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Golod Complexes

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Abstract

We will in this paper discuss the concept of a Golod complex, that is for the Stanley-Reisner ring of a simplicial complex Δ, $k[\Delta]$, the torsion $\mathrm{Tor}_i^{k[\Delta]}(k,k)$ is as large as it could possibly be, i.e. the Poincaré series of $k[\Delta]$ viewed as a local ring to be coefficient wise as large as possible. We will see a theorem which gives a purely combinatorial requirement for this otherwise algebraic notion. We will then spend some time studying the concept of shellability both as a shellable complex and of an EL-shellable poset. We will use these notions to be able to calculate the dimensions of homology groups of order complexes of certain interval posets. Now let Δ_d^{2d} be a simplicial complex on $2d$ vertices, with all faces of cardinality less than d and no other faces. We will use the combinatorial requirement for Golodness of a complex and our discussion of shellability to show that Δ_d^{2d} is a Golod complex for all $d > 1$.

Contents

Chapter 1

Preliminaries

1.1 Simplicial complexes

We will start with some rather basic definitions of a simplicial complex which is the objects we will study throughout the paper; these definitions can be found in e.g. [St]

Definition 1.1 (Simplicial complexes)**.**

- *We write* $\mathbf{n}$ *for the set* $\{1, 2, \ldots, n\}$
- *A simplicial complex* Δ *on the vertex set* V*, where* V *is finite, is a subset of the power set of* V*, such that if* $\tau \subset \sigma \in \Delta$ *then* $\tau \in \Delta$*.*
- *For a simplicial complex* Δ*, we call elements of* Δ *for faces. If* σ *is an inclusionwise maximal face of* Δ *we say that* σ *is a facet of* Δ*. We say that* Δ *is a* $(d-1)$*-complex if* $\max\{|F| \big| F \textit{is a facet of } \Delta\} = d$*.*
- *For a simplicial complex* Δ *the minimal non-faces of* Δ*,* $\mathcal{M}(\Delta)$*, is the set of inclusionwise least elements of* $\mathcal{P}(V) - \Delta$
- *We write* Δ_i^n *for the simplicial complex on the vertex set* $\mathbf{n}$ *where* $\Delta_i^n := \{\sigma | \sigma \subset \mathbf{n},\ |\sigma| < i\}$*.*
- *Given a simplicial complex* Δ *on the vertex set* $V = \{v_i, \ldots, v_n\}$ *we call* $|\Delta| \subset \mathbb{R}^n$ *the geometric realization of* Δ *if for each* $\{v_{i_0}, \ldots, v_{i_r}\} \in \Delta$ *the convex hull of* $e_{i_0}, \ldots, e_{i_r} \subset |\Delta|$*, where* $e_1, \ldots, e_n \in \mathbb{R}^n$ *with* $e_2 - e_1, \ldots, e_n - e_1$ *being linear independent. We equip* $|\Delta| \subset \mathbb{R}^n$ *with the subspace topology.*

It is trivial that any two geometric realizations of a simplicial complex are homeomorphic and there is thus no need to distinguish between them. Given a complex Δ we can always rename its vertices so the vertex set becomes $\mathbf{n}$ for some $n \in \mathbb{N}$. We will also need the homology of simplicial complexes, which we will define in the usual sense[1]. The reason for the definition of the complex Δ_i^n is that these complexes are rather simple and we will be able to prove that for a certain class of these Δ_d^{2d}, with $d > 1$ the complex is Golod over any field k.

We will for our main theorem need to be able to make rings out of our complexes, this we will do in the following way

Definition 1.2. *Let Δ be a simplicial complex on the vertex set $V = \{v_1, \ldots, v_n\}$, then the Stanley-Reisner ring of Δ, $k[\Delta]$ for a field k, is the ring $k[x_1, \ldots, x_n]/I_\Delta$, being the polynomial ring modded with the ideal which contains $x_{i_1} \cdots x_{i_r}$ if $\{v_{i_1}, \ldots, v_{i_r}\} \notin \Delta$*

This allows us to talk about algebraic notions of our simplicial complexes in much the same way as the geometric realization allows us to talk about topological properties of our otherwise combinatorial complexes.

1.2 Partially ordered sets

We will also need another class of combinatorial objects closely related to the simplicial complexes above, see for instance the definition of the order complex. These definitions are likewise taken from [St]

Definition 1.3 (Partially ordered sets)**.**

- *A set P is called a poset, if it is partially ordered by some relation $\leq$ which is reflexive, anti-symmetric and transitive.*
- *We call $m = (c_1, c_2, \ldots)$ a chain in the poset P if $c_i \in P$ for all i, and $c_1 < c_2 < \ldots$, we partially order the chains of a poset by for $m' = (c'_1, c'_2, \ldots)$ then $m \leq m' \Leftrightarrow c'_i = c_{n_i}$ for all i and some sequence n_i.*
- *For a finite poset P we define the order-complex of P, $\Delta(P)$, to be the simplicial complex on the vertex set P, where the faces of $\Delta(P)$ are the chains of P.*
- *For a poset P the set $\mathcal{M}(P)$ is the set of all maximal chains of P. If P is finite this is the facets of $\Delta(P)$*
- *The edges of P, $\mathcal{E}(P)$, is the set of elements of the form $(p_i < p_j)$ where $p_i, p_j \in P$ with $p_i < p_j$. If P is finite this is exactly the edges (faces of cardinality 2) of $\Delta(P)$.*
- *A poset P is called bounded if it has a unique minimal and maximal element called respectively $\hat{0}$ and $\hat{1}$. The proper part of a bounded poset,*

[1] Again see [St]

$\bar{P}$, is the poset $P - \{\hat{0}, \hat{1}\}$, and we define $\widehat{P} = P \cup \{\hat{0}, \hat{1}\}$ with $\hat{0}$ and $\hat{1}$ being the respectively unique minimal and unique maximal elements of P should they exists, otherwise they are elements added such that they become the unique minimal respectively maximal elements.

- *For a poset P we can have intervals, so for $a, b \in P$ we define both the open and closed (and semi-open) intervals in the usual sense. We will note in which poset the interval is, if doubt could arise, so we would write $[a, b]_P$ for the closed interval from a to b in P.*
- *For a poset P we define the n'th reduced homology group of P with coefficients in some group G, $\tilde{H}_n(P, G)$, to be the n'th reduced homology group of the simplicial complex $\Delta(P)$ with coefficients in G.*

We will also need some rather specific posets, i.e. the coordinate and diagonal lattices defined below. These play a very important role in our main theorem. But we will start with a simple definition of a lattice

Definition 1.4. *A poset P is called a lattice if $\forall a, b \in P$ there is are unique elements $u, v \in P$ such that $u \geq a$, $u \geq b$ with no $c \in P$ such that $u > c > a$ and $u > c > b$, and $v \leq a$, $v \leq b$ with no $c \in P$ such that $v < c < a$ and $v < c < b$.*

Meaning a poset is lattices if there for all two elements are a unique supremum and infimum.

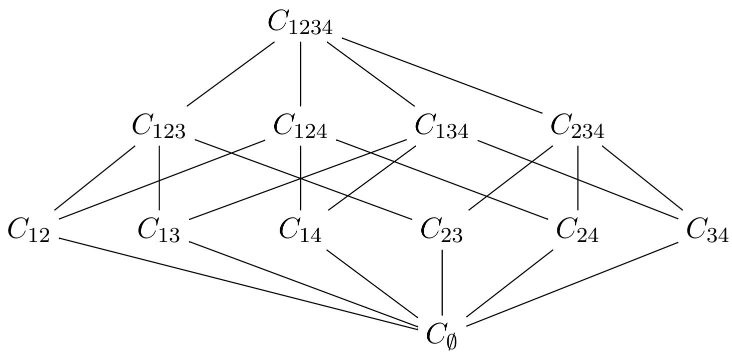
Definition 1.5.

- *For $\sigma \subset \mathbf{n}$, we define $C_\sigma := \{(x_1, \ldots, x_n) \in \mathbb{C}^n | \forall i \in \sigma : x_i = 0\}$*
- *For the simplicial complex Δ, on the vertex set $\mathbf{n}$, the coordinate arrangement lattice, $\mathcal{L}_{\mathcal{C}(\Delta)}$, is the lattice consisting of all intersections of C_σ for $\sigma \in \mathcal{M}(\Delta)$ ordered by reverse inclusion, where the empty intersection is viewed as $C_\emptyset = \mathbb{C}^n$.*
- *For $\sigma \subset \mathbf{n}$, we define $D_\sigma := \{(x_1, \ldots, x_n) \in \mathbb{C}^n | \forall i, j \in \sigma : x_i = x_j\}$*
- *For $\sigma_1, \ldots, \sigma_r \subset \mathbf{n}$ disjoint sets, we write $D_{\sigma_1 | \ldots | \sigma_r}$ for the intersection $\bigcap_{i=1}^{r} D_{\sigma_i}$*
- *For the simplicial complex Δ, on the vertex set $\mathbf{n}$, the diagonal arrangement lattice, $\mathcal{L}_{\mathcal{D}(\Delta)}$, is the lattice consisting of all intersections of D_σ for $\sigma \in \mathcal{M}(\Delta)$ ordered by reverse inclusion, where the empty intersection is viewed as $D_\emptyset = \mathbb{C}^n$.*
- *For each simplicial complex Δ on the vertex set $\mathbf{n}$ we have a map $\epsilon : \mathcal{L}_{\mathcal{D}(\Delta)} \to \mathcal{L}_{\mathcal{C}(\Delta)}$ given by for $D_{\sigma_1 | \ldots | \sigma_r} \in \mathcal{L}_{\mathcal{D}(\Delta)}$ then $\epsilon(D_{\sigma_1 | \ldots | \sigma_r}) = C_{\sigma_1 \cup \ldots \cup \sigma_r}$*

- *We have a map $c : \mathcal{L}_{\mathcal{D}(\Delta)} \to \mathbb{N}_0$ given by for $D_{\sigma_1|\ldots|\sigma_r} \in \mathcal{L}_{\mathcal{D}(\Delta)}$ then $c(D_{\sigma_1|\ldots|\sigma_r}) = r$. In other words, this function counts the number of disjoint subsets in the index of the element.*

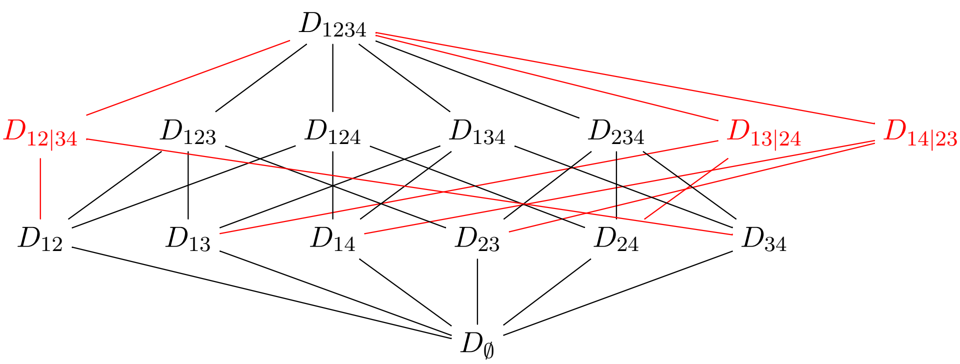
We can now see that for all C in $\mathcal{L}_{\mathcal{C}(\Delta)}$ there is a unique $\sigma \subset \mathbf{n}$ such that $C = C_\sigma$. Likewise for all D in $\mathcal{L}_{\mathcal{D}(\Delta)}$ there are unique $\sigma_1, \ldots, \sigma_r \subset \mathbf{n}$ such that $D = D_{\sigma_1|\ldots|\sigma_r}$. It is trivially clear that $C_\sigma \cap C_\tau = C_{\sigma\cup\tau}$, for $\sigma, \tau \subset \mathbf{n}$, and likewise if $\sigma \cap \tau \neq \emptyset$ then $D_\sigma \cap D_\tau = D_{\sigma\cup\tau}$. It is clear that ϵ is a surjective map, since for $C \in \mathcal{L}_{\mathcal{C}(\Delta)}$ then there is $\sigma_1, \ldots, \sigma_r \in \mathcal{M}$ such that $\bigcap_{i=1}^r C_{\sigma_i} = C$, but then trivially $\bigcap_{i=1}^r D_{\sigma_i} \in \mathcal{L}_{\mathcal{D}(\Delta)}$ with $\epsilon(\bigcap_{i=1}^r D_{\sigma_i}) = C$.

So let's give an example as to how these posets might look. So we will draw the Hasse-diagrams, that is a graph with the vertices being the elements of the poset, P and $\mathcal{E}(P)$ being the edges, of both the coordinate- and the diagonal-intersection lattice for Δ_2^4. We will denote $C_{\{1,2\}}$ with the simpler C_{12} to ease notation and the same for the diagonal elements. So first the poset $\mathcal{L}_{\mathcal{C}(\Delta_2^4)}$, clearly $\mathcal{M}(\Delta_2^4) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$ and hence we get



(1.1)

Now lets draw the Hasse-diagram of $\mathcal{L}_{\mathcal{D}(\Delta_2^4)}$, I will color the edges and vertices not mirrored in $\mathcal{L}_{\mathcal{C}(\Delta_2^4)}$ red, to high light the differences between the two posets.



(1.2)

As we can see these posets rather quickly grow rather large.

1.3 Golod complexes

To define what we mean with a Golod complex we will need some definitions and results on the Poincaré series. But first we will need something as elementary as a local ring.

Definition 1.6. *A ring R is said to be local if it has a unique maximal left ideal I. We call the field R/I the residue field.*

Now trivially we can view R/I as an R-module. We can therefore give the following definition of the Poincaré series from [GrTh]

Definition 1.7. *Given a local ring, R, with residue field, k, we define the Poincaré series of R, $P(R)$, to be the formal power series $\sum b_p Z^p$ for $b_p = Tor_p^R(k,k)$*

Now we would like to study the Poincaré series of the Stanley-Reisner ring of a simplicial complex. So for a simplicial complex Δ on a vertex set V and a field k, we can easily see that $k[\Delta]$ is local since the ideal generated by V is maximal. We are now able to give the following theorem from [Av]

Theorem 1.8. *For a simplicial complex Δ on a vertex set with n vertices and a field k, the Poincaré series of $k[\Delta]$ is always coefficient wise less than or equal to*

$$\frac{(1+Z)^n}{1 - Z \cdot (H_*(K^{k[\Delta]})(Z) - 1)} \tag{1.3}$$

where $H_*(K^{k[\Delta]})(Z)$ is the polynomial with the n'th coefficient $\dim_k H_n(K^{k[\Delta]})$ with $K^{k[\Delta]}$ being the Koszul chain complex of $k[\Delta]$.

In other words for any simplicial complex, the Poincaré series of its Stanley-Reisner ring is bounded. We will now define a complex to be Golod over a field, if it is in some sense maximal with respect to the boundary. Notice this is not the usual definition of a Golod complex, but the usual definition would require to much algebraic theory for this project and hence we will instead use the following equivalent definition.

Definition 1.9. *A simplicial complex Δ is said to be a Golod complex over k if*

$$P(k[\Delta]) = \frac{(1+Z)^n}{1 - Z \cdot (H_*(K^{k[\Delta]})(Z) - 1)} \tag{1.4}$$

with the same naming as in theorem 1.8.

Normally when giving the definition of a Golod complex will rely on a definition of a Golod ring, which is a ring with the largest possible Poincaré series, and then a complex is called Golod if $k[\Delta]$ is a Golod ring. I have here avoided that definition as to spare myself some algebraic considerations. For more information on Golod rings the reader is referred to [Av]. Now this definition of a Golod ring is rather bulky since $P(k[\Delta])$ is rather difficult to calculate since it relies on finding a resolution of k as a $k[\Delta]$-module, which might be of infinite length. We will therefore need another characterization of a Golod complex which only relies on pure combinatorics. The following theorem follows from [Be], and it is to this theorem we will dedicate the bulk of this paper

Theorem 1.10. *The following are equivalent for a simplicial complex Δ on a vertex set $\mathbf{n}$, where $\forall i \in \mathbf{n}: \ \{i\} \in \Delta$ and a field k*

(1) Δ is a Golod complex over k.

(2) for every $C \in \mathcal{L}_{\mathcal{C}(\Delta)}$ there is an equality of polynomials

$$\dim_k \tilde{H}_*((\mathbb{C}^n, C)_{\mathcal{L}_{\mathcal{C}(\Delta)}}; k)(z) = \sum_{D \in \epsilon^{-1}(C)} (-z)^{c(D)-1} \dim_k \tilde{H}_*((\mathbb{C}^n, D)_{\mathcal{L}_{\mathcal{D}(\Delta)}}; k)(z)$$

where $\dim_k \tilde{H}_*((\mathbb{C}^n, C)_{\mathcal{L}_{\mathcal{C}(\Delta)}}; k)(z)$ is the polynomial with variable z and the n'th coefficient $\dim_k \tilde{H}_n((\mathbb{C}^n, C)_{\mathcal{L}_{\mathcal{C}(\Delta)}}; k)$, and likewise $\dim_k \tilde{H}_*((\mathbb{C}^n, D)_{\mathcal{L}_{\mathcal{D}(\Delta)}}; k)(z)$ is the polynomial with variable z where the n'th coefficient is $\dim_k \tilde{H}_n((\mathbb{C}^n, D)_{\mathcal{L}_{\mathcal{D}(\Delta)}}; k)$.

This theorem gives a purely combinatorial requirement for the Golodness of a ring. So let's see an example of how a use of this theorem might go. Lets take Δ_2^4 from above. Now I will only check the combinatorial requirement for C_{1234} since the other cases are trivial. Now we can see that the geometric realization of the order complex of $(C_\emptyset, C_{1234})$ is homotopic equivalent to $S^1 \vee S^1 \vee S^1$ and hence $\dim_k \tilde{H}_*((\mathbb{C}^n, C_{1234})_{\mathcal{L}_{\mathcal{C}(\Delta_2^4)}}; k)(z) = 3z$. We now that $\epsilon^{-1}(C_{1234}) = D_{1234}, D_{12|34}, D_{13|24}, D_{14|23}$. Again we see that the realization of the order complex of $(D_\emptyset, D_{1234})$ is homotopic equivalent to $S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1$ and hence $\dim_k \tilde{H}_*((\mathbb{C}^n, D_{1234})_{\mathcal{L}_{\mathcal{D}(\Delta_2^4)}}; k)(z) = 6z$. Lastly we see that for $\sigma = \{12\}, \{13\}, \{14\}$ that the realization of the order complex of $(D_\emptyset, D_{\sigma|\sigma^c}$ is homotopic equivalent to S^0, and hence that $\dim_k \tilde{H}_*((\mathbb{C}^n, D_{\sigma|\sigma^c})_{\mathcal{L}_{\mathcal{D}(\Delta_2^4)}}; k)(z) = 1$. So we get the equation (2) to be

$$3z = 6z - z \cdot 1 - z \cdot 1 - z \cdot 1 \tag{1.5}$$

which is true, and hence Δ_2^4 is a Golod complex. We will later see that this is true for all complexes on the form Δ_d^{2d}. But for now we will use the theorem for a more trivial case.

Corollary 1.11. *A simplicial complex* Δ *on the vertex set* $\mathbf{n}$ *with the property that* $\forall \sigma \in \mathcal{M}(\Delta): \; |\sigma| \geq \left\lfloor \frac{n}{2} \right\rfloor + 1$, *then* Δ *is a Golod complex over any field* k.

Proof. Now let Δ be a complex fulfilling the requirement of the corollary. Now all the minimal faces of the complex consist of over half the elements of vertex set. That means for all $\sigma, \tau \in \mathcal{M}(\Delta) : \sigma \cap \tau \neq \emptyset$. So there are no elements in $\mathcal{L}_{\mathcal{D}(\Delta)}$ on the form $D_{\sigma_1|\ldots|\sigma_n}$. Now this means that if for $D, D' \in \mathcal{L}_{\mathcal{D}(\Delta)}$ $\epsilon(D) = \epsilon(D')$, then we have $D = D'$ so since we already knew that ϵ was surjective, it is therefore bijective. So we need to check that for all $C \in \mathcal{L}_{\mathcal{C}(\Delta)}$ we have

$$\dim_k \tilde{H}_*((\mathbb{C}^n, C)_{\mathcal{L}_{\mathcal{C}(\Delta)}}; k)(z) = \dim_k \tilde{H}_*((\mathbb{C}^n, \epsilon^{-1}(C))_{\mathcal{L}_{\mathcal{D}(\Delta)}}; k)(z)$$

since for all $D \in \mathcal{L}_{\mathcal{D}(\Delta_i^n)}$ $c(D) = 1$. Now take $D_\sigma, D_\tau \in \mathcal{L}_{\mathcal{D}(\Delta_i^n)}$ such that $D_\sigma < D_\tau$, this means that $\sigma \subset \tau$ and hence that $\epsilon(D_\sigma) = C_\sigma < C_\tau = \epsilon(D_\tau)$, and hence the poset $\mathcal{L}_{\mathcal{D}(\Delta)}$ and $\mathcal{L}_{\mathcal{C}(\Delta)}$ are the same poset up to a renaming of the elements. And hence the requirement is trivial. □

Chapter 2

Helpful results

To prove that Δ_d^{2d} is a Golod complex that is, fulfills the theorem 1.10 (2) requirement we will need a way of computing the dimensions of the homology groups of certain intervals. Now in general I know of no method to do this, but if we could recognize the order complexes of the intervals as something nice, as for example a wedge of spheres, and find some way to count the number of spheres with dimension, then we would be able to check the requirement quite easily. So we will in this chapter state some definitions and theorems about shellability of complexes, in the hope that this will aid us to do exactly as we want. This is entire chapter is full of results taken from [BjWa], and you are asked to go there for proofs and further discussions of the results.

2.1 Introduction to shellability

Before we can begin stating what we mean with shellability we will need this rather basic definition, of intervals with respect to the inclusion order.

Definition 2.1. *For a set F, we write $\bar{F}$ for the set $\{\sigma|\sigma \subset F\}$ i.e. the power set of F.*

The definition might seem redundant, but this will lead the following to confirm with [BjWa].

Definition 2.2. *A simplicial complex is called shellable if its facets can be arranged in a linear order $F_1, F_2, \ldots, F_t$ in such a way that the subcomplex $(\bigcup_{i=1}^{k-1} \bar{F}_i) \cap \bar{F}_k$ is pure, that is all its facets have equal cardinality, and a $(|F_k|-2)$-complex for all $k = 2, 3, \ldots, t$. Such an ordering is called a shelling.*

So the definition means that we are able to build our complexes by adding facets in the shelling order, such that they intersect our part complex in a face of cardinality one less. Notice this is the unpure version of the definition which allows for greater flexibility later on. Since we do not need to require the facets have the same cardinality.

2.2 The h integer array

Now to state more firmly what we want of this new concept of shellability is two-fold. First we would like that geometric realizations of a shellable complex i homotopic equivalent to a wedge of spheres since we rather trivially know how homology interacts with both wedge-sums and spheres. But knowing just that, will do us a small help if we have no way of counting both the number of spheres and their respective dimensions in the wedge-sum. We therefore introduce the following:

Definition 2.3. *For a $(d-1)$complex Δ, let*

(i) $f_{i,j}$ *= number of faces* $\sigma \in \Delta$ *of cardinality* j *and with* $i = \delta(\sigma) := \max\{|\tau| | \sigma \subset \tau \in \Delta\}$

(ii)

$$h_{i,j} = \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} f_{i,k}$$

(iii) The triangular integer arrays

$$\mathbf{f} = (f_{i,j})_{0 \leq j \leq i \leq d}$$

and

$$\mathbf{h} = (h_{i,j})_{0 \leq j \leq i \leq d}$$

are called the f*-triangle and the* h*-triangle of* Δ*, respectively.*

Notice again that this is a generalization of the normal **f**- and **h**-vectors which we usually see with simplicial complexes. We would hope that these to new integer arrays somehow would be able to tell us how many spheres there are in our wedge-sum, and luckily they are:

Theorem 2.4. *Let* Δ *be shellable* $(d-1)$*-complex. Then the geometric realization of* Δ *has the homotopy type of a wedge of spheres, consisting of* $h_{j,j}$ *copies of the* $(j-1)$*-sphere for* $1 \leq j \leq d$

Corollary 2.5. *Let* Δ *be shellable complex, then* $\dim_k \tilde{H}_{j-1}(\Delta; k) = h_{j,j}$

The corollary follows from the well known facts that $\tilde{H}_d(S^d, k) = k$ and that $\tilde{H}_d(\bigvee_{v \in V} T_v) = \bigoplus_{v \in V} \tilde{H}_d(T_v)$, where S^d is the d-sphere, with $d > 0$ and for T_v some pointed topological space. We also know that $H_0(S^0, k) = k$, and that a d-sphere doesn't have homology in other than the d'th dimension. Furthermore it is well known that homology is stable under homotopy equivalences and that a simplicial complex and its geometric realization have the same homology.

2.3 Lexicographic shelling

Now all this new theory will not help us out unless we are able to recognize a complex as shellable. Now remember that in our case we want to recognize an interval of a posets order complex as shellable, so let's look at some theory for how this is done. So we will define a map from a poset with certain properties, and if they are fulfilled the poset will be shellable.

Definition 2.6. *For a poset P, then*

(i) an edge labeling of P is a map $\lambda : \mathcal{E}(P) \to \Lambda$, for some poset Λ

(ii) for a maximal chain $m = (c_1, c_2, \ldots)$ in P, and an edge labeling λ we write $\lambda(m) = (\lambda((c_1 < c_2), \lambda((c_2 < c_3)), \ldots)$.

(iii) an edge labeling λ is called an edge rising labeling (ER-labeling) if for every interval $[x, y]$, with $x, y \in P$, there is an unique maximal chain $m = (c_1, c_2, \ldots)$ with a rising label, that is $\lambda((c_1 < c_2) < \lambda((c_2 < c_3)) < \ldots$ in Λ.

(iv) an ER-labeling is called an edge lexicographic labeling (EL-labeling) if for every interval $[x, y]$ in P the unique maximal chain with rising label is lexicographic first among the maximal chains in the interval.

(v) a bounded poset P is called EL-shellable if it admits an EL-labeling.

We will define what we mean with lexicographic in (iv) of the definition above.

Definition 2.7. *For $a = (a_1, a_2, \ldots)$ and $a' = (a'_1, a'_2 \ldots)$ with $\forall i \in \mathbb{N} : a_i, a'_i \in P$ for some poset P. We say that a is lexicographic before a' if $\exists j$ such that $\forall i < j$ $a_i = a'_i$ and $a_j < a'_j$ in the poset P. Now if a and a' are of different length in the sense that $a = (a_1, a_2, \ldots, a_k)$ and $a' = (a'_1, a'_2, \ldots, a'_l)$ then we say that a is lexicographic before a' if $\exists j$ such that $\forall i < j$ $a_i = a'_i$ and $a_j < a'_j$ in the poset P or if $\forall i \leq k$ $a_i = a'_i$ and $k < l$.*

Theorem 2.8. *If a bounded poset P is EL-shellable, then $\Delta(\bar{P})$ is shellable*

So that means if we want to check if some order complex is shellable it is enough to check that if we add minimal and maximal elements that there is some EL-shelling. We will see later that this is quite handy since, it for some posets, are a rather easy thing to construct.

Chapter 3

The curious case of Δ_d^{2d}

We will here show using theorem 1.10 (2) that Δ_d^{2d} is a Golod complex. We will do this by studying the diagonal- and coordinate arrangement lattices, showing that the interesting intervals are shellable with the help of EL-shellability and then using the **h** array to rewrite the equality of polynomials.

Proposistion 3.1. *Δ_d^{2d} is a Golod complex over any field k, for $d > 1$.*

To prove this we will need some further results, so let's start with two lemmas. Here we will use the theory stated in chapter 2 to show that the relevant intervals for our complex are shellable.

Lemma 3.2. *$[\mathbb{C}^n, C_{\mathbf{2d}}]_{\mathcal{L}_{\mathcal{C}(\Delta_k^n)}}$ is EL-shellable for $k > 1$*

Proof. We see that this is a bounded poset and identify $\mathbb{C}^n$ with $C_\emptyset$. We define an edge labeling λ by $\lambda(C_\sigma < C_\tau) = \max(\tau - \sigma)$. Now clearly for all $\sigma \subset \mathbf{n}$ with $k \leq |\sigma|$, $C_\sigma \in [\mathbb{C}^n, C_{\mathbf{n}}]_{\mathcal{L}_{\mathcal{C}(\Delta_k^n)}}$

Now take arbitrary $C_\sigma, C_\tau \in (\mathbb{C}^n, C_{\mathbf{n}}]_{\mathcal{L}_{\mathcal{C}(\Delta_k^n)}}$, now all chain labels of maximal chains in the interval $[C_\sigma, C_\tau]$ will be permutations of the elements $\tau - \sigma$, and all these permutations are also the chain labels, so trivially they can only be arranged in a rising way in one way, and this way will also be lexicographic first.

Now take the interval $[C_\emptyset, C_\tau]$, take $i_1 < \ldots < i_k$ to be the first k elements of τ. Since $\sigma = \{i_1, \ldots, i_k\}$ consists of k elements and is a subset of $\mathbf{n}$ we know that $C_\sigma \in [C_\emptyset, C_\tau]$, now from above we know that there is a chain m which is uniquely rising maximal and lexicographic first in the interval $[C_\sigma, C_\tau]$, lets call $\lambda(m) = (j_1, \ldots, j_r)$. Now define m' to be the maximal chain from $C_\emptyset$ to C_τ by first going to C_σ and then follow m. Now since $\sigma \cup \{j_1, \ldots, j_r\} = \tau$ by the way we have defined λ we know that $i_k < j_1$, and hence $\lambda(m') = (i_k, j_1, \ldots, j_k)$ is rising. Since $i_1 < \ldots < i_k$ was the first k elements of τ, then all other elements of k elements $\gamma \subset \tau$ we will get that $\lambda(C_\emptyset < C_\gamma) > \lambda(C_\emptyset < C_\sigma)$, and hence m' will be lexicographic first among the maximal chains.

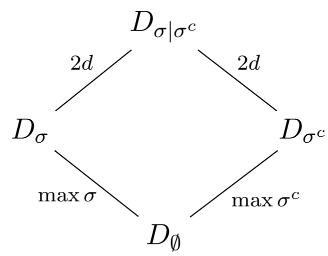
Now take a rising maximal chain m' from $C_\emptyset$ to C_τ, lets call the second element of the chain C_σ, then clearly $|\sigma| = k$, now the maximal chain m'' from C_σ to C_τ, which is just m' just skipping $C_\emptyset$ must be rising as well, lets assume $\lambda(m'') = (j_1, \ldots, j_r)$ then we know that $\sigma \cup \{j_1, \ldots, j_r\} = \tau$, but since the $\sigma \cap \{j_1, \ldots, j_r\} = \emptyset$, and since m' was rising we know that for $i_k = \max \sigma - \emptyset$ $i_k < j_1$, and hence σ must consist of the first k elements of τ, we therefore have uniqueness of the rising chain and hence λ is an EL-shelling. □

And again much in the same way as before we will se that also $\mathcal{L}_{\mathcal{D}(\Delta^{2d}_d}$ is EL-shellable. Even our shelling map will be taken as before and expanded to a new shelling map.

Lemma 3.3. *The poset $\mathcal{L}_{\mathcal{D}(\Delta^{2d}_d)}$ is EL-shellable.*

Proof. We see that this is a bounded poset and identify $\mathbb{C}^n$ with $D_\emptyset$. We define an edge labeling λ by $\lambda(D_\sigma < D_\tau) = \max \tau - \sigma$, and for any $\sigma \subset \mathbf{2d}$ with cardinality d, we define $\lambda(D_\sigma < D_{\sigma|\sigma^c}) = \lambda(D_{\sigma|\sigma^c} < D_{\mathbf{2d}}) = 2d$. Now since this is almost the same labeling as in lemma 3.2, for interval which does not contain elements of the form $D_{\sigma|\sigma^c}$ with the same argument as before λ fulfills the EL-requirement. Trivially the EL-requirement is fulfilled for the intervals $[D_\sigma, D_{\sigma|\sigma^c}]$ and $[D_{\sigma|\sigma^c}, D_{\mathbf{2d}}]$, for all $\sigma \subset \mathbf{2d}$ with d elements, since they both only contain on edge.

Now take the interval $[D_\emptyset, D_{\sigma|\sigma^c}]$ for some $\sigma \subset \mathbf{2d}$ with d elements. The Hasse-diagram of it will look like this



(3.1)

with the edge labeling written on. We clearly see that $2d$ is an element in either σ or σ^c. So assume without loss of generality that $2d \notin \sigma$, then $\max \sigma < 2d$ and hence the left chain in the diagram above will be rising and lexicographic first. Since pr assumption $2d \in \sigma^c$ we can see that the left chain has the label $(2d, 2d)$ and is therefore not rising, and hence the EL-criteria is fulfilled in this case.

Now take $D_\tau \in \mathcal{L}_{\mathcal{D}(\Delta^{2d}_d)}$, such that there is some $\sigma \subset \mathbf{2d}$ with d elements such that $D_{\sigma|\sigma^c} \in [D_\tau, D_{\mathbf{2d}}]$, then either $D_\tau = D_\emptyset$ or $\tau = \sigma$ or $\tau = \sigma^c$. So lets cover the case were $\tau = \sigma$ or $\tau = \sigma^c$. Now we can assume without loss of generality that $\tau = \sigma$, and take m to be the maximal chain which goes $(D_\sigma, D_{\sigma|\sigma^c}, D_{\mathbb{2d}})$, then the label of m is $\lambda(m) = (2d, 2d)$, and hence m is clearly not a rising chain. Now we know since λ is almost the same as in lemma 3.2 that there is a chain m' which is rising, doesn't go through $D_{\sigma|\sigma^c}$ and is the lexicographic first among the maximal chains which don't. Now assume that D_δ is the second element of the chain m', now if $\max(\delta-\sigma) = 2d$, then since the label of m' is rising then $\delta = \mathbf{2d}$, but this can't be since $d > 1$, and since δ is the second element it must contain $d + 1$ elements. Hence m' will be lexicographic before m.

Now take the interval $[D_\emptyset, D_{\mathbf{2d}}]$. All maximal chains through $D_{\sigma|\sigma^c}$ will have the label $(i, 2d, 2d)$ for some $i = d, d+1, \ldots, 2d$ and are therefore not rising. Again we can by the same arguments as in the coordinate lattice case state that there is a unique rising maximal chain, m' which doesn't go through $D_{\sigma|\sigma^c}$ which is lexicographic before all other maximal chains not going through $D_{\sigma|\sigma^c}$. So let m be a maximal chain through $D_{\sigma|\sigma^c}$, with $\lambda(m) = (i, 2d, 2d)$ and let $\lambda(m') = (j_1, j_2, \ldots, j_{d+1})$. Now if $j_1 < i$ we are done, and m' will be before m in the lexicographic order. Now assume $j_1 > i$, but we saw in lemma 3.2 that j_1 would be d since the only rising maximal chain form $D_\emptyset$ to $D_{\mathbf{2d}}$ started with D_σ where σ consisted of the first d elements, and since d is the lowest label on any edge from $D_\emptyset$ we have a contradiction. So assume that $i = j_1$. Now the chain m' consists of $d + 2$ elements, so if the $j_2 = 2d$ then since m' is rising there will be no j_k, $k > 2$, but that would mean that the third element of m' was the last, now since $d > 1$ this cannot be true, and hence $j_2 < 2d$, and therefore m' is before m in the lexicographic order.

We have now covered all possible intervals and can therefore conclude that λ is an EL-shelling □

And we are now ready to prove our proposition. We will start by arguing in the various trivial cases first, and then see that we can use the two lemmas and the results we know about shellability to come to the conclusion.

Proof of proposition 3.1. Now take any field k. I will use theorem 1.10 (2) for this, and will therefore need to verify, that for all $C \in \mathcal{L}_{\mathcal{C}(\Delta_d^{2d})}$ that

$$\tilde{H}_*((\mathbb{C}^n, C)_{\mathcal{L}_{\mathcal{C}(\Delta_d^{2d})}}; k)(z) = \sum_{D \in \epsilon^{-1}(H)} (-z)^{c(D)-1} \tilde{H}_*((\mathbb{C}^n, D)_{\mathcal{L}_{\mathcal{D}(\Delta_d^{2d})}}; k)(z)$$

Now assume that $C \neq C_{\mathbf{2d}}$, trivially there is some $\sigma \subset \mathbf{2d}$, such that $C = C_\sigma$, but then since $\sigma \neq \mathbf{2d}$ we see that $\epsilon^{-1}(C_\sigma) = \{D_\sigma\}$, since that trivially $\epsilon(D_\sigma) = C_\sigma$ and assume there was an element of the form $D_{\tau_1|\ldots|\tau_n}$ such that $\epsilon(D_{\tau_1|\ldots|\tau_n}) = C_\sigma$, but this would imply that $\bigcup_{i=1}^n \tau_i = \sigma$, but all the τ_i are disjoint and contain at least d elements, and hence $|\sigma| \geq n \cdot d$, which cannot be if $n \neq 1$. Furthermore we see that $(\mathbb{C}^n, C)_{\mathcal{L}_{\mathcal{C}(\Delta_d^{2d})}}$ is the same poset up to renaming as $(\mathbb{C}^n, \epsilon^{-1}(C))_{\mathcal{L}_{\mathcal{D}(\Delta_d^{2d})}}$, and hence the requirement is trivial.

So now take $C_{\mathbf{2d}} \in \mathcal{L}_{\mathcal{C}(\Delta_d^{2d})}$, we see that $\epsilon^{-1}(C_{\mathbf{2d}}) = \{D_{\mathbf{2d}}\} \cup \{D_{\sigma|\sigma^c} | \sigma \subset \mathbf{2d},\ |\sigma| = d\}$. Now we know from lemma 3.2 and theorem 2.8 that the order complex of $(C_\emptyset, C_{\mathbf{2d}})_{\mathcal{L}_{\mathcal{C}(\Delta_d^{2d})}}$ is shellable. Likewise we know from lemma 3.3 and theorem 2.8 that the order complex of $(D_\emptyset, D_{\mathbf{2d}})_{\mathcal{L}_{\mathcal{D}(\Delta_d^{2d})}}$ is shellable. Now from (3.1) we can see that the chain complex of $(D_\emptyset, D_{\sigma|\sigma^c})_{\mathcal{L}_{\mathcal{D}(\Delta_d^{2d})}}$, for $\sigma \subset \mathbf{2d}$ with d elements, is two disjoint points and hence trivially shellable, since any ordering of the points fulfills definition 2.2. Now this means that we can use corollary 2.5 to restate the requirement of theorem 1.10 (2) in the following way: For all $j \in \mathbb{N}$

$$h_{j,j}^{C_{\mathbf{2d}}} = \sum_{D \in \epsilon^{-1}(C_{\mathbf{2d}})} (-1)^{c(D)-1} h_{j-c(D)+1,j-c(D)+1}^{D} \tag{3.2}$$

where $h_{i,j}^D$ is $h_{i,j}$ for the order complex of $(D_\emptyset, D)_{\mathcal{L}_{\mathcal{D}(\Delta_d^{2d})}}$ and likewise $h_{j,j}^{C_{\mathbf{2d}}}$ is $h_{j,j}$ integer for the order complex of $(C_\emptyset, C_{\mathbf{2d}})_{\mathcal{L}_{\mathcal{C}(\Delta_d^{2d})}}$. Now with same naming convention for the f integers we can use definition 2.3 (ii) to rewrite this with the left hand side of (3.2) being

$$\sum_{k=0}^{j} (-1)^{j-k} \binom{j-k}{j-k} f_{j,k}^{C_{\mathbf{2d}}} = \sum_{k=0}^{j} (-1)^{j-k} f_{j,k}^{C_{\mathbf{2d}}} \tag{3.3}$$

and the right hand side of (3.2) being

$$\begin{aligned}
&\sum_{D \in \epsilon^{-1}(C_{\mathbf{2d}})} (-1)^{c(D)-1} \sum_{k=0}^{j-c(D)+1} (-1)^{j-c(D)+1-k} \binom{j-c(D)+1-k}{j-c(D)+1-k} f_{j-c(D)+1,k}^{D} \\
&= \sum_{D \in \epsilon^{-1}(C_{\mathbf{2d}})} (-1)^{c(D)-1} \sum_{k=0}^{j-c(D)+1} (-1)^{j-c(D)+1-k} f_{j-c(D)+1,k}^{D} \\
&= \sum_{D \in \epsilon^{-1}(C_{\mathbf{2d}})} \sum_{k=0}^{j-c(D)+1} (-1)^{j-k} f_{j-c(D)+1,k}^{D}
\end{aligned} \tag{3.4}$$

Now fix $j \neq 2$ and lets study $f_{j-c(D_{\mathbf{2d}})+1,k}^{D_{\mathbf{2d}}} = f_{j,k}^{D_{\mathbf{2d}}}$, now all maximal chains in the interval $(D_\emptyset, D_{\mathbf{2d}})_{\mathcal{L}_{\mathcal{D}(\Delta_d^{2d})}}$ which goes through elements on the form $D_{\sigma|\sigma^c}$ for σ having d elements, are of length 2, and hence since the rest of poset is like that of $(C_\emptyset, C_{\mathbf{2d}})_{\mathcal{L}_{\mathcal{C}(\Delta_d^{2d})}}$, the only difference between $f_{j,k}^{D_{\mathbf{2d}}}$ and $f_{j,k}^{C_{\mathbf{2d}}}$ is in the case of $j = 2$. Now $f_{2,k}^{D_{\mathbf{2d}}}$ counts the number of faces which are at most a face of a one-simplex. Now assume that we have such an element, then it is a chain in the interval $(D_\emptyset, D_{\mathbf{2d}})_{\mathcal{L}_{\mathcal{D}(\Delta_d^{2d})}}$. If it does not go through $D_{\sigma|\sigma^c}$, then there will be a mirror chain in $(C_\emptyset, C_{\mathbf{2d}})_{\mathcal{L}_{\mathcal{C}(\Delta_d^{2d})}}$, so $f_{2,k}^{D_{\mathbf{2d}}}$ is equal to $f_{2,k}^{C_{\mathbf{2d}}}$ plus the number of elements of cardinality k containing $D_{\sigma|\sigma^c}$ which are not also in a facet not containing $D_{\sigma|\sigma^c}$. Of these there are only the following faces of cardinality one: $\{D_{\sigma|\sigma^c}\}$, with $\sigma \subset \mathbf{2d}$ $|\sigma| = d$, and of these there are $\frac{\binom{2d}{2d}}{2}$, i.e. one for every other subset of cardinality d of $\mathbf{2d}$. Of cardinality two there are only the faces $\{D_\sigma, D_{\sigma|\sigma^c}\}$ of which there are $\binom{2d}{d}$, one for each subset of cardinality d of $\mathbf{2d}$. So all in all we see that

$$f_{j,k}^{D_{\mathbf{2d}}} = \begin{cases} f_{j,k}^{C_{\mathbf{2d}}} & \text{when } j \neq 2 \vee k \neq 1,2 \\ f_{2,1}^{C_{\mathbf{2d}}} + \frac{\binom{2d}{2}}{2} & \text{when } j = 2 \wedge k = 1 \\ f_{2,2}^{C_{\mathbf{2d}}} + \binom{2d}{2} & \text{when } j = 2 \wedge k = 2 \end{cases} \tag{3.5}$$

So let us now turn our attention to $f_{j-c(D_{\mathbf{2d}})+1,k}^{D_{\sigma|\sigma^c}} = f_{j-1,k}^{D_{\sigma|\sigma^c}}$. Now this complex consists of two disjoint points, and therefore rather trivial we see that

$$f_{j-1,k}^{D_{\sigma|\sigma^c}} = \begin{cases} 1 & \text{when } j = 2 \wedge k = 0 \\ 2 & \text{when } j = 2 \wedge k = 1 \\ 0 & \text{else} \end{cases} \tag{3.6}$$

Now since there are $\frac{\binom{2d}{d}}{2}$ elements on the form $D_{\mathbf{2d}}$, we get the right side of the equality (3.2) by way of the rewrite of (3.4) as for $j \neq 2$

$$\sum_{k=0}^{j} (-1)^{j-k} f_{j,k}^{C_{\mathbf{2d}}} + 0 \tag{3.7}$$

i.e. the left hand side of (3.2) by way of (3.3), now in the case of $j = 2$ we get

$$\sum_{D=D_{\mathbf{2d}} \vee D=D_{\sigma|\sigma^c}} \sum_{k=0}^{2-c(D)+1} (-1)^{j-k} f_{j-c(D)+1,k}^{D} \tag{3.8}$$

$$= \sum_{k=0}^{2-c(D_{\mathbf{2d}})+1} (-1)^{j-k} f_{j-c(D_{\mathbf{2d}})+1,k}^{D_{\mathbf{2d}}} + \frac{\binom{2d}{d}}{2} \sum_{k=0}^{2-c(D_{\sigma|\sigma^c})+1} (-1)^{j-k} f_{j-c(D_{\sigma|\sigma^c})+1,k}^{D_{\sigma|\sigma^c}}$$

$$= \sum_{k=0}^{2} (-1)^{j-k} f_{j,k}^{D_{\mathbf{2d}}} + \frac{\binom{2d}{d}}{2} \sum_{k=0}^{1} (-1)^{j-k} f_{j-1,k}^{D_{\sigma|\sigma^c}}$$

$$= \sum_{k=0}^{2} (-1)^{j-k} f_{j,k}^{C_{\mathbf{2d}}} - \frac{\binom{2d}{d}}{2} + \binom{2d}{2} + \frac{\binom{2d}{d}}{2}(1-2)$$

$$= \sum_{k=0}^{2} (-1)^{2-k} f_{j,k}^{C_{\mathbf{2d}}} + 0 \tag{3.9}$$

i.e. exactly the left hand side of (3.2) by way of (3.3).

We have now shown that the complex Δ_d^{2d} fulfills (2) in theorem 1.10, and hence it is a Golod complex over k. □

3.1 Further problems

I will end this project with some further problems which have occurred to me.

Several other complexes which have studied to some degree seems to have intervals of their coordinate- and diagonal intersection lattices which are homotopic equivalent to a wedge of spheres, and hence easy to calculate the homology of. Is there some characterization of a complex with these types of coordinate- and diagonal intersection lattices?

When working with simplicial complexes with 4 vertices, all of them faces of the complex; I found no immediate non-Golod complex. The examples I did both the coordinate- and diagonal intersection lattices were wedges of spheres, with the diagonal intersection lattice having one extra sphere for each element of the form $D_{\sigma|\sigma^c}$, which, due to the nature of $(D_\emptyset, D_{\sigma|\sigma^c})$ being the zero sphere, made the polynomials in theorem 1.10 (2) match. So one could wonder if there is no non-Golod complex on 4 vertices, or if not what the minimum number of vertices required are before non-Golod complexes start showing. We know due to corollary 1.11 that complexes on 1, 2 and 3 vertices were all the vertices are faces of the complex always are Golod, since if they have non-faces they are of rather large cardinality.

The edge labeling we used in lemma 3.2 worked in more cases that the one we used in lemma 3.3, so one could wonder if you could expand the EL-labeling from 3.3 to at least Δ_i^n, and thusly maybe be able to prove that Δ_i^n is a Golod complex, which we know is true from [GrTh], using theorem 1.10, and hence expanding our main result.

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