \mathrm{map}(BY, BY)_{Bg} \times \mathrm{map}(BS, BS)_{Bh}
\end{aligned}$$

where the second homeomorphism is provided by adjointness and the homotopy equivalence is induced by composition with evaluation maps.

Moreover, composition with the projections Bq_1 and Bq_2 induce maps

$$\mathrm{map}(BY \times BS, BY \times BS)_{Bg \times Bh} \xrightarrow{\underline{Bq_2}} \mathrm{map}(BY \times BS, BX_2)_{Bf \circ Bq_1} \underset{\simeq}{\xleftarrow{\overline{Bq_1}}} \mathrm{map}(BX_1, BX_2)_{Bf}$$

where $\underline{Bq_2}$ is a fibration with fiber BK_2 by Lemma 2.4 and $\overline{Bq_1}$ is a homotopy equivalence by Lemma 2.10. □

COROLLARY 4.5. *Let $f \in \varepsilon_{\mathbf{Q}}(X_1, X_2)$ be a rational isomorphism with $\Lambda(f) = (g, h) \in \varepsilon_{\mathbf{Q}}(Y) \times \varepsilon_{\mathbf{Q}}(S)$. Then:*

(1) *f is an isogeny if and only if g is an isomorphism.*

(2) *f is an isomorphism if and only if g and h are isomorphisms.*

Proof. Diagram (7) induces a fibration

$$K_2/Z(g)K_1 \to Y/gY \times S/hS \to X_2/fX_1$$

where the fiber has no homotopy in degree > 1, Y/gY is simply connected, and (Lemma 2.9) $S/hS \simeq B(\ker \check{h})$. The corollary follows by inspecting the exact homotopy sequence for this fibration. □

In short form the content of Theorem 4.3 and Corollary 4.5 is that

$$\varepsilon_{\mathbf{Q}}(X_1, X_2) \subseteq \varepsilon_{\mathbf{Q}}(Y) \times \varepsilon_{\mathbf{Q}}(S), \quad \varepsilon_{\mathbf{Q}}(X) \subseteq \varepsilon_{\mathbf{Q}}(Y) \times \varepsilon_{\mathbf{Q}}(S)$$

$$\mathrm{Cov}(X_1, X_2) \subseteq \mathrm{Out}(Y) \times \mathrm{Cov}(S), \quad \mathrm{Out}(X) \subseteq \mathrm{Out}(Y) \times \mathrm{Out}(S)$$

is the set of pairs (g,h) for which $Z(g)(K_1) \subseteq K_2$ and diagram (8) commutes. Here, $\mathrm{Out}(Y) \subseteq \mathrm{Rep}(Y, Y)$ denotes the group of conjugacy classes of automorphisms of Y, i.e. the group of invertible elements of the monoid $\varepsilon_{\mathbf{Q}}(Y)$. In the second column, where the assumption is that $X_1 = X = X_2$, the inclusion respects the monoid or group structure. The lower line represents a direct translation of Baum's expression [3, Corollary 6] for the set of isogenies between two locally isomorphic compact connected Lie groups. For instance,

$$\varepsilon_{\mathbf{Q}}(Y \times S) \cong \varepsilon_{\mathbf{Q}}(Y) \times \varepsilon_{\mathbf{Q}}(S) \cong (PY \times S)$$

where $PY = Y/Z(Y)$. Note also that up to isomorphism the only isogenies onto $X := Y \times S/(K, \varphi)$ are of the form

$$Y \times S/(L, \psi) \xrightarrow{(\mathrm{id}_Y, h)} Y \times S/(K, \varphi)$$

where $L < K$, $h \in \mathrm{Cov}(S)$, and $h \circ \psi = \varphi | L$.

Write $X_1 \geq X_2$ if there exists an isogeny $X_1 \to X_2$.

Proposition 4.6. *If $X_1 \geq X_2$ and $X_2 \geq X_1$ then $X_1 \cong X_2$.*

Proof. Let $(g_1, h_1): X_1 \to X_2$ and $(g_2, h_2): X_2 \to X_1$ be isogenies where $g_1, g_2 \in \mathrm{Out}(Y)$ and $h_1, h_2 \in \mathrm{Cov}(S)$ as in Theorem 4.3. Observe that $Z(g_1)$ and $Z(g_2)$ are isomorphisms and that $Z(g_1) | K_1$ takes K_1 isomorphically to K_2. Injectivity of the p-discrete torus $\breve{S}$ implies that the commutative diagram

$$\begin{array}{ccc} K_1 & \xrightarrow{\varphi_1} & S \\ {\scriptstyle Z(g_1)|K_1}\downarrow \cong & & \downarrow \\ K_2 & \longrightarrow & S \end{array}$$

can be completed by some homomorphism $h: S \to S$ and an application of Nakayama's lemma [15, Lemma 3.2] shows that h may be chosen to be an isomorphism. Then $(g_1, h): X_1 \to X_2$ is an isomorphism of p-compact groups. □

In contrast, two compact connected Lie groups may cover each other without being isomorphic [3].

Definition 4.7. *The local isomorphism system* of the connected p-compact group X is the set of isomorphism classes of p-compact groups locally isomorphic to X equipped with the ordering $\geq$.

This definition is a direct copy of the equivalent notion from the category of compact connected Lie groups [3, Definition 8].

For example, let Y be the p-compact group $SU(p)_p^{\wedge}$. The local isomorphism system of $Y \times S$, where S is a p-compact torus of rank 1, has the form

$$Y \times S \to U(p)_p^{\wedge} \to PY \times S$$

and the local isomorphism system of $Y \times Y$ has the form

$$\begin{array}{ccccc} & & PY \times Y & & \\ & \nearrow & & \searrow & \\ Y \times Y & & & & PY \times PY \\ & \searrow & & \nearrow & \\ & & Y \times Y/\Delta & & \end{array}$$

where $\Delta < Z(Y) \times Z(Y) \cong \mathbb{Z}/p \times \mathbb{Z}/p$ is the diagonal subgroup.

5. CENTRALIZERS OF RATIONAL AUTOMORPHISMS

The purpose of this section is to investigate centralizers of endomorphisms that are rational isomorphisms. It is shown that any nontrivial endomorphism of a connected simple p-compact group is a rational isomorphism and in case p divides the Weyl group order even a homotopy equivalence.

Let X be a p-compact group with maximal torus $i: T \to X$. The associated Weyl space $\mathscr{W}_T(X)$ is the group-like topological monoid of self-maps of BT over BX [7, Definition 9.2]. Define [7, Definition 9.8] $BN(T)$ to be the total space in the fibration

$$BT \to BN(T) \to B\mathscr{W}_T(X) \simeq BW_T(X)$$

for which the monodromy action is the inclusion homomorphism $\Omega B\mathscr{W}_T(X) = \mathscr{W}_T(X) \hookrightarrow \mathrm{aut}(BT)$ of the Weyl space monoid into the group-like topological monoid $\mathrm{aut}(BT)$ of self-homotopy equivalences of BT. Let also

$$BT \to BN_p(T) \to BW_p$$

be the restriction of $BN(T)$ to the classifying space of a p-Sylow subgroup W_p of $W_T(X)$. The loop space $N(T) = \Omega BN(T)$ is called the normalizer and $N_p(T) = \Omega BN_p(T)$ the p-normalizer of T in X. Note that, unless $W_T(X)$ happens to be a p-group, the normalizer $N(T)$ will not be a p-compact group. The p-normalizer $N_p(T)$, however, is always a p-compact toral group. Sine the action of $\mathscr{W}_T(X)$ on BT respects the map $Bi: BT \to BX$, the homomorphism $i: T \to X$ extends to $T \to N_p(T) \to N(T) \xrightarrow{j} X$ and [7, Proposition 9.9] $N_p(T) \xrightarrow{j_p} X$ is a monomorphism.

Suppose $f: X \to X$ is an endomorphism of X. The space of all lifts in the diagram

$$\begin{array}{ccccc} & & & & BN(T) \\ & & & \nearrow & \downarrow{\scriptstyle Bj} \\ BN(T) & \xrightarrow[Bj]{} & BX & \xrightarrow[Bf]{} & BX \end{array}$$

is the homotopy fixed point space $(X/N(T))^{hfj(N(T))}$.

LEMMA 5.1. *If the endomorphism $f: X \to X$ is a rational isomorphism, then* $(X/N(T))^{hfjN(T)} \simeq *$.

Proof. The universal covering map $BT \to BN(T)$ induces a covering map

$$W_T(X) \to (X/T)^{hT} \to (X/N(T))^{hT}$$

where the total space is homotopy equivalent (Theorem 3.6) to the homotopically discrete fiber. Thus the base space $(X/N(T))^{hT}$ and [7, Lemma 10.5] $(X/N(T))^{hN(T)} = ((X/N(T))^{hT})^{hW_T(X)}$ are contractible. □

In particular there exists a self-map $BN(f)$, unique up to vertical homotopy, such that the diagram

$$\begin{array}{ccc} BN(T) & \xrightarrow{BN(f)} & BN(T) \\ {\scriptstyle Bj}\downarrow & & \downarrow{\scriptstyle Bj} \\ BX & \xrightarrow[Bf]{} & BX \end{array}$$

commutes. In view of the proposition below and the results of [10], it seems not totally unlikely that the monoid homomorphism

$$N : \varepsilon_{\mathbf{Q}}(X) \to [BN(T), BN(T)]$$

is injective.

PROPOSITION 5.2. *Let X be a connected p-compact group and $f_0, f_1 : X \to X$ two endomorphisms of X. If $BN(f_0)$ and $BN(f_1)$ are homotopic, then $H^*_{\mathbf{Q}_p}(Bf_0) = H^*_{\mathbf{Q}_p}(Bf_1)$ and $f_0 \circ \xi$ and $f_1 \circ \xi$ are conjugate for any homomorphism $\xi : G \to X$ from a p-compact toral group G into X.*

Proof. Since $H^*_{\mathbf{Q}_p}(Bj) : H^*_{\mathbf{Q}_p}(BX) \to H^*_{\mathbf{Q}_p}(BN(T))$ is an isomorphism [7, Theorem 9.7], the first assertion immediately follows.

Because $\pi_1(X/N(T))$ maps onto $\pi_0(N(T))$ we may assume by adjust Bf_j by a vertical homotopy to achieve that $\pi_1(BN(f_j))$ preserves the p-Sylow subgroup $W_p = \pi_1(BN_p(T)) < \pi_1(BN(T)) = W_T(X)$. Let $BN_p(f_j) : BN_p(T) \to BN_p(T)$ be the unique based map that covers $BN(f), j = 0, 1$. Since $BN(f_0)$ and $BN(f_1)$ are homotopic, $BN_p(f_1) = w \circ BN_p(f_0)$ for some covering translation w of the covering $BN_p(T) \to BN(T)$.

Let $\xi : G \to X$ be a homomorphism from a p-compact toral group G into X. Since the Euler characteristic of $X/N_p(T)$ is prime to p [7, Proof of 2.3] there exists by [17, Lemma 3.13] or [8, Proposition 2.14] a lift $\zeta : G \to N_p(T)$ such that ξ and $j_p \circ \zeta$ are conjugate. Then $Bf_0 \circ B\xi \simeq Bf_0 \circ Bj_p \circ B\zeta \simeq Bj_p \circ BN_p(f_0) \circ B\zeta \simeq Bj_p \circ w \circ BN_p(f_1) \circ B\zeta \simeq Bj_p \circ BN_p(f_1) \circ B\zeta \simeq Bf_1 \circ Bj_p \circ B\zeta \simeq Bf_1 \circ B\xi$. □

Now follows another observation with regard to the loop map $N(f)$ associated to the rational automorphism f.

LEMMA 5.3. *Let $f : X \to X$ be a rational isomorphism and $BN(f)$ the self-map of $BN(T)$ that covers Bf (Lemma 5.1). Then precomposition with $BN(f)$*

$$\overline{BN(f)} : \operatorname{map}(BN(T), BX)_{Bj} \to \operatorname{map}(BN(T), BX)_{Bf \circ Bj}$$

is a homotopy equivalence.

Proof. Any lift $T(f) : T \to T$ to the maximal torus is an epimorphism (Theorem 3.6) so the induced map

$$\overline{BT(f)} : BC_X(iT) \to BC_X(fiT)$$

is a homotopy equivalence by [7, Lemma 7.5] or [8, Lemma 10.3]. Note that both $BC_X(iT) = \operatorname{map}(BT, BX)_{Bi}$ and $BC_X(fiT) = \operatorname{map}(BT, BX)_{Bf \circ Bi}$ are $W_T(X)$-spaces in that the homotopy orbit spaces $BC_X(iT)_{hW_T(X)}$ and $BC_x(fiT)_{hW_T(x)}$ exist. The homotopy equivalence $\overline{BT(f)}$ induces a homotopy equivalence

$$BC_X(iT)^{hW_T(X)} \to BC_X(fiT)^{hW_T(X)}$$

of homotopy fixed point spaces. In particular, the component $\text{map}(BN(T), BX)_{Bj}$ of the domain is mapped by a homotopy equivalence, identifiable to $\overline{BN(f)}$, to the component $\text{map}(BN(T), BX)_{Bf\circ Bj}$ of the target. □

The analogous statement with the normalizer $N(T)$ replaced by the p-normalizer $N_p(T)$ also holds.

Definition 5.4. A connected p-compact group X is *simple* if $\pi_2(BT) \otimes \mathbb{Q}$ is a simple $\mathbb{Q}_p[W_T(X)]$-module and *semisimple* if its fundamental group $\pi_1(X)$ is finite.

Since $(\pi_2(BT) \otimes \mathbb{Q})^{W_T(X)} \cong \pi_1(Z(X)) \otimes \mathbb{Q} \cong \pi_1(X) \otimes \mathbb{Q}$ by [17, Proposition 5.1, Corollary 5.2], any simple p-compact group is semisimple.

Simplicity is, of course, a very distinguished property of a p-compact group as shown by the below theorem which in the case of compact Lie groups follows from the classification theorem [11, Theorem 2].

THEOREM 5.5. *Let X be a connected simple p-compact group and $f: X \to X$ a nontrivial endomorphism. Then f is a rational isomorphism.*

Proof. Suppose $f: X \to X$ is not a rational isomorphism. Let $T(f): T \to T$ be an endomorphism that covers f as in Theorem 3.5. It follows from admissibility that the image of $H^2_{\mathbb{Q}_p}(BT(f))$ is a $W_T(X)$-submodule of $H^2_{\mathbb{Q}_p}(BT)$. Since f is not a rational isomorphism and $H^2_{\mathbb{Q}_p}(BT)$ an irreducible $W_T(X)$-representation, $T(f)$ is trivial. Then f itself is trivial by Corollary 6.7 below. □

Under the additional assumption that p divides the order of the Weyl group an even stronger statement, which may be seen as a generalization of Ishiguro's theorem [9], holds.

THEOREM 5.6. *Let X be a connected simple p-compact group where p divides the order of the Weyl group $W_T(X)$. Then $\varepsilon_{\mathbb{Q}}(X) = \text{Out}(X)$.*

Proof. Let $f: X \to X$ be a rational isomorphism and $BN(f): BN(T) \to BN(T)$ the map (Lemma 5.1) that covers $Bf: BX \to BX$. Then $\pi_1(BN(f))$ is an isomorphism (namely conjugate to $W_T(T(f))$ where $BT(f): BT \to BT$ covers Bf). Thus (some map homotopic to) $BN(f)$ lifts to self-map $BN_p(f)$ of the covering space $BN_p(T)$ of $BN(T)$. The discrete approximation $\check{N}_p(f)$ to $N_p(f)$ is a self-map

$$\begin{array}{ccccccccc}
1 & \to & \check{T} & \to & \check{N}_p(T) & \to & W_p & \to & 1 \\
 & & \downarrow{\scriptstyle \check{N}_p(f)|\check{T}} & & \downarrow{\scriptstyle \check{N}_p(f)} & & \cong\downarrow{\scriptstyle W_T(T(f))} & & \\
1 & \to & \check{T} & \to & \check{N}_p(T) & \to & W_p & \to & 1
\end{array}$$

of the group extension inducing the Postnikov tower for $BN_p(T)$.

Choose an element $w \in W_p$ of order p (remember that p divides the order of $W_T(X)$) with a lifting $g \in \check{N}_p(T)$. Since the discrete approximation $\check{T}$ to T is p-divisible, $g^p = t^p$ for some

$t \in \check{T}$. Then $gt^{-1} \in \check{N}_p(T)$ has order p and projects onto w; denote also by w this lift gt^{-1} of w. Note that $\check{N}_p(f)(w) \neq 1$.

Let $\langle w \rangle \subseteq \check{N}_p(T)$ be the cyclic group generated by w and $B(j_p | \langle w \rangle)$ the monomorphism $B\langle w \rangle \xrightarrow{B(\subseteq)} BN_p(T) \xrightarrow{j_p} BX$. In the commutative diagram

$$\begin{array}{ccccc} B\langle w \rangle & \xrightarrow{B(\subseteq)} & BN_p(T) & \xrightarrow{B(N_p(f))} & BN_p(T) \\ & \searrow^{B(j_p|\langle w \rangle)} & \downarrow^{Bj_p} & & \downarrow^{Bj_p} \\ & & BX & \xrightarrow[Bf]{} & BX \end{array}$$

the composition of the two top horizontal maps is a monomorphism since $\check{N}_p(f) | \langle w \rangle$ is a monomorphism [17, Proposition 3.4]. As also the vertical homomorphism j_p is monomorphic, $f \circ j_p | \langle w \rangle$ is a monomorphism.

Since X is connected, the monomorphism $j_p | \langle w \rangle$ factors through $i: T \to X$ (use [7, Proposition 5.5, Proposition 8.11] to see this) and we obtain a homotopy commutative diagram

$$\begin{array}{ccccc} & & BT & \xrightarrow{BT(f)} & BT \\ & \nearrow^{Bk} & {}^{Bi}\downarrow & & \downarrow^{Bi} \\ B\langle w \rangle & \xrightarrow[B(j_p|\langle w \rangle)]{} & BX & \xrightarrow[Bf]{} & BX \end{array}$$

of maps of classifying spaces. By assumption $\pi_2(BT) \otimes \mathbb{Q}$ is an irreducible $W_T(X)$-module, so $\pi_2(BT(f))$ is scalar multiplication by some p-adic integer λ; i.e. $BT(f) = \psi^\lambda$. We must show that λ is a p-adic unit for then f will be an automorphism (Theorem 3.6).

Assume the converse, i.e. that λ is divisible by p. Then, since w has order p, $BT(f) \circ Bk$ is null homotopic so also $Bf \circ B(j_p | \langle w \rangle)$ is null homotopic. This contradicts [7, Proposition 5.4] the fact that $f \circ j_p | \langle w \rangle$ is monomorphism. □

The above proof is a transcription of that of [10, Proposition 1.3].

We now turn to the case where the prime p does not divide the order of the Weyl group.

An important result [8, Theorem 1.3] asserts that the natural map

$$BZ(X) \to \operatorname{map}(BX, BX)_{B1}$$

of the center $Z(X)$ into the centralizer of the identity map, denoted 1, of X is a homotopy equivalence. The following proposition explicitly computes this center as well as the centralizer of any rational automorphism of X provided the order of the Weyl group is prime to p.

PROPOSITION 5.7. *Let X be a connected p-compact group with Weyl group order prime to p and let $f: X \to X$ be a rational isomorphism. Then precomposition with Bf induces a homotopy equivalence*

$$BC_X(X) = \operatorname{map}(BX, BX)_{B1} \xrightarrow{\overline{Bf}} BC_X(fX) = \operatorname{map}(BX, BX)_{Bf}$$

of mapping space components. Here,

$$BC_X(X) \simeq K(\pi_2(BT)^{W_T(X)}, 2),$$

i.e. $C_X(X)$ and $C_X(fX)$ are p-compact tori. If, additionally, X is semisimple, then $C_X(X)$ and $C_X(fX)$ are trivial p-compact groups.

Proof. Composition with the maps of the relation $Bf \circ Bj = Bj \circ BN(f)$ induce a commutative diagram

$$\begin{array}{ccc} \operatorname{map}(BX,BX)_{Bf} & \xrightarrow[\simeq]{\overline{Bj}} & \operatorname{map}(BN(T),BX)_{Bf\circ Bj} \\ \overline{Bf}\uparrow & & \overline{BN(f)}\uparrow\simeq \\ \operatorname{map}(BX,BX)_{B1} & \xrightarrow[\overline{Bj}]{\simeq} & \operatorname{map}(BN(T),BX)_{Bj} \end{array}$$

of maps between mapping spaces. Since p is prime to the order of the Weyl group of X,

$$H^*(BX;\mathbb{F}_p) \cong H^*(BT;\mathbb{F}_p)^{W_T(X)} \cong H^*(BN(T);\mathbb{F}_p)$$

by [6, Theorems 2.10–2.11]. Thus $Bj\colon BN(T) \to BX$ is an $H^*\mathbb{F}_p$-equivalence and the maps $\overline{Bj}$ are homotopy equivalences. Since also $\overline{BN(f)}$ is a homotopy equivalence by Lemma 5.3, $\overline{Bf}$ is a homotopy equivalence and so it only remains to compute $\operatorname{map}(BN(T),BX)_{Bj}$ which is a path-component of the space of sections of the fibration

$$BC_X(T) \to BC_X(T)_{hW_T(X)} \to BW_T(X)$$

with fiber $BC_X(T) \simeq BT$. Since p is prime to $|W_T(X)|$, so that $H^{>0}(W_T(X);\, \pi_2(BT)) = 0$, [15, Theorem 6.3] shows that

$$\operatorname{map}(BN(T),BX)_{Bj} = K(\pi_2(BC_X(T))^{W_T(X)},2)$$

where the action $\operatorname{map}(BT,BX)_{Bi} \times W_T(X) \to \operatorname{map}(BT,BX)_{Bi}$ is composition with elements from the Weyl space. Choose a based map $w\colon BT \to BT$ realizing some $w \in W_T(X)$ and a based map $w^{-1}\colon BT \to BT$ which is a homotopy inverse to w. Then the homotopy commutative diagram of homotopy equivalences

$$\begin{array}{ccccc} BT & \xleftarrow[\simeq]{e_{BT}} & \operatorname{map}(BT,BT)_{B1} & \xrightarrow[\simeq]{\overline{Bi}} & \operatorname{map}(BT,BX)_{Bi} \\ \downarrow^{w^{-1}} & & \downarrow & & \downarrow^{\bar{w}} \\ BT & \xleftarrow[e_{BT}]{\simeq} & \operatorname{map}(BT,BT)_{B1} & \xrightarrow[\overline{Bi}]{\simeq} & \operatorname{map}(BT,BX)_{Bi} \end{array}$$

where the middle vertical map is conjugation by w and e_{BT} is evaluation at the base point, shows that the above right action of $w \in W_T(X)$ on $\operatorname{map}(BT,BX)_{Bi}$ identifies to the standard left action of w on BT.

Thus $Z(X) \cong C_X(fX)$ is always a p-compact torus in this nonmodular case. If X is semisimple, the center $Z(X)$ is also finite [17, Theorem 5.3]; hence trivial. □

Combining Theorem 5.6 covering the modular case and Proposition 5.7 covering the nonmodular case, we obtain the following corollary.

COROLLARY 5.8. *Suppose that $f\colon X \to X$ is a nontrivial endomorphism of a connected simple p-compact group X with nontrivial Weyl group. Then pre- and post-composition with Bf induce maps*

$$\operatorname{map}(BX,BX)_{B1} \xrightarrow{\underline{Bf}} \operatorname{map}(BX,BX)_{Bf} \xleftarrow{\overline{Bf}} \operatorname{map}(BX,BX)_{B1}$$

that both are homotopy equivalences.

Proof. If the prime p divides $|W_T(X)|$, Bf is a homotopy equivalence by Theorem 5.6 and if not, both centralizers are contractible by Proposition 5.7. □

In the Lie group case the above corollary is contained in [11, Theorem 3].

6. A TRIVIALITY CRITERION

The purpose of this section is to prove that any p-compact group homomorphism that vanishes on all elements is trivial.

A homomorphism $f: X \to Y$ between p-compact groups is said to vanish on all elements if

$$\mathbb{Z}/p^n \xrightarrow{\xi} X \xrightarrow{f} Y$$

is trivial for any $n \geq 1$ and any homomorphism $\xi: \mathbb{Z}/p^n \to X$. (A homomorphism is trivial if the corresponding map of classifying spaces is null homotopic.)

THEOREM 6.1. *Let $f: X \to Y$ be a homomorphism that vanishes on all elements. Then f is trivial.*

The proof uses an inductive principle introduced in [8, Section 9] and codified in the concept of a saturated class. A class $\mathscr{C}$ of p-compact groups is said to be saturated [8, Definition 9.1] if:

(1) $\mathscr{C}$ is closed under isomorphisms.

(2) The trivial p-compact group belongs to $\mathscr{C}$.

(3) If the identity component X_0 of X is in $\mathscr{C}$, then $X \in \mathscr{C}$.

(4) If X is connected and $X/Z(X) \in \mathscr{C}$, then $X \in \mathscr{C}$.

(5) If X is connected and has trivial center, and $C \in \mathscr{C}$ for all p-compact groups C with $\mathrm{cd}_{\mathbb{F}_p}(C) < \mathrm{cd}_{\mathbb{F}_p}(X)$, then $X \in \mathscr{C}$.

The point is that the only saturated class is [8, Theorem 9.2] the class of all p-compact groups.

Now let $\mathscr{C}$ be the class of all p-compact groups X with the property that any homomorphism defined on X that vanishes on all elements is trivial. The strategy is to show that $\mathscr{C}$ is saturated. Obviously, $\mathscr{C}$ is closed under isomorphisms.

LEMMA 6.2. *Any p-compact toral group belongs to $\mathscr{C}$.*

Proof. Suppose that X is a p-compact toral group and $f: X \to Y$ a homomorphism that vanishes on all elements. Then $\ker f = X$ so f is trivial by [7, Lemma 7.7]. □

LEMMA 6.3. *Let X be a p-compact group with identity component X_0 and let $f: X \to Y$ be a homomorphism that vanishes on all elements. If $X_0 \in \mathscr{C}$, then f is trivial.*

Proof. Since $X_0 \in \mathscr{C}$ and f vanishes on all elements, the composition $X_0 \to X \to Y$ is trivial and f admits (Corollary 2.11) a factorization $X \to \pi_0(X) \to Y$ through the finite p-group $\pi_0(X)$ of components. I claim that $\pi_0(X) \to Y$ vanishes on all elements; hence (Lemma 6.2) is trivial.

To see this, let $\mathbb{Z}/p^n \to \pi_0(X)$ be any group homomorphism. Recall that the p-normalizer admits a monomorphism $j_p: N_p(T) \to X$ with $\pi_0(j_p)$ an epimorphism; see [17, Corollary 3.9] or [8, Remark 2.11]. Choose an $m \geq n$ and a homomorphism $\mathbb{Z}/p^m \to \pi_0(N_p(T))$ making

$$\begin{array}{ccc} \mathbb{Z}/p^m & \rightarrow & \pi_0(N_p(T)) \\ {\scriptstyle \mathrm{mod}\, p^n} \downarrow & & \downarrow {\scriptstyle \pi_0(j_p)} \\ \mathbb{Z}/p^n & \rightarrow & \pi_0(X) \end{array}$$

commutative. Here, the top horizontal homomorphism lifts to the p-discrete approximation $\check{N}_p(T)$ as in the proof of Theorem 5.6. Compose this lift with the monomorphism j_p into X to obtain the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}/p^m & \to & X & \xrightarrow{f} & Y \\ {\scriptstyle \bmod p^n}\downarrow & & \downarrow & \nearrow & \\ \mathbb{Z}/p^n & \to & \pi_0(X) & & \end{array}$$

of p-compact group homomorphisms. The composition of the two horizontal homomorphisms at the top is trivial since f vanishes on all elements. Hence also $\mathbb{Z}/p^n \to \pi_0(X) \to Y$ is trivial by [7, Lemma 7.7]. □

For any p-compact group X, put $PX = X/Z(X)$.

LEMMA 6.4. *Let X be a connected p-compact group and $f: X \to Y$ a homomorphism that vanishes an all elements. If $PX \in \mathscr{C}$, then f is trivial.*

Proof. Since the center $Z(X) \in \mathscr{C}$ by Lemma 6.3, f factors (Lemma 2.10) through a homomorphism $PX \to Y$ defined on PX. I claim that this factorization vanishes on all elements and hence is trivial.

Let $\mathbb{Z}/p^n \to PX$ be a homomorphism. Since PX is connected, the orientable fibration $BZ(X) \to BX \to BP(X)$ is classified by some k-invariant $k: BP(X) \to B^2Z(X)$. The obstruction to lifting $\mathbb{Z}/p^n \to PX$ to X is the restriction of k to $\mathbb{Z}/p^n$ corresponding to a cohomology class in $H^2(\mathbb{Z}/p^n; \check{Z}(X))$. Assume the order of this obstruction is p^m (any element of this cohomology group has finite order). Then it is possible to find a homomorphism $\mathbb{Z}/p^{m+n} \to X$ such that

$$\begin{array}{ccccc} \mathbb{Z}/p^{m+n} & \to & X & \xrightarrow{f} & Y \\ {\scriptstyle \bmod p^n}\downarrow & & \downarrow & \nearrow & \\ \mathbb{Z}/p^n & \to & PX & & \end{array}$$

commutes up to conjugacy. Since f vanishes on all elements, the composition of the two horizontal homomorphisms at the top is trivial. Hence also $\mathbb{Z}/p^n \to PX \to Y$ is trivial. □

LEMMA 6.5. *Let X be a connected p-compact group with trivial center and $f: X \to Y$ a homomorphism that vanishes on all elements. If all p-compact groups C with $\mathrm{cd}_{\mathbb{F}_p}(C) < \mathrm{cd}_{\mathbb{F}_p}(X)$ are in $\mathscr{C}$, then f is trivial.*

Proof. The category $\mathbb{A}_X$ has as objects all pairs (V, v) where V is a nontrivial elementary abelian p-group and $v: V \to X$ a conjugacy class of monomorphisms $V \to X$. The morphisms of $\mathbb{A}_X$ are injections over X. For any object (V, v) of $\mathbb{A}_X$, let

$$e(V, v): BC_X(vV) \to BX$$

be the evaluation monomorphism. Naturality of these maps makes them combine to a map

$$e: \operatorname{hocolim} BC_X(vV) \to BX$$

where the homotopy colimit is taken over (the opposite of) $\mathbb{A}_X$. The map e is an $H^*\mathbb{F}_p$-equivalence [8, Theorem 8.1] so we have homotopy equivalences

$$\mathrm{map}(BX, BY) \underset{\simeq}{\xrightarrow{\bar{e}}} \mathrm{map}(\operatorname{hocolim} BC_X(vV), BY) \simeq \operatorname{holim} \mathrm{map}(BC_X(vV), BY)$$

for any p-compact group Y.

Consider now a homomorphism $f: X \to Y$ that vanishes on all elements. For any object (V, v) of $\mathbb{A}_X$ also $f \circ e(V, v): C_X(vV) \to Y$ vanishes on all elements. The assumption that X has trivial center implies [17, Theorem 4.4] that the monomorphism $v: V \to X$ is not central and hence [7, Proposition 6.14, Remark 6.15] that $\mathrm{cd}_{\mathbb{F}_p}(C_X(vV)) < \mathrm{cd}_{\mathbb{F}_p}(X)$ so that $C_X(vV) \in \mathscr{C}$ by assumption. Hence $f \circ e(V, v)$ is trivial.

Let $R \subseteq \mathrm{Rep}(X, Y)$ denote the set of those homomorphisms $X \to Y$ that vanish on all elements and let $\mathrm{map}(BX, BY)_R$ be the corresponding union of path-components of $\mathrm{map}(BX, BY)$. By the above, $\mathrm{map}(BX, BY)_R$ is homotopy equivalent to a union of components of the homotopy inverse limit

$$\operatorname{holim} \mathrm{map}(BC_X(vV), BY)_0 \tag{9}$$

of spaces of null homotopic maps. Since trivial homomorphisms are central [7, Proposition 10.1], the inclusions $BY \to \mathrm{map}(BC_X(vV), BY)_0$ are homotopy equivalences providing a natural transformation from the constant functor BY to the functor that takes (V, v) to $\mathrm{map}(BC_X(vV), BY)_0$. Hence [5, Lemma XI. 5.6] the homotopy inverse limit (9) is homotopy equivalent to the space of maps from the nerve of the category $\mathbb{A}_X$ to BY. But since the nerve has the $\mathbb{F}_p$-cohomology of a point [8, Proposition 8.2], we conclude that (9) is homotopy equivalent to BY. In particular, $\mathrm{map}(BX, BY)_R$ is connected, meaning that the set R contains only the trivial homomorphism $0 \in \mathrm{Rep}(X, Y)$. □

Together with the induction principle, Lemmas 6.3–6.5 prove Theorem 6.1.

Two easy corollaries can be obtained by combining Theorem 6.1 with the facts that any element of an arbitrary p-compact group is conjugate to an element of the p-normalizer and any element of a connected p-compact group is conjugate to an element of the maximal torus.

COROLLARY 6.6. *Let X be a p-compact group with maximal torus T and p-normalizer $N_p(T)$ and let $f: X \to Y$ be any p-compact group homomorphism. If $f | N_p(T)$ is trivial, then f is trivial.*

Proof. Since $X/N_p(T)$ has Euler characteristic prime to p, any homomorphism $\mathbb{Z}/p^n \to X$ factors through $N_p(T)$ by [17, Lemma 3.13] or [8, Proposition 2.14]. As $f | N_p(T)$ is trivial, this shows that $f | \mathbb{Z}/p^n$ is trivial and hence f itself is trivial by Theorem 6.1. □

COROLLARY 6.7. *Let X be a connected p-compact group and $T \to X$ a maximal torus. Then the following are equivalent*:

(1) *f is trivial,*

(2) *$f | T$ is trivial,*

(3) $\bar{H}^*_{\mathbb{Q}_p}(Bf) = 0$,

for any homomorphism $f: X \to Y$.

Proof. Let $i: T \to X$ and $j: S \to Y$ be maximal tori and choose (Theorem 3.5) a homomorphism $\varphi: T \to S$ such that $f \circ i$ and $j \circ \varphi$ are conjugate.

Let $\varphi^*: U \to V$ be the induced linear map of $U := H^2_{\mathbb{Q}_p}(BS)$ into $V := H^2_{\mathbb{Q}_p}(BT)$. As in the proof of Theorem 3.6, $\varphi^*(U)$ is a $W_T(X)$-submodule and $\varphi^*(S[U]^{W_S(Y)}) = S[\varphi^*(U)]^{W_T(X)}$. This equation shows that $\bar{H}^*_{\mathbb{Q}_p}(B\varphi) \circ \bar{H}^*_{\mathbb{Q}_p}(Bj) = 0$ if and only if $\bar{H}^*_{\mathbb{Q}_p}(B\varphi) = 0$ from which the equivalence of (2) and (3) follows.

Assuming (2), let $\xi: \mathbb{Z}/p^n \to X$ be any homomorphism from a cyclic p-group into X. Choosing a homomorphism $\zeta: \mathbb{Z}/p^n \to T$ with $i \circ \zeta$ conjugate to ξ shows that $Bf \circ B\xi \simeq Bf \circ Bi \circ B\zeta \simeq *$. Hence f is trivial by Theorem 6.1. □

Together with the Sullivan conjecture for p-compact groups [8, Theorem 9.3], Corollaries 6.6 and 6.7 represent a generalization of [11, Theorem 3.11].

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