# Euler Characteristic of Subgroup Categories

**Bachelor Thesis in Mathematics.**

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**Abstract**

Let $G$ be a finite group and let $S_G^{p+*}$ be the associated Brown poset consisting of all non-trivial p-subgroups of $G$ ordered by inclusion. Let $|G|_p$ denote the maximal power of $p$ dividing $|G|$. We prove that Brown's theorem, stating that $|G|_p$ divides the reduced Euler characteristic of the poset $S_G^{p+*}$, is equivalent to a theorem by Frobenius' which says that $|G|_p$ divides the number of p-singular elements in $G$. In order to show this we introduce the concept of the Euler characteristic of a finite category and use this definition to find an Euler characteristic for subgroup categories of $G$. We will be particularly interested in the orbit category of $G$.

**Resumé**

Lad $G$ være en endelig gruppe og lad $S_G^{p+*}$ være den partielt ordenede mængde der består af alle ikke-trivielle p-undergrupper af $G$ ordnet efter inklusion. Lad $|G|_p$ betegne den maksimale potens af $p$ der går op i $|G|$. Vi viser at Browns sætning, som siger, at $|G|_p$ går op i den reducerede Euler karakteristik af $S_G^{p+*}$, er ækvivalent med en sætning vist af Frobenius, som siger at $|G|_p$ går op i antallet af p-singulære elementer i $G$. For at vise dette introducerer vi konceptet Euler karakteristikken af en endelig kategori. Vi bruger denne definition til at finde en Euler karakteristik for undergruppe-kategorier af $G$. Specielt vil vi kigge på orbit kategorien af $G$.

# 1 Introduction

Given a finite group $G$ we look at the Brown poset of this group $S_G^{p+*}$ which consists of all non-trivial p-subgroups of $G$ ordered by inclusion. We look at this poset in hope that it will tell us something about our group. When we view this poset as a category we can realize it to get a topological space and use tools from topology to get information about our group $G$. As a topological invariant we can use the Euler characteristic since it depends only on the homotopy-type of a given space.
Let $|G|_p$ denote the maximal power of $p$ dividing $|G|$. We will prove that the Euler characteristic of $S_G^{p+*}$ is congruent to 1 mod $|G|_p$, a theorem first proved by Brown in 1975, is equivalent to a theorem shown by Frobenius in 1907, namely that given a group $G$, $|G|_p$ divides the number of p-singular elements in $G$.

In this paper we look at a rather new definition of the Euler characteristic of a category taken from Tom Leinster's article "The Euler characteristic of a category" and use this definition to show that the two theorems are equivalent. To do this we first need some basic knowledge about categories, which will be the subject in the first chapter. We only include the definitions and theorems we need for later use. This chapter does not include many examples as we will see examples of their application later on. In the second chapter we introduce the realization of a category and the connection between categories and topology. In the third chapter we define the Euler characteristic of a category. We show that for a poset $P$ we can take the geometric realization and find the Euler characteristic in the usual way or we can view it as a category and find the Euler characteristic in the sense of Leinster. We show that these to numbers will be equal. In the fourth chapter we will look at homotopy equivalences of subgroup-categories of $G$ and we will find an Euler characteristic for the orbit category $O_G^p$. We will obtain the following

$$\sum_{[H]} \frac{-\tilde{\chi}(S^{p+*}_{O_G(H)})}{|O_G(H)|}|G| = |G_p|$$

where $H$ is a p-subgroup of $G$. This will show that theorems of Frobenius and Brown are indeed equivalent.

# Contents

# 2 Categories

In this chapter we will introduce some basic category theory. We will look at categories, functors and the relationship between functors, all of which we will need later. This chapter is based on Mac Lane's "Categories for the working mathematician."

## 2.1 Categories

**Definition 2.1.** *A category consists of a collection of objects and a collection of morphisms that satisfy the following conditions :*

- *To every morphism $f$ there is an object $a = dom\ f$, called the domain of $f$ and an object $b = cod\ f$ called the codomain of $f$. We write $f : a \to b$.*
- *For every object $a$ there is a morphism called the identity morphism of $a$ denoted $1_a : a \to a$ satisfying $1_a \circ f = f$ and $g \circ 1_a = g$ for all morphisms $g$ and $f$ satisfying $codf = a$ and $domg = a$.*
- *If $f$ and $g$ are morphisms with $cod\ f = dom\ g$ then the composition $g \circ f$ is a morphism with $dom(g \circ f) = dom\ f$ and $cod(g \circ f) = cod\ g$.*
- *The composition of morphisms is associative, that is $(f \circ g) \circ h = f \circ (g \circ h)$ for composable morphisms $f, g, h$.*

For a category $C$ we let $obC$ denote the objects of $C$ and $C(a, b)$ the morphisms from $a$ to $b$ in $C$ and $\zeta(a, b)$ is the number of morphisms from $a$ to $b$. Two objects $a$ and $a'$ are said to be isomorphic if there exist two morphisms $f : a \to a'$ and $g : a' \to a$ with $g \circ f = 1_a$ and $f \circ g = 1_{a'}$.

**Definition 2.2.** *We call a category small if both the collection of objects and the collection of morphisms are sets. If this is not the case, we call it large.*

We call a category $C$ discrete if the only morphisms are the identity morphisms.

### Example

1. The category of sets, **Set**, has as objects sets, and morphisms are function between sets. This is a large category.

2. **Grp** denotes the category of groups and group homomorphisms.

3. Topological spaces and all continuous maps between them form a category, **Top**.

4. **0** denotes the empty category with no objects and no morphisms. **1** is the category with one object and one morphism, the identity morphism.

5. We can consider every partially ordered set, $P$, as a category. The objects are the elements of $P$, and there is exactly one morphism from $x$ to $y$ if $x \leq y$. The relation $x \leq x$ gives us the identity morphism and transitivity gives the composition. The category of all partially ordered sets is denoted **Poset**.

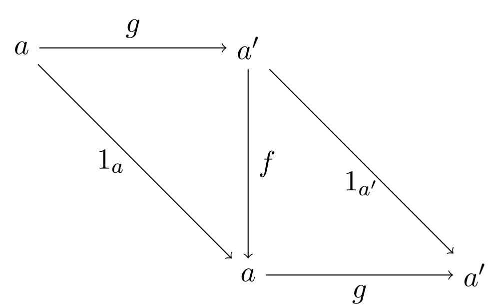
**Definition 2.3.** *For a category $C$ we denote by $C^{op}$ the category $C$ where we have reversed all the morphisms.*

**Definition 2.4.** *Let $C$ be a category.*

1. *An object $a \in C$ is said to be initial if for every object $c \in C$ there is a unique morphism $a \to c$.*

2. *An object $b \in C$ is said to be terminal if for every object $c \in C$ there is a unique morphism $c \to b$.*

Initial and terminal objects are unique up to isomorphism. To show this let $a$ and $a'$ be two initial objects. Then we have the following commutative diagram




and we see that $1_a = f \circ g$ and $1_{a'} = g \circ f$ hence $a$ and $a'$ are isomorphic. In **Grp**, the trivial group is both an initial and a terminal object.

**Definition 2.5.** *A morphism from an object to itself is called an endomorphism. If all endomorphisms in a category are also isomorphisms, we call the category an EI-category.*

**Definition 2.6.** *Given two categories $A$ and $B$ we can construct the product category $A \times B$. The objects in $A \times B$ are pairs $(a, b)$ where $a$ is an object of $A$ and $b$ is an object of $B$. The morphisms $(a, b) \to (a', b')$ are pairs $(f, g)$ where $f : a \to a'$ and $g : b \to b'$. Composition is given coordinate wise as in $A$ and $B$.*

## 2.2 Functors

We now look at functors which are mappings of categories that respect domain and codomain of morphisms and preserve the structure of composition and identities. Then we look at the relationship between them.

**Definition 2.7.** *For categories $C$ and $D$, a functor $T : C \to D$, is a map which assigns to each object $c$ in $C$ an object $Tc$ in $D$, and to each arrow $f : c \to c'$ in $C$ an arrow $Tf : Tc \to Tc'$ in $D$, such that*

- *$T$ preserves composition: $T(g \circ f) = Tg \circ Tf$ for every composable pair of morphisms $f, g$ in $C$.*
- *$T$ preserves identity: $T(1_c) = 1_{Tc}$.*

**Examples of functors**

1. For two categories $C$ and $D$ we can construct a functor $F : C \to D$ that maps every object of $C$ to a fixed object $d_0 \in D$ and every morphism of $C$ to the identity morphism on $d_0$.
2. The map $U$ : **Grp** $\to$ **Set** that assigns to each group the set of its elements and to each group homomorphism itself regarded as a function is a functor, called the forgetful functor.

We can compose functors. Given functors $T : C \to D$ and $S : D \to E$ we can define the composite functor $ST : C \to E$ by $c \mapsto S(Tc)$ and $f \mapsto S(Tf)$ on objects $c$ and morphisms $f$ in $C$. For every category $C$ there is the identity functor $1_C : C \to C$.

We can consider the category of small categories **Cat** whose objects are all small categories and whose morphisms are functors between categories.

**Definition 2.8.** *Let $C$ and $D$ be categories. Suppose we have functors $F : C \to D$ and $G : D \to C$ such that $GF = 1_C$ and $FG = 1_D$ . Then we call $F$ and $G$ isomorphisms and say that $C$ and $D$ are isomorphic.*

This is a very strong condition as it only allows the category to differ in the notation of their objects and morphisms. Weaker conditions such as equivalence and adjunction will turn out to be sufficient for our purposes.

**Definition 2.9.** *A functor $F : C \to D$ is full when to every pair $c, c'$ of objects in $C$ and to every morphism $g : Fc \to Fc'$, there is a morphism $f : c \to c'$ in $C$ with $g = Ff$.*
*A functor $F : C \to D$ is faithful when for every pair $c, c'$ of objects in $C$ and every pair of morphisms $f_1, f_2 : c \to c'$ in $C$, the equality $Ff_1 = Ff_2 : Tc \to Tc'$ implies $f_1 = f_2$.*

If $f$ is an isomorphism in $C$ then $F(f)$ is also an isomorphism in $D$ by the definition of a functor. The converse holds if $F$ is fully faithful. Indeed, let $c$ and $c'$ be objects of $C$ and let $f : c \to c'$ be a morphism in $C$. Let $F(f)$ be the morphism between $F(c)$ and $F(c')$ and assume $F(f)$ is an isomorphism with inverse $h$. Then $F(f) \circ h = id_{F(c')}$ and $h \circ F(f) = id_{F(c)}$ Since $F$ is full there is a morphism $g$ in $C$ with $F(g) = h$. We can write $F(f \circ g) = F(f) \circ F(g) = F(f) \circ h = id_{Fc'}$. Since $F$ is faithful we get that $f \circ g = id_{c'}$ and in the same way we get that $g \circ f = id_c$.

**Definition 2.10.** *A subcategory $A$ of $C$ is a collection of some of the objects and some of the morphisms of $C$ such that to each morphism in $A$, $A$ contains both the domain and codomain and for each object $a$ in $A$, $A$ contains the identity morphism and for each pair of composable morphisms $A$ also contains their composition. These conditions makes $A$ into a category.*

The inclusion functor $K : A \to C$ sends each object to itself and each morphism to itself. We say that a subcategory is full if the inclusion functor is full.

**Definition 2.11.** *For a category $C$, a skeleton of $C$ is any full subcategory $A$ such that each object in $C$ is isomorphic to exactly on object of $A$.*

A category is called skeletal when any two isomorphic objects are identical, i.e. when the category is its own skeleton. We will denote any skeleton of $C$ by $sk(C)$ or $[C]$.

**Definition 2.12.** *Let $F, G : C \to D$ be two functors. A natural transformation $t : F \to G$ assigns to each object $c \in ob(C)$ a morphism $t_c : F(c) \to G(c)$ such that for every morphism $f : c \to c'$ in $C$ the diagram*

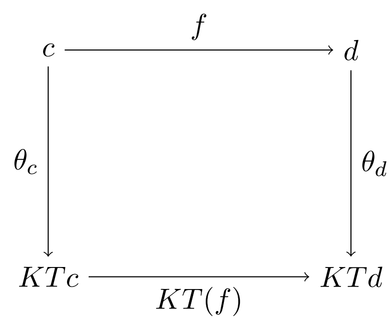
$$\begin{array}{ccc} F(c) & \xrightarrow{t_c} & G(c) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(c') & \xrightarrow[t_{c'}]{} & G(c') \end{array}$$

*commutes. If for every object $c$ in $C$ the morphism $t_c$ is an isomorphism in $D$ we call $t$ a natural isomorphism.*

**Definition 2.13.** *Let $C$ and $D$ be categories. Suppose we have functors $F : C \to D$ and $G : D \to C$ and suppose we have two natural isomorphisms $1_C \to GF$ and $FG \to 1_D$. Then the categories $C$ and $D$ are said to be equivalent and we call $F$ an equivalence.*

**Theorem 2.14.** *A category $C$ is equivalent to any of its skeletons.*

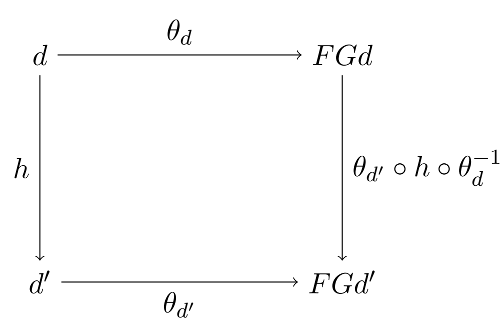
*Proof.* Let $A$ be a skeleton of $C$, and let $K : A \to C$ be the inclusion. For each object $c$ in $C$ we have that $c \cong Tc$ for a unique object $Tc \in A$. We call this isomorphism $\theta_c$. We can make a functor $T : C \to A$ which sends each object $c$ of $C$ to the representative in $A$. For a morphism $f : c \to d$ in $C$ we define $Tf = \theta_d f \theta_c^{-1}$. This makes the following diagram commute




Hence $\theta$ is a natural isomorphism between the functors $KT$ and $1_C$. And $TK = 1_D$, so $K$ is an equivalence and $C$ and $A$ are equivalent. □

**Theorem 2.15.** *$F : C \to D$ is an equivalence if and only if $F$ is full and faithful and essentially surjective on objects, meaning that each object $d \in D$ is isomorphic to $Fc$ for some object $c \in C$*

*Proof.* $\Leftarrow$ : We want to find a functor $G : D \to C$ and natural isomorphisms $\theta : 1_D \to FG$ and $\varepsilon : 1_C \to GF$.
For each object $d \in D$, there exists $c \in C$ such that $d \cong Fc$. Define $G(d) = c$ and let $\theta_d$ be this isomorphism. For each $h : d \to d'$ the following diagram commutes




Since $F$ is full and faithful, there is a unique $G(h) : G(d) \to G(d')$. Then $E$ is a functor with $FG(h) = \theta_{d'} h \theta_d^{-1}$ which makes $\theta$ natural.
We now need to find $\varepsilon : 1_C \to GF$. Let $c$ be an object in $C$. Consider $\theta_{Fc} : F(c) \cong FGF(c)$. Since $F$ is full and faithful, there is a unique $\varepsilon_c : c \cong GFc$ with $F(\varepsilon_c) = \theta_{Fc}$. And since $\theta$ is natural we know that the following

commutes

$$\begin{array}{ccc} Fc & \xrightarrow{Ff} & Fc' \\ \downarrow{\scriptstyle \theta_{Fc}} & & \downarrow{\scriptstyle \theta_{Fc'}} \\ FGFc & \xrightarrow[FGFf]{} & FGFc' \end{array}$$

Hence

$$\begin{aligned} FGFf \circ \theta_{Fc} &= \theta_{Fc'} \circ Ff \\ F(GFf \circ \varepsilon_c) &= F(\varepsilon_{c'} \circ f) \\ GFf \circ \varepsilon_c &= \varepsilon_{c'} \circ f \end{aligned}$$

so $\varepsilon$ is natural and we have an equivalence.
$\Rightarrow$ : Since $F : C \to D$ is an equivalence there exists a functor $G : D \to C$ and two natural isomorphisms $GF \to 1_C$ and $FG \to 1_D$. Hence each object $d \in D$ has the form $d \cong FGd$ for some $c = Gd \in obC$.
Let $\theta : GF \cong 1_C$ be the natural isomorphism making the following diagram commute

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \\ \downarrow{\scriptstyle \theta_{c}} & & \downarrow{\scriptstyle \theta_{c'}} \\ GFc & \xrightarrow[GFf]{} & GFc' \end{array}$$

Assume $Ff = Ff'$. Then $GFf = GFf'$ and by the commutativity of the square $f = {\theta_{c'}}^{-1} GF(f)\theta_c = f'$, hence $F$ is faithful. By symmetry, $G$ is also faithful.
To show that $F$ is full, consider any $h : Fc \to Fc'$ and let $f = \theta_{c'}^{-1} Gh\theta_c$. The above diagram also commutes if we replace $Ff$ by $h$, thus we see that $GFf = Gh$, and since $G$ is faithful, we get that $Ff = h$, so $F$ is full. □

**Theorem 2.16.** *An equivalence of categories induces an isomorphism between their skeletons.*

*Proof.* Suppose we have an equivalence $F : C \to D$. For an object $c$ in $C$ we write $[c] = \{c' \in C \mid c \cong c'\}$. Define $F_* : [c] \to [Fc]$. Since $F$ is an equivalence, $[c] = [c'] \Leftrightarrow [Fc] = [Fc']$. Hence $F_*$ is an isomorphism. □

**Definition 2.17.** *We call a category $C$ thin if for any two objects $a, b \in C$ there is a most one morphism from $a$ to $b$.*

Notice that if $C$ is a thin category then $sk(C)$ is a poset. This follows from the fact that if $f : a \to b$ and $g : b \to b$ in $sk(C)$ then the composition $fg = 1_b$ and $gf = 1_a$, hence $a$ and $b$ are isomorphic and then $a = b$.

**Definition 2.18.** *An adjunction between two categories $C$ and $D$ consists of two functors $F : C \to D$ and $G : D \to C$ and two natural transformations $\varepsilon : FG \to 1_D$ and $\eta : 1_C \to GF$ called the counit and unit, respectively, such that the compositions*

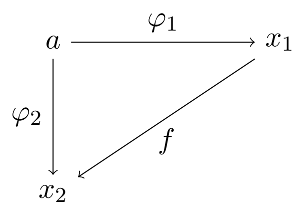
$$F \xrightarrow{\varepsilon F} FGF \xrightarrow{F\eta} F$$
$$G \xrightarrow{G\eta} GFG \xrightarrow{\varepsilon G} G$$

*are both the identity. We say that $G$ is left adjoint to $F$.*

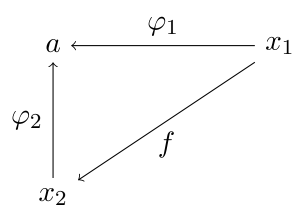
An adjunction like this induces an isomorphism between the morphism sets $D(Fc, d) \cong C(c, Gd)$ for any $c \in obC$ and $d \in obD$. For every $g : Fc \to d$ in $D$ and $f : c \to Gd$ in $C$, we can define $\varphi(g) = Gg \circ \eta_c$ and this is an isomorphism with inverse $\psi(f) = \varepsilon_d \circ Ff$.

## 2.3 Slice and coslice category

**Definition 2.19.** *For a category $C$, we denote by $a/C$ the coslice category of $C$ under $a$. It has as objects all morphisms in $C$ with domain $a$. The morphisms between two objects of $a/C$, say $\varphi_1 : a \to x_1$ and $\varphi_2 : a \to x_2$, are any morphisms $f \in C(x_1, x_2)$ that makes the following diagram commute*




*The slice category $C/a$ has as objects all morphisms in $C$ with codomain $a$. The morphisms between two objects of $C/a$, say $\varphi_1 : x_1 \to a$ and $\varphi_2 : x_2 \to a$, are any morphisms $f \in C(x_1, x_2)$ that makes the following diagram commute*




*$a//C$ is the full subcategory of $a/C$ with objects all non-isomorphisms with domain $a$. Similarly, $C//a$ is the full subcategory of $C/a$ with objects all non-isomorphisms with codomain $a$.*

Notice that $1_a : a \to a$ is an initial object in $a/C$ and a terminal object in $C/a$.

**Example**
If we consider a poset $P$ as a category and an object $K \in P$, then the slice category $P/K = \{H \in P \mid H \leq K\}$.

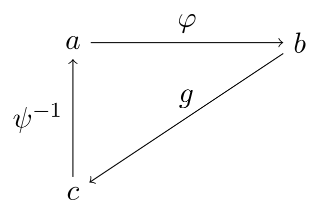
**Definition 2.20.** *Let $C$ be a finite category and let $S$ and $T$ be two full subcategories of $C$. If*

$$a \in ob(S) \text{ and } C(a,b) \neq \emptyset \Rightarrow b \in ob(S)$$
$$C(a,b) \neq \emptyset \text{ and } b \in ob(T) \Rightarrow a \in ob(T)$$

*holds for all $a, b \in ob(C)$ then we call $S$ a left ideal in $C$ and $T$ a right ideal in $C$.*

**Proposition 2.21.** *For an EI-category $a//C$ is a left ideal in $a/C$*

*Proof.* Assume $\varphi : a \to b$ is an object in $a//C$. Then $\varphi$ is a non-isomorphism in C. Let $\psi : a \to c$ be an element of $a/C$. Assume also that there is a morphism $g \in (a/C(\varphi, \psi))$. Assume for contradiction that $\psi$ is an isomorphism. Then we have the following commutative diagram




and since we C is an EI-category $\varphi$ would also be an isomorphism. Hence $\psi \in ob(a//C)$ and $a//C$ is a left ideal in $a/C$. □

# 3 The Geometric realization of a category

In this section we see how categories produce topological spaces and functors produce continuous maps between these topological spaces. We also see that a natural transformation between two functors produces a homotopy between the maps defined by the functors. My treatment of the subject is based on Ib Madsen's lecture notes and on Quillen's article "Higher algebraic K-theory". Once again this is mostly a preliminary presentation and also includes some theorems stated without proofs.

## 3.1 The Geometric realization of a category

Let C be a small category. The topological space associated to a category is constructed in two steps. First we define the nerve of a category which is a simplicial set and then we define the geometric realization of this simplicial set which gives us a topological space. We denote by $NC$ the nerve of $C$, which is a simplicial set. The n-simplices are the compositions $x_0 \to ... \to x_n$ in C. The 0-simplices are the objects of $C$. The geometric realization of the nerve, $BC$, is called the classifying space of $C$: This is a CW-complex and the n-cells are in a one-to-one correspondence with the n-simplices of the nerve where non of the arrows are the identity. If an n-simplicex contains the identity as one of its arrows we call it degenerate.

For an integer $n \geq 0$ we define the category $[n]$ which has as objects $\{i \in \mathbb{Z} \mid 0 \leq i \leq n\}$ and there is exactly one morphism from $i$ to $j$ provided $i \leq j$. The functors from $[n]$ to $[m]$ consist of weakly increasing functions $\theta : [n] \to [m]$. The category $\Delta$ has as objects $[n]$ and the morphisms from $[n]$ to $[m]$ are the functors from $[n]$ to $[m]$.

A simplicial set is a functor $X[-] : \Delta^{op} \to \mathbf{Set}$. The morphisms in $\Delta$ are generated by the face maps $d^i : [n-1] \to [n]$, the map that skips $i$ and the degeneracy maps $s^i : [n+1] \to [n]$, the map that repeats $i$. Define $d_i : X[n] \to X[n-1]$ and $s_i : X[n] \to X_[n+1]$. These maps satisfy the simplicial identities [7]

$$d_i d_j = d_{j-1} d_i \text{ for } i < j$$
$$s_i s_j = s_{j+1} s_i \text{ for } i \leq j$$
$$d_i s_j = \begin{cases} s_{j-1} d_k & \text{if } i < j \\ 1 & \text{if } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1 \end{cases}$$

The nerve $NC[-]$ of a category $C$ is the simplicial set given by $NC[n] = \{c_0 \to c_1 \cdots \to c_n\} = \{F : [n] \to C\}$ and for $\theta : [n] \to [m]$ we define $\theta^* : NC[m] \to NC[n]$ given by $\theta^*([m] \xrightarrow{F} C) = [n] \xrightarrow{\theta} [m] \xrightarrow{F} C$.

Simplicial sets form a category **sSet**. The morphisms $f[-] : X[-] \to Y[-]$ are the functions $f[n] : X[n] \to Y[n]$ that make the diagrams commute

$$\begin{array}{ccc} X[m] & \xrightarrow{f[m]} & Y[m] \\ \downarrow{\scriptstyle \theta^*} & & \downarrow{\scriptstyle \theta^*} \\ X[n] & \xrightarrow[f[n]]{} & Y[n] \end{array}$$

for each $\theta : [n] \to [m]$ in $\Delta$. A functor $F : C_0 \to C_1$ induces a map of simplicial sets $NF[-] : NC_0[-] \to NC_1[n]$. In degree $n$ we have $F(c_0 \to \cdots \to c_n) = Fc_o \to \cdots \to Fc_n$ and this makes the above diagrams commute.

Let $\Delta^n$ be the standard n-simplex

$$\{(t_0, \cdots t_n) \in \mathbf{R}^{n+1} \mid \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}$$

A morphism $\theta : [n] \to [m]$ induces a continuous map $\theta_* : \Delta^n \to \Delta^m$ given by $\theta_*(\sum t_i e_i) = \sum t_i e_{\theta(i)}$

**Example**
If we have a functor $\theta : [1] \to [2]$ defined by $\theta(0) = 0$ and $\theta(1) = 1$, then we have a continuous function $\theta_* : \Delta^1 \to \Delta^2$ that maps $1e_0 + 0e_1 + 0e_2 + \cdots$ to $1e_{\theta(0)} + 0e_{\theta(1)} + 0e_2 + \cdots = 1e_0 + 0e_2 + 0e_1 + \cdots$ and $0e_0 + 1e_1 + 0e_2 + \cdots$ to $0e_{\theta(0)} + 1e_{\theta(1)} + 0e_2 + \cdots = 0e_0 + 1e_2 + 0e_1 + \cdots$.

We now associate to each simplicial set a topological space, its geometric realization.

$$|X[-]| = \coprod_{n=0}^{\infty} X[n] \times \Delta^n / \sim$$

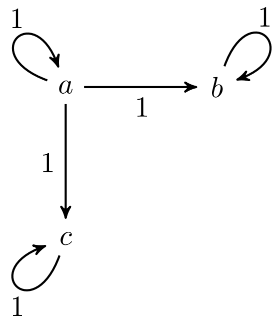
where $(x_n, d^i(t_{n-1})) \sim (d_i x_n, t_{n-1})$ for $x_n \in X[n], t_{n-1} \in \Delta^{n-1}$ and $(x_n, s^i(t_{n+1})) \sim (s_i x_n, t_{n+1})$ or $x_n \in X[n], t_{n+1} \in \Delta^{n+1}$.

A morphism $f[-] : X[-] \to Y[-]$ of simplicial sets induces a continuous map of geometric realizations $|f[-]| : |X[-]| \to |Y[-]|$ induced from $\coprod f[-] \times id : \coprod X[n] \times \Delta^n \to \coprod Y[n] \times \Delta^n$

To sum up: a functor $F : C_0 \to C_1$ induces a continuous map $BF : BC_0 \to BC_1$. The composition $C_0 \xrightarrow{F} C_1 \xrightarrow{G} C_2$ is given by $B(G \circ F) \to BG \circ BF$ and for the identity functor $1_C : C \to C$ we have $B(1_C) = 1_{BC}$. Hence B is a functor from **Cat** to **Top**

One can show that the resulting topological space is a CW-complex and the n-cells are in a one-to-one correspondence with the non-degenerate n-simplices of the nerve. [7]

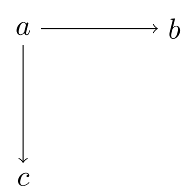
**Example**
Consider the following category C




The nerve of this category is

$$\begin{aligned}
NC[0] &= \{a, b, c\} \\
NC[1] &= \{a \to a,\ b \to b,\ c \to c,\ a \to b,\ a \to c\} \\
NC[2] &= \{a \to a \to a,\ b \to b \to b,\ c \to c \to c,\ a \to b \to b, \\
&\qquad\qquad a \to c \to c,\ a \to a \to b,\ a \to b \to b, \} \\
\vdots \quad &\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{aligned}$$

and so on.
In the realization $|NC[-]|$, the relation $(x_n, s^i(t_{n+1})) \sim (s_i x_n, t_{n+1})$ will collapse all chains containing the identities $\{a \to a,\ b \to b,\ c \to c\}$ to the points $\{a,\ b,\ c\}$ and the relation $(x_n, d^i(t_{n-1})) \sim (d_i x_n, t_{n-1})$ tells us to glue the simplices together such that the resulting space looks like this




**Example**
The cyclic group of order 2, $C_2$, form a category with one object and two morphisms, the identity, 1 and $g$ where $gg = 1$. The nerve $NC_2[n]$ has exactly non-degenerate simplex for each $n \geq 0$ and the resulting space $BC_2$ has one n-cell in each dimension $n \geq 0$, hence is homeomorphic to $\mathbb{R}P^\infty$.

## 3.2 Properties of the geometric realization

**Theorem 3.1.** *[7] Milnor: The geometric realization preserves product, that is*

$$|(X \times Y)[-]| \to |X[-]| \times |Y[-]|$$

*is a homeomorphism.*

We have to be a bit careful choosing our topology on the target, but if $X[-]$ and $Y[-]$ are both countable we can equip it with the usual product topology.

The classifying space of the category **[1]** is homeomorphic to the unit interval $I = [0, 1]$.

**Corollary 3.2.** *A functor $F : C \times \mathbf{[1]} \to D$ induces a homotopy of classifying spaces $BF : BC \times I \to BD$*

A natural transformation $t$ between functors $F_0, F_1 : C \to D$ is equivalent to a functor

$$F : C \times \mathbf{1} \to D$$

Defined as $F(c, 0) = F_0(c), F(c, 1) = F_1(c)$ for all $c \in obC$
The morphisms of $C \times \mathbf{[1]}$ are generated by

$$f \times id_0 : (c, 0) \to (c', 0)$$
$$f \times id_1 : (c, 1) \to (c', 1)$$
$$id_c \times \iota : (c, 0) \to (c, 1)$$

since all morphisms can be constructed by composing the above. We define

$$F(f \times id_0) = F_0(f) : F_0c \to F_0c'$$
$$F(f \times id_1) = F_1(f) : F_1c \to F_1c'$$
$$F(id_c \times \iota) : t_c : F_0c \to F_1c$$

An arbitrary morphism in $C \times \mathbf{[1]}$ is the composition

$$(f, \iota) = (id, \iota) \circ (f, id_0) = (f, id_1) \circ (id, \iota)$$

F is well-defined since the diagram

$$\begin{array}{ccc} F_0(c) & \xrightarrow{F_0(f)} & F_0(c') \\ {\scriptstyle t_c}\downarrow & & \downarrow{\scriptstyle t_{c'}} \\ F_1(c) & \xrightarrow[F_1(f)]{} & F_1(c') \end{array}$$

commutes.
On the other hand, given a functor $F : C \times \mathbf{1} \to D$, the two functors $F_0, F_1 : C \to D$ given by $F_0 = F|_{C\times\{0\}}$ and $F_1 = F|_{C\times\{1\}}$ we have the following commutative diagram

$$\begin{array}{ccc} F_0(c) & \xrightarrow{F(f, id_0)} & F_0(c') \\ {\scriptstyle F(id_c, \iota)}\downarrow & & \downarrow{\scriptstyle F(id_{c'}, \iota)} \\ F_1(c) & \xrightarrow[F(f, id_1)]{} & F_1(c') \end{array}$$

Hence the two functors are related by the natural transformation

$$t_c = F|_{\{c\}\times\iota} : C \times \{0\} \to D \times \{1\}$$

It follows from Theorem 3.1 that a natural transformation $t : F_0 \to F_1$ between functors $F_0, F_1 : C \to D$ induces a homotopy from $BF_0$ to $BF_1$.

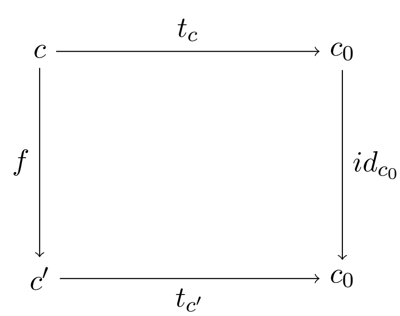
**Theorem 3.3.** *An adjunction between categories induces a homotopy equivalence.*

*Proof.* Let F and G, $C \underset{F}{\overset{G}{\leftrightarrows}} D$, be functors such that we have natural transformations $1_C \xrightarrow{\sim} GF$ and $1_D \xrightarrow{\sim} FG$. Then we have homotopies $1_{BC} \simeq BG \circ BF$ and $1_{BD} \simeq BF \circ BG$ and BC and BD are homotopy equivalent. $\square$

We call a functor $F$ a homotopy equivalence if it induces a homotopy equivalence of classifying spaces. We call a category $C$ contractible if the classifying space is contractible.

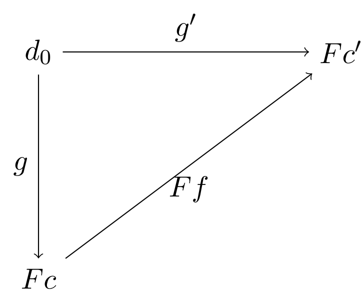
**Corollary 3.4.** *If a category $C$ has either an initial or a terminal object, then it is contractible.*

*Proof.* We prove the case where $C$ has a terminal object. Assume $C$ has a terminal object $c_0$. Let $*$ denote the 1-category with one object and one morphism $1_*$. We have functors $F : C \to \{*\}$ and $G : * \to C$. The functor $F$ sends all the objects of $C$ to the only object in $*$ and all morphisms to $1_*$. G sends the only object in $*$ to the terminal object $c_0$ and the only morphism to $1_{c_0}$. The composition $F \circ G = 1_*$ and the composition $G \circ F$ send all objects in C to $c_0$. We have a natural transformation $t : 1_C \xrightarrow{\sim} G \circ F$. For all objects $c \in C$ define $t_c : c \to c_0$ to be the unique morphism into $c_0$. This is a natural transformation since the diagram




commutes for all $f : c \to c'$. So we have $BF \circ BG = B(F \circ G) = id_{B(*)}$ and a homotopy between $id_C$ and $BG \circ BF = B(G \circ F)$. Hence $BC$ is homotopy equivalent to $B(*)$ which is a single point. □

**Definition 3.5.** *Let $F : C \to D$ be a functor. For an object $d_0 \in D$ the category $d_0/F$ has as its object all morphisms in $D$ with domain $d_0$ and codomain $Fc$ for some object $c \in C$. The morphisms from $g : d_0 \to Fc$ to $g' : d_0 \to Fc'$ are the morphisms $f : c \to c'$ in $C$ that make the following diagram commute*




When $F$ is the identity functor $d_0/F$ is the coslice category.
We will need the following result shown by Quillen.

**Theorem 3.6.** *[9](Theorem A) If the category $d/F$ is contractible for all $d \in D$ then $F$ is a homotopy equivalence.*

# 4 The Euler characteristic of a category

This chapter is based on Tom Leinster's article. We define the Euler characteristic of a category and investigate some of its properties. A category $C$ is called finite if both the set of objects and the set of morphisms are finite. We will only consider finite categories in this section.

## 4.1 Weighting and Coweighting

**Definition 4.1.** *Let $A$ be a finite category. A weighting on $A$ is a function $k^\bullet : obA \to \mathbb{Q}$ such that for all objects $a \in A$*

$$\sum_{b \in obB} \zeta(a,b)k^b = 1$$

.

*A coweighting $k_\bullet$ is a function $k_\bullet : obA \to \mathbb{Q}$ such that*

$$\sum_{a \in obA} k_a \zeta(a,b) = 1$$

.

One can also think of weightings and coweightings as matrices. If we arrange $\zeta(a,b)$ in an $n \times n$ matrix a weighitng is a column vector satisfying

$$\begin{pmatrix} \zeta(a,a) & \zeta(a,b) & \cdots \\ \zeta(b,a) & \zeta(b,b) & \cdots \\ \vdots & \vdots & \\ & & \end{pmatrix} \begin{pmatrix} k^a \\ k^b \\ k^c \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}$$

and a coweighting is a row vector satisfying

$$\begin{pmatrix} k_a & k_b & k_c & \cdots \end{pmatrix} \begin{pmatrix} \zeta(a,a) & \zeta(a,b) & \cdots \\ \zeta(b,a) & \zeta(b,b) & \cdots \\ \vdots & \vdots & \\ & & \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots \end{pmatrix}$$

.

If the matrix of $\zeta$ is invertible the category $C$ is said to have Möbius inversion. We call the inverse, $\mu = \zeta^{-1}$, the Möbius function of C. A category $C$

has Möbius inversion if and only if it has a unique weighting and coweighting.

**Examples**

1. Let $C$ be the following category

   To find a weighting for this category we need to find a column vector satisfying

$$\begin{pmatrix} 3 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} k^X \\ k^Y \\ k^Z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

   We see that a weighting for $C$ is $\begin{pmatrix} k^X \\ k^Y \\ k^Z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

2. If a category $C$ has an initial object $a$, then

$$\begin{pmatrix} k_a & k_b & k_c & \cdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots \end{pmatrix}$$

   is a coweighting for $C$ and if $C$ has a terminal object $a$, then

$$\begin{pmatrix} k^a \\ k^b \\ k^c \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

   is a weighting for $C$.

For a finite category $C$ we could also have defined weightings and coweightings such that it would be constant on isomorphism classes.
Let $[C]$ be any skeletal category of C and let $\zeta([a],[b])$ be the number of morphisms from $[a],[b] \in [C]$.

**Definition 4.2.** *A weighting for* $[C]$ *is a function* $l^{\bullet} : [C] \to \mathbb{Q}$ *such that for all* $[a] \in [C]$ *we have*

$$\sum_{[b]\in[C]} \zeta([a],[b])l^{[b]} = 1$$

The category $C$ has a weighting if and only if $[C]$ has a weighting and we have

$$k^{[b]} = \sum_{b \in [b]} k^b \text{ and } k^b = \frac{k^{[b]}}{|[b]|}$$

We could have done the same for coweightings.

## 4.2 Euler Characteristic

**Lemma 4.3.** *Let $A$ be a finite category, $k^\bullet$ a weighting for $A$ and $k_\bullet$ a coweighting for $A$. Then $\sum_a k^a = \sum_a k_a$.*

*Proof.* $\sum_b k^b = \sum_b (\sum_a k_a \zeta(a,b)) k^b = \sum_a k_a (\sum_b \zeta(a,b) k^b) = \sum_a k_a$ □

**Definition 4.4.** *Let $A$ be a finite category that admits both a weighting and a coweighting. Its Euler Characteristic is given by*

$$\chi(A) = \sum_a k_a = \sum_a k^a$$

*for any weighting and coweighting.*

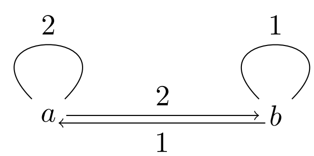
Any category C with Möbius inversion has Euler characteristic, $\chi(C) = \sum_{a,b} \mu(a,b)$, the sum of the entries in $\mu(C)$.

**Examples**

1. If $A$ is a finite discrete category then $\chi(A) = |obA|$ since for all objects $a \in A$ $k^a = 1$ will be a weighting.

2. If a finite category $C$ has an initial or terminal object then $\chi(C) = 1$.

3. The category




   does not admit a weighting or a coweighting, hence it does not possess Euler Characteristic as defined here.

**Proposition 4.5.** *If $F : A \to B$ is an equivalence, then $\chi(A) = \chi(B)$.*

*Proof.* We know that $F$ induces an isomorphism $F_* : sk(A) \to sk(B)$ so $\chi(sk(A)) = \chi(sk(B))$. And we know that $\chi(A) = \chi(sk(A))$ since

$$\sum_{[a] \in [A]} k^{[a]} = \sum_{a \in A} k^a$$

Hence $\chi(A) = \chi(B)$. □

**Proposition 4.6.** *Let $A$ and $B$ be finite categories. If there is an adjunction $A \overset{F}{\underset{G}{\rightleftarrows}} B$ and both $A$ and $B$ have Euler characteristic, then $\chi(A) = \chi(B)$.*

*Proof.* Let $A \overset{F}{\underset{G}{\rightleftarrows}} B$ be an adjunction. We have a natural transformation $1_A \to GF$ and a natural transformation $FG \to 1_B$. We know that $\zeta(Fa, b) = \zeta(a, Gb)$ for all objects $a \in A$ and $b \in B$. Let $k_\bullet$ be a coweighting on $A$. For every $b$ we have $1 = \sum\limits_a k_a \zeta(a, Gb)$. We can write

$$\begin{aligned}\chi(B) = \sum_b k^b &= \sum_b k^b (\sum_a k_a \zeta(a, Gb)) \\ &= \sum_a k_a (\sum_b \zeta(a, Gb) k^b) \\ &= \sum_a k_a (\sum_b \zeta(Fa, b) k^b) \\ &= \sum_a k_a \\ &= \chi(A)\end{aligned}$$

□

## 4.3 Euler Characteristic of a Finite Poset

**Proposition 4.7.** *A finite poset $S$ has Euler Characteristic $\sum\limits_{n \geq 0} (-1)^n c_n$ where $c_n$ is the number of chains in $S$ of length $n$.*

*Proof.* We look at the Möbius function $\mu(a, b)_{a,b \in S} = \mu(S) = \zeta^{-1}(a, b)_{a,b \in S}$ for a finite poset. For $a, b \in S$ we have that

$$\zeta(a, b) = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}$$

Hence, $\mu(S) = \zeta^{-1}(S) = (E + (\zeta - E))^{-1} = E - (\zeta - E) + (\zeta - E)^2 - \cdots$ where E is the identity matrix.

The matrix $\zeta(a, b)$ can be arranged such that we have one in the diagonal and zero below and 1 above if $a \leq b$. Notice that $(\zeta - E)(a, b)$ is the number of 1-simplices from $a$ to $b$ in $S$ and $(\zeta - E)^k(a, b)$ is the number of k-simplicies from $a$ to $b$ in $S$ for $k \geq 0$. Hence $\sum_{a,b \in S} (\zeta - E)^k(a, b)$ is the number of k-simplicies in S.

So $\chi(S) = \sum\limits_{a,b} \mu(a, b) = \sum\limits_{a,b} E(a, b) - \sum\limits_{a,b} (\zeta - E)(a, b) + \sum\limits_{a,b} (\zeta - E)^2(a, b) ... = \sum\limits_k (-1)^k c_k$ where $c_k$ is the number of k-simplicies in $S$. □

For a finite poset $S$, we can find the usual Euler characteristic of the classifying space of $S$ or we can see $S$ as a category and find the Euler characteristic as defined in this chapter. The above result tells us that the two Euler characteristics will be the same.

**Example**
We have seen that the cyclic group $C_2$ is a category with one object and two morphisms and that $BC_2 = \mathbb{R}P^\infty$. This is not a finite CW-complex and traditionally it does not have Euler characteristic but according to our new definition the Euler characteristic of this category is $\chi(C_2) = 1/2$.

## 4.4 Euler Characteristic of Slice Categories

We will show how the Euler characteristic can be expressed using the slice and coslice category.

**Lemma 4.8.** *If $S$ is a left ideal in $C$ then any weighting for $C$ restricts to a weighting for $S$. And if $T$ is a right ideal in $C$ then any coweighting for $C$ restricts to a coweighting for $T$*

*Proof.* Let $S$ be a left ideal in $C$. If $a \in obS$ and an object $b \in obC$ is not in $S$ then $\zeta(a,b) = 0$. Hence a weighting $k^\bullet : C \to \mathbb{Q}$ restricts to a weighting $k^\bullet : S \to \mathbb{Q}$ where we have deleted $k^b$. The proof for coweighting is similar. □

**Lemma 4.9.** *Let $C$ be a finite category admitting a weighting: $k^\bullet : ob(C) \to \mathbb{Q}$ and let $a \in ob(C)$. Then $k^{cod(\bullet)} : ob(a/C) \to \mathbb{Q}$ is a weighting for the coslice category $a/C$.*

*Proof.* The objects in $a/C$ are exactly the morphisms in C with domain $a$. So we can partition the objects of $a/C$ by their codomain $ob(a/C) = \coprod_{b \in ob(C)} C(a,b)$. Also the set of morphisms from b to c in C is the union of all morphisms from $\varphi$ to $\psi$ in $a/C$ where $\varphi : a \to b$ and $\psi : a \to c$ in C. This means that for $\varphi \in C(a,b)$ and an object $c \in C$ we can partition the morphisms from b to c in C into $a/C$ morphisms: $C(b,c) = \coprod_{\psi \in C(a,c)} (a/C)(\varphi, \psi)$. We have to show that

$$\sum_{\psi \in ob(a/C)} \mid (a/C)(\varphi,\psi) \mid k^{cod(\psi)} = 1$$

.

We use the above to conclude :

$$\begin{aligned}\sum_{\psi\in ob(a/C)} \mid (a/C)(\varphi,\psi) \mid k^{cod(\psi)} &= \sum_{b\in ob(C)}\sum_{\psi\in C(a,b)} \mid (a/C)(\varphi,\psi) \mid k^{cod(\psi)} \\ &= \sum_{b\in ob(C)} \mid C(cod(\varphi),b) \mid k^b \\ &= 1\end{aligned}$$

□

If $C$ is a finite EI-category we can arrange the objects such that the matrix of $\zeta$ for the skeletal category $[C]$ is an upper triangle matrix. Indeed, if we have morphisms $f : a \to b$ and $g : b \to a$, then $a \cong b$ since all morphisms from an object to itself are isomorphisms. Hence an EI-category has a unique weighting and coweighting that is constant on isomorphism classes.

**Definition 4.10.** *For a category $C$ we denote by $\tilde{\chi}(C)$ the reduced Euler characteristic $\chi(C) - 1$.*

**Theorem 4.11.** *Let $C$ be a finite EI-category and let $k^\bullet$ and $k_\bullet$ be the weighting and coweighting that is constant on isomorphism classes of object in $C$. Then for $a, b \in ob(C)$*

$$k^a = \frac{-\tilde{\chi}(a//C)}{\mid [a] \mid\mid C(a) \mid} \text{ and } k_b = \frac{-\tilde{\chi}(C//b)}{\mid [b] \mid\mid C(b) \mid}$$

*is a weighting and a coweighting for $C$, respectively. The Euler characteristic for $C$ is*

$$\sum_{[a]\in[C]} \frac{-\tilde{\chi}(a//C)}{\mid C(a) \mid} = \sum_{[b]\in[C]} \frac{-\tilde{\chi}(C//b)}{\mid C(b) \mid}$$

*Proof.* Since $C$ is a finite EI-category, $a/C$ and $a//C$ are also finite EI-categories and thereby possess Euler characteristics. Since $a//C$ is a left ideal in $a/C$ the weighting for $a/C$ restricts to a weighting for $a//C$. The category $a/C$ has an initial object, and therefore it has Euler characteristic

1:

$$\begin{aligned} 1 &= \sum_{\varphi \in Ob(a/C)} k^{cod(\varphi)} \\ &= |[a]||C(a)|k^a + \sum_{\varphi \in Ob(a//C)} k^{cod(\varphi)} \\ &= |[a]||C(a)|k^a + \chi(a//C) \end{aligned}$$

since the weighting is constant on isomorphism classes. The proof for coweightings is similar. □

# 5 Equivalence of Brown and Frobenius

For a finite group $G$ and a prime $p$, we let $|G|_p$ be the maximal power of $p$ dividing $|G|$ and $|G|_{p'}$ be the non-p-part of the order of $G$, hence $|G| = |G|_p|G|_{p'}$. We write $G_p$ for the set of p-singular elements in $G$, that is the set of elements in $G$ of order some power of $p$ and we let 1 denote the identity element of the group. For a finite group $G$, we denote by $S_G^p$ the Brown poset of $G$. It consists of all p-subgroups of $G$ ordered by inclusion. $S_G^{p+*}$ denotes the poset of all non-trivial p-subgroups of $G$.
The purpose of this chapter is to show that Brown's theorem which states $|G|_p$ divides $\tilde{\chi}(S_G^{p+*})$ [1] is equivalent to a theorem proved by Frobenius [2] that $|G|_p$ divides $|G_p|$
This chapter is based on Jesper Møllers article "Euler Characteristic of Equivariant Subcategories" and "Homotopy Equivalences between p-Subgroup Categories" by Gelvin and Møller.

## 5.1 The orbit category

Let$G$ be a finite group. The orbit category $O_G$ is a category whose objects are all subgroups $H$ of $G$ and the morphisms between two subgroups $H$ and $K$ are the sets

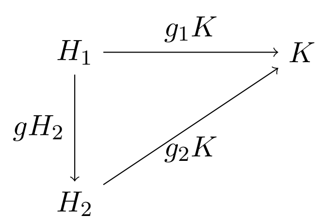
$$\begin{aligned} O_G(H,K) &= N_G(H,K)/K \\ &= \{g \in G \mid H^g \leqq K\}/K \\ &= \{g \in G \mid g^{-1}Hg \leqq K\}/K \\ &\text{and} \\ O_G(H) &= N_G(H)/H \\ &= \{g \in G \mid H^g \leqq H\}/H \\ &= \{g \in G \mid g^{-1}Hg \leqq H\}/H \end{aligned}$$

This is a category since the identity $1 \in N_G(H)/H$ and the composition of morphisms is given by the following: Let $H$, $K$ and $J$ be subgroups of $G$ and let $x \in N_G(H,K)$ and let $y \in N_G(K,J)$. Then $xy \in N_G(H,J)$ and is a morphism from $H$ to $J$ since $(xy)^1Hxy \leqq y^{-1}Ky \leqq J$. We could equivalently have defined the orbit category $O_G$ as a category whose objects are the G-sets $G/H$ for all subgroups $H$ of $G$, and the morphisms $G/H \to G/K$

are G-maps. The restriction that this has to be a well-defined G-map will lead to the same morphism set as before.

The orbit category is an EI-category. This follows from the fact that if $gH \in O_G(H)$ then $g \in N_G(H) \Leftrightarrow g^{-1}Hg \leqq H$ and since $G$ is finite we have that $g^{-1}Hg = H$. Hence $gHg^{-1} = gg^{-1}Hgg^{-1} = H$ so $g^{-1} \in N_G(H)$ and $g^{-1}H \in O_G(H)$

For a subgroup $K \leqq G$ we will also need to look at the slice category $O_G/K$ with objects being morphisms in $O_G$ with codomain $K$ and where morphisms from $g_1K : H_1 \to K$ to $g_2K : H_2 \to K$ are the morphisms $gH_2 \in O_G(H_1, H_2)$ that makes the following diagram commute




That is, the morphisms from $g_1K : H_1 \to K$ to $g_2K : H_2 \to K$ is the set $\{g \in G \mid H_1^g \leqq H_2, g_1K = gg_2K\}/H_2$. This is well-defined. The category $O_G//K$ has as objects all morphisms $H \to K$ such that $|H| < |K|$, otherwise we would have an isomorphism $gK : H \to K$. Notice that this is a thin category. In the same way we can define the coslice category $K/O_G$ and $K//O_G$.

We want to find weightings and coweightings for the orbit category. To start with we look at the category $O_V$, where $V$ is an elementary abelian p-group, and find a coweighting for this category.

**Definition 5.1.** *Let $C_p$ be the cyclic group of order $p$. A group $G$ is an elementary abelian p-group if it is of the form $G = C_p \times C_p \times \cdots \times C_p$. We call the number of copies of $C_p$ the dimension of $G$.*

**Lemma 5.2.** *Let $V$ be an elementary abelian group and let $U$ be a subgroup* *of V. The function* $K_\bullet : ob(O_V) \to \mathbb{Q}$

$$K_U = \begin{cases} \mid V \mid^{-1} & \textit{if dimU=0} \\ (p-1) \mid V \mid^{-1} & \textit{if dimU=1} \\ 0 & \textit{otherwise} \end{cases}$$

*is a coweighting for* $O_V$.

*Proof.* Let $U_1$ and $U_2$ be two subgroups of $V$. If $U_2$ is a proper subgroup of $U_1$ then there are no morphisms from $U_1$ to $U_2$. So it is enough to show that $\sum_{U_1 \leq U_2} K_{U_1} |O_V(U_1, U_2)| = 1$. Since $V$ is abelian there are $\frac{|V|}{|U_2|}$ morphisms from $U_1$ to $U_2$. The statement is then that $\sum_{U_1 \leq U_2} K_{U_1} \frac{|V|}{|U_2|} = 1$. When $U_2 = \{1\}$ this is true. An elementary abelian group $(C_p)^n$ has $p^n - 1$ elements of order $p$ and each of them generates a subgroup of order p. But every subgroup of order $p$ has $p-1$ generators hence the number of distinct p-subgroups is $\frac{p^n-1}{p-1}$. Therefore if $|U_2| = p^n$ :

$$\begin{aligned} \sum_{U_1 \leq U_2} K_{U_1} \frac{|V|}{|U_2|} &= \frac{1}{|V|} \frac{|V|}{|U_2|} + \sum_{U_1 \leq U_2 \text{ of order p}} \frac{(p-1)}{|V|} \frac{|V|}{|U_2|} \\ &= \frac{1}{|U_2|} + \sum_{\frac{p^n-1}{p-1}} \frac{(p-1)}{|U_2|} \\ &= 1 \end{aligned}$$

So $K_U$ is a coweighting for $V$. □

**Example**

Let $G$ be the elementary abelian group $C_2 \times C_2$. The orbit category $O_G$ has as objects the following subgroups $C_2 \times C_2, C_2, C_2, C_2, 1$. For two subgroups $H$ and $K$ with $K < H$ the morphism set $O_G(H, K)$ is empty. When $H \leqq K$, the morphism set is $\{gK \in G \mid g^{-1}Hg \leqq K\} = G/K$ since G is an abelian group. For example the morphisms from any subgroup H to $C_2 \times C_2$ is the set $O_G(H, C_2 \times C_2) = \{1\}$. The function

$$K_U = \begin{cases} 1/4 & \text{if dimU=0} \\ 1/4 & \text{if dimU=1} \\ 0 & \text{otherwise} \end{cases}$$

is a coweighting for $G$.

Let $O_V^{[1,V)}$ be the full subcategory of $O_V$ that consists of all but the terminal object $V$.

**Theorem 5.3.** *The Euler characteristic of the category $O_V^{[1,V)}$ is*

$$\chi(O_V^{[1,V)}) = \begin{cases} 0 & \textit{if dim}\, V\textit{=0} \\ p^{-1} & \textit{if dim}\, V\textit{=1} \\ 1 & \textit{dim}\, V\textit{>1} \end{cases}$$

*Proof.* When the dimension of $V$ is 0 $O_V^{[1,V)}$ is the 0-category and has Euler characteristic 0. When $V$ has dimension 1, $V$ has only one proper subgroup, the trivial group. It has dimension 0 so the assertion follows from the above Lemma 5.2. When $dimV > 1$ the coweighting for $O_V$ restricts to a coweighting for $O_V^{[1,V)}$ because $O_V^{[1,V)}$ is a right ideal in $O_V$. By the Lemma 5.2 $K_V = 0$, so we have that $\chi(O_V) = \chi(O_V^{[1,V)})$. $O_V$ has a terminal object, hence $\chi(O_V^{[1,V)}) = 1$ □

## 5.2 Homotopy equivalences of the orbit category

In this section we will categories that are homotopy equivalent to the orbit category. First we will need some results from algebra.

**Lemma 5.4.** *Let $G$ be a group. The number of p-singular elements in $G$ is*

$$|G_p| = 1 + \sum_{1<C\leq G} (1-p^{-1})|C| = p^{-1} + \sum_{1\leq C\leq G} (1-p^{-1})|C|$$

*where $C$ is a cyclic p-subgroup of $G$.*

*Proof.* Instead of counting the p-singular elements in $G$, we count the cyclic p-subgroups generated by the p-singular elements of $G$ and use that for a cyclic group $C_{p^i}$, we know the number of elements of order $p^i$. First, the subgroup generated by 1 is the trivial group and it has one element. Two distinct cyclic subgroups of order $p$ cannot share any element except for 1 and in general two distinct cyclic groups of order $p^i$ cannot share an element of order $p^i$ since this element will generate the entire subgroup. There are

$(p-1)$ elements of order $p$ in each $C_p$. For a subgroup of order $p^i$ we can use Eulers $\varphi$ function to count the elements of order $p^i$.

$$\varphi(p^i) = p^{i-1}(p-1)$$

hence the elements of $C_{p^i}$ of order $p^i$ are $p^{i-1}(p-1)$. Thus

$$|G_p| = 1 + \sum_{1<C\leq G} (1-p^{-1})|C|$$

□

**Definition 5.5.** *For a group G, the Frattini subgroup of G is the intersection of all maximal subgroups of G and it is denoted* $\Phi(G)$.

$\Phi(G)$ is a normal subgroup. One can show that for a p-group $P$, $\Phi(P)$ is the smallest subgroup such that $P/\Phi(P)$ is elementary abelian.

**Lemma 5.6.** *A p-group P is cyclic if and only if* $P/\Phi(P)$ *is one-dimensional.*

*Proof.* Let $P$ be a cyclic p-group that is $P \cong C_{p^i}$ for some $i$. Then $\Phi(C_{p^i}) = C_{p^{i-1}}$ and $C_{p^i}/\Phi(C_{p^i}) = C_p$ hence the Frattini quotient is one dimensional. Let now $P$ be any p-group, $P = C_{p^{i_1}} \times C_{p^{i_2}} \times \cdots \times C_{p^{i_r}}$ and assume $P/\Phi(P) \cong C_p$. Then the order of $|\Phi(P)| = \frac{p^{i_1}p^{i_2}\ldots p^{i_r}}{p} = p^{i_1-1}p^{i_2}\cdots p^{i_r}$. Since $\Phi(P)$ is the smallest subgroup making $P/\Phi(P)$ elementary abelian, $P$ has to be cyclic. If not, the subgroup $N = C_{p^{i_1-1}} \times C_{p^{i_2-1}} \ldots C_{p^{i_r-1}}$ is a smaller subgroup making the quotient $P/N$ elementary abelian. □

Also every subgroup $Q$ of $P$ containing the Frattini subgroup is normal. This is because we have a surjective group homomorphism $P \xrightarrow{\varphi} P/\Phi(P)$ and since $P/\Phi(P)$ is abelian the subgroup $Q/\Phi(P)$ is normal in $P/\Phi(P)$ and hence $Q\Phi(P)$ is normal in $P$.

Let $O_P^{[\Phi(P),P)}$ be the orbit category with objects proper subgroups of $P$ containing $\Phi(P)$ and let $O_{P/\Phi(P)}^{[1,P/\Phi(P))}$ be the category with objects all proper subgroups of $P/\Phi(P)$

**Lemma 5.7.** *Let P be a finite p-group. There is an isomorphism between the categories*

$$O_P^{[\Phi(P),P)} \xrightarrow{\approx} O_{P/\Phi(P)}^{[1,P/\Phi(P))}$$

*Proof.* We will show that the objects and the morphisms are in a one-to-one correspondence. The objects are in a one-to-one correspondence since we can use the fact that for a normal subgroup $N$ of $G$, every subgroup of $G/N$ is of the form $H/N$ where $H$ is a subgroup of $G$ containing $N$. There is also a one-to-one correspondence between the morphisms of the two categories. Let $Q_1$ and $Q_2$ be two subgroups of $P$ containing $\Phi(P)$ with $Q_1 \leqq Q_2$. Since every subgroup of $P$ containing the Frattini subgroup is normal we have that the morphism set from $Q_1$ to $Q_2$ in $O_P^{[\Phi(P),P)}$ is the set

$$N_{O_P^{[\Phi(P),P)}}(Q_1, Q_2)/Q_2 = P/Q_2 = \frac{P/\Phi(P)}{Q_2/\Phi(P)}$$

The morphism set from $Q_1/\Phi(P)$ to $Q_2/\Phi(P)$ in $O_{P/\Phi(P)}^{[1,P/\Phi(P))}$ is also the set $\frac{P/\Phi(P)}{Q_2/\Phi(P)}$. Thus we have an isomorphism. □

Now we look at the orbit category $O_G^p$ of a finite group $G$, which has as objects all p-subgroups of $G$ and find a coweighting for this category.

**Theorem 5.8.** *The function*

$$k_K = \begin{cases} |G|^{-1} & \text{if } K = \{1\} \\ |G|^{-1}(1-p^{-1})|K| & \text{if } K > 1 \text{ is cyclic} \\ 0 & \text{otherwise} \end{cases}$$

*is a coweighting for* $O_G^p$ *and the Euler characteristic is*

$$\chi(O_G^p) = \frac{|G_p|}{|G|}$$

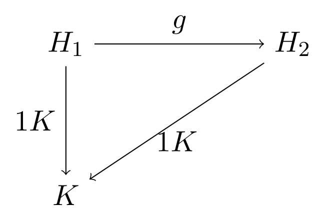
*Proof.* We use Lemma 4.11 to find a coweighting for $O_G^p$ defined as

$$k_K = \frac{-\tilde{\chi}(O_G^p//K)}{|[K]||O_G^p(K)|}$$

where $K$ is a p-subgroup of $G$ and $[K]$ is the isomorphism class of $K$ in $O_G^p$, that is the set $\{K^g \mid g \in G\}$.
First we show that we have equivalences of categories

$$i_K : O_K \to O_G^p/K \text{ and } i_K^* : O_K^{[1,K)} \to O_G^p//K$$

First we define $i_K$. The element $1 \in N_G(H,K)/K$. Let $H$ be an object in $O_K^p$. The functor $i_K$ assigns to every object $H$ in $O_K^p$, the morphism $1K : H \to K$. For a morphism $fH_2 \in O_G^p(H_1, H_2)$ we have to assign a morphism from $1K : H_1 \to K$ and $1K : H_2 \to K$, e.i. a morphism $g : H_1 \to H_2$ that makes the following diagram commute




If we let $g = f$, the diagram commutes since $1K = K = gK$ because $g \in K$. We can restrict this functor to $i_K^* : O_K^{[1,K)} \to O_G^p//K$ since $O_K^{[1,K)}$ only contains subgroup $H < K$
By construction the functor $i_K^*$ is full and faithful and essentially surjective on objects, and by Theorem 2.15 it is an equivalence. We now know that $O_K^{[1,K)}$ and $O_G//K$ have identical Euler characteristic.

We claim that we have an adjunction between the categories $O_K^{[1,K)}$ and $O_K^{[\Phi(K),K)}$. We want to show that $L$ is left adjoint to $R$, where

$$O_K^{[1,K)} \underset{L}{\overset{R}{\leftrightarrows}} O_K^{[\Phi(K),K)} \overset{\approx}{\leftarrow} O_{K/\Phi(K)}^{[1,K/\Phi(K))}$$

and $L$ and $R$ are given by $L(Q) = Q\Phi(K)$ and $R(Q) = Q$ for any subgroup $Q \leqq K$.
We first check that these are indeed functors. Since the Frattini subgroup is a normal subgroup $Q\Phi(K)$ is again a subgroup of $K$ containing $\Phi(K)$, hence is an object of $O_K^{[\Phi(K),K)}$. A morphism $g : H \to J$ in $O_K^{[1,K)}$ will also be a morphism from H to K in $O_K^{[\Phi(K),K)}$. This holds since

$$g^{-1}H\Phi(K)g = g^{-1}Hgg^{-1}\Phi(K)g \leqq J\Phi(K)$$

This is also well-defined since

$$O_K^{[1,K)}(H,J) = \{gJ \mid g^{-1}Hg \leqq J\}$$

and

$$O_K^{[\Phi(K),K)}(H\Phi(K), J\Phi(K)) = \{gJ\Phi(K) \mid g^{-1}H\Phi(K)g \leqq J\Phi(K)\}$$

and if $gJ = g'J$ then $g^{-1}g' \in J$ so $g^{-1}g' \in J\Phi(K)$ that is $gL\Phi(K) = g'J\Phi(K)$.
The functor $R$ does nothing to objects and morphisms. We have to show that there are natural transformations $s : LR \to 1_{O^{[\Phi(K),K)}_{\Phi(K)}}$ and $t : 1_{O^{[1,K)}_{K}} \to RL$.
The first is easy since it is actually equal to the identity functor.
For the second one we have to show that for all object $H$ in $O^{[1,K)}_{K}$ there exists a morphism $t_H : H \to H\Phi(K)$ such that for every $gJ : H \to J$ the following diagram commutes

$$\begin{array}{ccc} H & \xrightarrow{\quad gJ \quad} & J \\ {\scriptstyle t_H}\downarrow & & \downarrow{\scriptstyle t_J} \\ H\Phi(K) & \xrightarrow[\quad gJ\Phi(K) \quad]{} & J\Phi(K) \end{array}$$

We can choose $t_H = 1H\Phi(K)$ since $1^{-1}H1 = H \leqq H\Phi(K)$ and this will make the diagram commute since $t_H gJ\Phi(K) = gJ\Phi(K) = gt_J J\Phi(K)$
Thus there is an adjunction between the two categories and they have identical Euler characteristic $\chi(O^p_g//K) = \chi(O^{[1,K)}_K$.
We use the Orbit-Stabilizer Theorem ([4] Chapter II, Theorem 4.3) to obtain the following equality $|[K]||O^p_G(K)| = [G : K]$. Hence

$$k_K = \frac{-\tilde{\chi}(O^p_G//K)}{|[K]||O^p_G(K)|} = \frac{-\tilde{\chi}(O^{[1,V)})}{|G : K|}$$

where $V = K/\Phi(K)$ is the Frattini quotient of K.
Since $V$ is elementary abelian we can use Theorem 5.3 and we see that we only get a contribution to $k_K$ when the dimension of $K/\Phi(K)$ is 0 or 1 since

$$-\tilde{\chi}(O^{[1,V)}_V) = \begin{cases} 1 & \text{if dimV=0} \\ 1 - p^{-1} & \text{if dimV=1} \\ 0 & \text{dimV>1} \end{cases}$$

If the dimension of $K/\Phi(K)$ is 0, then $k_K = 1/|G|$. Since $K$ is cyclic if and only if $V$ is one-dimensional we have that

$$k_K = \frac{-\tilde{\chi}(O^{[1,V)})}{|G:K|} = \frac{(1-p^{-1})}{|G:K|} = |K|(1-p^{-1})|G|^{-1}$$

. Hence a coweighting for $O_G^p$ is

$$k_K = \begin{cases} |G|^{-1} & \text{if } K = \{1\} \\ |G|^{-1}(1-p^{-1})|K| & \text{if } K > 1 \text{ is cyclic} \\ 0 & \text{otherwise} \end{cases}$$

for any p-subgroup $K$ of $G$. To find the Euler characteristic of this category we have to take the sum of the values of the coweightings and we can use Lemma 5.4 to conclude that $\chi(O_G^p) = \frac{|G_p|}{|G|}$ □

Now we find a weighting for the category $O_G^p$ and use the fact that for a category admitting both a weighting and a coweighting their sum must be the same.

**Theorem 5.9.** *The function*

$$k^H = \frac{-\tilde{\chi}(S^{p+*}_{O_G(H)})}{\mid G:H\mid}$$

*is a weighting for $O_G^p$ and the Euler characteristic is given by*

$$\chi(O_G^p) = \sum_H \frac{-\tilde{\chi}(S^{p+*}_{O_G(H)})}{\mid G:H\mid} = \sum_{[H]} \frac{-\tilde{\chi}(S^{p+*}_{O_G(H)})}{\mid O_G(H)\mid}$$

*where $H$ is a p-subgroup of $G$ and $[H]$ is the conjugacy class of $H$ .*

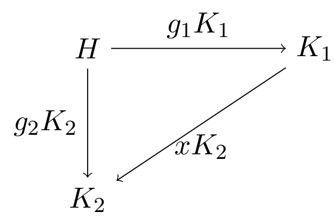
*Proof.* Again we use Lemma 4.11 to define a weighting for $O_G^p$.

$$k^H = \frac{-\tilde{\chi}(H//O_G^p)}{|[H]||O_G(H)|} = \frac{-\tilde{\chi}(H//O_G^p)}{|G:H|}$$

We want to show that the categories $H//O_G^p$ and $S^{p+*}_{O_G(H)}$ are homotopy equivalent.

We start by looking at a functor $r_H : H/O_G^p \to S^p_{O_G(H)}$. An object in $H/O_G^p$ is a morphism in $O_G^p(H,K)$ where $H$ and $K$ are p-subgroups of $G$. Let

$gK \in O_G^p(H,K) = N_G(H,K)/K$. The functor $r_H$ sends takes this object to $N_{{}^gK}(H)/H$. This is an object in $S^p_{O_G(H)}$ since it is a p-subgroup of $O_G^p(H)$ as ${}^gK = gKg^{-1}$ is a subgroup of G. Let $xK_2$ be a morphism from $g_1K_1$ to $g_2K_2$ in $H/O_G^p$ that is a morphism in $O_G^p$ that makes the following diagram commute

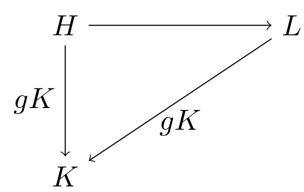



Then $x^{-1}K_1x \leqq K_2$ and $g_1xK_2 = g_2K_2$ i.e. $g_2^{-1}g_1x \in K_2$. Since $S^{p+*}_{O_G(H)}$ is a poset $r_H$ can only send morphisms to the inclusion, so we have to make sure that $r_H(g_1K_1) \leqq r_H(g_2K_2)$ if there is a morphism between them in $H/O_g^p$, i.e. that $N_{{}^{g_1}K_1}(H)/H \leqq N_{{}^{g_2}K_2}(H)/H$. This holds since ${}^{g_1}K_1 \leqq {}^{g_2}K_2$. This is true because $K_1^x \leqq K_2 \Leftrightarrow {}^{x^{-1}}K_1 \leqq K_2 \Leftrightarrow {}^{g_2^{-1}g_1xx^{-1}}K_1 \leqq {}^{g_2^{-1}g_1x}K_2 = K_2$. Hence $r_H$ is a functor.

We now want to show that $r_H$ is a homotopy equivalence. We show that for all $\bar{L} \leqq N_G(H)/H$ the category $\bar{L}/r_H$ has an initial object and hence is contractible. Then we use Quillen's theorem A to conclude that $r_H$ is a homotopy equivalence.

Let L be an object in $S^p_{O_G(H)}$ that is let L be a p-subgroup satisfying $H \leqq L \leqq N_G(G)$. Let $\bar{L} = L/H$ be the image of $L$ in $N_G(H)/H$. The category $\bar{L}/r_H$ with objects all morphisms in $S^p_{O_G(H)}$ with domain $\bar{L}$ and codomain $r_H(gK)$ for all objects $gK \in O_G^p(H,K)$. Hence $\bar{L}/r_H$ is the full subcategory of $O_G^p(H)/H$ generated by all morphisms $gK \in O_G^p(H,K)$ such that $L \leqq N_{{}^gK}(H)$.

The inclusion $1L : H \to L$ is an object in $\bar{L}/r_H$ since $L = N_L(H)$. We want to show that this object is initial. Let $gK$ be any object in $\bar{L}/r_H$ that is $L \leqq N_{{}^gK}(H)$. The morphism $gK : H \to K$ extends to a morphism $gK : L \to K$ because $L^g \leqq N_{{}^gK}(H)^g = N_K(H^g) \leqq K$. We have the following commutative diagram in $\bar{L}/r_H$




This shows that the object, the inclusion $H \to L$, is an initial object in the category $\bar{L}/r_H$ and it is thereby contractible. By Theorem 3.6 it follow that $r_H$ is a homotopy equivalence and $H/O_G^p \simeq S^p_{O_G(H)}$.
$r_H$ restricts to a homotopy equivalence $r_H^* : H//O_G^p \to S^{p+*}_{O_G(H)}$ hence $H//O_G^p \simeq S^{p+*}_{O_G(H)}$. We know that any category is equivalent to its skeleton and that an equivalence of categories induces a homotopy equivalence between their classifying spaces, hence $sk(H//O_G^p) \simeq H//O_G^p \simeq S^{p+*}_{O_G(H)}$ thus their usual Euler characteristic are identical. We know that $H//O_G^p$ is a thin category so both $sk(H//O_G^p)$ and $S^{p+*}_{O_G(H)}$ are posets hence these Euler characteristics equals the ones defined by Leinster and we get that $\chi(H//O_G^p)) = \chi(S^{p+*}_{O_G(H)})$.

Now we can write $k^H = \frac{-\tilde{\chi}(H//O_G^p)}{|[H]||O_G(H)|} = \frac{-\tilde{\chi}(S^{p+*}_{O_G(H)})}{|[H]||O_G(H)|}$ This weighting is constant on conjugacy classes, so the Euler characteristic is $\chi(O_G^p) = \sum_H k^H = \sum_{[H]} \mid [H] \mid k^H = \sum_{[H]} \frac{-\chi(S^{p+*}_{O_G(H)})}{|O_G(H)|}$. □

By combining Theorem 5.9 and Theorem 5.8 we obtain

$$\sum_{[H]} \frac{-\tilde{\chi}(S^{p+*}_{O_G(H)})}{|O_G(H)|}|G| = |G_p|$$

Notice that the contribution from the trivial subgroup $H = 1$ is $-\tilde{\chi}(S^{p+*}_G)$ since $O_G(1) = G$. So we can rewrite the above to

$$|G_p| + \tilde{\chi}(S^{p+*}_G) + \sum_{[H]\neq 1} \frac{\tilde{\chi}(S^{p+*}_{O_G(H)})}{|O_G(H)|_p} \frac{|G|}{|O_G(H)|_{p'}} = 0 \qquad (1)$$

We will use this expression to show that Frobenius' and Brown's theorems are equivalent.

**Theorem 5.10.** *Given (1), Frobenius' and Brown's theorems are equivalent.*

*Proof.* Assume Frobenius theorem is true. By induction we can show that $\tilde{\chi}(S^{p+*}_{O_G(H)})$ divides $|O_G(H)|_p$.

Notice that for the trivial group $G = \{1\}$ we have that $\tilde{\chi}(S^{p+*}_G) = -1$ since this category is empty and $|G|_p = 1$ hence $|G|_p$ divides $\tilde{\chi}(S^{p+*}_G)$. Assume $|H|_p$ divides $\tilde{\chi}(S^{p+*}_H)$ for all group H of order $< |G|$. Then $\tilde{\chi}(S^{p+*}_{O_G(H)})$ also divides $|O_G(H)|_p$ for $H \neq \{1\}$ since $|O_G(H)| < |N_G(H)| \leq |G|$.

Also $\frac{|G|}{|O_G(H)|_{p'}}$ is an integer divisible by $|G|_p$. Since every term is divisible by $|G|_p$ so is $\tilde{\chi}(S^{p+*}_G)$. This is Brown's theorem.

Assume Browns theorem holds. Then $\frac{\tilde{\chi}(S^{p+*}_{O_G(H)})}{O_G(H)}$ is an integer and $\frac{|G|}{|O_G(H)|_{p'}}$ is an integer divisible by $|G|_p$ and by assumption $\tilde{\chi}(S^{p+*}_G)$ is divisible by $|G|_p$. Then so is $|G_p|$. This is Frobenius' theorem. □

**Example**

We look at the group $S_3 \times S_3$. This group has order 36, thus $|G|_2 = 2^2 = 4$. The poset $S^{2+*}_G$ consists of the subgroups of order 2 and 4. There are 15 subgroups of order 2 and 9 subgroups of order 4. Each of the subgroups of order 4 contains 3 subgroups of order 2. This means that $|S_3 \times S_3|$ has 24 0-simplices and 27 1-simplices. The reduced Euler characteristic is $\chi(\tilde{S^{2+*}_G}) = -1 + 24 - 27 = -4$, which is congruent to 0 mod $|G|_p = 4$.

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