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Duality theorems for simplicial complexes

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Abstract

This thesis deals with the homology and cohomology groups of simplicial complexes, and especially with the two duality theorems which willdemonstrate links between the two notions. For this we will need some results, and we will therefore prove the long exact sequence for the reduced relative homology group; and to link the homology and cohomolgy to our geometric notion of a simplicial complex we will see that these are connected to the Euler characteristic of the complex. Then by rewriting the relative homology group we will be able to prove the Alexander duality which is the major theorem of this thesis. We will use the result that if realizations of two different simplicial complexes are homotopic then their homology and cohomology groups are isomorphic in order to prove another duality, by showing that there is a different complex constructed by the nonfaces of our complex which is homotopic to the Alexander dual of the complex. For this we shall need some homotopy theorems which will be duly proven.

Resumé

Dette projekt beskæftiger sig med homologi- og kohomologigrupperne af simplicielle komplekser, og specielt med to dualitets sætninger som vil vise forbindelsen mellem disse to strukturer. For at gøre dette vil vi bevise den lange eksakte sekvens for den reducerede relative homologigruppe, og for at forbinde homologi med kohomologi til vores geometriske intuition for simplicielle komplekser vil vi vise, at begge begreber er forbundet til Euler-karakteristikken af et kompleks. Vi vil så ved at omskrive på den relative homologi gruppe være i stand til at vise Alexander dualiteten, som er hovedsætningen i dette projekt. Vi vil benytte resultatet, at hvis realisationer af forskellige simplicielle komplekser er homotope da implicerer det, at deres homologi- og kohomologigrupper er isomorfe, til at vise en anden dualitet, ved at vise at der er et kompleks konstrueret af ikke simplekser fra vores kompleks som er homotopt til det Alexander duale. Til dette skal vi benytte nogle homotopi sætninger, som vi så vil bevise.

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Chapter 1

Preliminaries

1.1 Preface

This bachelor thesis was was started with the intent of giving a solely combinatorial proof of the Alexander duality. My thesis advisor did not know of such a proof but thought it possible for me to construct one. I therefore spent the first six weeks reading up on simplicial homology theory and trying to wrap my head around the nature of the combinatorical proofs, together with the more topological proofs of the Alexander duality that we knew to exist. It had been an agreement, that if at any point I felt that I would not in time be able to produce the proof I would instead write my thesis on another subject where I would be able to use the knowledge gained in my attempt at proving the duality. The day before a meeting with my advisor I decided that if I did not see more progress I would throw in the towel, and with that thought I stumpled unto [3], an article wherein Björner and Tancer gave a short and rather elegant proof for the duality. Björners proof used several of the ideas I myself halfbaked had tried. I then spent the next couple of weeks on writing both the proof and the definitions and results necessary for the proof to work. This is what appears as chapters one till three. Hereafter followed the search for what would make up the rest of the thesis. First my advisor and I tried to find a connection between the vertex coloring of a simplicial complex and its dual, but we found none, and after a week or so we gave up. But the minor detour ended up in my reading a wide array of articles on vertex colorings. In the end I found an article, [2], with the apt name "Note on a Combinatorial Application of Alexander Duality". The main result of [2] was related to the μ-operator on posets, but it also had a theorem which in this thesis is called, rather unimaginatively, the second duality. Again I needed some more results and these appear as chapters four and five of my thesis.

I have tried to include the results necerssary for my main theorems, but as always one has to stop somewhere. I have therefore chosen to consider as well known results and defintions of courses I have taken. These mostly include algebra and topology, but there is also at least one result from linear algebra. Likewise I have excluded the proof of the topological invariance of both homology and cohomology groups and instead I have only sketched. This I was advised to do, since they are rather long and technical and are so well known that it should be unthinkable that I should not see it proven. Lastly I will not for any specific space prove that it is contractible, since this tend to be a tad technical and is not very combinatorial which is the focus of the thesis. Instead I will rely on some topological intuition.

1.2 Acknowledgements

Most of the theorems, definitions and proofs presented throughout the thesis are at least inspired from other sources, and I'm most grateful to the authors of these. For most results I will mention their source, so that a reader can give credit where credit is due. Therefore at the beginning of each section I will state my major inspirational source and also note it if some result is from other sources.

1.3 Simplicial Complexes

This section is mostly written with help from Jesper Michael Møllers hand-written notes, but I have also used [4] and [6]. Lemma 1.8 is completly of my own design.

1.3.1 Definition

We wish to formally define the simplicial complexes to which most of the rest of the thesis will be devoted. We want to somehow give combinatorial sets somehow equivalent to our notion of structures composed of generalized triangles. It is quite clear that in two dimensions such a structure is a triangle in the normal sense, composed of the convex hull of three points in the plane. So we wish to somehow identify the triangle with the set consisting of the three vertices. Now it is also intuitively relatively clear that a line segment is a triangle-like structure in one dimension and that the triangle in the plane somehow consist of three line segments i.e. three "triangles" of dimension one lower. These lines egments seems to be the convex hull of two points in the plane, so we could identify them with the set of these two points. Now it seems clear that whenever I have one of these sets which I think of as triangle structures all subsets seem to be the triangle structures of which the original structure is composed of. This line of thoughts leads us to the following definition

Definition 1.1. *Given a finite ordered set V, we will say that K is a simplicial complex on the vertex set V, if K is a subset of the powerset of V, and that if whenever $\sigma \in K$, and $\tau \subset \sigma$ then $\tau \in K$. We will call elements of K simplices or faces.*

This raises some issues. Firstly we notice that we require V to be ordered. This doesn't seem to stem from our geometric intuition, but it will prove itself quite useful later, when we will be able to talk about the i'the vertex of a simplex. Secondaryly we see that if a simplicial complex K is non-empty then $\emptyset \in K$, and we will recognize the difference between $\emptyset$ as a simplicial complex and $\{\emptyset\}$ as a simplicial complex. We will be needing some notation for the rest of the thesis and hence:

Definition 1.2.

- *The set N_+ for $N \in \mathbb{N}_0$ is the set $\{0, 1, \ldots, N\}$ ordered by the "$<$"-relation.*
- *For a finite set V, then $D[V]$ is the power set of V viewed as a simplicial complex.*
- *For a simplicial complex K on the vertex set V, with a simplex σ, we write $\sigma = \{v_0, v_1, \ldots, v_n\}$ in an unique way due to the ordering, so that $v_0 < v_1 < \ldots < v_n$.*
- *Let B be a partially ordered by $\prec$, then if B is finite we will define the order complex $\Delta(B)$ to be the simplicial complex where if*

$$b_0 \prec b_1 \prec \ldots \prec b_n \tag{1.1}$$

  is a chain in B then $\{b_0, b_1, \ldots, b_n\} \in \Delta(B)$. Its vertex set will be B ordered in such a way that it preserves the chains of B.
- *If K is a simplicial complex then K_n, is the set of all faces σ of K, such that $|\sigma| - 1 = n$.*
- *If K is a simplicial complex then a subcomplex is a simplicial complex L such that $L \subset K$.*

Definition 1.3. *For a simplicial complex K, then we define the reduced Euler characteristic as*

$$\tilde{\chi}(K) = \sum_i (-1)^i |K_i|. \tag{1.2}$$

Clearly this is a finite sum since K was finite, and it is trivially linked to the classic notion of *Vertics - Edges + Faces.*

We will also need some special maps between simplicial complexes:

Definition 1.4. *Let K, L be simplicial complexes on the vertex sets V_K, V_L. Then a function $f : V_K \to V_L$ is said to extend to a simplicial map $g : K \to L$ if $\forall \sigma \in K$, with $\sigma = \{v_0, \dots, v_n\}$ then $\{f(v_0), \dots, f(v_n)\} \in L$; and then we define $g(\sigma) = \{f(v_0), \dots, f(v_n)\}$. We say that two simplicial complexes are simplicial-isomorphic if g is bijective.*

It is clear that the extended simplicial map g is bijective if and only if f is.

1.3.2 Duality

We will here introduce the notion of the Alexander dual of a simplicial complex. This is defined as all the complements in the vertex set to non-faces of the complex it self. This seems a rather odd definition, and indeed it is, since it doesn't seem to be linked to any geometric intuition. But it will still turn out most useful, since it is in the statement of the Alexander duality.

Definition 1.5. *For a simplicial complex K, on the vertex set V we define the Alexander dual of K, $K^\vee$, to be*

$$K^\vee = \{V - \sigma | V \supset \sigma \notin K\} \tag{1.3}$$

Now we would wish that $K^\vee$ would be a simplicial complex as well, and indeed this holds:

Lemma 1.6. *If K is a simplicial complex on the vertex set V, then $K^\vee$ is a simplicial complex on the vertex set V*

Proof. It is trivial to see that all elements of $K^\vee$ are subset of V and hence elements of the powerset and therefore if $K^\vee$ is a simplicial complex, it can be seen as having V as a vertex set. Suppose there is $\tau \subset \sigma \in K^\vee$, now we know there is $V \supset \gamma \notin K$, such that $\sigma = V - \gamma$, then $\gamma \cup (\sigma - \tau)$ is not in K, since γ is not, and K is stable under subsets, and hence $V - (\gamma \cup (\sigma - \tau)) \in K^\vee$, but $V - (\gamma \cup (\sigma - \tau)) = ((V - \gamma) - \sigma) \cup \tau = \tau$, and hence $K^\vee$ is stable under intersection and therefore is a simplicial complex. □

We would also very much prefer if our newfound notion was self dual, meaning that $(K^\vee)^\vee = K$, and again this is a true statement, but to prove it, we will need a bit of help:

Lemma 1.7. *Let K be a simplicial complex on the vertex set V, then*

$$K^\vee = D[V] - \{V - \sigma | \sigma \in K\}. \tag{1.4}$$

Proof. We will need to prove the two inclusions, and we therefore start by taking an arbitrary $\tau \in D[V] - \{V - \sigma | \sigma \in K\}$, then we know that $V - \tau \notin K$, since if it was, then τ would not be in the set, and hence $V - (V - \tau) \in K^\vee$ and hence $\tau \in K^\vee$. For the other inclusion we take arbitrary $\tau \in K^\vee$, then there is $V \supset \sigma \notin K$, such that $\tau = V - \sigma$, but trivially $\sigma \in D[V]$, and there is no $\gamma \in K$, such that $V - \gamma = \tau$, and hence $\tau \in D[V] - \{V - \sigma | \sigma \in K\}$, and hence we are done. □

Lemma 1.8. *Let K be a simplcial complex on a vertex set V, then*

$$(K^\vee)^\vee = K. \tag{1.5}$$

Proof. We can use the lemma above to write

$$(K^\vee)^\vee = D[V] - \{V - \sigma | \sigma \in K^\vee\}. \tag{1.6}$$

Take $\sigma \notin K$, then $V - \sigma \in K^\vee$, and hence $V - (V - \sigma) \notin (K^\vee)^\vee$, and therefore $(K^\vee)^\vee \subset K$. Now take $\sigma \notin (K^\vee)^\vee$, then there is $\tau \in K^\vee$ such that $\sigma = V - \tau$, but since $\tau \in K^\vee$ there is $\gamma \notin K$ such that $V - \gamma = \tau$, but $\sigma = (V - (V - \gamma)) = \gamma \notin K$, and hence we are done. □

Chapter 2

Homology and chain complexes

2.1 Chain complex

This section is written with inspiration from Jesper Michael Møllers hand-written notes, and from [4] and [6]. Theorem 2.2 is of my own design.

Let G_n for $n \in \mathbb{Z}$ be groups and $f_n : G_n \to G_{n-1}$ be homomorphisms, then

$$(G, f) := \ldots \overset{f_{n+1}}{\to} G_n \overset{f_n}{\to} G_{n-1} \overset{f_{n-1}}{\to} \ldots \tag{2.1}$$

is a chain complex if $\mathrm{Im} f_{n+1} \subset \ker f_n$, for all n. We say that (G, f) is exact if $\mathrm{Im} f_{n+1} = \ker f_n$.

Definition 2.1. *The homology of a chain complex (G, f) is the sequence of groups $H_n(G) := \ker f_n / Im\ f_{n+1}$*

It is clear that the homology group somehow measures how close a chain complex is to being exact. It is also clear that given two chain complexes (G, f) and (G', f'), and homomorphisms $\phi_n : G \to G'$ such that $\phi_{n-1} \circ f_n = f'_n \circ \phi_n$ then there is $\phi_n^H : H_n(G) \to H_n(G')$ given by for a $z \in \ker f_n$ then $\phi_n^H([z]) = [\phi(z)]$. This is clearly well defined since

$$f'_n(\phi_n(z)) = \phi_{n-1}(f_n(z)) = \phi_{n-1}(0) = 0 \tag{2.2}$$

it is clear that if all ϕ are isomorphisms then $H(G) \cong H(G')$ since ϕ_n can be restricted to an isomorphism from $\ker f_n \to \ker f'_n$ and likewise for the images.

We will be using the following result a lot in the rest of the thesis since it greatly eases the definition of homomorphisms

Theorem 2.2. *If G is a free group with the set of generators $g = \{g_0, \ldots, g_n\}$, H an abelian group, then every homomorphism from G to H is uniquely determined by the values it takes on g*

Proof. Let $f: G \to H$ be a homomorphism, then take an arbitrary element in G, it can be written on the form $a_0g_0 + \ldots + a_ng_n$, for $a_0, \ldots, a_n \in \mathbb{Z}$, then $f(a_0g_0 + \ldots + a_ng_n) = a_0f(g_0) + \ldots + a_nf(g_n)$. $\square$

This means that whenever we are defining homomorphisms between free groups it is enough for us to define how they act on the set of generators.

We will wish to define the dual of a chain complex since this will be needed twice below, in the definition of the rank of a finitely generated abelian group and in the definition of cohomology.

Definition 2.3. *Let G_n be a group and let H be a group, then $Hom(G_n, H)$ is the set of homomorphies from G_n to H. This is a group with the usual function addition.*

Let (G, f) be a chain complex, we can now define functions

$$\tilde{f}_n : \text{Hom}(G_{n-1}, H) \to \text{Hom}(G_n, H) \tag{2.3}$$

given as follows: For $\psi \in \text{Hom}(G_{n-1}, H)$ and $g \in G_n$ then $\tilde{f}_n(\psi)(g) = \psi(f_n(g))$. It is clear from this that

$$(\tilde{f}_{n+1} \circ \tilde{f}_n)(\psi)(g) = \tilde{f}_{n+1}(\psi(f_n(g))) = \psi(f_n(f_{n+1}(g))) = \psi(0) = 0 \tag{2.4}$$

so we see that

$$\ldots \overset{\tilde{f}_n}{\to} \text{Hom}(G_n, H) \overset{\tilde{f}_{n+1}}{\to} \text{Hom}(G_{n+1}, H) \overset{\tilde{f}_{n+2}}{\to} \ldots \tag{2.5}$$

is a chain complex.

2.2 Homology

This section is written with help from [4], where the definition of homology is stated. The proof of theorem 2.9 is inspired by a proof strategy found in section 3.4 of [5], but is proved by me. Likewise lemma 2.6 is only stated in [4], and the proof is adapted from a similar one from [6] with some added inputs from Sune Kristen Jakobsen and espicially Kristian Knudsen Olesen. The fact that the homology group is stable under different orderings of the vertex set, is my own proof.

2.2.1 Preliminaries

Let K be a simplicial complex. Then for all $n \in \mathbb{Z}$ we can construct the abelian group $\mathbb{Z}K_n$, given in the natural way

$$\mathbb{Z}K_n := \{p_0\sigma_0 + \ldots + p_m\sigma_m | p_0, \ldots, p_m \in \mathbb{Z},\ \sigma_0, \ldots, \sigma_m \in K_n\} \tag{2.6}$$

with the natural addition

$$(p_0\sigma_0 + \ldots + p_m\sigma_m) + (p'_0\sigma_0 + \ldots + p'_m\sigma_m) = (p_0 + p'_0)\sigma_0 + \ldots + (p_m + p'_m)\sigma_m \quad (2.7)$$

We obviously identify $\mathbb{Z}\emptyset$ with the trivial group, which we call 0, and $\mathbb{Z}\{\emptyset\}$ with $\mathbb{Z}$.

We now wish to define functions $\partial_n : \mathbb{Z}K_n \to \mathbb{Z}K_{n-1}$, such that we get a chain complex from our simplicial complex. Since simplicial complexes are stable under subsets, it is clear that if we take a basis element $\{v_0, \ldots, v_n\} \in K_n$ then if we remove a single vertex from the simlpex we get a member of K_{n-1}. So we define $\partial_n : \mathbb{Z}K_n \to \mathbb{Z}K_{n-1}$ by for $\{v_0, v_1, \ldots, v_n\} = \sigma \in K_n$

$$\partial_n(\sigma) = \sum_{v_i \in \sigma} (-1)^i(\sigma - \{v_i\}) \quad (2.8)$$

By the argument above this is well defined, and further more if $K_n = \emptyset$, or $K_n = \{\emptyset\}$ then this clearly still works. We now wish to make sure that $(\mathbb{Z}K, \partial)$ is a chain complex, so we need to check that for arbitrary $\sigma \in K_n$, given as above, then $(\partial_{n-1} \circ \partial_n)(\sigma) = 0$. So

$$\begin{aligned}(\partial_{n-1} \circ \partial_n)(\sigma) &= \partial_{n-1}(\sum_{v_i \in \sigma} (-1)^i(\sigma - \{v_i\})) && (2.9)\\ &= \sum_{v_i \in \sigma} \sum_{v_j \in \sigma, j<i} (-1)^i(-1)^j(\sigma - \{v_i\} - \{v_j\}) && (2.10)\\ &\quad + \sum_{v_i \in \sigma} \sum_{v_j \in \sigma, j>i} (-1)^i(-1)^{j-1}(\sigma - \{v_i\} - \{v_j\}) && (2.11)\\ &= 0 && (2.12)\end{aligned}$$

So from this it follows that

$$\ldots \stackrel{\partial_{n+1}}{\to} \mathbb{Z}K_n \stackrel{\partial_n}{\to} \mathbb{Z}K_{n-1} \stackrel{\partial_{n-1}}{\to} \ldots \quad (2.13)$$

is a chain complex, which leads us to the main definition of this section:

2.2.2 The reduced homology

Definition 2.4. *Let K be a simplicial complex and let ∂ be given as above then we call $H_n(\mathbb{Z}K)$ the n'th reduced homology group of K, written as $\tilde{H}_n(K)$ i.e. $\tilde{H}_n(K) = \ker \partial_n / Im\partial_{n-1}$*

Take a simplicial complex K on the vertex set V, and let V' be V with a different ordering, now we would very much like that homology didn't depend on the ordering of V. Now it is clear that a new ordering of a set can be seen as a permutation of the set, and all permutations can be composed of neighbour-switches. So we need to show that if K is a simplicial complex and we alter the ordering on two elements so they now come in the other order $\tilde{H}_n(K)$ stays the same. Take an element $\sigma = \{v_0, \ldots, v_i, v_{i+1}, \ldots, v_n\} \in K_n$, then

$$\partial_n(\sigma) = \sum_{v_j \in \sigma} (-1)^j \sigma - \{v_j\} \tag{2.14}$$

and let $\sigma' = \{v_0, \ldots, v_{i+1}, v_i, \ldots, v_n\}$, then

$$\partial_n(\sigma') = \sum_{v_j \in \sigma'} (-1)^j \sigma' - \{v_j\} \tag{2.15}$$

Now it is easy to see that all generator elements of $\partial_n(\sigma)$ without both of the switched vertices v_i, v_{i+1}, have the opposite sign in $\partial_n(\sigma')$. And generator elements with both switched elements keep its sign. Let $\phi_n : \mathbb{Z}K_n \to \mathbb{Z}K_n$, where the second K_n is ordered with the switched vertices, be given for $\sigma \in K_n$ by

$$\phi_n(\sigma) = \begin{cases} \sigma & \text{if } \sigma \text{ contains both the switched vertices} \\ -\sigma & \text{else} \end{cases} \tag{2.16}$$

Now it is clear that this commutes for all n

$$\begin{array}{cccc} \ldots \stackrel{\partial_{n+1}}{\to} & \mathbb{Z}K_n \stackrel{\partial_n}{\to} & \mathbb{Z}K_{n-1} \stackrel{\partial_{n-1}}{\to} & \ldots \\ & \phi_n \downarrow & \phi_{n-1} \downarrow & \\ \ldots \stackrel{\partial_{n+1}}{\to} & \mathbb{Z}K_n \stackrel{\partial_n}{\to} & \mathbb{Z}K_{n-1} \stackrel{\partial_{n-1}}{\to} & \ldots \end{array}$$

and hence $\tilde{H}_n(K)$ is independent of the ordering of K. It is also clear that if K, L are simplical-isomorphic then $\tilde{H}_n(K) = \tilde{H}_n(L)$.

We will now give a result to demonstrate the usefulness of the homology definition. We want to link the reduced homology groups to the more geometric notion of reduced Euler characteristic. But for this we will need a short result, which will come in handy in the section on cohomology as well. This result and the following will require the notion of rank of a group generated by finitely many abelian groups, so

Definition 2.5. *Let A be a finitely generated abelian group, then the rank of A is defined to be*

$$\mathit{rank} \dim_{\mathbb{Q}} \mathit{Hom}(A, \mathbb{Q}) \tag{2.17}$$

It is clear that this is well defined since $\text{Hom}(A, \mathbb{Q})$, can be seen as a vector space over the rationals, with the normal function addition and scalar multiplication. The helpful lemma is then

Lemma 2.6. *If* $0 \to A \overset{f_A}{\to} B \overset{f_B}{\to} C \to 0$ *is exact, and* A, B, C *are finitely generated abelian groups then* $rankB = rankA + rankC$

Proof. So let $0 \to A \overset{f_A}{\to} B \overset{f_B}{\to} C \to 0$ be exact, then it is clear that f_A is injective since its kernel is trivial. We now wish to show that

$$0 \to \mathrm{Hom}(C, \mathbb{Q}) \overset{\tilde{f}_B}{\to} \mathrm{Hom}(B, \mathbb{Q}) \overset{\tilde{f}_A}{\to} \mathrm{Hom}(A, \mathbb{Q}) \to 0 \tag{2.18}$$

is exact. Take arbitary $\psi \in \mathrm{Hom}(C, \mathbb{Q})$ such that $\tilde{f}_B(\psi) = 0$, meaning that $\forall b \in B$, we have $\psi(f_B(b)) = 0$, but since f_B is surjective it maps to all elements of C, and hence $\psi = 0$, so $\tilde{f}_B$ is injective. Now take $\psi \in \mathrm{Hom}(B, \mathbb{Q})$ such that $\tilde{f}_A(\psi) = 0$, meaning that $\psi \circ f_A = 0$ hence we can induce a homomorphism $\psi' : B/f_A(A) \to \mathbb{Q}$, we can likewise induce an isomorphism from f_B $f'_B : B/f_A(A) \to C$, since the original chain was exact, now let $\phi : C \to \mathbb{Q}$, be given by $\phi = \psi' \circ (f'_B)^{-1}$, then we clearly see that $\tilde{f}_B(\phi) = \psi$. Now take an arbitary $\psi \in \mathrm{Hom}(A, \mathbb{Q})$, we define a set X, where elements of X, are on the form (H, ϕ), where $f_A(A) \subset H \subset B$, with H a group and ϕ a homomorphism from H to $\mathbb{Q}$.It is now clear that X is non-empty since $(f_A(A), \psi \circ f_A^{-1}) \in X$, since f_A is injective. We partially order X, by writing that

$$(H, \phi) \leq (H', \phi') \Leftrightarrow H \subset H' \text{ and } \phi = \phi'|_H \tag{2.19}$$

Now for an arbitary chain in X

$$(H_1, \phi_1) \leq (H_2, \phi_2) \leq \ldots \tag{2.20}$$

Then clearly $H = \bigcup_{i=1}^{\infty} H_i$ is a subgroup of B and $\phi : H \to \mathbb{Q}$ is a homomorphism given as follows: For $x \in H$, then $\phi(x) = \phi_n(x)$ for $x \in H_n$. And hence $(H, \phi) \in X$, and for all $i \in \mathbb{N}$ then $(H_i, \phi_i) \leq (\hat{H}, \hat{\phi})$. We can now use Zorns lemma to state that there is a maximal element $(\hat{H}, \hat{\phi}) \in X$, and we wish to show that $\hat{H} = B$. Assume for contradiction that there is $b \in B - \hat{H}$, then clearly the set of $k \in \mathbb{Z}$ such that $k \cdot b \in \tilde{H}$ is an ideal in $\mathbb{Z}$, and since $\mathbb{Z}$ is a prime ideal domain there is $k_0 \in \mathbb{Z}$ such that (k_0) is this ideal. Now clearly we can pick $y \in \mathbb{Q}$ such that $\hat{\phi}(k_0 \cdot b) = y \cdot k_0$. Now take the group $K = \left\langle \hat{H}, b \right\rangle$ being the subgroup of B generated by $\hat{H}$ and b. Now define $\beta : K \to \mathbb{Q}$ given as: For $h \in H$ and $n \in \mathbb{Z}$ $\beta(h + n \cdot b) = \hat{\phi}(h) + y \cdot k_0$. Now this is well defined since if we take $n \in \mathbb{Z}$ such that $n \cdot b \in \hat{H}$, but then there is $m \in \mathbb{Z}$ such that $n = mk_0$, then $\beta(nb) = \hat{\phi}(nb) = \hat{\phi}(mk_0 b) = m\hat{\phi}(k_0 b) = myk_0 b = nb$. So β is a homomorphism, and $\beta|_{\hat{H}} = \hat{\phi}$ and hence $(\hat{H}, \hat{\phi}) \leq (K, \beta)$, but this is a contradiction and hence $\hat{H} = K$, and therefore $\hat{\phi} \in \mathrm{Hom}(B, \mathbb{Q})$ such that $\tilde{f}_A(\hat{\phi}) = \psi$. And now we know that

$$0 \to \mathrm{Hom}(C, \mathbb{Q}) \overset{\tilde{f}_B}{\to} \mathrm{Hom}(B, \mathbb{Q}) \overset{\tilde{f}_A}{\to} \mathrm{Hom}(A, \mathbb{Q}) \to 0 \tag{2.21}$$

is exact. It is trivial to see that $\tilde{f}_A$ and $\tilde{f}_A$ can be seen as linear functions and hence we can now use the dimension theorem of linear algebra and state that

$$\begin{aligned}\text{rank}B &= \dim_{\mathbb{Q}} \text{Hom}(B, \mathbb{Q} = \text{rg}\tilde{f}_A + \dim_{\mathbb{Q}} \ker \tilde{f}_A && (2.22)\\ &= \dim_{\mathbb{Q}} \text{Im}\tilde{f}_A + \dim_{\mathbb{Q}} \text{Im}\tilde{f}_B = \dim_{\mathbb{Q}} \text{Hom}(A, \mathbb{Q}) + \dim_{\mathbb{Q}} \text{Hom}(C, \mathbb{Q})\\ &= \text{rank}A + \text{rank}C && (2.23)\end{aligned}$$

And hence we are done. □

Theorem 2.7. *For a simplicial complex K, then*

$$\tilde{\chi}(K) = \sum_i (-1)^i rank\tilde{H}_i(K). \tag{2.24}$$

Proof. We wish to show that with i as the inclusion map

$$0 \to \ker \partial_n \xrightarrow{i} \mathbb{Z}K_n \xrightarrow{\partial_n} \text{Im}\partial_n \to 0 \tag{2.25}$$

is exact. This is clear since i is injective so its kernel is just 0, and the kernel of ∂_n is by definition $\ker \partial_n$, and the kernel of something which maps to the trivial group is the group itself and hence it is exact, so by lemma 2.6 we see that

$$\text{rank}\mathbb{Z}K_n = \text{rank} \ker \partial_n + \text{rank}\, \text{Im}\partial_n \tag{2.26}$$

Likewise we wish to convince ourselves that

$$0 \to \text{Im}\partial_{n+1} \xrightarrow{i} \ker \partial_n \xrightarrow{q} \tilde{H}_n \to 0 \tag{2.27}$$

, with i as the inclusion map, which makes sense since ∂ is a chain map and q as the quotient map, is exact. Again since i is injective its kernel is 0, and the kernel of the quotient map is the thing you mod away, in this case $\text{Im}\partial_{n+1}$, and since q is surjective its image is all of $\tilde{H}_n(K)$, which is exactly what gets mapped to 0, so this is exact as well. Now we want to know the rank of $\mathbb{Z}K_n$, but since any homomorphism between $\mathbb{Z}K_n$ and $\mathbb{Q}$ is going to be uniquely determined by its values on elements of K_n, then we trivially see that the rank is equal to the cardinality. Combining lemma 2.6 with the results above

$$|K_n| = \text{rankIm}\partial_{n+1} + \text{rankIm}\partial_n + \text{rank}\tilde{H}_n(K) \tag{2.28}$$

And this gives that

$$\begin{aligned}\tilde{\chi}(K) &= \sum_i (-1)^i \text{rankIm}\partial_{n+1} + \text{rankIm}\partial_n + \text{rank}\tilde{H}_n(K) && (2.29)\\ &= -\text{rankIm}\partial_{-1} + (-1)^N \text{rankIm}\partial_{N+1} + \sum_{i=-1}^{N} \tilde{H}_i(K) && (2.30)\end{aligned}$$

where $N \in \mathbb{N}_0$ such that $K_j = \emptyset$ for all $j \geq N$. But obviously $\text{rankIm}\partial_{-1} = \text{rankIm}\partial_{N+1} = 0$ and hence we are done. □

Another result about the reduced homology which we will need later in the thesis is that the full simplicial complex (the complex which includes the vertex set as a simplex) has the trivial reduced homology group for all n.

Definition 2.8. *A simplicial complex K is called acyclic if $\tilde{H}_n(K) = 0$, for all $n \in \mathbb{Z}$*

With this terminology we want to prove that

Lemma 2.9. *$D[N_+]$ is acyclic for all $N \in \mathbb{N}_0$*

Proof. We will prove this by induction, and therefore let $N = 0$, then trivially $\tilde{H}_n(D[0_+]) = 0$ since the chain complex is $\ldots \to 0 \to \mathbb{Z}\{0\} \to \mathbb{Z} \to 0 \to \ldots$. Now assume that $D[M_+]$ is acyclic for all $M \leq N$. Take $z \in \mathbb{Z}D[N+1_+]_n$ such that $\partial_n(z) = 0$, we know that $z = p_0\sigma_0 + \ldots + p_m\sigma_m$, but for each σ_j either $\sigma_j \in D[N_+]_n$ or there is some $\tau_j \in D[N_+]_{n-1}$ such that

$$\sigma_j = \tau_j \cup \{N+1\} \tag{2.31}$$

So we can now rewrite z as

$$z = a_0\sigma_0 + \ldots + a_t\sigma_t + a_{t+1}\tau_1 \cup \{N+1\} + \ldots + a_{t+s}\tau_s \cup \{N+1\} \tag{2.32}$$

for $a_0, \ldots, a_{t+s} \in \mathbb{Z}$, $\sigma_0, \ldots, \sigma_t \in D[N_+]_n$ and $\tau_1, \ldots, \tau_s \in D[N_+]_{n-1}$. We see that $\partial_n(\tau_j \cup \{N+1\}) = \partial_{n-1}(\tau_j) \cup \{N+1\} + (-1)^n\tau_j$, where $\partial_{n-1}(\tau_j) \cup \{N+1\}$ means that we add $N+1$ to each simplex we get from ∂. We see that

$$\begin{aligned}
0 &= \partial_n(z) \\
&= a_0\partial_n(\sigma_0) + \ldots + a_t\partial_n(\sigma_t) + a_{t+1}\partial_n(\tau_1 \cup \{N+1\}) + \ldots + a_{t+s}\partial_n(\tau_s \cup \{N+1\}) \\
&= \partial_n(a_0\sigma_0 + \ldots + a_t\sigma_t) + \partial_{n-1}(a_{t+1}\tau_1 + \ldots + a_{t+s}\tau_s) \cup \{N+1\} \\
&\qquad + (-1)^n a_{t+1}\tau_1 + \ldots + a_{t+s}\tau_s
\end{aligned}$$

From this it is clear that $\partial_{n-1}(a_{t+1}\tau_1 + \ldots + a_{t+s}\tau_s) \cup \{N+1\} = 0$ since these are the only elements which contain $N+1$ as an element in the simplexes. But if that is true then it must follow that $\partial_{n-1}(a_{t+1}\tau_1 + \ldots + a_{t+s}\tau_s) = 0$, and since $D[N_+]$ is acyclic its chain complex must be exact, and hence there must be an $x \in D[N_+]_n$ such that $\partial_n(x) = a_{t+1}\tau_1 + \ldots + a_{t+s}\tau_s$. But with this we find that

$$\begin{aligned}
0 &= \partial_n Z \\
&= a_0\partial_n(\sigma_0) + \ldots + a_t\partial_n(\sigma_t) + a_{t+1}\partial_n(\tau_1 \cup \{N+1\}) + \ldots + a_{t+s}\partial_n(\tau_s \cup \{N+1\}) \\
&= a_0\partial_n(\sigma_0) + \ldots + a_t\partial_n(\sigma_t) + (-1)^n a_{t+1}\tau_1 + \ldots + a_{t+s}\tau_s \\
&= a_0\partial_n(\sigma_0) + \ldots + a_t\partial_n(\sigma_t) + (-1)^n\partial_n(x)
\end{aligned} \tag{2.33}$$

But $a_0\sigma_0 + \ldots + a_t\sigma_t + (-1)^n x \in D[N_+]_n$ which is acyclic and hence there is $y \in D[N_+]_{n+1}$ such that $\partial_{n+1}(y) = a_0\sigma_0 + \ldots + a_t\sigma_t + (-1)^n x$, and we see that $y + x \cup \{N+1\} \in D[N+1_+]_{n+1}$, and that

$$\begin{aligned}\partial_{n+1}(y + x \cup \{N+1\}) &= \partial_{n+1}(y) + \partial_{n+1}(x \cup \{N+1\}) && (2.34)\\ &= \partial_{n+1}(y) + \partial_n(x) \cup \{N+1\} + (-1)^{n+1}x && (2.35)\\ &= a_0\sigma_0 + \ldots + a_t\sigma_t + (-1)^n x + \partial_n(x) \cup \{N+1\} + (-1)^{n+1}x\\ &= a_0\sigma_0 + \ldots + a_t\sigma_t + (a_{t+1}\tau_1 + \ldots + a_{t+s}\tau_s) \cup \{N+1\}\\ &= z && (2.36)\end{aligned}$$

And hence we see that for an arbitrary $[z] \in \tilde{H}_n(D[N+1_+])$, then

$$[z] = [\partial_{n+1}(y + x \cup \{N+1\})] = [0] \qquad (2.37)$$

, and hence $D[N+1_+]$ is acyclic, and this concludes the proof. □

This lemma might seem at the moment somewhat arbitrary, but it will come in most handy when we prove the Alexander duality. However this will require a great deal more theory, so we will carry on:

2.3 Cohomology

This section too is written with [4]. Theorem 2.11 is stated without proof in [8].

2.3.1 Preliminaries

Again we will now turn our attention to the group of homomorphisms from a group to a given set, this time the set of integers. We wish to again construct a chain complex somehow related to our simplicial complex. This time we want it to be the dual of the homology so to speak. As we have seen earlier if one has homomorphisms between groups it is quite easy to construct homeomorphisms between the set of homomophisms into a given group. So let ∂_n be given as in the previous section, and let K be a simplicial complex, then define $\delta_n : \operatorname{Hom}(\mathbb{Z}K_{n-1}, \mathbb{Z}) \to \operatorname{Hom}(\mathbb{Z}K_n, \mathbb{Z})$, given as follows: For an arbitrary homomorphy $\psi : \mathbb{Z}K_{n-1} \to \mathbb{Z}$, then $\delta_n(\psi) = \psi \circ \partial_n$. This is clearly well defined since both ∂ and ψ are homomorphisms and hence so is their composition. Then clearly we find that

$$(\delta_{n+1} \circ \delta_n)(\psi) = \delta_{n+1}(\psi \circ \partial_n) = (\psi \circ \partial_n) \circ \partial_{n+1} = \psi \circ \mathbf{0} = \mathbf{0} \qquad (2.38)$$

with $\mathbf{0}$ being the trivial homomorphy. Then except for the numbering which seems to go the other way than usually, we find that

$$\ldots \overset{\delta_n}{\to} \operatorname{Hom}(\mathbb{Z}K_n, \mathbb{Z}) \overset{\delta_{n-1}}{\to} \operatorname{Hom}(\mathbb{Z}K_{n-1}, \mathbb{Z}) \overset{\delta_{n-1}}{\to} \ldots \qquad (2.39)$$

Is a chain complex.We can now as previously move on to a definition.

2.3.2 The reduced cohomology

Definition 2.10. *Let K be a simplicial complex, and δ_j given as above, then the n'th reduced cohomology group of the complex K is*

$$\tilde{H}^n(K) := \ker \delta_{n+1}/Im\delta_n. \tag{2.40}$$

This definition seems very close to our reduced homology groups, and as we will see throughout the thesis this is quite true; there are several duality theorems (one of which is the Alexander duality) linking homology and cohomology of different objects together. Like homology cohomology like homology is independent of the ordering of the vertex set. We will not show this since the proof is rather similar to the argument for homology. We will start by showing that as homology the cohomology has the same link to the geometric notion of a simplicial complex, since:

Theorem 2.11. *For a simplicial complex K, then*

$$\tilde{\chi}(K) = \sum_i (-1)^i rank\tilde{H}^i(K). \tag{2.41}$$

The proof for this is completely analogous to that for the homology case and therefore I will omit the proof. The only thing that changes slightly is that we would need to realize that $\text{rankHom}(\mathbb{Z}K_n, \mathbb{Z}) = |K_n|$, but this is clear since a homology from a free group to $\mathbb{Z}$ is merely a function from the set of generators to $\mathbb{Z}$.

2.4 Relative homology

This section is likewise inspired by [4]. Corollary 2.15 is from [3]. The result (2.64) is stated in [3], but proved by me.

2.4.1 Preliminaries

Our reason for studing simplicial complexes is that, given a topological space it is often quite easy to triangulate it and then find a simplicial complex, the homology of which we can then study. But sometimes it is not quite clear how this triangulation should go. It is *e.g.* clear that a ball in three dimensions seems to be equivalent to $D[3_+]$, but what about the ball where we remove the sphere? It is not quite clear what simplicial complex it should be. We can easily see that the sphere should be $D[3_+] - \{0, 1, 2, 3\}$, but since $\{\{0, 1, 2, 3\}\}$ isn't a simplicial complex then our methods so far are at a loss. We could let the homology be the quotient between the homology groups, but this isn't always well defined so instead we construct the following:

For K, A simplicial complexes with $A \subset K$ then we have

$$q : \ \mathbb{Z}K_n \to \mathbb{Z}K_n/\mathbb{Z}A_n \tag{2.42}$$

being the quotient map. Then define

$$\partial_n^* : \mathbb{Z}K_n/\mathbb{Z}A_n \to \mathbb{Z}K_{n-1}/\mathbb{Z}A_{n-1} \tag{2.43}$$

given by for $x \in \mathbb{Z}K_n/\mathbb{Z}A_n$ then take $y \in \mathbb{Z}K_n$ such that $q(y) = x$, and then $\partial_n^*(x) = q(\partial_n(y))$. We need to check that this is well defined, so let $y, y' \in \mathbb{Z}K_n$ such that $q(y) = q(y') = x$, then there is $a \in \mathbb{Z}A_n$ such that $y = y' + a$. Since A is a simplicial complex, and hence stable under subsets it is clear that $\partial_n(a) \in \mathbb{Z}A_{n-1}$, and hence

$$q(\partial_n(y)) = q(\partial_n(y' + a)) = q(\partial_n(y') + \partial_n(a)) \tag{2.44}$$
$$= q(\partial_n(y')) + q(\partial_n(0)) = q(\partial_n(y')) \tag{2.45}$$

If we take arbitrary $x, x' \in \mathbb{Z}K_n/\mathbb{Z}A_n$, with $y, y' \in \mathbb{Z}K_n$ such that $q(y) = x$ and $q(y') = x'$ then

$$\partial_n^*(x) + \partial_n^*(x') = q(\partial_n(y)) + q(\partial_n(y')) = q(\partial_n(y) + \partial_n(y')) \tag{2.46}$$
$$= q(\partial_n(y + y')) = \partial_n^*(x + x') \tag{2.47}$$

Thus ∂_n^* is a homomorphism. Since we wish to construct a chain complex we need to check that the image is in the kernel for our new function so let $x \in \mathbb{Z}K_n/\mathbb{Z}A_n$ be arbitrary, and $y \in \mathbb{Z}K_n$ be so that $q(y) = x$ then

$$(\partial_{n-1}^* \circ \partial_n^*)(x) = \partial_{n-1}^*(q(\partial_n(y))) = q((\partial_{n-1} \circ \partial_n)(y)) = q(0) = 0 \tag{2.48}$$

We are now ready to define our new notion of relative homology:

2.4.2 The reduced relative homology

Definition 2.12. *Let K, A be simplicial complexes with $A \subset K$ and let ∂_n^* be given as above, then the n'th reduced relative homology group between K and A is $\tilde{H}_n(K, A) = \ker \partial_n^* / Im\partial_{n+1}^*$.*

Again we will abstain from showing that this group is independent of the ordering of the vertex set. Again the argument would be rather similar to the that for homology case. This relative homology tends to come in handy especially for calculating various homology groups, since the next theorem we will show gives an exact sequence linking the reduced homology groups of K and A to the their relative homology. For this we will need a function $\varphi : H_n(K, A) \to H_{n-1}(A)$, so we will construct that.

Remark 2.13. *from this point on we will stop indexing our functions ∂_n, ∂_n^* and δ_n since it is very seldom that the indices do anything but hinder both the imagination and notation of proofs and constructions. We will however include the index if somewhere it should be important.*

Consider the following diagram

$$\begin{array}{ccccccccc}
\dots & \overset{\partial}{\to} & \mathbb{Z}A_{n+1} & \overset{\partial}{\to} & \mathbb{Z}A_n & \overset{\partial}{\to} & \mathbb{Z}A_{n-1} & \overset{\partial}{\to} & \dots \\
 & & i\downarrow & & i\downarrow & & i\downarrow & & \\
\dots & \overset{\partial}{\to} & \mathbb{Z}K_{n+1} & \overset{\partial}{\to} & \mathbb{Z}K_n & \overset{\partial}{\to} & \mathbb{Z}K_{n-1} & \overset{\partial}{\to} & \dots \\
 & & q\downarrow & & q\downarrow & & q\downarrow & & \\
\dots & \overset{\partial_*}{\to} & \mathbb{Z}K_{n+1}/\mathbb{Z}A_{n+1} & \overset{\partial_*}{\to} & \mathbb{Z}K_n/\mathbb{Z}A_n & \overset{\partial_*}{\to} & \mathbb{Z}K_{n-1}/\mathbb{Z}A_{n-1} & \overset{\partial_*}{\to} & \dots
\end{array}$$

where the rows are chain complexes, i is the inclusion map, and q the quotient map. It is rather trivial that $i \circ \partial = \partial \circ i$, and likewise from the definition that $\partial^* \circ q = q \circ \partial$, and hence the diagram commutes which is a rather useful fact, for then we can define i^H and q^H as we did in section 2.1.

Theorem 2.14. ***Long exact sequence:*** *If K, A are simplicial complexes with $A \subset K$ then the following is an exact sequence for the functions given above*

$$\dots \overset{\varphi}{\to} \tilde{H}_n(A) \overset{i^H}{\to} \tilde{H}_n(K) \overset{q^H}{\to} \tilde{H}_n(K, A) \overset{\varphi}{\to} \tilde{H}_{n-1}(A) \overset{i^H}{\to} \dots \tag{2.49}$$

The following proof is a diagram chase, and hence it is most useful to keep the diagram above in mind as one reads it.

Proof. We will do this proof one step at a time, checking one inclusion after the other

- $\mathrm{Im} i^H \subset \ker q^H$

For arbitrary $[a] \in \tilde{H}_n(A)$, then $q^H \circ i^H([a]) = q^H([i(a)]) = [q(i(a))] = [0]$, and hence we have the inclusion.

- $\mathrm{Im} q^H \subset \ker \varphi$

Take arbitrary $[x] \in \tilde{H}_n(K)$, then we know that $\partial(x) = 0$, so if $q(x) = z$, then $\varphi(q^H([x])) = [0]$.

- $\mathrm{Im}\varphi \subset \ker i^H$

For arbitrary $[z] \in \tilde{H}_n(K, A)$, then there exists $x \in \mathbb{Z}K_n$ such that $q(x) = z$, but then $i^H(\varphi[z]) = i^H([a]) = [i(a)] = [\partial(x)] = 0$.

- $\ker q^H \subset \mathrm{Im} i^H$

Fix $x \in \ker q^H$, then we know there is $z \in \mathbb{Z}K_{n+1}/\mathbb{Z}A_{n+1}$, such that $q(x) = \partial_* z$, since q is surjectiv we get a $x' \in \mathbb{Z}K_{n+1}$, such that $q(x') = z$. We now see that $q(x - \partial(x')) = q(x) - \partial(q(x')) = \partial_*(z) - \partial_*(z) = 0$, so $x - \partial(x') \in \ker q = \mathrm{Im} i$, hence $\exists a \in \mathbb{Z}A_n$ such that $i(a) = x - \partial(x')$. We see that $\partial(i(a)) = \partial(x - \partial(x')) = \partial(x) = 0$, and since i is injectiv then $a \in \ker \partial$, and hence $[a] \in \tilde{H}_n(A)$, and we find that $i^H([a]) = [i(a)] = [x - \partial(x')] = [x]$.

- $\ker\varphi \subset \text{Im}q^H$

Take $[z] \in \ker\varphi$ then $\exists a \in \mathbb{Z}A_n$ such that $\varphi([z]) = [0] = [\partial a]$, now take $x \in \mathbb{Z}K_n$ such that $q(x) = z$ and see that $\partial(x - i(a)) = \partial(x) - i(\partial(a)) = 0$, due to the definition of the φ function, we find $[x - i(a)] \in \tilde{H}_n(K)$ and $q^H([x - i(a)]) = [q(x) - q(i(a))] = [z]$, hence $\ker q = \text{Im } i$.

- $\ker i^H \subset \text{Im}\varphi$

Fix $a \in \ker i^H$, then $i(a) = \partial(x)$, for some $x \in \mathbb{Z}K_n$, and

$$\partial_*(q(x)) = q(\partial(x)) = q(i(a)) = 0 \tag{2.50}$$

so $[q(x)] \in \tilde{H}_n(K, A)$, and from this it clearly follows that $\varphi([q(x)]) = [a]$. We have now shown all the necessary inclusions. This concludes the proof. □

As promised above, the concept of the reduced relative homology is quite handy for calculating homology groups, and here we will give a short corollary to show the usefulness, the result will be used in the proof of the Alexander duality.

Corollary 2.15. *Let K be a simplicial compleks on the vertex set of V then it follows that $\tilde{H}_{n-1}(K) \cong \tilde{H}_n(D[V], K)$ for all n*

Proof. Trivially $K \subset D[V]$, so hence the relative homology group is defined. It is also clear that we can relabel V with N_+ for a suitable $N \in \mathbb{N}_0$ since V is finite and hence $\tilde{H}_n(D[V]) = 0$ for all n due to lemma 2.9. If we use this result in the long exact sequence above then we find that for all n, $0 \to \tilde{H}_n(D[V], K) \overset{\varphi}{\to} \tilde{H}_{n-1}(D[V], K) \to 0$ is exact. This implies that φ is injectiv since its kernel must be the image of 0, hence 0. And it is surjective since its image must be the kernel of a homomorphy mapping into the trivial group, and hence we are done. □

In this section we will need another small result to prove the Alexander duality. Since later it will make our lives easier we want to invent a new notation and prove it equivalent to the one we have already introduced. So let A, K be simplicial complexes such that $A \subset K$ then we will use the groups $\mathbb{Z}(K_n - A_n)$ where "$-$" is the set theoretical difference. We wish to make a chain complex so we will need a function. Therefore we will define $d : \mathbb{Z}(K_n - A_n) \to \mathbb{Z}(K_n - A_n)$ given as follows: For $\sigma \in K_n - A_n$ then

$$d(\sigma) = \sum_{v_i \in \sigma, \sigma - \{v_i\} \notin A} (-1)^i \sigma - \{v_i\} \tag{2.51}$$

Then we have that

$$d \circ d(\sigma) = d\left(\sum_{\substack{v_i \in \sigma \\ \sigma - \{v_i\} \notin A}} (-1)^i \sigma - \{v_i\}\right) \tag{2.52}$$

$$= \sum_{\substack{v_i \in \sigma \\ \sigma - \{v_i\} \notin A}} (-1)^i \sum_{\substack{v_j \in \sigma - \{v_i\} \\ \sigma - \{v_i, v_j\} \notin A}} (-1)^j \sigma - \{v_i, v_j\} \tag{2.53}$$

$$= \sum_{\substack{v_i \in \sigma \\ \sigma - \{v_i\} \notin A}} \sum_{\substack{v_j \in \sigma, v_j < v_i \\ \sigma - \{v_i, v_j\} \notin A}} (-1)^i (-1)^j \sigma - \{v_j, v_i\} \tag{2.54}$$

$$+ \sum_{\substack{v_i \in \sigma \\ \sigma - \{v_i\} \notin A}} \sum_{\substack{v_j \in \sigma, v_j > v_i \\ \sigma - \{v_i, v_j\} \notin A}} (-1)^i (-1)^j \sigma - \{v_j, v_i\} \tag{2.55}$$

$$= 0 \tag{2.56}$$

So we now have a chain complex, which we wish to show to be isomorphic to the relativ homology group, so we need an isomorphism

$$f : \mathbb{Z}(K_n - A_n) \to \mathbb{Z}K_n / \mathbb{Z}A_n \tag{2.57}$$

such that $\partial^* \circ f = f \circ d$. Now let $\sigma \in K_n - A_n$ then trivially $\sigma \in K_n$ and hence $q(\sigma) \in \mathbb{Z}K_n/\mathbb{Z}A_n$ with q being the quotient map, so let $f(\sigma) = q(\sigma)$.

$$(\partial^* \circ f)(\sigma) = \partial^*(q(\sigma)) = q(\partial(\sigma)) = q(\sum_{v_i \in \sigma} (-1)^i (\sigma - \{v_i\})) \tag{2.58}$$

$$= \sum_{v_i \in \sigma} (-1)^i q((\sigma - \{v_i\})) \tag{2.59}$$

$$= \sum_{\substack{v_i \in \sigma \\ \sigma - \{v_i\} \notin A}} (-1)^i q((\sigma - \{v_i\})) \tag{2.60}$$

$$(f \circ d)(\sigma) = f\left(\sum_{\substack{v_i \in \sigma \\ \sigma - \{v_i\} \notin A}} (-1)^i \sigma - \{v_i\}\right) \tag{2.61}$$

$$= \sum_{\substack{v_i \in \sigma \\ \sigma - \{v_i\} \notin A}} (-1)^i q(\sigma - \{v_i\}) \tag{2.62}$$

We need to show that f is an isomorphism, but clearly it is injective since other than elements of $\mathbb{Z}A_n$ the only thing that maps to 0 is 0 itself, so its kernel is trivial. If we take arbitrary $[x] \in \mathbb{Z}K_n/\mathbb{Z}A_n$, then there is $x' \in \mathbb{Z}(K_n - A_n)$ and $a \in \mathbb{Z}A_n$ such that $x' + a = x$, but then

$$[x] = [x' + a] = [x'] \tag{2.63}$$

and then $f(x') = [x]$, and hence f is surjective. We now see that

$$\tilde{H}_n(K, A) = \ker d_n / \mathrm{Im}\ d_{n+1}. \tag{2.64}$$

Chapter 3

The Alexander duality

This entire chapter is inspired by [3], the only exception being corollary 3.3, which is stated with an alternative proof in [8], but this proof is by me.

3.1 The Theorem

3.1.1 Preliminaries

We are now ready to both state and prove the Alexander duality. As we saw in the section on cohomology there is a close link between homology and cohomology, and it is one of these connections between the two notions we wish to prove. The theorem states that

Theorem 3.1. ***Alexander duality:*** *For a simplicial complex K, on a vertex set V, with $|V| - 1 = N$ the following statement holds*

$$\tilde{H}_i(K) \cong \tilde{H}^j(K^\vee) \tag{3.1}$$

whenever $i + j = N - 2$

There are several versions of the theorem, all being equivalent. The one above is the formulation I have chosen to prove.

We will start by sketching out the proof, so that when the formal proof is given below we will not loose our way. We remember that for a simplicial complex K the elements of $K^\vee$ are complements of non-simplexes of K. So if somehow we could construct a complement map from the set of non-simplexes it would seem doable. But we seem to already have something that more or less is this, namely the relative group $\mathbb{Z}D[V]/\mathbb{Z}K_n$, which has a homology isomorphic to K's reduced homology due to the long exact sequence. So we wish to map these elements to homologies from $\mathbb{Z}K_n^\vee$ to $\mathbb{Z}$. It is clear that every element in $\text{Hom}(\mathbb{Z}K_n^\vee, \mathbb{Z})$ can be described as the sum of functions $\mathbb{1}_\tau : \mathbb{Z}K_n^\vee \to \mathbb{Z}$, for $\tau \in K_n^\vee$ given as follows: If $x = a_0\sigma_0 + \ldots + a_{m-1}\sigma_m + a_m\tau$ then $\mathbb{1}_\tau(x) = a_m$. So we could hope that if we map $\sigma \in \mathbb{Z}D[V]/\mathbb{Z}K_n$ to $\mathbb{1}_{V-\sigma}$ then everything would work out nicely. This does not hold, but if we remember to do something about the sign of the function it will commute with δ and ∂_* and then we will be done.

3.1.2 The proof

Proof of the Alexander duality. Firstly we will need that the group

$$A := \{\sum_{\sigma \in K_n} a_\sigma \mathbb{1}_\sigma | a_\sigma \in \mathbb{Z}\} \tag{3.2}$$

with $\mathbb{1}_\sigma$ defined as above is equal to $\mathrm{Hom}(\mathbb{Z}K_n, \mathbb{Z})$. It is rather trivial that $A \subset \mathrm{Hom}(\mathbb{Z}K_n, \mathbb{Z})$. And if we have $\psi \in \mathrm{Hom}(\mathbb{Z}K_n, \mathbb{Z})$, then ψ is uniquely determined by it actions on elements in K_n and hence we can compose it of elements of A and therefore $\{\sum_{\sigma \in K_n} a_\sigma \mathbb{1}_\sigma | a_\sigma \in \mathbb{Z}\} = \mathrm{Hom}(\mathbb{Z}K_n, \mathbb{Z})$, since their composition rules are trivially identical.

Now let K be a simplicial complex on the vertex set V, with $|V| - 1 = N$, then clearly we can rename the vertices such that $K \subset D[N_+]$ then we wish to define a homomorphy so that for $\sigma \in (D[N_+]_{i+1} - K_{i+1})$ then we define

$$f : \mathbb{Z}(D[N_+]_{i+1} - K_{i+1}) \to \left\{ \sum_{\sigma \in K^\vee_{N-i-2}} a_\sigma \mathbb{1}_\sigma \middle| a\sigma \in \mathbb{Z} \right\} \tag{3.3}$$

$$f : \sigma \mapsto \Pi_{v_i \in \sigma} (-1)^{v_i} \mathbb{1}_{V-\sigma} \tag{3.4}$$

Then this is well defined due to the definition of $K^\vee$. It is rather trivially a homomorphy, and it is clear that the only thing f maps to the constant function 0 is 0, so it is injective. Given a homomorphy we can decompose it into a sum of $a_\sigma \mathbb{1}_\sigma$, for suitable σ's and $a_\sigma \in \mathbb{Z}$, and hence f is surjective.

So for $\sigma \in (D[N_+]_{i+1} - K_{i+1})$ and $\tau \in K^\vee_{N-i-2}$ we find that

$$
\begin{aligned}
(\phi \circ d)(\sigma)(\tau) &= \phi\left(\sum_{v_i \in \sigma, \sigma - \{v_i\} \notin K} (-1)^i \sigma - \{v_i\}\right)(\tau) && (3.5)\\
&= \sum_{v_i \in \sigma, \sigma - \{v_i\} \notin K} (-1)^i \Pi_{j \in \sigma - \{v_i\}} (-1)^j \mathbb{1}_{N_+ - (\sigma - \{v_i\})}(\tau) && (3.6)\\
&= \sum_{v_i \in \sigma, \sigma - \{v_i\} \notin K, \tau = N_+ - (\sigma - \{v_i\})} (-1)^i \Pi_{j \in \sigma - \{v_i\}} (-1)^j && (3.7)\\
&= \sum_{v_i \in \sigma, \tau - \{v_i\} = N_+ - \sigma} (-1)^i \Pi_{j \in \sigma - \{v_i\}} (-1)^j && (3.8)\\
(\delta \circ \phi)(\sigma)(\tau) &= \delta(\Pi_{i \in \sigma} (-1)^i \mathbb{1}_{N_+ - \sigma})(\tau) && (3.9)\\
&= \Pi_{i \in \sigma} (-1)^i \mathbb{1}_{N_+ - \sigma}(\partial(\tau)) && (3.10)\\
&= \Pi_{i \in \sigma} (-1)^i \mathbb{1}_{N_+ - \sigma}\left(\sum_{v_i \in \tau} (-1)^i \tau - \{v_i\}\right) && (3.11)\\
&= \Pi_{i \in \sigma} (-1)^i \sum_{v_i \in \tau, \tau - \{v_i\} = N_+ - \sigma} (-1)^i && (3.12)
\end{aligned}
$$

It is clear that if there are no $v_i \in \tau$ such that $\tau - \{v_i\} = N_+ - \sigma$ then both sums are empty and hence $(\phi \circ d)(\sigma)(\tau) = 0 = (\delta \circ \phi)(\sigma)(\tau)$. So assume that there is $v_i \in \tau$ such that $\tau - \{v_i\} = N_+ - \sigma$, then clearly $v_i = v_k \in \sigma$ for some k, and v_i is unique, then we find that

$$
\begin{aligned}
(\phi \circ d)(\sigma)(\tau) &= (-1)^k \cdot \Pi_{j \in \sigma - \{v_k\}} (-1)^j = \Pi_{j \in \sigma, j < v_k} (-1) \cdot \Pi_{j \in \sigma - \{v_k\}} (-1)^j\\
(\delta \circ \phi)(\sigma)(\tau) &= \Pi_{j \in \sigma} (-1)^j \cdot (-1)^i = \Pi_{j \in \sigma} (-1)^j \cdot \Pi_{l \in N_+ - \sigma, l < v_k} (-1) && (3.13)
\end{aligned}
$$

It is clear by simple arithmetic that these to products are identical, and therefore

$$\ker d_{i+1} / \mathrm{Im}\ d_{i+2} \cong \ker \delta_{N-i-1} / \mathrm{Im} \delta_{N-i-2} \tag{3.14}$$

and now we are done since

$$
\begin{aligned}
\tilde{H}_i(K) &\overset{\text{Cor. 2.9}}{\cong} \tilde{H}_{i+1}(D[N_+], K) \overset{(2.64)}{\cong} \ker d_{i+1} / \mathrm{Im} d_{i+2} && (3.15)\\
&\overset{(3.14)}{\cong} \ker \delta_{N-i-1} / \mathrm{Im} \delta_{N-i-2} = \tilde{H}^{N-i-2}(K^\vee) && (3.16)
\end{aligned}
$$

□

Corollary 3.2. *Let K be a simplicial complex on the vertex set V, with $|V| - 1 = N$. Then*

$$\tilde{H}_i(K^\vee) \cong \tilde{H}^j(K) \tag{3.17}$$

whenever $i + j = N + 2$

Proof. This trivially follows from the fact that $(K^\vee)^\vee = K$ and from the Alexander duality. □

Corollary 3.3. *Let K be a simplicial complex, with N_+ as the vertex set, then $\tilde{\chi}(K) = (-1)^{N-2}\tilde{\chi}(K^\vee)$*

Proof. We have that $\tilde{\chi}(K) = \sum_i (-1)^i \text{rank} \tilde{H}_i(K)$ and using the Alexander duality we have that

$$\tilde{\chi}(K) = \sum_i (-1)^i \text{rank} \tilde{H}_i(K) = \sum_i (-1)^i \text{rank} \tilde{H}^{N-2-i}(K^\vee) \tag{3.18}$$

$$= \sum_i (-1)^{N-2-i} \text{rank} \tilde{H}^i(K^\vee) = (-1)^{N-2} \sum_i (-1)^{-i} \text{rank} \tilde{H}^i \tag{3.19}$$

$$= (-1)^{N-2} \sum_i (-1)^i \text{rank} \tilde{H}^i = (-1)^{N-2} \tilde{\chi}(K^\vee) \tag{3.20}$$

and hence we are done. □

Chapter 4

Second duality

4.1 Topological invariance

This section is a sketch of proofs and theorems found in [6].

In this chapter, in pursuit of a different duality between homology and cohomology, we willl try to realize our simplicial complexes as topological spaces. For this we will need some basic definitions.

Definition 4.1. *Let $\{a_0, a_1, \ldots, a_n\}$ be a set of points in $\mathbb{R}^N$ for some N, then they are geometrically independent, if $a_1 - a_0, \ldots, a_n - a_0$ are linearly independent in the usual linear algebra way.*

Given a set of geometrically independent points, $\{a_0, a_1, \ldots, a_n\}$, their convex hull is the set of points $\sum_{i=0}^{n} t_i a_i$, where $t_i \in \mathbb{R}$ and are non-negative, and $\sum_{i=0}^{n} t_i = 1$. Given a convex hull of $\{a_0, a_1, \ldots, a_n\}$, it faces are the convex hulls of subsets of $\{a_0, a_1, \ldots, a_n\}$

It is clear that for a point in a convex hull the t_i's are uniquely determined. We now need a topological equivalent to our simplicial complexes, so we define.

Definition 4.2. *A simplicial polytope, $\mathcal{P}$ is a subset of the powerset $\mathbb{R}^N$ for some N consisting of convex hulls of finite set of points such that*

- *For any convex hull σ in $\mathcal{P}$ then all faces of σ are also in $\mathcal{P}$.*
- *All intersections between two elements of $\mathcal{P}$ are faces of both elements of $\mathcal{P}$.*

We will write $\|\mathcal{P}\| := \bigcup_{s \in \mathcal{P}} s$, and call this object the realization of $\mathcal{P}$. We give to this space the following topology: A subset A of $\|\mathcal{P}\|$ is closed if $A \cap s$ is closed in the standard topology on $\mathbb{R}^N$ for all $s \in \mathcal{P}$.

Now it appears that somehow $\mathcal{P}$ is closely related to our simplicial complexes. The simplicial polytope and the simplicial complex actually seems to be equivalent since given a simplicial complex K, then take a function from $f:\ K_0 \to \mathbb{R}^N$ such that the images of the vertices are geometrically independent, then let

$$\mathcal{K} = \{\text{convex hull of}\{f(v_{i_0}),\ldots,f(v_{i_n})\}|\{v_{i_0},\ldots,v_{i_n}\}\in K\} \tag{4.1}$$

This is clearly a simplicial polytope. Given a simplicial polytope $\mathcal{P}$ we can define

$$P = \{\{a_{i_0,\ldots,a_{i_n}}\}|\text{The convex hull of } \{a_{i_0,\ldots,a_{i_n}}\} \in \mathcal{P}\} \tag{4.2}$$

and this is clearly a simplicial complex. Throughout the following chapter we will also view a simplicial complex as a simplicial polytope, and, *e.g.*, write $\|K\|$ for the geometricale realization of the simplicial polytope which corresponds to the simplicial complex $K \neq \{\emptyset\}$.

For simplicial complexes K, L let $f:\ K \to L$, be a simplicial map, then clearly $f^*:\ \|K\| \to \|L\|$ given as follows: For an element in $x \in \|K\|$ then there is a simplex in K, $\{a_0,\ldots,a_n\}$ such that x in the convex hull of this, then $x = \sum_{i=0}^n t_i a_i$ then $f^*(x) = \sum_{i=0}^n t_i f(a_i)$. This is clearly welldefined and continuous. It is also clear that if f was simplicial-isomorphy then $\|K\|, \|L\|$ are homeomorphic.

We will now sketch the way to prove the following result:

Theorem 4.3. *Let K, L be simplicial complexes such that $\|K\|$ is homotopic equivalent to $\|L\|$, then $\tilde{H}_n(K) \cong \tilde{H}_n(L)$*

Now we would wish for that if $f:\ \|K\| \to \|L\|$ was continuous then we would be able to induce a simplicial-map between K, L but this is not the case. Then we would like, that f induces some homomorphy between $\tilde{H}_n(K)$ and $\tilde{H}_n(L)$ this is not directly true either, but by going slightly out of our way we will be able to make a isomorphy to some "subdivision" of K's homology group from K's homology and then a homomorphy onwards to the homology of L. We will need the following

Definition 4.4. *Let K be a simplicial complex and let v be a vertex of K, then the st (v, K), or just st v if the complex is clear, is the subset of $\|K\|$ given by the union of the interiors of the simplices of K that have v as a vertex.*

That this is such an important definition seems surprising but it shall prove most useful.

Definition 4.5. *Let K, L be simplicial complexes and let $h:\ \|K\| \to \|L\|$ be continuous, then we say that h satisfies the star condition with respect to K and L if for each vertex $\{v\} \in K$ there is a vertex $\{w\} \in L$, such that $h(st\ v) \subset st\ w$.*

It can be proven that if $h : \|K\| \to \|L\|$ satisfies the star condition then it induces a simplicial map $K \to L$, but it is also clear that not all maps satisfy the star condition. Now let the subdivision of a simplicial complex K be $\mathrm{sd}K = \Delta(K - \emptyset)$, where the K on the right hand sight is viewed as a poset ordered by the subset-ordering. Then one could prove that for a continuous map $f : \|K\| \to \|L\|$ there exists $N \in \mathbb{N}_0$ such that there is a continuous map $f' : \|\mathrm{sd}^N K\| \to \|L\|$ such that f' has a simplicial approximation from $\mathrm{sd}^N K \to L$. Now it is provable that there is a isomorphy $\tilde{H}_n(K) \to \tilde{H}_n(\mathrm{sd}^N K)$, and so for any continuous map we can think of its induced simplicial map, since we can merely use the simplicial map induced by the function from some subdevision, and one can therefore show that the homology group of a simplicial complex is stable under homeomorphisms of its realization.

Now to prove theorem 4.3, one first needs to realize that there is a simplicial complex, which can be realized as the space $\|K\| \times I$, with $I = [0, 1]$. Then one can show that if two continuous maps between realizations are homotopic then their induced homomorphisms between the homology groups are equal. Because then if K, L realize as homotopic equivalent spaces then the homotopic equivalences induces inverse homomorphisms whose composition is the identity i.e. they are isomorphisms.

The same process will clearly show us that likewise cohomology is stable under homotopy. And we can now define the reduced homology and cohomology of a topological space X, which is homotopic to some realization of a simplicial complex K

Definition 4.6. *If a topological space is homotopic to some realization of some simplicial complex K, then we define the n'th reduced homology group of X: $\tilde{H}_n(X) = \tilde{H}_n(K)$ and like wise the n'th reduced cohomology group of X: $\tilde{H}^n(X) = \tilde{H}^n(K)$*

4.2 Homotopic theorems

This section is inspired by [1], with the exception of lemma 4.14, which is only stated there, but is proved in [9], and theorem 4.13 which is adapted from a proof in [7].

4.2.1 Preliminaries

In this section we will prove some theorems bearing on the question: When are realizations of different complexes homotopic. For this we will have use of some definitions.

Definition 4.7. *Let K be a simplicial complex and T a topological space. Let $C : K \to \mathbb{P}(T)$ such that if $\sigma \subset \tau$ for $\sigma, \tau \in K$ then $C(\sigma) \in C(\tau)$.*

A mapping $f: \|K\| \to T$ *is said to be carried by* C *if for all* $\sigma \in K$, $f(\|\sigma\|) \subset C(\sigma)$.

We will often use that maps of interest between simplicial complexes has a continous realization which is carried by something mapping to a contractible space.

Definition 4.8. *Let* K *be a simplicial complex and* P *a poset ordered by* $\prec$ *then* $f: K \to P$ *is po-simplicial if* $\sigma \in K$*, with* $\sigma = \{v_0, \ldots, v_n\}$*, then* $f(v_{i_0}) \prec \ldots \prec f(v_{i_n})$*, where* v_{i_j} *is an unique element in* $\{v_0, \ldots, v_n\}$.

In other words a po-simplicial function maps simplices to chains.

Definition 4.9. *Given a family of sets* $(A_i)_{i \in I}$ *indexed by a finite ordered set* I*, the nerve of the family is the simplicial complex* $\mathcal{N}(A_i)$ *defined on the vertex set* I*, such that* $\sigma \in \mathcal{N}(A_i)$ *if* $\sigma \subset I$ *and* $\bigcap_{i \in \sigma} A_i \neq \emptyset$*, where we define* $\bigcap_{i \in \emptyset} A_i := \bigcup_{i \in I} A_i$

It is clear that this is a simplicial complex since, if $\sigma \in \mathcal{N}(A_i)$ and $\tau \subset \sigma$ then $\bigcap_{i \in \sigma} A_i \subset \bigcap_{i \in \tau} A_i$, so $\tau \in \mathcal{N}(A_i)$.

Definition 4.10. *Let* P *be a poset ordered by* $\prec$*, a subset* C *is called a crosscut if*

1. *no two elements in* C *are comparable*
2. *for every finite chain* $a_0 \prec \ldots \prec a_n$*, with* $a_0, \ldots, a_n \in P$*, there is some* $a \in C$ *such that* a *is comparable to each* a_i.
3. *for all* $A \subset C$ *for which there is an element* $p \in P$ *such that* $a \prec p$ *for all* $a \in A$*, then there should be a unique least upper bound for* A *in* P*. Like wise if there is* $p \in P$ *such that* $p \prec a$ *for all* $a \in A$*, then there should be a unique greatest lower bound for* A *in* P.

Definition 4.11. *If* C *is a finite crosscut then* $\Gamma(P, C)$ *determines a simplicial complex consisting of all subsets of* C *such that there is an upper or a lower bound in* P

We will also be needing a more topological result and for that a definition

Definition 4.12. *A cell is a topological space homeomorphic with the unit ball* B^n *for some* $n \in \mathbb{N}_0$

Theorem 4.13. *If* X *is a cell, and* T *is a contractible space, then if* f *is a continuous function from the boundary of* X *to* T*, then* f *extends to a continuous function* $\tilde{f}: X \to T$.

Proof. For X being homeomorph with B^0 this is a trivial statement since then X consists of a single point, and hence is equal to its own boundary. Next let $n \geq 1$. Let $h: X \to B^n$ be the homeomorphism, and let F be the homotopy between id_T and the constant map. Now clearly h maps the boundary of X to S^{n-1}. Now define $H: S^{n-1} \times I \to T$ by $H(x,t) = F((f \circ h^{-1})(x), t)$, then clearly this is a homotopy between $f \circ h^{-1}$ and the constant map. Now let $q: S^{n-1} \times I \to B^n$, be given by $q(x,t) = (1-t)x$, clearly this is continuous, closed and surjective, so it is a quotient map collapsing $S^{n-1} \times \{1\}$ to 0. Since H is constant on $S^{n-1} \times \{1\}$, then clearly it induces a continuous function $k: B^n \to T$ via the quotient map. Now let $\tilde{f} = h^{-1} \circ k$, then it is clear that for any element x in the boundary of X, it holds that $\tilde{f}(x) = f(x)$. □

4.2.2 Combinatorical homotopy theorems

We will be needing a lemma for use in the following theorem, namely:

Lemma 4.14. *Let K be a simplicial complex, let T a topological space and let $C: K \to \mathbb{P}(T)$ be a carrier such that $C(\sigma)$ is contractible for all $\sigma \in K$. Then:*

1. *If $f, g: \|K\| \to T$ are both carried by C, then f and g are homotopic.*
2. *There exists a mapping $f: \|K\| \to T$ carried by C.*

Proof. Let C be a carrier from K, such that $C(\sigma)$ is contractible for all $\sigma \in K$.

We will start by proving (1). Suppose that $f, g: \|K\| \to T$ are both carried by C, the we will construct a homotopy $F: \|K\| \times I \to T$. Now we have that $f(\|\{v\}\|) \in C(\{v\})$ and $g(\|\{v\}\|) \in C(\{v\})$ for all $v \in K_0$. Now we can consider $\|K\| \times I$ to be a collection of cells of the form $\|\sigma\| \times I$, for $\sigma \in K$, and a function on $\|K\| \times I$ is continuous if and only if it is coninuous on each $\|\sigma\| \times I$. Now it is clear that the boundry of $\|\{v\}\| \times I$ is given by $\|\{v\}\| \times \{0\} \cup \|\{v\}\| \times \{1\}$. Further let $F(\|\{v\}\|, 0) = f(\|\{v\}\|)$, and $F(\|\{v\}\|, 1) = g(\|\{v\}\|)$, and now by theorem 4.13 we can continuously extend $F: \|\{v\}\| \times \{0,1\} \to T$ to $\|\{v\}\| \times I$. Now suppose that F is defined and continuous on $\|K\| \times \{0,1\} \cup \|K_{\leq n}\| \times I$, with $K_{\leq n} = \bigcup_{i=-1}^{n} K_i$, and that $F(\|\sigma\| \times I) \subset C(\sigma)$. Then take $\tau \in K_{n+1}$, then the boundary of $\|\tau\| \times I$ is in $\|K\| \times \{0,1\} \cup \|K_{\leq n}\| \times I$, and since for all proper faces σ of τ then $C(\sigma) \subset C(\tau)$, it is clear that F taken on the boundry of $\tau \times I$ is continuous and a subset of $C(\tau)$ which is contractible and hence, F can be extended continuously to $\|\tau\| \times I$, and per induction we are now done.

We will now prove (2) by inductively constructing f as a continuous function. For $\{v\} \in K_0$ then $f(\|\{v\}\|)$ should be any point in $C(\{v\})$. Suppose now that f is continuously defined on $\|K_{\leq n}\|$, with $f(\|\sigma\|) \subset C(\sigma)$, for all $\sigma \in K_{\leq n}$. Now take $\tau \in K_{n+1}$, then for each simlex σ of τ we have that

$$f(\|\sigma\|) \subset C(\sigma) \subset C(\tau) \tag{4.3}$$

and hence f on the boundary of τ is a subset of $C(\tau)$, so since the realization of a simplex is a cell we can now extend f continuously to $\|\tau\|$, in such a way so $f(\|\tau\|) \subset C(\tau)$. And since f is continuous on K, if and only if it is continuous on each simplex of K, we are done. □

The next theorem will be actually be needed for proving the duality, which is the goal of this chapter, but also to prove some further theorems.

Theorem 4.15. *Let K be a simplicial complex, let P be a finite poset and let $f: K \to \Delta(P)$ a simplicial map. Define $P_{\geq x}$ to be all chains in P consisting of elements all greater than equal to x. Suppose that $\|f^{-1}(P_{\geq x})\| \subset \|K\|$, for all $x \in P$, is contractible. Then f induces homotopy equivalence between $\|K\|$ and $\|\Delta(P)\|$.*

Proof. Suppose that $f^{-1}(P_{\geq x})$ are contractible, then define

$$C: \Delta(P) \to \mathbb{P}(\|K\|) \tag{4.4}$$

given by $C(\sigma) = \|f^{-1}(P_{\geq \min \sigma})\|$. By part two of lemma 4.14 then there is a continuous g carried by C. Now let $C': \Delta(P) \to \mathbb{P}(\|\Delta(P)\|)$ given by $C'(\sigma) = \|P_{\geq \min \sigma}\|$. Now it is clear that $C'(\sigma)$ is contractible for all $\sigma \in \Delta(P)$ since it is all simplices of $\Delta(P)$ with elements greater than or equal to $\min \sigma$. It is clear that $f \circ g$ is carried by C' since g is carried by C, and $f \circ C$ is C', and trivially likewise the identity map on $\|\Delta(P)\|$ is carried, and hence by lemma 4.14 2 we get that $f \circ g$ is homotopic to the identity. Let $C'': \Delta(P) \to \mathbb{P}(\|K\|)$ given as follows: For $\sigma \in K$ then $C''(\sigma) = \|f^{-1}(P_{\geq \min f(\sigma)})\|$, then $C''(\sigma)$ is contractible by assumption and clearly it carries $g \circ f$ since $f(\sigma) \subset \|P_{\geq \min f(\sigma)}\|$ and g is carried by C. Likewise the identity map on $\|K\|$ is carried by C'' and hence by lemma 4.14 $g \circ f$ is homotopic to the identity and therefore $\|\Delta(P)\|$ is homotopic equivalent to $\|K\|$. □

Theorem 4.16. *Let K be a simplicial complex and let $(K_i)_{i \in I}$, be a finite family of simplicial complexes such that $K = \bigcup_{i \in I} K_i$. Suppose that every nonempty finite intersection $K_{i_1} \cap \ldots \cap K_{i_n}$ is contractible, then $\|K\|$ and $\|\mathcal{N}(K_i)\|$ are homotopic equivalent.*

Proof. Let $f: K \to \mathcal{N}(K_i)$, where $K, \mathcal{N}(K_i)$ are both viewed as posets ordered by subset-ordering; and for $\sigma \in K$ let $f(\sigma) = \{i \in I | \sigma \in K_i\}$. Then $f': \Delta(K) \to \mathcal{N}(K_i)$ given as follows: For $\{\sigma_0, \ldots, \sigma_n\} \in \Delta(K)$ then

$$f'(\{\sigma_0, \ldots, \sigma_n\}) = \{f(\sigma_0), \ldots, f(\sigma_n)\} \tag{4.5}$$

is well defined and simplicial. Now take arbitrary $\tau \in \mathcal{N}(K_i)$ viewed as a subset. Then $f'^{-1}(P_{\geq\tau}) = \bigcap_{i\in\tau} K_i$ since elements that are mapped to τ or greater must at least be in all the same K_i as τ. But now the result follows from theorem 4.15 and the assumption. □

Corollary 4.17. *If C is a crosscut of a poset P, then $\|\Gamma(P, C)\|$ and $\|\Delta(P)\|$ are homotopic equivalent.*

Proof. For any $x \in C$ let $K_x := \Delta(P_{\leq x} \cup P_{\geq x})$, then it is clear that $(K_x)_{x\in C}$ is a covering of $\Delta(P)$, i.e., $\Delta(P) = \bigcup_{x\in C} K_x$ since every element in P is comparable to some element in C. If we take $A \subset C$ such that $\bigcap_{x\in A} K_x \neq \emptyset$, that means there is either an element $p_1 \in P$ such that $p_1 \geq k$ for all $k \in A$, or there is $p_0 \in P$ such that $p_0 \leq k$ for all $k \in A$. If p_1 exists there is a least such we will call this element $\hat{p}_1$, since C is a crosscut, and hence $\bigcap_{x\in A} K_x$ is contractible because it consists of the vertices with $\hat{p}_1$ as the least element in each. If p_0 exists there is a $\hat{p}_0$ which is the greatest element smaller than all $k \in A$, and then $\bigcap_{x\in A} K_x$ is contractible since it consists of all simpleces with $\hat{p}_0$ as the greatest element. Now we need to show that $\Gamma(P, C) = \mathcal{N}(K_x)$, but this is clear since $A \subset C$ has $\bigcap_{x\in A} K_x \neq \emptyset$ if and only if it either has a lower or greater bound, and hence we are done. □

4.3 Duality

This section is inspired by [2] where the 4.18 is stated. I have followed the same proof-strategy as in [2], but the proof is deviced by me.

4.3.1 Preliminaries

In this section we will finally be able to state and prove the second duality of this thesis. So let K be a simplicial complex on the vertex set V, then let C be the set of subsetwise least nonfaces of K, meaning that

$$C := \min(D[V] - K) \tag{4.6}$$

, and let K^Γ be the simplicial complex such that $\{c_0, \ldots, c_n\} \in K^\Gamma$ for $c_0, \ldots, c_n \in C$, whenever $c_0 \cup \ldots \cup c_n \neq V$. This is clearly a simplicial complex since if $c_0 \cup \ldots \cup c_n \neq V$, then no subset of $\{c_0, \ldots, c_n\}$ will have union equal to V. Now it is clear that this complex is not the same as the Alexander dual complex.

Now we can state the that theorem we want to prove:

Theorem 4.18. *Let K be a simplicial complex on the vertex set V, with $|V| - 1 = N$, then $\tilde{H}_i(K) \cong \tilde{H}^j(K^\Gamma)$ whenever $i + j = N - 2$.*

It is clear that this is somehow related to the Alexander duality since the two statements are almost identical. The only difference is that here we have replaced $K^\vee$ with K^Γ. And that is actually the heart of the proof: Since the reduced cohomology is stable under homotopy equivalences, then we wish to show that there is a homotopy equivalence between $\|K^\vee\|$ and $\|K^\Gamma\|$, and for this we will need the theorems from the previous subsection.

4.3.2 Proof

Proof of theorem 4.18 . Let $\partial D[V] := D[V] - V$ and $\mathrm{sd}K^\vee := \Delta(K^\vee - \emptyset)$, where $K^\vee - \emptyset$ is viewed as a poset ordered by the subset-ordering. Now define

$$f:\ \mathrm{sd}K_0^\vee \to \partial D[V] - K \tag{4.7}$$

such that for any $\sigma \in \Delta(K^\vee - \emptyset)_0$ then $f(\sigma) = V - \sigma$. This is well defined since $\sigma \neq \emptyset$ and since $\sigma \in K^\vee - \emptyset$ and hence $V - \sigma \notin K$. Now we see that if $\{\sigma_0, \dots, \sigma_n\} \in \mathrm{sd}K^\vee$, then $\sigma_0 \subset \dots \subset \sigma_n$, but then $f(\sigma_n) \subset \dots \subset f(\sigma_0)$, so $\{f(\sigma_n), \dots, f(\sigma_0)\} \in \Delta(\partial D[V] - K)$. Now take an arbitrary $x \in \partial D[V] - K$ then $f^{-1}((D[V] - K)_{\geq x})$ consists of all simplices in $\mathrm{sd}K^\vee$ where $V - x$ is greater or equal to all elements in the simplex, and hence this is contractible. We can now by theorem 4.15 conclude that $\|\mathrm{sd}K^\vee\|$ and $\|\Delta(\partial D[V] - K)\|$ are homotopic equivalent.

Now let C be given as the set of minimal non-faces of K. We now whish to show that C is a crosscut of $\partial D[V] - K$. It is clear that since the elements of C are minimal no two distinct elements are related. It is clear that given an arbitrary element p in $\partial D[V] - K$ there must always be some element in C such that it is smaller than p, since C consists of minimal elements. Given an arbitrary nonempty subset A of C then there can't be a lower bound for A in $\partial D[V] - K$ since C concists of minimal elements. It is clear that if $\bigcup_{c \in A} c \neq V$, then A has an upper bound, namely $\bigcup_{c \in A} c \in \partial D[V] - K$, if $\bigcup_{c \in A} c = V$, then A is unbounded. We now have that C is a crosscut of $\partial D[V] - K$. And it is clear that $\Gamma(\partial D[V] - K, C) = K^\Gamma$, and hence we now have that $\|\Delta(\partial D[V] - K)\|$ is homotopic to $\|K^\Gamma\|$.

And hence we get by use of the Alexander duality that

$$\tilde{H}_i(K) \cong \tilde{H}^j(K^\vee) \cong \tilde{H}^j(\mathrm{sd}K^\vee) \cong \tilde{H}^j(\Delta(\partial D[V] - K)) \cong \tilde{H}^j(K^\Gamma) \tag{4.8}$$

whenever $i + j = N - 2$. □

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