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Introduction to Characteristic Classes

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Abstract

This project represents an introduction to the theory of characteristic classes of real and complex vector bundles, that is, cohomology classes which are assigned naturally to the base spaces of vector bundles and which encode information of such bundles. Beginning in the first chapter with a summary of the basics of vector bundles, the project then turns in chapter 2 to the classification of vector bundles on a paracompact space, a necessary result to define the notion of a characteristic class. Once we have proven this result, we invest chapters 3 and 4 to a systematic study of two important types of characteristic classes associated to real vector bundles, namely, the Stiefel-Whitney classes and the Euler class. The last chapter treats mainly characteristic classes of complex vector bundles. In particular, in this chapter we describe a way of constructing the Chern classes which will also yield us an alternate construction of the Stiefel-Whitney classes. This chapter and the whole of the project concludes with the description of the cohomology rings of the infinite complex and real Grassmannian manifolds using the coefficients $\mathbb{Z}$ and $\mathbb{Z}_2$ respectively, and with a brief introduction to Pontrjagin classes. Throughout this project all spaces are assumed to be Hausdorff.

Contents

1 Basics about Vector Bundles

In this first chapter we introduce the most basic concepts and results concerning vector bundles that will be used through out the rest of this monograph. Proofs in general will be omitted in this chapter since the main focus of this work is to treat characteristic classes of vector bundles. Nevertheless, the proofs can be found in complete detail in [5].

1.1 Definitions and Basic Examples

Definition 1.1. *A* ***real vector bundle*** η *over a topological space* B *consists of the following:*

1. *a topological space* $E = E(\eta)$ *called the* ***total space****,*
2. *a continuous map* $p : E \to B$ *called the* ***projection map****, and*
3. *for each* $b \in B$*,* $p^{-1}(b)$ *is a finite dimensional real vector space*

Furthermore, this definition is subject to the following ***local triviality condition****: For each point* $b_0 \in B$ *there exists an open neighborhood* U *of* b_0*, an integer* $n \geq 0$*, and a homeomorphism* $h : U \times \mathbb{R}^n \to p^{-1}(U)$ *such that for all* $b \in U$ *the restriction of* h *to* $b \times \mathbb{R}^n$ *is a vector space isomorphism between* $\mathbb{R}^n$ *and* $p^{-1}(b)$*.*

Complex vector bundles are defined in the exact same way with the only difference that we use complex Euclidean vector spaces as fibers.

A pair (U, h) as described above is called a ***local trivialization*** for η about b_0. A vector bundle is called trivial if there exists a local trivialization (U, h) with $U = B$.

The set $p^{-1}(b)$ is called the ***fiber over b*** and it will be also denoted sometimes as $F_b(\eta)$. Observe that by the local triviality condition we have that the correspondce between $b \in B$ and $\dim p^{-1}(b)$ is a continuous map. In particular, this correspondence is constant on connected components of the base space B. Through out most of this monograph, we shall be mainly interested on vector bundles with constant fiber dimension. Such vector bundles are usually called $n-$***vector bundles*** or $n-$***plane bundles*** where n is the dimension of the fibers.

Let $\eta = (E, B, p)$ and $\delta = (E'.B', p')$ be two $n-$vector bundles.

Definition 1.2. A ***bundle map*** between η and δ is a pair of maps $F : E \to E'$ and $f : B \to B'$ such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \downarrow p & & \downarrow p' \\ B & \xrightarrow{f} & B' \end{array}$$

and such that F is a linear isomorphism between the vector spaces $p^{-1}(b)$ and $p'^{-1}(f(b))$ for all $b \in B$. The map $f : B \to B'$ is usually called the ***base space map***.

Now assume that both η and δ are defiined over the same base space B.

Definition 1.3. η and δ are said to be ***isomorphic***, written $\eta \cong \delta$, if there exists a bundle map (F, f) between η and δ such that $f = Id_B$ and such that F is an homeomorphism.

The following lemma however tells us that in the previous definition we can ommit the condition of F being a homemorphism.

Lemma 1.4. *Let η and δ be two $n-$vector bundles over B and let (F, f) be a bundle map with $f = Id_B$. Then F is necessarily a homeomorphism. Hence η is isomorphic to δ.*

The following examples of real vector bundles will be of vital importance throughout the rest of our work.

Example 1.5. The *real projective space*, denoted by $\mathbb{R}P^n$, is the set of all unordered pairs $\{x, -x\}$ where x is in the unit sphere $S^n \subset \mathbb{R}^{n+1}$. We give the space $\mathbb{R}P^n$ the quotient topology.
Let $E(\gamma_1^n)$ be the subset of $\mathbb{R}P^n \times \mathbb{R}^{n+1}$ consisting of all points $(\{x, -x\}, v)$ with $v \in \operatorname{span}\{x\}$ and let $p : E(\gamma_1^n) \to \mathbb{R}P^n$ be the map defined as $p(\{x, -x\}, v) = \{x, -x\}$. We prove the following claim:

Claim: $\gamma_1^n = (E(\gamma_1^n), \mathbb{R}P^n, p)$ is locally trivial.

Proof. Let U be an open set in S^n which doesn't contain pairs of antipodal points, and let $U' \subset \mathbb{R}P^n$ be the image of U under the quotient map $q : S^n \to \mathbb{R}P^n$. Since U doesn't have any pairs of antipodal points we have that the restriction of q on U is an homeomorphism between U and U'. Furthermore, U' is open since $q^{-1}(U')$ is the union of U and $-U$. Define $h : U' \times \mathbb{R} \to p^{-1}(U')$ by $h(\{x, -x\}, t) = (\{x, -x\}, tx)$. Since clearly this map is an homeomorphism and a vector space isomorphism when restricted to fibers we conclude that γ_1^n is locally trivial. $\square$

The bundle γ_1^n is usually called the ***canonical line bundle*** over $\mathbb{R}P^n$.

We have the following important result concerning these bundles γ_1^n.

Proposition 1.6. *The bundles γ_1^n are not trivial for all $n \geq 1$.*

Definition 1.7. A ***cross section*** of a vector bundle $\eta = (E(\eta), B, p)$ is a continuous function $s : B \to E(\eta)$ such that $ps = Id_B$, that is, $s(b)$ lies in $p^{-1}(b)$.

A collection of sections $s_1, \ldots, s_m$ is said to be ***no-where dependent*** if $s_1(b), \ldots . s_m(b)$ is linearly independent for each $b \in B$.
The applicability of these notions becomes clear with the following proposition.

Proposition 1.8. *A $n-$vector bundle η is trivial if and only if η admits n no-where dependent sections $s_1, \ldots, s_n$.*

The proof of this result follows from lemma 1.4 as it is explained in [5].

1.2 Construction of Vector Bundles

In this section we shall discuss operations that yield us new vector bundles. We begin with the notion of a ***pull-back*** which shall be of great importance when we discuss the classificaton of vector bundles in the next chapter. Let $f : B \to B'$ be a continuous map and let $\eta = (E', B', p')$ be a $n-$vector bundle defined over B'. Consider the subspace E of $B \times E'$ consisting of all pairs (b, e) with $e \in p'^{-1}(f(b))$. Furthermore, let $p : E \to B$ be the usual projection map and consider the triple $f^*\eta = (E, B, p)$. We have the following result.

Proposition 1.9. *$f^*\eta = (E, B, p)$ satisfies the local triviality condition and hence $f^*\eta$ is a $n-$vector bundle, which is usually called the **pull-back** of η by f.*

Details of this proof can be found in [5]. The importance of this notion lies on the fact that, at least for paracompact base spaces, *any $n-$vector bundle is the pull-back* of a particular $n-$bundle, called the *universal $n-$plane bundle.* A result related to this concept is the following:

Proposition 1.10. *Let (F, f) be a bundle map between vector bundles $\eta = (E, B, p)$ and $\delta = (E', B', p')$. Then, we have that $\eta \cong f^*\delta$.*

Another important related proposition, whose proof is left as an excercise to the reader, is the following

Proposition 1.11. *Let $f : B \to B_1$ be a continuous map and let ξ_1 and ξ_2 be two $n-$vector bundles on B_1 such that $\xi_1 \cong \xi_2$. Then, we have that $f^*\xi_1 \cong f^*\xi_2$.*

Finally, observe that if $f : B_1 \to B$ is a continuous map and if ξ is a bundle on B, then the function $(Id, f) : f^*\xi \to \xi$ is clearly a bundle map.

Observe that propostion 1.11 allows us to define a contravariant functor $\mathrm{Vect}_n(-)$ from the category of topological spaces **Top** to the category **Set** where an object $\mathrm{Vect}_n(B)$ is the set of isomorphism classes of $n-$vector bundles defined on B and where a map $f : B \to B_1$ induces a function of sets $f^* : \mathrm{Vect}_n(B_1) \to \mathrm{Vect}_n(B)$ given by $\xi \to f^*\xi$.

Cross-products. If $\eta = (E', B', p')$ is a $n-$vector bundle and $\delta = (E'', B'', p'')$ is a $m-$vector bundle we define their ***cross-product*** by taking $B = B' \times B''$, $E = E' \times E''$ and $p : E \times E' \to B \times B'$ with $p(e_1, e_2) = (p'(e_1), p''(e_2))$. The triple $\eta \times \delta = (E, B, b)$ satisfies also the local triviality condition and thus it is a $(n + m)-$vector bundle.

Whitney sum of vector bundles. With the notions of pull-backs and cross-products we can define in a formal way an *'adding'* operation between vector bundles defined over the same base space. More explicitely, consider two vector bundles $\eta = (E', B, p')$ and $\delta = (E'', B, p'')$ of fiber dimension n and m respectively over the same base space B. By taking the diagonal map $d : B \to B \times B$ and pulling back $\eta \times \delta$ by d we obtain a new $(n + m)-$vector bundle $d^*(\eta \times \delta)$ over B. This new vector bundle, usually denoted by $\eta \oplus \delta$, is called the ***Whitney sum*** of δ and η.
As a consequence of proposition 1.4 we have the following proposition concerning Whitney sums

Proposition 1.12. *Let η be a vector bundle over some base space B. If η_1 and η_2 are two sub-bundles of η such that $F_b(\eta) = F_b(\eta_1) \oplus F_b(\eta_2)$ for all $b \in B$ then we have $\eta \cong \eta_1 \oplus \eta_2$*

Proof. Consider the map $f : E(\eta_1 \oplus \eta_2) \to E(eta)$ defined as $f(e_1, e_2) = e_1 + e_2$. Then, by 1.4 f is a bundle isomorphism. □

The previous proposition raises the following question: *if we have a vector bundle η over a space B and a sub-bundle ξ, is it always possible to find another sub-bundle ξ' such that $\eta \cong \xi \oplus \xi'$?* We shall show in one of the following sections that it is possible to find such a complement if η has a matric.

1.3 Continuous Functors

The Whitney sum falls in a more general method of constructing new vector bundles out of old ones. To describe this general method consider the category of finite dimensional (real or complex) vector spaces $\mathcal{V}$ and all the isomorphisms between such vector spaces. A ***continuous covariant functor*** $T : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is a covariant functor from the category $\mathcal{V} \times \mathcal{V}$ to $\mathcal{V}$ such that for any finite dimensional vector spaces V_1, V_2, W_1, W_2 we have that the map

$$T(\cdot,\cdot) : \mathrm{Hom}(V_1, W_1) \times \mathrm{Hom}(V_2, W_2) \to \mathrm{Hom}(T(V_1, V_2), T(W_1, W_2))$$

is continuous. Direct sums is a particular example of a continuous functors. An analogous definition holds completely when we consider the $n-$fold product category $\mathcal{V} \times \ldots \times \mathcal{V}$.

Let now $T : \mathcal{V} \times \ldots \times \mathcal{V} \to \mathcal{V}$ be a continuous functor on m factors and consider m vector bundles $\xi_1, \ldots \xi_m$ defined over a common base space B. For each $b \in B$ consider the vector space $F_b = T(F_b(\xi_1), \ldots, F_b(\xi_m))$. Let E be the disjoint union of all such vector spaces and define $\pi : E \to B$ by $\pi(F_b) = b$. We have then the following theorem

Theorem 1.13. *There exists a canonical topology for E so that E is the total space of a vector bundle with projection π and with fibers F_b*

This theorem enable us now to apply fiber wise other such vector space operations such as the tensor product $\otimes$, taking the dual $\mathrm{Hom}(\cdot, \mathbb{C})$ or $\mathrm{Hom}(\cdot, \mathbb{R})$, etc, in order to obtain new vector bundles.

1.4 Hermitian and Riemannian Metrics

We begin this section with the following definition

Definition 1.14. Let ξ be a complex vector bundle over B. A ***hermitian metric*** for ξ is a continuous function $u : E(\xi \oplus \xi) \to \mathbb{C}$ such that when restricted to each fiber F we have that u_F is an inner product for F.

In a totally analogous manner we define ***riemannian metrics*** for real vector bundles.

Let $\eta = (E, B, p)$ be a $n-$vector bundle with a metric u and let $\xi = (E', B, p')$ be a sub-bundle of η. For each fiber $F(b)$ of η let $F^{\perp}(b)$ be the orthogonal complement of $p'^{-1}(b)$. Denote by $E(\xi^{\perp})$ the union of all such sets $F^{\perp}(b)$ and let $p^{\perp} : E(\xi^{\perp}) \to B$ be the usual projection. Then, we have the following result concerning $\xi^{\perp} = (E(\xi^{\perp}), B, p^{\perp})$

Proposition 1.15. *$E(\xi^{\perp})$ is the total space of a sub-bundle $\xi^{\perp}$ of η. Furthermore, we have that $\eta \cong \xi \oplus \xi^{\perp}$.*

Definition 1.16. $\xi^{\perp}$ is called the ***orthogonal complement*** of ξ in η.

1.5 Specifics about Complex Vector Bundles

In this section we shall introduce some terminology concerning complex vector bundles which we shall need for the rest of this project.

Definition 1.17. Let $\xi = (E, B, p)$ be a complex $n-$vector bundle. The ***conjugate vector bundle*** of ξ, denoted by $\overline{\xi}$, is the vector bundle with the same underlying structure (E, B, p) as ξ but where scalar multiplication on each fiber of $\overline{\xi}$ is given by $\lambda \cdot a := \overline{\lambda} a$ where $\lambda \in \mathbb{C}$ and $a \in E$.

A particular important application of the notion of the conjugate of a vector bundle is summarized in the following proposition

Proposition 1.18. *Let $\xi = (E, b, p)$ be a complex line bundle with a hermitian metric $u : E(\xi \oplus \xi) \to \mathbb{C}$. Then, the complex line bundle $\xi \otimes \overline{\xi}$ is a trivial line bundle.*

Proof. A metric on ξ induces a continuous function $f : \xi \otimes \overline{\xi} \to B \times \mathbb{C}$ by the relation $f(a \otimes b) = (p(a), u(a, b))$. It is clear that this map preserves projections and it is not hard to verify that it is indeed linear on each fiber. Furhtermore, since clearly f induces a surjective map $\xi_b \otimes \xi_b^* \to b \times \mathbb{C}$ when restricted to each fiber then we must have that f is an isomorphism when restricted to fibers and thus it is a bundle isomorphism. $\square$

Definition 1.19. Let $\xi = (E, B, p)$ be a complex $n-$vector bundle. Then, the ***underlying real vector bundle*** ξ_R of ξ is the $2n$-real vector bundle with the same underlying structure (E, B, p) but where in each fiber F_b we just consider the real vector space structure.

1.6 Basics about Paracompact Spaces

We finish this first chapter by introducing the concept of a paracompact space and state the results which are relevant for the theory of vector bundles.

Definition 1.20. A space B is a ***paracompact space*** if it is a Hausdorff space and if for every open covering $\{U_i\}_{i \in I}$ of B, there exists an open covering $\{V_j\}_{j \in F}$ which

1. is a refinement of $\{U_i\}_{i \in I}$, and
2. is *locally finite*, that is, each $b \in B$ has an open neighborhood $V(b)$ which intersects only finitely many of the sets V_j.

Proposition 1.21. *A paracompact space is a normal space.*

For a proof of this proposition see [6].

Proposition 1.22. *The inductive limit of compacts spaces is a paracompact space.*

The following lemma will be used several times through out this project and its proof can be found in [5].

Lemma 1.23. *For any fiber bundle ξ over a paracompact space B, there exists a locally finite covering of B by countably many open sets $U_1, U_2, \ldots,$ so that ξ_{U_i} is trivial for each i.*

REMARK: For the rest of this project, we shall assume that *all spaces are paracompact spaces*. This is not such a harsh restriction since nearly all familiar topological spaces are paracompact (for example, all CW-complexes are paracompact), and furthermore, many of the arguments that appear in the literature of characteristic classes require that the base space of a vector bundle ξ has a countable covering $\{U_i\}_i$ such that ξ_{U_i} is trivial for all i.

2 Classification of Vector bundles

2.1 The Homotopy Theorem

Throughout this chapter we shall denote **Top** the category of paracompact spaces and by **Set** the category of sets. Furthermore, let us denote by $\mathrm{Vect}_n(B)$ the set of isomrphism classes of $n-$plane bundles over a paracompact space B. We shall do everything in this chapter for real vector bundles, since the proofs for compex vector bundles are completely analogous.

We start off this section by proving that the pullbacks of homotopic maps are isomorphic. This result, called the ***homotopy theorem*** will follow from the two following auxiliary lemmas.

Lemma 2.1. *If $\eta = (E, I, \pi)$ is an n-vector bundle defined over the unit interval $I = [0, 1]$, then η is a trivial bundle.*

Proof. For the vector bundle η it is possible to find a partition

$$0 = r_0 < \ldots < r_m = 1$$

such that $E_{[r_{i-1}, r_i]}$ is a trivial bundle. On $[r_0, r_1]$ we can contruct n linearly independent sections $s_{1,1}, \ldots, s_{1,n} : I \to E$, and furthermore, on $[r_1, r_2]$ we can construct n linearly independent sections $s_{2,1}, \ldots, s_{2,n} : I \to E$ such that $s_{1,i}(r_1) = s_{2,i}(r_1)$ for all $i = 1, \ldots, n$. Following this procedure inductively we obtain n linearly independent sections $s_1, \ldots, s_n : I \to E$ and thus η is a trivial bundle. □

Our main result will be a particular case of the following slightly more technical result.

Lemma 2.2. *Let B be a paracompact space and let $\xi = (E, B \times I, p)$ be a n vector bundle over $B \times I$. Then, the restrictions ξ_0 and ξ_1 over $B \times \{0\}$ and $B \times \{1\}$ respectively are isomorphic bundles when considered as bundles over B.*

Proof. We divide this proof into several steps:

Step 1: We begin by proving that we can find a locally finite open cover $\{U_i\}_{i \in F}$ for B such that $\xi_{U_i \times I}$ is a trivial bundle for each i. Fixing a point $b \in B$, we can find for each $t \in I$ open neighborhoods $U(t)$ and $V(t)$ of b and t respectively such that $\xi_{U(t) \times V(t)}$ is a trivial bundle. By the compactness of I we can find finitely many points $t_1, \ldots, t_m$ in I such that $\bigcup V(t_i) = I$. Denote by $U(b)$ the finite intersection $\bigcap_i U(t_i)$. Using the open sets $V(t_i)$ we can find a partition $0 = r_0 < \ldots < r_p = 1$ of I such that ξ is trivial on each set $U(b) \times [r_{i-1}, r_i]$. It follows now using a totally analogous argument used in the previous lemma that there exists n linearly independent sections $s_1, \ldots, s_n : U(b) \times I \to E$, which would make the restriction of ξ trivial over $U(b) \times I$. We complete this first step by observing that, given that the space B is paracompact, we can find a locally finite open covering $\{U_i\}_{i \in F}$ which refines $\{U(b)\}_{b \in B}$.

Step 2: Consider now the map $r : B \times I \to B \times I$ defined as $r(b,t) = r(b,1)$. The vital step of this proof will be to construct a bundle map $R : E \to E$ which has r as a base space map. Consider the locally finite open covering $\{U_i\}_{i \in F}$ of B which was constructed in the previous step. For all $i \in F$ we denote by $h_i : U_i \times I \times \mathbb{R}^n \to p^{-1}(U_i \times I)$ the local trivialization on $U_i \times I$. Since the space B is both paracompact and Hausdorff there exists locally finite open coverings $\{W_i\}_{i \in F}$ and $\{O_i\}_{i \in F}$ such that $\overline{O_i} \subset W_i$ and $\overline{W_i} \subset V_i$ for all $i \in F$ (See [M], page 294) and continuous functions $\alpha_i : B \to [0,1]$ such that $\alpha^{-1}(\{1\}) = \overline{O_i}$ and $\alpha^{-1}(\{0\}) = B - W_i$. For all $i \in F$ define $f_i : U_i \times I \times \mathbb{R}^n \to U_i \times I \times \mathbb{R}^n$ as $f_i(b,t,v) = (b, max\{t, \alpha_i(b)\}, v)$. Clearly these maps f_i are continuous. Finally, for all $i \in F$ define $R_i : E \to E$ in the following way:

$$R_i(x) = \begin{cases} x & \text{if } x \in E - p^{-1}(U \times I) \\ h_i f_i h_i^{-1}(x) & \text{if } x \in p^{-1}(U \times I) \end{cases}$$

Clearly R_i is continuous in $E - \overline{p^{-1}(W_i \times I)}$ since in this subset is just the identity map and the continuity of f_i implies that R_i is continuous in $p^{-1}(U_i \times I)$ and thus R_i is a bundle map. Giving F a well-ordering, the desired cover for $r : B \times I \to B \times I$ is the composition of all the maps R_i according to the ordering given to F. This map is well defined since every point $x \in B$ is contained in only finitely many of the open sets U_i and the fact that each x is in some O_i implies that R indeed covers r. For each $b \in B$, there exists an open neighborhood $V(b)$ which intersects finitely many of the sets U_i. Thus, on the open sets $p^{-1}(V(b) \times I)$ we have that R is continuous, which imples that R is continuous on all of E.

Step 3: The final step of this proof is to observe that when restricted to ξ_0, R_{ξ_0} is a biyective bundle map and the base space map r_{ξ_0} is such that $r_{\xi_0}(b,0) = (b,1)$. Thus, by proposition 1.10, we obtain that $E\xi_0$ and ξ_1 are equivalent bundles when viwed as bundles over B.

□

We are now ready to prove the promised result

Theorem 2.3. *The pullbacks of an $n-$plane bundle $\xi = (E,B,p)$ along homotopic maps $f_0, f_1 : A \to B$ are isomorphic.*

Proof. Let $F : A \times I \to B$ be a homotopy between f_0 and f_1. Then, the restrictions of $F^*\xi$ over $A\times\{0\}$ and $A\times\{1\}$ can be identified with the pullbacks $f_0^*\xi$ and $f_1^*\xi$ respectively. The result now follows from the previous lemma. $\square$

An immediate result of this theorem is the following corollary.

Corollary 2.4. *Any $n-$plane bundle defined over a contractible paracompact space B is trivial.*

The aim in the following sections, and really the heart of this chapter, will be to construct for any natural number n a vector bundle γ_n, called the ***'universal bundle'*** or ***'tautological bundle'***, over a space $G_n(\mathbb{R}^\infty)$ such that for any n-vector bundle ξ over a paracompact space B we can construct a bundle map $f : \xi \to \gamma_n$ unique up to homotopy. Thus, for any paracompact space B there is a one to one correspondence between $\mathrm{Vect}_n(B)$ and $[B, G_n(\mathbb{R}^\infty)]$. Furhtermore, it will be shown that this correspondence constitutes a natural equivalence between the functors $\mathrm{Vect}_n(-) : \mathbf{Top} \to \mathbf{Set}$ and $[-, G_n(\mathbb{R}^\infty)] : \mathbf{Top} \to \mathbf{Set}$.

2.2 Grassmann Manifolds and Universal Bundles

We begin this subsection by constructing the base space for our universal bundle. This base space $G_n(\mathbb{R}^\infty)$ will be the inductive limit of certain compacts manifolds $G_n(\mathbb{R}^{n+k})$, called Grassmann manifolds, making our space $G_n(\mathbb{R}^\infty)$ paracompact according to proposition 1.22. So let us begin by contructing the spaces $G_n(\mathbb{R}^{n+k})$.

Let $n \le q$ and consider the n-fold cross product $\mathbb{R}^q \times \ldots \times \mathbb{R}^q$. Let $V_n(\mathbb{R}^q)$ be the subspace conformed by all n-frames in $\mathbb{R}^q$ and let $V_n^0(\mathbb{R}^q)$ be the subspace of all orthonormal n-frames in $\mathbb{R}^q$. Since for any n-frame $(v_1, \ldots, v_n)$ in $\mathbb{R}^q$ we can find open neighborhoods $U_1, \ldots, U_n$ of $v_1, \ldots, v_n$ such that $U_1 \times \ldots \times U_n \subset V_n(\mathbb{R}^q)$ we have that $V_n(\mathbb{R}^q)$ is an open subset of $\mathbb{R}^q \times \ldots \times \mathbb{R}^q$ and thus a manifold. Furthermore, taking the map $F : (\mathbb{R}^q)^n \to M_n(\mathbb{R})$ defined as $F(v_1, \ldots, v_n) = (v_i \cdot v_j)_{i,j}$ it is easy to see that $V_n^0(\mathbb{R}^q) = F^{-1}(Id_n)$ making $V_0^n(\mathbb{R}^q)$ a closed subset of $(\mathbb{R}^q)^n$ and thus a compact subspace since it is also bounded. Let now $G_n(\mathbb{R}^q)$ be the set of all $n-$planes in $\mathbb{R}^q$. The function $q : V_n(\mathbb{R}^q) \to G_n(\mathbb{R}^q)$ which assigns to a $n-$frame the $n-$plane which it spans is surjective and thus we can topologize $G_n(\mathbb{R}^q)$ with the quotient topology. Analogously, we could have also given $G_n(\mathbb{R}^q)$ the quotient topology induced by the function $q_0 : V_n^0(\mathbb{R}^q) \to G_n(\mathbb{R}^q)$. Nevertheless, the commutativity of the following diagram shows that both quotient topologies are the same

$$\begin{tikzcd}
V_n^0(\mathbb{R}^q) \arrow[r, "i"] \arrow[dr, "q_0"'] & V_n(\mathbb{R}^q) \arrow[r, "GS"] \arrow[d, "q"] & V_n^0(\mathbb{R}^q) \arrow[dl, "q_0"] \\
& G_n(\mathbb{R}^q) &
\end{tikzcd}$$

where GS denotes the Gram-Schmidt process. Observe that by taking $n = 1$ we have that $G_1(\mathbb{R}^q) = \mathbb{R}P^{q-1}$.

We now stray slightly away from the main goal of this chapter to prove the following proposition concerning the topological nature of $G_n(\mathbb{R}^q)$.

Proposition 2.5. *$G_n(\mathbb{R}^q)$ is a compact topological manifold of dimension $n(q-n)$.*

Proof. The compactness of $G_n(\mathbb{R}^q)$ follows easily from the fact that $G_n(\mathbb{R}^q)$ is the quotient of $V_n^0(\mathbb{R}^q)$.
Let us now prove that $G_n(\mathbb{R}^q)$ is a Hausdorff space. To do this it is enough to show that any two points in $G_n(\mathbb{R}^q)$ are seperated by a real valued function. For a fixed $w \in \mathbb{R}^q$ let $d_w : G_n(\mathbb{R}^q) \to \mathbb{R}$ be the function which assigns to $X \in G_n(\mathbb{R}^q)$ the square of the distance of w to X. If $x_1, \ldots, x_n$ is an orthonormal basis for X then the identity

$$d_w(X) = (w \cdot w)^2 - (w \cdot x_1)^2 + \ldots + (w \cdot x_n)^2$$

implies that the composition

$$V_n^0(\mathbb{R}^q) \xrightarrow{q_0} G_n(\mathbb{R}^q) \xrightarrow{d_w} \mathbb{R}$$

is continuous and by properties of the quotient topology d_w is also continuous. If X and Y are two different $n-$planes in $\mathbb{R}^q$ and if w is a vector in X but not in Y then clearly $d_w(X) \neq d_w(Y)$. Thus, $G_n(\mathbb{R}^q)$ is a Haussdorf space.
Now we prove that each $n-$plane $X \in G_n(\mathbb{R}^q)$ has an open neighborhood homeomorphic to $\mathbb{R}^{n(q-n)}$. It will be convenient for this proof to view $\mathbb{R}^q$ as the direct sum $X \oplus X^\perp$. Let $\pi : \mathbb{R}^q \to X$ be the orthogonal projection onto X and consider the set U of all $n-$planes Y which are mapped isomorphically onto X through π, or equivalently, all $n-$planes Y such that $Y \cap X^\perp = \{0\}$. U is an open set since if $x_1, \ldots, x_n$ is an $n-$frame which spans an $n-$plane $X_1 \in U$ then it is possible to find open neighborhoods $U_1, \ldots, U_n$ of $x_1, \ldots, x_n$ such that $U_1 \times \ldots \times U_n \subset q^{-1}(U)$. Then, $q^{-1}(U)$ is an open set which by definition implies that U is also an open set. The key observation in order to prove that U is homeomorphic to $\mathbb{R}^{n(q-n)}$ is that each $Y \in U$ determines a linear transformation $T_Y : X \to X^\perp$ and that each linear map $T : X \to X^\perp$ determines an $n-$plane $Y[T] \in U$. Indeed, if $Y \in U$ then each $y \in Y$ can be written uniquely as $y = x + x'$ with $x \in X$ and $x' \in X^\perp$ and the association $T_y(x) = x'$ constitutes a linear homomorphism $T_Y : X \to X^\perp$. Conversely, if $T : X \to X^\perp$ is a linear map then the set $Y_T = \{x + T(x) : x \in X\}$ is an $n-$plane such that $Y \cap X^\perp = \{0\}$. Furthermore, it is easy to verify that the functions $Y \to T_Y$ and $T \to Y_T$ are mutually inverse. We denote now by $T : U \to \mathrm{Hom}(X, X^\perp)$ the function $Y \to T_Y$ and we shall now prove that this is in fact a homeomorphism, which would make U homeomorphic to $\mathbb{R}^{n(q-n)}$.Take a fixed basis $x_1, \ldots, x_n$ for X. Instead of proving directly that $T : U \to \mathrm{Hom}(X, X^\perp)$ is continuous we shall prove the continuity of the map

$S : U \to (X^{\perp})^n$ defined as $S(Y) = (T_Y(x_1), \ldots, T_Y(x_n))$. In order to show that S is continuous we consider the following continuous maps:

- The orthogonal projection $\pi : \mathbb{R}^q \to X$ onto X.
- The orthogonal projection $\pi' : \mathbb{R}^q \to X^{\perp}$ onto $X^{\perp}$.
- The map $D : q^{-1}(U) \to q^{-1}(U)$ which assigns to an $n-$frame $(y_1, \ldots, y_n)$ the unique $n-$frame $(y_1', \ldots, y_n')$ which spans $q((y_1, \ldots, y_n))$ and which satisfies $\pi(y_i') = x_i$ for all $i = 1, \ldots, n$.
- The map $P : q^{-1}(U) \to (X^{\perp})^n$ defined as $P(y_1, \ldots, y_n) = (\pi'(y_1), \ldots, \pi'(y_n))$.

Then, since clearly we have that $S \circ q(y_1, \ldots, y_n) = P \circ D(y_1, \ldots, y_n)$ it follows that $S \circ q$ is continuous and thus S is continuous. The continuity of S implies then the continuity of T.
In order to show that T^{-1} is continuous consider the map $D' : \mathrm{Hom}(X, X^{\perp}) \to q^{-1}(U)$ defined as $D'(T) = (x_1 + T(x_1), \ldots, x_n + T(x_n))$. Since we clearly have that $T^{-1} = q \circ D'$ we conclude that T^{-1} is also continuous.

□

We delay the construction of our universal bundle a little bit more by defining first on each Grassman manifold $G_n(\mathbb{R}^q)$ the following canonical $n-$vector bundle. Consider the subspace E of $G_n(\mathbb{R}^q) \times \mathbb{R}^q$ consisting of pairs (Y, v) with $v \in Y$ and define $\pi : E \to G_n(\mathbb{R}^q)$ as the obvious projection. The following proposition proves that the triple $\gamma_n^q = (E, G_n(\mathbb{R}^q), \pi)$ is indeed a vector bundle.

Proposition 2.6. $\gamma_n^q = (E, G_n(\mathbb{R}^q), \pi)$ *is a $n-$vector bundle over* $G_n(\mathbb{R}^q)$

Proof. All we have to show is that $\gamma_n^q = (E, B, \pi)$ satisfies the local triviality condition. Take a fixed $n-$plane $X \in G_n(\mathbb{R}^q)$ and consider the same open neighborhood U of X used in proposition 2.5, that is, all $n-$planes Y such that the projection of Y onto X is an isomorphism. Then clearly, if p_X is the orthogonal projection onto X, the map $h : \pi^{-1}(U) \to U \times X$ defined as $h(Y, v) = (Y, p_X(v))$ is a continuous map and an isomorphism on each fiber. Furthermore, the map $g : U \times X \to \pi^{-1}(U)$ given by $g(Y, x) = (Y, x + T_Y(x))$ is continuous and it is the inverse of h. Thus, $h : \pi^{-1}(U) \to U \times X$ is a local trivialization.

□

Now for the construction of our universal bundle. Identifying $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{n+1}$ in the canonical way for all n, we obtain the following inclusion of Grassmann manifolds

$$G_n(\mathbb{R}^n) \subset G_n(\mathbb{R}^{n+1}) \subset \ldots \subset G_n(\mathbb{R}^q) \subset \ldots$$

We define $G_n(\mathbb{R}^\infty)$ to be the union $\bigcup_q G_n(\mathbb{R}^q)$ topologized with the weak topology and the universal bundle γ_n is defined as the subspace E of $G_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty$ consisting of pairs (Y, v) with $v \in Y$. By proposition 1.22, we have that

$G_n(\mathbb{R}^\infty)$ is a paracompact space. Observe that by taking $n = 1$ we have that $G_1(\mathbb{R}^\infty) = \mathbb{R}P^\infty$. As we did for vector bundles on Grassmann manifolds we will prove that $\gamma_n = (E, G_n(\mathbb{R}^\infty), \pi)$ is a vector bundle. The proof of this proposition will follow from 2.6 and the following technical lemma, which is proved in [5], page 64.

Lemma 2.7. *Let $A_1 \subset A_2 \subset \ldots$ and $B_1 \subset B_2 \subset \ldots$ be sequences of locally compact spaces with direct limits A and B respectively. Then, the Cartesian product topology on $A \times B$ coincides with weak topology which is associated with the sequence $A_1 \times B_1 \subset A_2 \times B_2 \subset \ldots$.*

Proposition 2.8. *$\gamma_n = (E, G_n(\mathbb{R}^\infty), \pi)$ is a vector bundle over $G_n(\mathbb{R}^\infty)$.*

Proof. Let $X \in G_n(\mathbb{R}^\infty)$ and consider the set $U \subset G_n(\mathbb{R}^\infty)$ of all $n-$planes in $\mathbb{R}^\infty$ which project onto X under the orthogonal projection $p : \mathbb{R}^\infty \to X$. This set U is an open neighborhood of X since in the proof of proposition 2.5 it was shown that $U_k = U \cap G_n(\mathbb{R}^{n+k})$ is open for all k. Define $h : \pi^{-1}(U) \to U \times X$ as $h(Y, v) = (Y, p(v))$. It was shown in proposition 2.6 that the restriction of h on each U_k is continuous and thus applying lemma 2.7 we obtain that h is continuous. We define h^{-1} in a totally analogous manner as we did in 2.6 and applying again 2.7 we can prove that h^{-1} is continuous. Since h is clearly an isomorpism when restricted to fibers, we obtain that h is a local trivialization. $\square$

The reason for the vector bundle γ_n to be called 'universal' follows from the fact that any vector bundle $\xi = (E, B, p)$ over a paracompact space B can be mapped through a bundle map into γ_n. We show now how to construct such a map. The guiding line for such a construction will be the following somewhat technical lemma.

Lemma 2.9. *Let $\xi = (E, B, p)$ be an $n-$vector bundle. Then a map $f : E \to \mathbb{R}^\infty$ which is linear and injective on each fiber of E determines a bundle map $F : E \to \gamma_n$ through the identity $F(x) = (g(p(x)), f(x))$ with $g : B \to G_n(\mathbb{R}^\infty)$ defined as $g(b) = f(p^{-1}(\{b\}))$. Conversely, if $F : E \to \gamma_n$ is a bundle map and if $p_{\mathbb{R}^\infty} : G_n(\mathbb{R}^\infty) \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ is the projection onto $\mathbb{R}^\infty$ then $f : E \to \mathbb{R}^\infty$ defined as $f = p_{\mathbb{R}^\infty} \circ F$ is a map which maps every fiber of E injectively into $\mathbb{R}^\infty$.*

Proof. The latter statement is trivial. Suppose then that $f : E \to \mathbb{R}^\infty$ is a map which is linear and injective on each fiber. All that remains to be done in order to show that F is indeed a bundle map is to prove that $g : B \to G_n(\mathbb{R}^\infty)$ is continuous. Let $\{U_i\}_{i \in \mathbb{N}}$ be an open covering of B such that E is trivial on each U_i. Consider for each U_i the following functions:

- The local trivialization $h_i : U_i \times \mathbb{R}^n \to p^{-1}(U_i)$.
- $w_i : U_i \to (U_i \times \mathbb{R}^n)^n$ with $w(b) = ((b, e_1), \ldots, (b, e_n))$ where $e_1, \ldots, e_n$ is the canonical basis for $\mathbb{R}^n$.

- $H_i : (U_i \times \mathbb{R}^n)^n \to (E)^n$ with $H_i((b_1, v_1), \ldots, (b_n, v_n)) = (h_i(b_1, v_1), \ldots, h_i(b_n, v_n))$.
- $G_i : (p^{-1}(U_i))^n \to (\mathbb{R}^\infty)^n$ with $G_i(x_1, \ldots, x_n) = (f(x_1), \ldots f(x_n))$

Since the local trivializations h_i are isomorphisms when restricted to fibers and since $f : E \to \mathbb{R}^\infty$ is injective, we have that $G_i \circ H_i \circ w_i(b) \in V_n(\mathbb{R}^\infty)$ for all $b \in U_i$. Thus, when restricted to each U_i, g is equal to the composition $q \circ G_i \circ H_i \circ w_i$, which makes g continuous.

□

In the previous proof q denotes the map $q : V_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ which maps an $n-$frame of $\mathbb{R}^\infty$ to the $n-$plane which it spans. The continuity of this map follows from lemma 2.7.

Theorem 2.10. *If $\xi = (E, B, p)$ is an $n-$vector bundle over a paracompact space B then there exists a bundle map $F : \xi \to \gamma_n$.*

Proof. Let $\{U_i\}_i$ be a countable locally finite open cover of B such that E_{U_i} is trivial for each U_i (See proposition 1.23). Furthermore, let $\{V_i\}_i$ and $\{W_i\}_i$ be countable locally finite open covers such that $\overline{W_i} \subset V_i$ and $\overline{V_i} \subset U_i$ for each i (See [6], page 294). Since B is normal by proposition 1.21 there exists for each i a continuous map $\lambda_i : B \to [0, 1]$ such that $\lambda_i(\overline{W_i}) = 1$ and $\lambda_i(V_i^c) = 0$. Let $h_i : U_i \times \mathbb{R}^n \to p^{-1}(U_i)$ be again the local trivialization on U_i and denote by $r_i : U_i \times \mathbb{R}^n \to \mathbb{R}^n$ the projection map onto $\mathbb{R}^n$. For each i define $H_i : E \to \mathbb{R}^n$ as

$$H_i(x) = \lambda_i(p(x)) \cdot r_i h_i^{-1}(x)$$

for all $x \in p^{-1}(U_i)$ and $H_i(x) = 0$ for all $x \in p^{-1}(U_i)^c$. Clearly each H_i is continuous. Define finally $f : E \to \oplus_{i=1}^\infty \mathbb{R}^n = \mathbb{R}^\infty$ as $f(x) = (H_i(x))_{i=1}^\infty$. Since the open cover $\{V_i\}_i$ is locally finite we have that f is well defined. Furhtermore, since the local trivializations h_i are isomorphisms when restricted to fibers we obtain that f maps each fiber of E injectively into $\mathbb{R}^\infty$. Applying lemma 2.9 we conclude that there exists a bundle map $F : \xi \to \gamma_n$. □

Implicit in the proof of the previous theorem we have the following result which will be of great applicability in forthcoming sections

Proposition 2.11. *Every $n-$vector bundle ξ over a paracompact space B admits a metric. In particular, any sub-bundle of a bundle over a paracompact space has an orthogonal complement.*

Proof. Let $f : E(\xi) \to \mathbb{R}^\infty$ be the map constructed in the proof of theorem 2.10. Let $g : \mathbb{R}^\infty \oplus \mathbb{R}^\infty \to \mathbb{R}$ be the map defined as $g(v, w) = v \cdot w$, which is continuous since it is continuous on each subset $\mathbb{R}^n \oplus \mathbb{R}^n$ and define $\mu : E(\xi \oplus \xi) \to \mathbb{R}$ as $\mu(x, y) = g(f(x), f(y))$. Clearly μ is a continuous map and since f is injective we have that μ yields an inner product on each fiber of ξ.

□

In order to prove that there is a one to one correspondence between $\mathrm{Vect}_n(B)$ and $[B, G_n(\mathbb{R}^\infty)]$ for any paracompact space B we will prove that the map $F : \xi \to \gamma_n$ constructed in theorem 2.10 is unique up to homotopy. Even more so, we will prove that such a bundle map F is unique up to bundle homotopy, i.e., if $F_0 : \xi \to \gamma_n$ and $F_1 : \xi \to \gamma_n$ are two bundle maps then there exists a homotopy $H_t : \xi \to \gamma_n$ between F_0 and F_1 such that each H_t is also a bundle map for all $t \in [0, 1]$. In the proof of this result we will use the following lemma.

Lemma 2.12. *Scalar multiplication and addition are continuous operations in* $\mathbb{R}^\infty$.

The proof of this result is a direct application of lemma 2.7.

Theorem 2.13. *If $\xi = (E, B, p)$ is an $n-$vector bundle over a paracompact space B then any two bundle maps $F_0 : \xi \to \gamma_n$ and $F_1 : \xi \to \gamma_n$ are bundle homotopic.*

Proof. Let $f_0 : E \to \mathbb{R}^\infty$ and $f_1 : E \to \mathbb{R}^\infty$ be the maps induced respectively by F_0 and F_1 as it was explained in lemma 2.9. Let us assume first that $f_0(x)$ is not a negative multiple of $f_1(x)$ for any $x \in E$. For all $t \in [0, 1]$ define $h_t : E \to \mathbb{R}^\infty$ as $h_t(x) = (1 - t)f_0(x) + tf_1(x)$. Applying lemma 2.12 we have that h_t constitutes an homotopy between f_0 and f_1. Furthermore, by the condition that $f_0(x)$ is not a negative scalar multiple of $f_1(x)$ we have that h_t maps each fiber of E inyectively into $\mathbb{R}^\infty$. By lemma 2.9 the maps h_t induce bundle maps $H_t : \xi \to \gamma_n$. Now, if $\{U_i\}_i$ is a countable locally finite open cover of B such that E_{U_i} is trivial for each U_i and if g_0, g_1 and g_t are the base space maps of F_0, F_1 and H_t respectively then it is not hard to verify that g_t constitutes an homotopy between g_0 and g_1 when restricted to each U_i and thus it is an homotopy on all of B. Thus, the maps $H_t(x)$ can be expressed as $H_t(x) = (g_t(p(x)), h_t(x))$ and it follows that they constitute a bundle homotopy between F_0 and F_1.

To tackle the general case let $d_0 : \mathbb{R}^\infty \to \mathbb{R}^\infty$ and $d_1 : \mathbb{R}^\infty \to \mathbb{R}^\infty$ be the linear maps defined as $d_0(e_i) = e_{2i}$ and $d_1(e_i) = e_{2i+1}$ for all elements e_i of the canonical basis for $\mathbb{R}^\infty$. If F_0' and F_1' are the bundle maps induced respectively by $d_0 f_0$ and $d_1 f_1$ then by applying the previous case we obtain that $F_0 \cong F_0'$, $F_0' \cong F_1'$ and $F_1' \cong F_1$, where all the homotopy equivalences are actually bundle homotopy equivalences. Thus, F_0 and F_1 are bundle homotopic.

□

As a straight forward yet useful corollary of the previous result we have the following

Corollary 2.14. *Let B be a paracompact space. If $f : B \to G_n(\mathbb{R}^\infty)$ and $g : B \to G_n(\mathbb{R}^\infty)$ are two maps such that $f^*\gamma_n \cong g^*\gamma_n$ then $f \cong g$.*

2.3 Classification of Vector Bundles

For all paracompact spaces B define $\eta_B : \mathrm{Vect}_n(B) \to [B, G_n(\mathbb{R}^\infty)]$ to be the function which sends an equivalnece class $[\xi] \in \mathrm{Vect}_n(B)$ to the homotopy class

$[g]$ of the base space map of the bundle map $(F,g):\xi\to\gamma_n$ constructed in theorem 2.10. This function is well defined: Indeed, if $\xi_1=(E_1,B,p_1)$ and $\xi_2=(E_2,B,p_2)$ are two $n-$vector bundles over B such that $[\xi_1]=[\xi_2]$ then there exists a bundle map $(S,Id):\xi_1\to\xi_2$ such that S is an homeomorphism. If $(F_1,g_1):\xi_1\to\gamma_n$ and $(F_2,g_2):\xi_2\to\gamma_n$ are the bundle maps obtained from theorem 2.10 we have that $\eta_B([\xi_1])=[g_1]$ and $\eta_B([\xi_2])=[g_2]$. On the other hand, $(F_1\circ S^{-1},g_1):\xi_2\to\gamma_n$ is another bundle map from ξ_2 to γ_n and by theorem 2.13 we have that (F_2,g_2) and $(F_1\circ S^{-1},g_1)$ are bundle homotopic. In particular, we obtain that $[g_1]=[g_2]$. We have reached the point where we can prove the central result of this chapter.

Theorem 2.15. *The collection of maps* $\eta_B:\mathrm{Vect}_n(B)\to[B,G_n(\mathbb{R}^\infty)]$ *constitutes a natural equivalence between the contravariant functors* $\mathrm{Vect}_n(-)$ *and* $[-,G_n(\mathbb{R}^\infty)]$.

Proof. We have to show both that each η_B is biyective and that all of the maps η_B are natural.

- *Injectivity of* η_B: Suppose that $[\xi_1]$ and $[\xi_2]$ in $\mathrm{Vect}_n(B)$ are such that $\eta_B([\xi_1])=\eta_B([\xi_2])$, that is, if $(F_1,g_1):\xi_1\to\gamma_n$ and $(F_2,g_2):\xi_2\to\gamma_n$ are the bundle maps obtained by theorem 2.10 then $[g_1]=[g_2]$. By proposition 1.10 of chapter 1 we have that ξ_1 is isomorphic to $g_1^*\gamma_n$ and that ξ_2 is isomorphic to $g_2^*\gamma_n$. But by theorem 2.3 we must have that $g_1^*\gamma_n\cong g_2^*\gamma_n$ and thus $\xi_1\cong\xi_2$.
- *Surjectivity of* η_B: Let $[g]$ be any element in $[B,G_n(\mathbb{R}^\infty)]$ and consider the pull-back $g^*\gamma_n$. Then, we clearly have that $\eta_B([g^*\gamma_n])=[g]$.
- *Naturality of* η_B: Let B and A be two paracompact spaces and let $f:B\to A$ be any continuous map. We need to verify that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Vect}_n(A) \arrow[r, "\eta_A"] \arrow[d, "f^*"'] & {[A,G_n(\mathbb{R}^\infty)]} \arrow[d, "f^*"] \\
\mathrm{Vect}_n(B) \arrow[r, "\eta_B"] & {[B,G_n(\mathbb{R}^\infty))]}
\end{tikzcd}$$

  Let ξ be any $n-$vector bundle over A. Let $(F,g):\xi\to\gamma_n$ and $(F',g'):f^*\xi\to\gamma_n$ be the bundle maps obtained by theorem 2.10 and let $(S,f):f^*\xi\to\xi$ be the obvious bundle map between $f^*\xi$ and ξ. Thus we have that $\eta_B([f^*\xi])=[g']$ and $f^*(\eta_A([\xi]))=f^*([g])=[g\circ f]$. On the other hand, since (F',g') and $(F\circ S,g\circ f)$ are bundle homotopic we have in particular that $g'\cong g\circ f$ and thus $\eta_B([f^*\xi])=f^*(\eta_A([\xi]))$.

□

Remark 1: It is worth noting that everything that has been done in this chapter holds also for *complex* vector bundles by making the necessary modifications and we shall make use of this results throughout the rest of the project.

In particular, we can also define the complex Grassman manifolds $G_n(\mathbb{C}^q)$ with $n \leq q$ and $G_n(\mathbb{C}^\infty)$ and the canonical n-plane bundles γ_n^q and γ_n as we did in the real case. Furthermore, observe that by taking $n = 1$ we have that $G_1(\mathbb{C}^q) = \mathbb{C}P^{q-1}$ and $G_1(\mathbb{C}^\infty) = \mathbb{C}P^\infty$.

3 Characteristic Classes of Vector Bundles

3.1 Definition of Characteristic Classes

We start this chapter by defining the most central notion of this project, that of a characteristic class of $n-$vector bundles.

Definition 3.1. *Let R be a ring. A **characteristic class** c of degree q for $n-$vector bundles is a natural assigment of a cohomology class $c(\xi) \in H^q(B; R)$ to an $n-$bundle ξ over a paracompact space B, i.e., if $\xi_1 = (E_1, B_1, p_1)$ and $\xi_2 = (E_2, B_2, p_2)$ are two $n-$vector bundles and if $(F, f) : \xi_1 \to \xi_2$ is a bundle map then $f^*(c(\xi_2)) = c(\xi_1)$.*

Clearly if c is a characteristic class of degree q and if ξ_1 and ξ_2 are two isomorphic $n-$vector bundles over a paracompact space B then $c(\xi_1) = c(\xi_2)$. Thus a characteristic class c of degree q actually determines a natural tansformation $c_B : \mathrm{Vect}_n(B) \to H^q(B; R)$ between the contravariant functors $\mathrm{Vect}_n(-) :$ **Top** $\to$ **Set** and $H^*(-; R) :$ **Top** $\to$ **Set**.

Let Λ_q be the set of all characteristic classes of degree q for $n-$vector bundles. If c_1 and c_2 are in Λ_q then we define $c_1 + c_2$ as the natural transformation which assigns to an $n-$vector bundle $\xi = (E, B, p)$ the cohomology class $c_1(\xi) + c_2(\xi) \in H^q(B; R)$. Clearly each Λ_q becomes an abelian group with this operation. Furhtermore, if $c_1 \in \Lambda_q$ and $c_2 \in \Lambda_p$, define $c_1 \cdot c_2$ to be the natural transformation which assigns to each $n-$vector bundle $\xi = (E, B, p)$ the cohomology class $c_1(\xi) \cup c_2(\xi) \in H^{p+q}(B; R)$. With this operation, we obtain a graded ring $\Lambda = \bigoplus_q \Lambda_q$, the ***ring of characteristic classes*** for $n-$vector bundles over paracompact spaces. Notice that this ring has an identity element: Indeed, the assigment $\xi = (B, E, p) \to 1 \in H^0(B; R)$ is a characteristic class of degree 0 and it acts as an identity in the ring of characteristic classes.

To prove that characteristic classes for $n-$plane bundles do indeed exist take $k \in H^m(G_n(\mathbb{F}^\infty); R)$ to be any cohomology class of degree m of the infinite Grassmann manifold $G_n(\mathbb{F}^\infty)$ (Here $\mathbb{F}$ stands either for the complex or real numbers). If $\xi = (E, B, p)$ is a $n-$vector bundle over a paracompact space B then there exists a bundle map $(F, g) : \xi \to \gamma_n$ unique up to bundle homotopy. We denote by $k(\xi)$ the cohomology class $g^*(k) \in H^m(B; R)$. Note that the definition of $k(\xi)$ does not depend on the bundle map $(F, g) : \xi \to \gamma_n$. The existence of characteristic classes will then follow from the following lemma.

Lemma 3.2. *The assigment $\xi = (E, B, p) \to k(\xi))$ is natural for any cohomology class $k \in H^m(G_n(\mathbb{F}^\infty); R)$.*

Proof. Let $\xi_1 = (E_1, B_1, p_1)$ and $\xi_2 = (E_1, B_1, p_1)$ be two $n-$vector bundles and let $(F, g) : \xi_1 \to \xi_2$ be a bundle map. Let furthermore $(F_1, g_1) : \xi_1 \to \gamma_n$ and $(F_2, g_2) : \xi_2 \to \gamma_n$ be bundle maps whose existence is guaranteed by theorem 2.10. Since (F_1, g_1) and (F_2F, g_2g) are bundle homotopic we have in particular that $g_1 \cong g_2g$ and thus $g_1^* = g^*g_2^*$ which imples that $g^*(k(\xi_2)) = k(\xi_1)$. $\square$

From this lemma we thus have that the assigment $\xi \to k(\xi)$ is a characteristic class which we shall denote simply by $\overline{k}$. But we can prove an even stronger result.

Theorem 3.3. *The map* $\Gamma : H^*(G_n(\mathbb{F}^\infty); R) \to \Lambda$ *defined as* $\Gamma(k) = \overline{k}$ *is a ring isomorphism.*

Proof. Clearly two different cohomology classes of $G_n(\mathbb{F}^\infty)$ will determine different characteristic classes and hence the map Γ is injective. To prove surjectivity let c be some characteristic class of $n-$plane bundles and set $k = c(\gamma^n)$. Let $\xi = (E, B, p)$ be an arbitrary $n-$plane bundle and let $(F, g) : \xi \to \gamma^n$ be some bundle map. Then, by the naturality of c, we must have that $c(\xi) = g^*(c(\gamma^n)) = g^*(k) = k(\xi)$ and thus $c = k$. $\square$

The previous theorem implies that the behaviour of all characteristic classes for $n-$plane bundles is determined by the ring $H^*(G_n(\mathbb{F}^\infty); R)$. It is then natural to ask if the cohomology rings $H^*(G_n(\mathbb{F}^\infty); R)$ have an explicit form. In the last chapter of this project we will show that this can be done for real vector bundles in the case when the ring of coeffcients is $R = \mathbb{Z}_2$ and for complex vector bundles when $R = \mathbb{Z}$. It will be proven in the last chapter that the ring of characteristic classes for *real* $n-$vector bundles is equal to $\mathbb{Z}_2[\omega_1, \dots, \omega_n]$ where the characteristic classes $\omega_1, \dots, \omega_n$ are called the ***Stiefel-Whitney classes***, which will be the main focus of the rest of this chapter. Parallel to the real case, it will be proven also in the last chapter that when working with $\mathbb{Z}$ the ring of characteristic classes for complex $n-$vector bundles is equal to $\mathbb{Z}[c_1, \dots, c_n]$ where the classes c_i are the ***Chern classes***, which will be treated in absolute detail in the last chapter. But until then, we shall just focus on the real case and from now on all vector bundles are assumed to be real unless otherwise stated.

3.2 Stiefel-Whitney Classes

It is the aim of this section to introduce the most important characteristic classes for real vector bundles when working with the ring $\mathbb{Z}_2$: ***Stiefel-Whitney classes***. As it was hinted at the end of the previous section, such classes are the building blocks for all charactreisitic classes when dealing with $\mathbb{Z}_2$. In particular, one of the main objectives of this project will be to prove the following.

Theorem 3.4. *For real $n-$plane bundles ξ over base spaces B, $n \geq 0$, there are characteristic classes $\omega_i(\xi) \in H^i(B; \mathbb{Z}_2)$, called the* ***Stiefel-Whitney classes****. They satisfy and are uniquely determined by the following axioms:*

1. $\omega_0(\xi) = 1$ *and* $\omega_i(\xi) = 0$ *if* $i > dim\xi$.

2. $\omega_i(\xi \oplus \delta) = \sum_{j=0}^{i} \omega_j(\xi) \cup \omega_{i-j}(\delta)$.

3. $\omega_1(\gamma_1) \neq 0$ *for the universal line bundle defined on* $\mathbb{R}P^\infty$.

Furthermore, every mod 2 characteristic class for $n-$*plane bundles can be written uniquely as a polynomial in the Stiefel-Whitney classes* $\{\omega_1, \ldots, \omega_n\}$.

This theorem will be proven in the last chapter together with its complex counterpart. It is one of the features of this project to show that both Stiefel-Whitney and Chern classes have a common origin and that they can be treated simultaneously. Nevertheless, in this chapter and the next we discuss an alternative way of defining of Stiefel-Whitney classes relying on the Steenrod squaring operations and the Thom isomorphism. We close this section by proving the following trivial consequences of 3.4.

Corollary 3.5. *1. If* ϵ *is a trivial* $m-$*vector bundle over a space* B *then* $\omega_i(\epsilon) = 0$ *for all* $i > 0$.
2. If η *is a* $n-$*vector bundle over a space* B *then* $\omega_i(\eta \oplus \epsilon) = \omega_i(\eta)$ *for all* i.

Proof. 1. This is poved by just considering a bundle mab $(F, g) : \epsilon \to L$ where L is just an $m-$bundle over a point.
2. Just apply the previous result and axiom 2 of the Stiefel-Whitney classes.

□

3.3 Definition of The Stiefel-Whitney Classes

In this section we describe one way of proving the existence of the Stiefel-Whitney classes.

Let $\xi = (E, B, \pi)$ be an $n-$vector bundle. If F is a fiber of ξ let us denote by F_0 the set of all non-zero elements of the fiber and similarly, let us denote by E_0 the set of all non-zero elements of the total space E. Since for all $b \in B$ we have that the pair $(\pi^{-1}(b), \pi_0^{-1}(b))$ is homotopy equivalent to the pair $(\mathbb{R}^n, \mathbb{R}^n - 0)$ we obtain for all non-negative integers i that

$$H^i(\pi^{-1}(b), \pi_0^{-1}(b); \mathbb{Z}_2) = \left\{ \begin{array}{cl} 0 & \text{if } i \neq n \\ \mathbb{Z}_2 & \text{if } i = n \end{array} \right.$$

Less trivially however is the following:

$$H^i(E, E_0; \mathbb{Z}_2) = \left\{ \begin{array}{rl} 0 & \text{if } i < n \\ H^{i-n}(B; \mathbb{Z}_2) & \text{if } i \geq n. \end{array} \right.$$

This will follow from the following theorem, proved in the chapter 5, and which is one of the main tools when proving the existence of Stiefel Whitney Classes.

Theorem 3.6. *The group $H^i(E, E_0; \mathbb{Z}_2)$ is zero for $i < n$, and $H^n(E, E_0; \mathbb{Z}_2)$ contains a unique class u such that for each fiber $F = \pi^{-1}(b)$ the restriction*

$$u\,|(F, F_0) \in H^n(F, F_0; \mathbb{Z}_2)$$

is the unique non-zero class in $H^n(F, F_0; \mathbb{Z}_2)$. Furthermore, the correspondence $x \to x \cup u$ defines an isomorphism $\cdot \cup u : H^k(E; \mathbb{Z}_2) \to H^{k+n}(E, E_0; \mathbb{Z}_2)$ for every k.

The cohomology class $u \in H^i(E, E_0; \mathbb{Z}_2)$ is called the ***fundamental cohomology class*** of ξ.

Observe also that the the total space E deformation retracts onto the zero section of the bundle ξ which in turn is homeomorphic to $B(\xi)$ making the projection map π an homotopy equivalence and thus yielding that $H^k(E; \mathbb{Z}_2) \cong H^k(B; \mathbb{Z}_2)$ for all k. With this remark and the previous unproved theorem we obtain the following

Definition 3.7. The ***Thom Isomorphism*** is defined to be the following composition

$$H^k(B; \mathbb{Z}_2) \xrightarrow{\pi^*} H^k(E; \mathbb{Z}_2) \xrightarrow{\cup u} H^{k+n}(E, E_0; \mathbb{Z}_2)$$

The final ingredient that we will need to prove the existence of the Stiefel Whitney classes are the so-called ***Steenrod squaring operations*** in $H^*(E, E_0; \mathbb{Z}_2)$. The proof of the existence of such maps lies beyond the scope of this project and we shall content ourselves by stating the properties that determine such operations and subsequently make use of them. The Steenrod squaring operations are characterized by the follwing four basic properties:

1. For each pair $Y \subset X$ and each pair of integers n, i there exists an additive homomorphism

$$Sq^i : H^n(X, Y; \mathbb{Z}_2) \to H^{n+i}(X, Y; \mathbb{Z}_2)$$

2. *Naturality*: If $f : (X, Y) \to (X', Y')$ is map of pairs then

$$Sq^i \circ f^* = f^* \circ Sq^i$$

3. If $a \in H^n(X, Y; \mathbb{Z}_2)$, then $Sq^0(a) = a$, $Sq^n(a) = a \cup a$, and $Sq^i(a) = 0$ for $i > n$.
4. *The Cartan Formula*: The identity

$$Sq^k(a \cup b) = \sum_{i+j=k} Sq^i(a) \cup Sq^j(b)$$

is valid whenever $a \cup b$ is defined.

With all these tools at hand, we define for a vector bundle ξ over a base space B the ***Stiefel-Whitney class of degree*** k of ξ to be the cohomology class of B given by:

$$\omega_k(\xi) = \phi^{-1}(Sq^k(u))$$

where u is the fundamental cohomology class of ξ.

Before working any further with the Stiefel-Whitney classes it is useful to introduce the following terminology, Throughout the rest of this chapter it is understood that we are working with the ring $\mathbb{Z}_2$.

Definition 3.8. Let B be some topological space. $H^{\Pi}(B)$ will denote the set of all formal series $a = a_0 + a_1 + a_2 + \ldots$ where $a_i \in H^i(B)$. This set can be made into a ring with the degree-wise addition and the product defined as

$$(a_0 + a_1 + a_2 + \ldots)(b_0 + b_1 + b_2 + \ldots) =$$
$$(a_0b_0) + (a_1b_0 + a_0b_1) + (a_2b_0 + a_1b_1 + a_0b_2) + \ldots$$

Since we are working with $\mathbb{Z}_2$ this ring is commutative.

The *total Stiefel-Whitney class* of an $n-$plane bundle η over B is the element

$$\omega(\eta) = 1 + \omega_1(\eta) + \ldots + \omega_n(\eta)$$

With this terminology, the Whitney sum formula can now be simplified as $\omega(\eta \oplus \delta) = \omega(\eta)\omega(\delta)$.

An important property of the ring $H^{\Pi}(B)$ is summarized in the following proposition, whose proof can be found in [MS], page 40.

Proposition 3.9. *The set A of all infinite series $a = 1 + a_1 + a_2 + \ldots \in H^{\Pi}(B)$ with leading term 1 is a commutative group under multiplication.*

From now on, the inverse of an element $a \in A$ will be denoted by $\overline{a}$. All this considerations allows us to state and prove the following proposition.

Proposition (Whitney duality theorem) 3.10. *If τ_M is the tangent bundle of a smooth manifold in Euclidean space and υ is the normal bundle then*

$$\omega(\upsilon) = \overline{\omega}(\tau_M)$$

Proof. Since $\omega(\tau_M \oplus \upsilon) = 1$ by the Whitney sum formula and by 3.9 we have that $\omega(\upsilon) = \overline{\omega}(\tau_M)$

□

3.4 Verification of the Axioms

For the verification of the axioms for the Stiefel-Whitney classes it will be convenient to consider the ***total Steenrod squaring operation*** given by

$$Sq(a) = a + Sq^1(a) + Sq^2(a) + \ldots + Sq^n(a)$$

for some cohomology class $a \in H^n(X, Y; \mathbb{Z}_2)$. With this terminology, the Cartan formula is simplified as $Sq(a \cup b) = Sq(a) \cup Sq(b)$ and the total Stiefel-Whitney class of a vector bundle ξ can be expressed simply as $\omega(\xi) = \phi^{-1}(Sq(u))$, where we are viewing the Thom isomorphism as an isomorphism between the abelian groups $H^{\Pi}(B; \mathbb{Z}_2)$ and $H^{\Pi}(E, E_0; \mathbb{Z}_2)$.

Using this new terminology we prove the following lemma which will be used in the verification of the Whitney sum formula.

Lemma 3.11. *If $a \in H^n(A, A'; \mathbb{Z}_2)$ and $b \in H^n(B, B'; \mathbb{Z}_2)$ then $Sq(a \times b) = Sq(a) \times Sq(b)$*

Proof. Let $p_1 : (A \times B, A' \times B) \to (A, A')$ and $p_2 : (A \times B, A \times B') \to (B, B')$ be the usual projection maps. We have the following:

$$Sq(a \times b) = Sq(p_1^*(a) \cup p_2^*(b)) = Sq(p_1^*(a)) \cup Sq(p_2^*(b)) =$$
$$p_1^*(Sq(a)) \cup p_2^*(Sq(b)) = Sq(a) \times Sq(b)$$

□

Now we set out to verify the axioms which will characterize uniquely the Stiefel-Whitney classes.

Naturality: Let ξ and ξ' be two n-vector bundles and let $(G, f) : \xi \to \xi'$ be a bundle map. Clearly this map will induce a map $G : (E, E_0) \to (E', E_0')$ and the fundamental cohomology class $u' \in H^n(E', E_0'; \mathbb{Z}_2)$ will be mapped to a cohomology class $G^*(u') \in H^n(E, E_0; \mathbb{Z}_2)$. But since for each fiber F of ξ we have that $G : (F, F_0) \to (G(F), G(F)_0)$ is an homotopy equivalence we must have that $G^*(u')$ is the fundamental cohomology class u of ξ. This together with the fact that cup products are maintained by maps we obtain commutativity in the following diagram:

$$\begin{tikzcd}
H^{n+k}(E', E_0') \arrow[r, "G^*"] & H^{n+k}(E, E_0) \\
H^n(E') \arrow[r, "G^*"] \arrow[u, leftrightarrow, "\cdot\cup u'"] \arrow[d, leftrightarrow, "\pi^*"'] & H^n(E) \arrow[u, leftrightarrow, "\cdot\cup u"'] \arrow[d, leftrightarrow, "\pi^*"] \\
H^n(B') \arrow[r, "f^*"] & H^n(B)
\end{tikzcd}$$

The commutativity of this diagram gives us now that $f^*(\omega_k(\xi')) = \omega_k(\xi)$.

Axiom 1. Given an n-bundle ξ the pre-image of the fundamental class $u \in H^n(E, E_0; \mathbb{Z}_2)$ under the isomorphism

$$\cdot \cup u : H^0(E; \mathbb{Z}_2) \to H^n(E, E_0; \mathbb{Z}_2)$$

is the the unit element of the cohomology ring $H^*(E; \mathbb{Z}_2)$ and thus we obtain that $\omega_0(\xi) = \phi^{-1}(u) = 1$. Furthermore, if $m > n$ then by property 3 of the Steenrod squaring operations we obtain $Sq^m(u) = 0$ which implies that $\omega_m(\xi) = 0$.

Axiom 2: *Stiefel Whitney classes of Whitney sums*: Let $\xi = (E, B, p)$ and $\xi' = (E', B, p')$ be two vector bundles defined on B of dimension n and m respectively, and let u and u' be their corresponding fundamental cohomology classes. In order to verify this axiom we start off by proving that

$$\omega(\xi \times \xi') = \omega(\xi) \times \omega(\xi')$$

Let us then denote by ξ'' the vector bundle $\xi \times \xi'$, whose total space is $E'' = E \times E'$. Observe first that $E_0'' = E \times E_0' \cup E_0 \times E'$ and thus the relative cross product $u \times u'$ is an element of $H^{n+m}(E'', E_0'')$. We wish to show that $u \times u'$ is indeed the fundamental cohomology class u'' of ξ''. In order to do this, it just suffices to verify that the restriction of $u \times u'$ to each fiber F'' of ξ'' is the non-zero element of $H^{n+m}(F'', F_0'')$. So let F and F' be any two fibers of ξ and ξ' respectively, let $F'' = F \times F'$ and consider the following commutative diagram

$$\begin{tikzcd}
H^{n+m}(E \times E', E_0 \times E' \cup E \times E_0') \arrow[r] & H^{n+m}(F \times F', F_0 \times F' \cup F \times F_0') \\
H^n(E, E_0) \otimes H^m(E', E_0') \arrow[r] \arrow[u] & H^n(F, F_0) \otimes H^m(F', F_0') \arrow[u]
\end{tikzcd}$$

where the vertical maps are the ones induced by the relative cross product operation and the horizontal maps are restrictions. Since the right vertical map is an isomorphism by the relative version of the Kunneth theorem we have that $u_{(F,F_0)} \times u'_{(F',F_0')}$ is mapped to the generator of $H^{n+m}(F \times F', F_0 \times F' \cup F \times F_0')$ and by the commutativuty of the previous diagram we have that the restriction of $u \times u'$ to the fiber F is the generator of $H^{n+m}(F \times F', F_0 \times F' \cup F \times F_0')$. Since the fundamental cohomology class of ξ'' is unique we can conclude that $u'' = u \times u'$.

Our second step is to show that the Thom ismomorphisms ϕ, ϕ' and ϕ'' of ξ, ξ' and ξ'' are related via the formula $\phi''(a \times b) = \phi(a) \times \phi'(b)$. Indeed, if for any spaces $C \subset A$ and D we denote by $p_A : A \times D \to A$ $p_{(A,C)} : (A \times D, C \times D) \to (A, C)$ the usual projections then using the identity $u'' = u \times u''$ we have that

$$
\begin{aligned}
\phi''(a \times b) &= (a \times b) \cup (u \times u') = (\pi \times \pi')^*(a \times b) \cup (u \times u') \\
&= (\pi^*(a) \times \pi'^*(b)) \cup (u \times u') \\
&= (p_E^*(\pi^*(a)) \cup p_{E'}^*(\pi'^*(b))) \cup (p_{(E,E_0)}^*(u) \cup p_{(E',E_0')}^*(u')) \\
&= (p_E^*(\pi^*(a)) \cup p_{(E,E_0)}^*(u)) \cup (p_{E'}^*(\pi'^*(b)) \cup p_{(E',E_0')}^*(u')) \\
&= p_{(E,E_0)}^*(\pi^*(a) \cup u) \cup p_{(E',E_0')}^*(\pi'^*(b) \cup u') \\
&= \phi(a) \times \phi'(b)
\end{aligned} \tag{1}
$$

where the above equality holds for elements $a \in H^{\Pi}(B;\mathbb{Z}_2)$ and $b \in H^{\Pi}(B';\mathbb{Z}_2)$. Then, taking the total Stifel Whitney classes $\omega(\xi)$ and $\omega(\xi')$ we obtain $\phi''(\omega(\xi) \times \omega(\xi')) = \phi(\omega(\xi)) \times \phi'(\omega(\xi')) = Sq(u) \times Sq(u')$. On the other hand, by the definition of the Stiefel-Whitney classes we also have $\phi''(\omega(\xi \times \xi'')) = Sq(u'') = Sq(u \times u') = Sq(u) \times Sq(u')$, where the last equality holds by lemma 3.11. Since the map ϕ'' is inyective we must have that $\omega(\xi \times \xi') = \omega(\xi) \times \omega(\xi')$.

Taking now ξ and δ to be two vector bundles over a base space B, consider their Whitney sum $\xi \oplus \delta$, the vector bundle $\xi \times \delta$ on $B \times B$ and the diagonal map $d : B \to B \times B$. Then, by naturality of the Stiefel-Whitney classes we obtain that $d^*(\omega(\xi) \times \omega(\delta))) = \omega(\xi \oplus \delta)$. On the other hand, since $p_i d = Id_B$ where $p_i : B \times B \to B$ is the projection onto the $i-th$ component we obtain that $d^*(\omega(\xi) \times \omega(\delta))) = \omega(\xi)\omega(\delta)$ and thus

$$\omega(\xi \oplus \delta) = \omega(\xi)\omega(\delta)$$

Axiom 3. Let $\gamma_1^1 = (E, S^1, p)$ be the canonical line bundle on S^1 and consider the cohomology group $H^1(E, E_0; \mathbb{Z}_2)$. It is not hard to see that the subspace E_1 of E conformed by all vectors of norm ≤ 1 is homeomorphic to the Möbious band M and that E_0 deformation retracts to the boundary circle of M (See [5], page 17). Furthermore, by an excision argument we have that $H^1(M, \partial M; \mathbb{Z}_2) \cong H^1(\mathbb{R}P^2, O; \mathbb{Z}_2)$ where O is a subspace homeomorphic to D^2 and since O is contractible, we also have the isomorphism $H^1(\mathbb{R}P^2, O; \mathbb{Z}_2) \cong H^1(\mathbb{R}P^2; \mathbb{Z}_2)$. All the previous arguments yield thus the following chain of natural isomorphisms

$$H^1(E, E_0; \mathbb{Z}_2) \cong H^1(M, \partial M; \mathbb{Z}_2) \cong H^1(\mathbb{R}P^2, O; \mathbb{Z}_2) \cong H^1(\mathbb{R}P^2; \mathbb{Z}_2)$$

Denote by $f : H^1(E, E_0; \mathbb{Z}_2) \to H^1(\mathbb{R}P^2; \mathbb{Z}_2)$ the previously obtained isomorphism. Since the fundamental cohomology class u of γ_1^1 cannot be zero we must have that $f^{-1}(a) = u$ where a is the generator of $H^1(\mathbb{R}P^2; \mathbb{Z}_2)$. Then, since $a \cup a$ is non zero (since it is the generator of $H^2(\mathbb{R}P^2; \mathbb{Z}_2)$) we must have then that $u \cup u \neq 0$ which implies that $\omega_1(\gamma_1^1) \neq 0$.

Consider now the usual inclusion map $i : S^1 \to \mathbb{R}P^\infty$. Since we have that $i^*\gamma_1 \cong \gamma_1^1$ it follows that $i^*(\omega(\gamma_1)) = \omega(\gamma_1^1)$, and the fact that $\omega_1(\gamma_1^1) \neq 0$ implies that $\omega(\gamma_1)$ is the non-zero element of $H^1(\mathbb{R}P^\infty)$.

3.5 Examples of Computations

Now we turn our focus to actually computing Stiefel- Whitney classes for some particular vector bundles.

Example 3.12. Consider the unit sphere S^n in the Euclidean space $\mathbb{R}^{n+1}$. Since in this case we have that the normal bundle on S^n is trivial it follows by the Whitney duality theorem that $\omega(\tau_{S^n}) = 1$.

This example wasn't at all that interesting. Nevertheless, if we take the real projective spaces things start to shape up. For this examples, we shall need the following result.

Lemma 3.13. *The group $H^i(\mathbb{R}P^n)$ is isomorphic to $\mathbb{Z}_2$ for $0 \leq i \leq n$and is zero for higher values of i. Furthermore, if a denotes the non-zero element of $H^1(\mathbb{R}P^n)$ then each $H^i(\mathbb{R}P^n)$ is generated by the $i-$th fold cup product a^i.*

A proof of this result is usually given in any introductory text of algebraic topology.

Example 3.14. Taking the real projective space $\mathbb{R}P^n$ we have that the canonical inclusion map $j : S^1 \to \mathbb{R}P^n$ is such that $j^*\gamma_1^n = \gamma_1^1$. Since we have already that $\omega(\gamma_1^1) = 1 + a$ then clearly we must have $\omega_1(\gamma_1^n) = a$. This together with the first axiom of Stiefel-Whitney classes imply that $\omega(\gamma_1^n) = 1 + a$.

Example 3.15. Taking again the real projective space $\mathbb{R}P^n$ we have that the canonical line bundle γ_1^n is a subbundle of the trivial bundle ϵ^{n+1} defined over $\mathbb{R}P^n$. Thus, denoting γ_1^n temporarily as γ we obtain that

$$\omega(\gamma)\omega(\gamma^\perp) = 1$$

which implies that $\omega(\gamma^\perp) = 1 + a + \ldots + a^n$ since taking the product of this element with $1 + a$ gives us the identity element in $H^0(\mathbb{R}P^n)$.

Now we proceed to tackle a less trivial example.

Example 3.16. Let τ denote the tangent bundle of the real projective space $\mathbb{R}P^n$. We begin proving the following lemma (Compare with [5])

Lemma 3.17. *The tangent bundle τ of $\mathbb{R}P^n$ is isomorphic to* $\mathrm{Hom}(\gamma_1^n, \gamma^\perp)$.

Proof. Consider the canonical quotient map $q : S^n \to \mathbb{R}P^n$ and the induced map $Dq : \tau_{S^n} \to \tau$ between the corresponding tangent bundles. Then, it is not hard to verify that Dq is a 2 to 1 map where the points $(x, v), (-x, -v)$ of τ_{S^n} are sent to the same point in τ and thus τ can be identified as the set of points $\{(x, v), (-x, -v)\}$ with $x \cdot x = 1$ and $x \cdot v = 0$. Consider now the map $f : \tau \to \mathrm{Hom}(\gamma_1^n, \gamma^\perp)$ defined as $f(\{(x, v), (-x, -v)\}) = (\{x, -x\}, \alpha_{(x,v)})$ with $\alpha_{(x,v)} : F_{\{x,-x\}} \to (F_{\{x,-x\}})^\perp$ given by $\alpha_{(x,v)}(x) = v$. This map is continuous and clearly it is an isomorphism when restricted to fibers. Thus, by lemma 1.4 we have that f is a bundle isomorphism. $\square$

The following proposition will give us the Stiefel-Whitney classes of τ.

Proposition 3.18. *The Whitney sum $\tau \oplus \epsilon^1$, where ϵ^1 is the trivial line bundle, is isomorphic to the $(n+1)-$fold Whitney sum $\gamma_1^n \oplus \ldots \oplus \gamma_1^n$. Hence,*

$$\omega(\tau) = (1+a)^{n+1} = 1 + \binom{n+1}{1}a + \binom{n+1}{2}a^2 + \ldots + \binom{n+1}{n}a^n$$

Proof. The line bundle $\mathrm{Hom}(\gamma_1^n, \gamma_1^n)$ over $\mathbb{R}P^n$ has a nowhere zero section, namely $s : \mathbb{R}P^n \to E(\mathrm{Hom}(\gamma_1^n, \gamma_1^n))$ given by $s(\{x, -x\}) = \alpha_{\{x,-x\}}$ where $\alpha_{\{x,-x\}}$ is defined as $\alpha_{\{x,-x\}}(x) = x$, and thus $\mathrm{Hom}(\gamma_1^n, \gamma_1^n)$ is isomorphic to the trivial line bundle ϵ_1. Thus we have the following

$$\begin{aligned} \tau \oplus \epsilon^1 &\cong \mathrm{Hom}(\gamma_1^n, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_1^n, \gamma_1^n) \\ &\cong \mathrm{Hom}(\gamma_1^n, \gamma^\perp \oplus \gamma_1^n) \\ &\cong \mathrm{Hom}(\gamma_1^n, \epsilon^{n+1}) \\ &\cong \mathrm{Hom}(\gamma_1^n, \epsilon^1) \oplus \ldots \oplus \mathrm{Hom}(\gamma_1^n, \epsilon^1) \end{aligned} \tag{2}$$

Observe now that since γ_1^n has a riemannian metric, the function $\gamma_1^n \to \mathrm{Hom}(\gamma_1^n, \epsilon^1)$ defined as $v \to < \cdot, v >$ defines a bundle isomorphism and thus we obtain that $\tau \oplus \epsilon^1$ is isomorphic to the $(n+1)-$fold Whitney sum $\gamma_1^n \oplus \ldots \oplus \gamma_1^n$. □

3.6 Stiefel-Whitney Numbers

Stiefel-Whitney classes have an extensive application to give insight about problems in differential geometry as it can be consulted in texts such as [5] and [2]. Since the main topic of this project is to give a description of characteristic classes as it was explained in the first section of this chapter, we shall only limit ourselves to give one such application, namely, that the Stiefel-whitney classes of tangent bundles of smooth manifolds can be used to determine whether or not two smooth manifolds of the same dimension are cobordant, that is, if their disjoint union constitutes the boundary of a manifold one dimension higher. Namely, we have the following

Theorem 3.19. *Two smooth closed n-manifolds are cobordant if and only if their corresponding Stiefel-Whitney numbers are equal.*

In this section we shall make sense of this theorem and give a partial proof for it. Let M be a closed (compact and with no boundary) smooth n- dimensional manifold. By means of excicion arguments we have for any point of $x \in M$ that

$$H_i(M, M - \{x\}) = \begin{cases} 0 & \text{if } i \neq n \\ \mathbb{Z}_2 & \text{if } i = n \end{cases}$$

The generator μ_x of the homology group $H_n(M, M - \{x\})$ is called the $\mathbb{Z}_2-$orientation of M at x. For such manifolds we have the following

Proposition 3.20. *For a closed n dimensional smooth manifold M there exists a homology class $\mu_M \in H_n(M)$ such that $(i_x)_*(\mu_M) = \mu_x$ for all $x \in M$ where the map $i_x : (M, \oslash) \to (M, M-x)$ is the usual inclusion map.*

The homology class μ_M is defined as the ***fundamental homology class of M***. In a similar fashion, consider a compact n dimensional smooth manifold M with boundary ∂M and let $\mathbb{R}^n_{\geq 0}$ be the subset of $\mathbb{R}^n$ with $x_n \geq 0$. In a similar fashion we define $\mathbb{R}^n_{>0}$. For every point x in ∂M there is an open neighborhood V_x and a smooth map $f : \mathbb{R}^n_{\geq 0} \to V_x$ such that $f(\partial \mathbb{R}^n_{\geq 0}) = V_x \bigcap \partial M$ and $f(0) = x$. Thus, for every point x in ∂M we have that $H_n(V_x, V_x - x) \cong H_n(\mathbb{R}^n_{\geq 0}, \mathbb{R}^n_{\geq 0} - 0) = 0$. For such manifolds we have the following analogous version of 3.20.

Proposition 3.21. *1. If M is a compact manifold with boundary ∂M then there is a unique* ***fundamental homology class of M*** *$\mu_M \in H_n(M, \partial M)$ such that for all $x \in M - \partial M$ we have that $(i_x)_*(\mu_M) = \mu_x$ where $i_x : (M, \partial M) \to (M, M - x)$ is the usual inclusion map.*

2. If ∂ denotes the connecting homophormisms for the long exact sequence of the pair (M, ∂_M) then we have that $\partial(\mu_M) = \mu_{\partial M}$.

The proofs of the previous two results shall be ommited and can be found in [1], page 253, and [8], page 304.

Let M be any closed smooth n dimensional manifold and consider the tangent bundle τ_M. If $r_1, \ldots, r_n$ are non-negative integers such that $r_1 + 2r_2 + \ldots + nr_n = n$ then the cohomology class $\omega_1(\tau_M)^{r_1} \ldots \omega_n(\tau_M)^{r_n}$ is an element in $H^n(M)$ and thus we can evaluate the fundamnetal class μ_M on $\omega_1(\tau_M)^{r_1} \ldots \omega_n(\tau_M)^{r_n}$ to obtain $< \omega_1(\tau_M)^{r_1} \ldots \omega_n(\tau_M)^{r_n}, \mu_M >$ which is an element in $\mathbb{Z}_2$. This yield us the following definition

Definition 3.22. The ***Stiefel-Whitney number*** of M associated to the $n-$tuple $(r_1, \ldots, r_n)$ is the element $< \omega_1(\tau_M)^{r_1} \ldots \omega_n(\tau_M)^{r_n}, \mu_M >$.

We say that two closed $n-$dimensional manifolds M and M' have the same Stiefel-Whitney numbers when for all $n-$tuples $(r_1, \ldots, r_n)$ of non-negative integers satisfying $r_1 + 2r_2 + \ldots + nr_n = n$ we have that

$$< \omega_1(\tau_M)^{r_1} \ldots \omega_n(\tau_M)^{r_n}, \mu_M > = < \omega_1(\tau'_M)^{r_1} \ldots \omega_n(\tau'_M)^{r_n}, \mu'_M >$$

The following result and its proof represent the core of this section

Theorem 3.23. *Let B be a smooth n dimensional manifold with boundary M. Then, all the Stiefel-Whitney numbers of M are zero.*

Proof. By proposition 3.21 we have that the connencting homomorphism ∂ of the long extact sequence of homology groups of the pair (B, M) satisfies $\partial(\mu_B) = \mu_M$. Thus, for any cohomology class $v \in H^n(M)$ we have the following equalities

$$< v, \mu_M > = < v, \partial\mu_B > = < \delta v, \mu_B >$$

where δ is the connecting homomorphism $\delta : H^n(M) \to H^{n+1}(B, M)$. The second equality holds because of the following commutative diagram

$$\begin{array}{ccc} H^n(M) & \xrightarrow{\delta} & H^{n+1}(B,M) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ \mathrm{Hom}_{\mathbb{Z}_2}(H_n(M);\mathbb{Z}_2) & \xrightarrow{\partial^*} & \mathrm{Hom}_{\mathbb{Z}_2}(H_{n+1}(B,M);\mathbb{Z}_2) \end{array}$$

where h is just the map that maps a cohomology class $a \in H^n(X, A)$ to the element in $\mathrm{Hom}_{\mathbb{Z}_2}(H_n(X, A); \mathbb{Z}_2)$ defined $v \to < a, v >$ for all $v \in H_n(X, A)$ (See [1], page 200).

Using the Collar Neighborhood theorem (See [1], page 253) it can be proven that the tangent bundles τ_M and τ_B satisfy the following relation on M

$$(\tau_B)_M \cong \tau_M \oplus \epsilon_1$$

where ϵ_1 is the trivial line bundle on M. Thus the inclusion map $i : M \to B$ satisfies

$$i^*\omega(\tau_B) = \omega(\tau_M)\omega(\epsilon_1) = \omega(\tau_M)$$

Taking any $n-$tuple $(r_1, \ldots, r_n)$ of non-negative integers satisfying $r_1+2r_2+\ldots+nr_n = n$ we have by the previous equality

$$\begin{aligned} &< \omega_1(\tau_M)^{r_1} \ldots \omega_n(\tau_M)^{r_n}, \mu_M > = \\ &< i^*(\omega_1(\tau_B)^{r_1} \ldots \omega_n(\tau_B)^{r_n}), \partial\mu_B > = \\ &< \delta i^*(\omega_1(\tau_B)^{r_1} \ldots \omega_n(\tau_B)^{r_n}), \mu_B > = 0 \end{aligned} \tag{3}$$

where in the last equality we applied the exactness of the long exact sequence of the pair (B, M). This concludes the proof.

□

The converse of this result is a classic application of algebraic topology in geometry and its proof falls out a little bit beyond the scope of this project

Theorem(René Thom) 3.24. *Let M be a closed smooth $n-$dimensional manifold. If all the Stiefel-Whitney numbers of M are zero then $M = \partial B$ where B is a smooth $(n + 1)$-dimensional manifold.*

Let M_1 and M_2 be two closed $n-$dimensional smooth manifolds. We say that M_1 and M_2 are ***cobordant*** if there exists a smooth manifold B one dimension higher such that $M_1 \coprod M_2 = \partial B$. The following is now an immediate corollary of all the previous results

Theorem 3.25. *Two closed n-dimensional smooth manifolds M_1 and M_2 are cobordant if and only if all of their Stiefel-Whitney numbers are equal.*

4 Oriented Bundles, the Thom Isomorphism and the Euler Class

4.1 The Euler Class

In this chapter we introduce the class of oriented vector bundles and discuss their most relevant characteristic class: ***The Euler class***. The main theoretical ingredient to prove the existence of the Euler class will be Thom Ismorphism, which in the non-oriented case will yield us the existence of the Stiefel-Whitney classes. In the big scheme of this project, the relevance of this chapter lies on the fact that we will give a description of the cohomology ring $H^*(\mathbb{C}P^\infty;\mathbb{Z})$ in terms of characteristic classes. Such a description will be of great importance for the next chapter when we begin to construct the Chern classes. In this chapter, all vector bundles are assumed to be real unless otherwise specified.

Before defining oriented vector bundles properly we must first give a definiton of orientability for finite dimensional vector spaces in terms of (co)homology. Through out the rest of this chapter, for any vector bundle ξ and for any vector space V we denote respectively by E_0 and V_0 the total space E minus the zero section and the space V minus the origin. Also, in this chapter we shall always assume that we are working with the ring $R = \mathbb{Z}$ unless otherwise specified.

Definition 4.1. Let V be a real vector space of dimension n. An ***orientation*** for V is an equivalence of ordered bases, where two bases are said to be equivalent if and only if their transition matrix has positive determinant.

We would like however to translate this definiton of orientation to the context of algebraic topology. Thus we prove the following

Lemma 4.2. *A choice of orientation of $\mathbb{R}^n$ corresponds to a choice of one of the generators of the group $H^n(\mathbb{R}^n, \mathbb{R}^n_0)$.*

Proof. Consider the canonical basis $(e_1, \ldots, e_n)$ for $\mathbb{R}^n$ and let $\sigma_1 : \mathbb{R}^n \to \mathbb{R}^n$ be the map which translates the barycenter of the standard $n-$simplex $\Delta^n = (0, e_1, \ldots, e_n)$ to the origin. Then the map σ_1 is a singular $n-$simplex which represents an element in $Z^n(\mathbb{R}^n, \mathbb{R}^n_0)$ since the boundary of σ_1 would lie in $\mathbb{R}^n_0$. Furthermore, observe that since $(\mathbb{R}^n, \mathbb{R}^n_0)$ is homotopy equivalent to $(\sigma_1(\Delta^n), \partial\sigma_1(\Delta^n))$ and since $\sigma_1(\Delta^n)$ has a simplicial structure with only one $n-$simplex, namely itself, then we can conclude that the homology class of σ_1 generates $H_n(\mathbb{R}^n, \mathbb{R}^n_0)$. Take now the ordered basis $(e_1, \ldots, -e_n)$ of $\mathbb{R}^n$. Defining σ_2 analogously as we did in the previous step we obtain that the singular $n-$simplex σ_2 also generates $H_n(\mathbb{R}^n, \mathbb{R}^n_0)$. Consider finally the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ which leaves $e_1, \ldots, e_{n-1}$ fixed and sends e_n to $-e_n$. Then the map $\sigma_2 T \sigma_1^{-1}$ maps the homology class $[\sigma_1]$ to $[\sigma_2]$. Since the map induced on $H_n(\mathbb{R}^n, \mathbb{R}^n_0)$ by $\sigma_2 T \sigma_1^{-1}$ is multiplication by -1 (See [1], page 155, exercise 7) we obtain that $[\sigma_1]$ and $[\sigma_2]$ are additive inverses to each other. $\square$

As a corollary we obtain the following

Lemma 4.3. *A choice of orientation of V corresponds to a choice of one of the generators of the group $H_n(V, V_0)$.*

Observe that by the Universal Coefficient theorem (See [1], page 195) we have that $H^n(V, V_0) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(V, V_0), \mathbb{Z})$ and thus a preferred generator μ_V of $H_n(V, V_0)$ will induce an orientation λ_V in terms of cohomology, namely, the element $\lambda_V \in H^n(V, V_0)$ such that $< \lambda_V, \mu_V >= 1$. Conversely, a preferred generator in cohomology will induce a preferred generator in homology. From now on, when we talk about the orientation of a particular space V we mean a preferred generator of $H^n(V, V_0)$. Also, from now on the preferred orientation we consider for $\mathbb{R}^n$ is the generator for $H^n(\mathbb{R}^n, \mathbb{R}^n_0)$ induced by the canonical basis $(e_1, \ldots, e_n)$.

Definition 4.4. *Let $\xi = (E, B, p)$ be an $n-$vector bundle. An* ***orientation*** *for ξ is a function which assigns an orientation to each fiber F of ξ subject to the following* ***local compatability condition:*** *For every point $b_0 \in B$ there should exist an open neighborhood U of b_0 and a local trivialization $h : U \times \mathbb{R}^n \to p^{-1}(U)$ such that for each $b \in U$ the map $h_{|b \times \mathbb{R}^n}$ is orientation preserving considering the orientation for $\mathbb{R}^n$ specified above.*

From now on, an ***oriented*** $n-$ ***vector bundle*** will mean an $n-$vector bundle with an orientation. Observe that an equivalent way of stating the local compatability condition is that there exists sections $s_1, \ldots, s_n : U \to E$ such that $s_1(b), \ldots, s_n(b)$ determines the specified orientation in F_b for all $b \in B$.

An immediate yet useful consequence of the previous definition is the following.

Lemma 4.5. *Any trivial $n-$vector bundle $\epsilon = (E, B, p)$ is orientable.*

Observe that the local compatability condition given in the previous definition implies the following lemma.

Lemma 4.6. *Let $\xi = (E, B, p)$ be an oriented $n-$vector bundle. Then, for each $b \in B$ there exists an open neighborhood U of b for which there exists $u \in H^n(p^{-1}(U), p_0^{-1}(U))$ such that for each fiber F in $p^{-1}(U)$ the restriction $u_F \in H^n(F, F_0)$ of u in F is the orientation of F.*

Proof. Let $b_0 \in B$ and let U be an open neighborhood of b_0 for which the local compatibility condition is satisfied. Let b be any point in U, let F be the fiber over b and consider now the following commutative diagram

$$\begin{tikzcd}
H^n(\mathbb{R}^n, \mathbb{R}^n_0) \arrow[r, "\pi^*"] \arrow[d, "Id^*"] & H^n(U \times \mathbb{R}^n, U \times \mathbb{R}^n_0) \arrow[r, "(h^{-1})^*"] \arrow[d, "i^*"] & H^n(p^{-1}(U), p_0^{-1}(U)) \arrow[d, "i^*"] \\
H^n(\mathbb{R}^n, \mathbb{R}^n_0) \arrow[r, "\pi^*"] & H^n(b \times \mathbb{R}^n, b \times \mathbb{R}^n_0) \arrow[r, "(h_F^{-1})^*"] & H^n(F, F_0)
\end{tikzcd}$$

where $\pi : U \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection onto the second component. Then if u' is the preferred generator for $H^n(\mathbb{R}^n, \mathbb{R}^n_0)$, the cohomology class $u = (h^{-1})^* \pi^*(u')$ satisfies the required condition.

□

The importance of the Thom Isomorphism Theorem, to be proven in the next section, is that such a cohomology class exists globally for the whole vector bundle. Namely we have the following theorem.

Theorem 4.7. *Let $\xi = (E, B, p)$ be an oriented $n-$vector bundle. Then the cohomology group $H^n(E, E_0)$ is zero for $i < n$ and $H^n(E, E_0)$ contains a unique class u such that the restriction*

$$u_{(F,F_0)} \in H^n(F, F_0)$$

of u is equal to the preferred generator u_F for any fiber F of ξ. Futhermore, the correspondence $y \to y \cup u$ maps $H^k(E)$ isomorphically onto $H^{k+n}(E, E_0)$ for every integer k.

Assuming this result, we introduce the Euler class for an oriented bundle. Let $i^* : H^*(E, E_0; \mathbb{Z}) \to H^*(E; \mathbb{Z})$ be the map in cohomology induced by $i : E \to (E, E_0)$.

Definition 4.8. The ***Euler class*** of an oriented $n-$vector bundle $\xi = (E, B, p)$ is the cohomology class $e(\xi) = p^* i^*(u) \in H^n(B; \mathbb{Z})$.

We state now some basic properties of the Euler class.

Proposition 4.9. *If $f : B \to B'$ is covered by an orientation preserving bundle map $\xi \to \xi'$, then $e(\xi) = f^* e(\xi')$.*

The proof of this result follows from the fact that the fundamental cohomology class of ξ' is mapped to the fundamental cohomology class of ξ. In particular, by this proposition we have that the Euler class of any trivial bundle is zero.

Proposition 4.10. *If the orientation of ξ is reversed, then the Euler class $e(\xi)$ changes sign.*

This last proposition follows from the fact that a change of orientation on ξ implies that the fundamental cohomology class is then $-u$.

Proposition 4.11. *If η is an $n-$vector bundle with n odd then $e(\eta) + e(\eta) = 0$.*

Proof. For any vector bundle the map defined as $f : (b, v) \to (b, -v)$ is a bundle isomorphism. If n is odd then this map reverses the orietation and thus $f^*(e(\eta)) = -e(\eta)$. On the other hand, since the base space map is the identity, we must have that $e(\eta) = -e(\eta)$. □

The following proposition establishes a relationship between the Euler class and the Stiefel-Whitney classes.

Proposition 4.12. *The natural homomorphism* $\Gamma : H^n(B; Z) \to H^n(B; Z_2)$ *sends the Euler class to the top Stiefel-Whitney class.*

Proof. Since clearly $u \cup u = i^* u \cup u$, where $i : E \to (E, E_0)$ is the usual inclusion map, then by the definition of the Thom isomorphism we obtain that

$$\phi^{-1}(u \cup u) = e(\xi)$$

Since also the natural homomorphism $H^n(E, E_0; Z) \to H^n(E, E_0; Z_2)$ maps the fundamental cohomology class u to the corresponding fundamental cohomology class u in the non-oirented case, applying Γ to both sides of the last equality we obtain that

$$\Gamma(e(\xi)) = \phi^{-1}(Sq^n(u)) = \omega_n(\xi)$$

□

The Euler class satisfies almost the same multiplicative propeties as the Stiefel-Whitney Classes. In particular, we have the following proposition whose proof can be found in [5].

Proposition 4.13. *The Euler class of a Whitney sum is given by* $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$*. Also, the Euler class of a cartesian product is given by* $e(\xi \times \xi') = e(\xi) \times e(\xi')$*.*

We finish this chapter with the following proposition

Proposition 4.14. *If the oriented vector bundle* ξ *posseses a nowhere zero cross section* s*, then the Euler class must be zero.*

Proof. Since ξ is defined over paracompact space B then by 2.11 we have that ξ has an Euclidean metric. Let then ϵ be the trivial line bundle spanned by the cross-section s. Then $e(\xi) = e(\epsilon) \cup e(\epsilon^{\perp})$. Since $e(\epsilon) = 0$ we conclude our proof. □

4.2 The Thom Isomorphism

In this chapter we finally prove the Thom isomorphism theorem and establish the existence of the Stiefel Whitney classes and the Euler class. We shall prove only the oriented version of this theorem since the unoriented case is completely analogous.

Theorem (Thom Isomorphism) 4.15. *Let $\xi = (E, B, p)$ be an oriented $n-$vector bundle. Then the cohomology group $H^n(E, E_0; \mathbb{Z})$ is zero for $i < n$ and $H^n(E, E_0; \mathbb{Z})$ contains a unique class u such that the restriction of u*

$$u_{(F,F_0)} \in H^n(F, F_0; \mathbb{Z})$$

is equal to the preferred generator u_F for any fiber F of ξ. Futhermore, the correspondence $y \to y \cup u$ maps $H^k(E; \mathbb{Z})$ isomorphically onto $H^{k+n}(E, E_0; \mathbb{Z})$ for every integer k. In particular, $H^i(E, E_0) = 0$ for $i < n$.

Proof. We shall divide this proof into three steps.

Step 1. Suppose $\xi = (B, E = B \times \mathbb{R}^n, p)$ is a trivial vector bundle. Let $\mu \in H^n(\mathbb{R}^n, \mathbb{R}^n_0)$ be the fixed preferred generator for the cohomology group $H^n(\mathbb{R}^n, \mathbb{R}^n_0)$. By the relative version of the Kunnteth formula we have that the map

$$\gamma : H^0(B) \to H^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0)$$

given by $\alpha \to \alpha \times \mu$ is an isomorphism. For any point $b \in B$ let F^b be the fiber on b and consider the following commutative diagram

$$\begin{tikzcd}
H^n(E, E_0) \arrow[r, "i_{F_b^*}"] & H^n(F^b, F^b_0) \\
H^0(B) \arrow[u, "\gamma"] \arrow[r, "i_b^*"] & H^0(\{b\}) \arrow[u, "\gamma_b"']
\end{tikzcd}$$

where γ_b is the map obtained using the Kunneth theorem for $\{b\}$ and $(\mathbb{R}^n, \mathbb{R}^n_0)$. Observe that the preferred orientation for F^b_0 is $u_{F^b} = 1 \times \mu$ and by the commutativity of the previous diagram the element $1 \times \mu \in H^n(E, E_0)$ is such that $i_{F_b^*}(1 \times \mu) = u_{F^b}$. Furthermore, since $1 \in H^0(B)$ is the only element which restricts to 1 for each point $b \in B$ we conclude by the commutativity of the diagram that $1 \times \mu$ is the only element in $H^n(E, E_0)$ which satisfies the required property.

Applying yet again the relative version of the Kunneth Formula we have that the map $H^j(B) \otimes H^n(\mathbb{R}^n, \mathbb{R}^n_0) \to H^{j+n}(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0)$ given by $y \otimes \mu \to y \times \mu$ is an isomorphism. Then, any element in $H^{j+n}(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0)$ can be written uniquely as $y \times \mu$ for some $y \in H^j(B)$. Since by the Kunneth theorem we also have that any element in $H^j(B \times \mathbb{R}^n)$ can be written uniquely as $y \times 1$ with $y \in H^j(B)$ we can conclude that the assigment

$$y \times 1 \to (y \times 1) \cup (1 \times \mu) = y \times \mu$$

is an isomorphism. Thus the theorem is proven for trivial bundles.

Step 2: Suppose that B is the union of two open sets U and V such that the theorem holds for $E' = p^{-1}(U)$, $E'' = p^{-1}(V)$ and $E^{\cap} = p^{-1}(U \cap V)$. Consider the following Mayer-Vietoris sequence:

$$\ldots \to H^{i-1}(E^{\cap}, E_0^{\cap}) \to H^i(E, E_0) \to H^i(E', E_0') \oplus H^i(E'', E_0'') \to H^i(E^{\cap}, E_0^{\cap}) \to \ldots$$

By assumption, there exists fundamental cohomology classes u' and u'' for ξ_U and ξ_V respectively. By the uniquness of the fundamental cohomology class for $\xi_{U \cap V}$, the cohomology classes u' and u'' have the same image in $H^n(E^{\cap}, E_0^{\cap})$ and therefore there exists a cohomology class $u \in H^n(E, E_0)$ such that when restricted to U and V gives us u' and u'' respectively. Clearly the restriction of u on each fiber gives us the orientation of the vector bundle. Furthermore, this cohomology class u is unique given that $H^{n-1}(E^{\cap}, E_0^{\cap}) = 0$ by assumption. Consider now the Mayer-Vietoris sequence

$$\ldots H^{j-1}(E^{\cap}) \to H^j(E) \to H^j(E') \oplus H^j(E'') \to H^j(E^{\cap}) \to \ldots$$

with $i = n + j$. Mapping this sequence to the previous Mayer-Vietoris sequence by the correspondence $y \to y \cup u$ and applying the Five lemma it follows that $H^j(E) \cong H^{j+n}(E, E_0)$.

Case 3: Suppose that B has a finite open covering $U_1, \ldots, U_m$ such that the vector bundles ξ_{U_i} are trivial for each i. We will prove by induction on m that the result holds for this case. Clearly the result holds for $m = 1$ since this just means that ξ is a trivial bundle. Assume now that the result holds for $m - 1$. Then we will have that the theorem is true for the vector bundles $\xi_{U_1 \cup \ldots \cup U_{m-1}}$ and ξ_{U_m}. Applying case 2 we obtain the result for ξ.

In particular, this means that the previous result holds for compact spaces. For the general case, we recommend the reader to consult [5].

□

4.3 The Gynsin Sequence

Let $\xi = (E, B, p)$ be an oriented $n-$vector bundle and let E_0 be the total space with the zero section removed. Now that we have the Thom isomorphism at our disposal we shall give a tool which relates the cohomology groups of B and E_0. More explicitly, we have the following theorem.

Theorem(Gynsin) 4.16. *For an oriented $n-$vector bundle $\xi = (E, B, p)$ we have the following long exact sequence of cohomology groups*

$$\cdots \longrightarrow H^i(B) \xrightarrow{\cdot e(\xi)} H^{i+n}(B) \xrightarrow{p^*} H^{i+n}(E_0) \xrightarrow{\Gamma} H^{i+1}(B) \longrightarrow \cdots$$

Proof. We have the following commutative diagram with the top row exact

$$\begin{tikzcd}
\arrow[r] & H^{i+n}(E,E_0) \arrow[r, "j*"] & H^{i+n}(E) \arrow[r] & H^{i+n}(E_0) \arrow[r] & H^{i+n+1}(E,E_0) \arrow[r] & {} \\
\arrow[r] & H^i(B) \arrow[u, "\phi"] \arrow[r] & H^{i+n}(B) \arrow[u, "p^*"] \arrow[r, "p^*"] & H^{i+n}(E_0) \arrow[u, "Id"] \arrow[r, "\Gamma"] & H^{i+1}(B) \arrow[u, "\phi"] \arrow[r] & {}
\end{tikzcd}$$

Where j is the inclusion map. Since all the vertical maps are isomorphisms we also have then that the bottom row is exact. In addition, we also have the following calculation

$$\begin{aligned}
&(p^*)^{-1}j^*\phi(a) \\
&= (p^*)^{-1}j^*(p^*(a)u_\xi) \\
&= (p^*)^{-1}(p^*(a)j^*(u_\xi)) \\
&= a[(p^*)^{-1}j^*(u_\xi)] = a \cdot e(\eta)
\end{aligned}$$

This concludes our proof. □

4.4 The Cohomology Rings of Complex Projective Spaces

We want now with our available tools of characteristic classes to give a description of the cohomology rings of complex projective spaces. This will be of extreme importance when defining the Chern classes of complex vector bundles. As a first step, the following result, whose proof can be found in [5], says that the underlying real vector bundle of a complex vector bundle has a canonical preferred orientation.

Proposition 4.17. *If ω is a complex vector bundle over a base space B, then the underlying real vector bundle ω_R has a canonical preferred orientation and thus the Euler class $e(\omega_R) \in H^{2n}(B;\mathbb{Z})$ is well defined.*

Using a cellular cohomology argument we have that the cohomology groups of the complex projective space $\mathbb{C}P^n$ are the following:

$$H^k(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

Nevertheless, with our available tools we can refine this result one step further.

Theorem 4.18. *The cohomology ring $H^*(\mathbb{C}P^n;\mathbb{Z})$ is a truncated polynomial ring terminanting in dimension $2n$, and generated by the Euler class $e((\gamma_n^1)_R)$ where γ_n^1 is the canonical line bundle over $\mathbb{C}P^n$.*

Proof. Let $(\gamma_n^1)_R = (E, \mathbb{C}P^n, p)$ be the underlying real vector bundle of the canonical line bundle on $\mathbb{C}P^n$. Applying the Gynsin sequence to this real vector bundle we obtain

$$\cdots \longrightarrow H^{i+1}(E_0) \longrightarrow H^i(\mathbb{C}P^n) \xrightarrow{\cdot\cup e} H^{i+2}(\mathbb{C}P^n) \xrightarrow{p^*} H^{i+2}(E_0) \longrightarrow \cdots$$

All the points of the space $E_0 = E_0(\gamma_1^n)$ are of the form ($L =$*Line through the origin*, x) with $x \in L$ and thus E_0 can be identified with $\mathbb{C}^{k+1} - 0$ and thus has the same homotopy type as S^{2k+1}. From the Gynsin exact sequence we obtain then short exact sequences of the form

$$0 \longrightarrow H^i(\mathbb{C}P^n) \xrightarrow{\cdot\cup e} H^{i+2}(\mathbb{C}P^n) \longrightarrow 0$$

for $0 \le i \le 2k-2$. From these short exact sequences we obtain

$$H^0(\mathbb{C}P^n) \cong H^2(\mathbb{C}P^n) \cong \ldots \cong H^{2n}(\mathbb{C}P^n)$$

and that $H^i(\mathbb{C}P^n)$ is generated by $e((\gamma_1^n)_R)^i$ if $i \le n$. This completes the proof.

□

As a corollary of this result we have the following

Corollary 4.19. *$H^*(\mathbb{C}P^\infty;\mathbb{Z})$ is the polynomial ring generated by $e((\gamma_1)_R)$.*

Proof. Let $n \in \mathbb{N}$ be any natural number and consider the usual CW - complex structures for $\mathbb{C}P^n$ and $\mathbb{C}P^\infty$ (See [7], page 6). Then, it is clear that the inclusion map $i : \mathbb{C}P^n \to \mathbb{C}P^\infty$ induces a chain map $\phi_\bullet$ between the cellular co-chain complex of $\mathbb{C}P^n$ and $\mathbb{C}P^\infty$ such that $\phi_i = Id$ for $0 \le i \le 2n$ and $\phi_i = 0$ otherwise. Thus, $i^* : H^i(\mathbb{C}P^\infty) \to H^i(\mathbb{C}P^n)$ is a an isomorphism for $0 \le i \le 2n$, which implies in this case that $H^i(\mathbb{C}P^\infty)$ is generated by $e((\gamma_1)_R)^i$. Letting n tend to infinity we obtain the stated result. □

In the next chapter we shall also make use of the real counterpart of the previous proven result

Propostition 4.20. *$H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[a]$ where a is the non-zero element of $H^1(\mathbb{R}P^\infty;\mathbb{Z}_2)$.*

5 Chern Classes

This chapter is devoted to the analogs of the Stiefel-Whitney classes when working with complex vector bundles and when working with the ring of coefficients $\mathbb{Z}$: ***Chern classes***. It is the aim of this chapter to present a contruction of the Chern classes which also yields us an alternate way of constructing the Stiefel Whitney classes by restricitng all of the arguments to real vector bundles and the ring of coefficients $\Lambda = \mathbb{Z}_2$ and to prove the complex counterpart of theorem 3.4. Namely we will prove the following

Theorem 5.1. *For complex $n-$plane bundles ξ over base spaces B, $n \geq 0$, there are characteristic classes $c_i(\xi) \in H^{2i}(B;\mathbb{Z})$, called the **Chern classes**. They satisfy and are uniquely determined by the following axioms:*

1. $c_0(\xi) = 1$ *and* $c_i(\xi) = 0$ *if* $i > dim\xi$.
2. $c_i(\xi \oplus \delta) = \sum_{j=0}^{i} c_j(\xi) \cup c_{i-j}(\delta)$.
3. $c_1(\gamma_1)$ *is the element* $e((\gamma_1)_R)$ *for the universal line bundle defined on* $\mathbb{C}P^\infty$.

Furthermore, every integer characteristic class for complex $n-$plane bundles can be written uniquely as a polynomial in the Chern classes $\{c_1, \ldots, c_n\}$.

5.1 The Leray-Hirsh Theorem

The first theorem of this chapter will be the main theoretical tool for the joint construction of Chern and Stiefel-Whitney classes. We assume that the ring of coefficents Λ is either $\mathbb{Z}$ or $\mathbb{Z}_2$.

Theorem 5.2. *Let $p : E \to B$ be a bundle with typical fiber F which is of finite type, that is, trivial over a finite covering $U_1, \ldots, U_n$ with local trivializations $f_i : U_i \times F \to p^{-1}(U)$. Let E_0 be an open subspace of E, let $a_1, \ldots, a_r$ be homogeneous elements in $H^*(E, E_0; \Lambda)$ and let F_0 be an open subspace of F such that $f_i : (U_i \times F, U_i \times F_0) \to (p^{-1}(U), p^{-1}(U) \cap E_0)$ is an homeomorphism of pair of spaces. Furthermore, if $b \in U_i$, denote by $j_{b,i} : (F, F_0) \to (p^{-1}(b), p^{-1}(b) \cap E_0)$ the composition of maps*

$$(F, F_0) \xrightarrow{i_b} (b \times F, b \times F_0) \xrightarrow{f_i} (p^{-1}(b), p^{-1}(b) \cap E_0).$$

Then, if for all $b \in B$ we have that $j_{b,i}^*(a_1), \ldots, j_{b,i}^*(a_r)$ is a Λ-basis for $H^*(F, F_0; \Lambda)$ then $a_1, \ldots, a_r$ is a $H^*(B; \Lambda)-$basis for $H^*(E, E_0; \Lambda)$, where the $H^*(B; \Lambda)$ action is given by $b \cdot e = p^*(b) \cup e$.

Proof. The proof of this theorem will follow the same guidelines as the proof of the Thom Isomorphism theorem. We shall for this proof omit writing the coefficient ring Λ. We divide the proof in three steps.

Step 1: Take one of the sets U_i on which the bundle is trivial, denote $p^{-1}(U_i)$ by E_{U_i} and take the homeomorphism $f_i : (U_i \times F, U_i \times F_0) \to (E_{U_i}, E_{U_i} \cap E_0)$

which preserves projections onto U_i. Furthermore, denote by $j_{U_i} : E_{U_i} \to E$ the inclusion map of E_{U_i} into E. Observe that the cohomology class $j^*_{b,i}(a_j)$ in $H^*(F, F_0)$ doesn´t depend on the chice of the element $b \in U_i$. Then , using the Kunneth theorem and the fact that the elements $j^*_{b,i}(a_1), \ldots, j^*_{b,i}(a_r)$ are homogeneous in $H^*(F, F_0)$ we obtain that $j^*_{U_i}(a_1), \ldots, j^*_{U_i}(a_r)$ is a $H^*(U_i)-$basis for $H^*(E_{U_i}, E_{U_i} \cap E_0)$. Thus, the theorem holds on all the open sets U_i.

Step 2: Take now two of the open sets U_i and U_j with $i \neq j$. Clearly the result also holds for the open set $U_i \cap U_j$ and we shall argue that the result holds for $U_i \cup U_j$. Let $r(i)$ be the degree of the cohomology class a_i and consider for any open set U of B and any natural number $n \in \mathbb{N}$ the following two groups:

- $K^n(U) = \oplus_{1 \leq i \leq r} H^{n-n(i)}(U)$
- $L^n(U) = H^n(E_U, E_U \cap E_0)$

Furthermore, let $\theta_n : K^n(U) \to L^n(U)$ be the map defined as

$$\theta_n(b_1, \ldots, b_r) = \sum_{1 \leq j \leq r} p^*(b_i) \cup a_i$$

Observe that the result of this theorem holds on U if and only if θ_n is an isomorphism for all n. By the use of Mayer- Vietoris sequences we construct a commutative diagram with exact rows of the form:

$$\begin{tikzcd}
\cdots \arrow[r] & K^{n-1}(U_i) \oplus K^{n-1}(U_j) \arrow[r] \arrow[d, "\theta_{n-1} \oplus \theta_{n-1}"] & K^{n-1}(U_i \cap U_j) \arrow[r] \arrow[d, "\theta_{n-1}"] & K^n(U_i \cup U_j) \arrow[d, "\theta_n"] \\
\cdots \arrow[r] & L^{n-1}(U_i) \oplus L^{n-1}(U_j) \arrow[r] & L^{n-1}(U_i \cap U_j) \arrow[r] & L^n(U_i \cup U_j)
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r] & K^{n}(U_i) \oplus K^{n}(U_j) \arrow[r] \arrow[d, "\theta_{n} \oplus \theta_{n}"] & K^{n}(U_i \cap U_j) \arrow[r] \arrow[d, "\theta_{n}"] & \cdots \\
\cdots \arrow[r] & L^{n}(U_i) \oplus L^{n}(U_j) \arrow[r] & L^{n}(U_i \cap U_j) \arrow[r] & \cdots
\end{tikzcd}$$

Then, by the Five-Lemma we obtain that the result also holds for $U_i \cup U_j$.

Step 3: Finally, since $B = U_1 \cup \ldots \cup U_n$, an easy induction argument will yield us the result on all of B.

□

Remark. By the use of spectral sequences, the Leray-Hirch theorem can be proved for bundles which are not of necessarily of finite type, although the techniches that require such a proof fall out of the scope of this project. Nevertheless we shall use this more general result throughout this chapter.

5.2 The Projective Bundle

This section is devoted to a particular *fiber bundle* which can be associated to any vector bundle and which will play a central in both the definition of Chern and Stiefel-Whitney classes and to prove that such characteristic classes are uniquely determined by their axioms. As a means of simplification, from now on we shall restrict ourselves only to *complex* vector bundles since the arguments in the real case are merely simplifications of the arguments that follow.

Definition 5.3. *Let ξ be a $n-$vector bundle $\xi = (E, B, p)$. Let E_0 be the subspace obtained when removing from E the zero section and let $p_0 = p_{E_0}$. Let E' be the quotient space obtained by identifying in E_0 points in a fiber which lie in the same one dimensional linear subspace and let $q : E' \to B$ be the map that factorizes $p_0 : E_0 \to B$. We define the* ***projective bundle*** *$P\xi$ associated to ξ to be the fiber bundle $q : E' \to B$.*

We have the following lemma concerning this fiber bundle.

Lemma 5.4. *For any $n-$vector bundle $\xi = (E, B, p))$ we have that $P\xi$ is locally trivial.*

Proof. Let U be an open subset of B where E_U is trivial. Let $Q : E_0 \to E'$ be the quotient map with which we obtain E' and let $\overline{Q} : U \times \mathbb{C}_0^n \to U \times \mathbb{C}P^{n-1}$ be the obvious quotient map. If $h_U : U \times \mathbb{C}_0^n \to p_0^{-1}(U)$ is a local trivialization for E_0 on U then define $f_U : U \times \mathbb{C}P^{n-1} \to q^{-1}(U)$ as the map which factorizes the composition Qh_u. Thus, we have the following commutative diagram:

$$\begin{tikzcd}
U \times \mathbb{C}_0^n \arrow[r, "\overline{Q}"] \arrow[d, leftrightarrow, "h_U"'] & U \times \mathbb{C}P^{n-1} \arrow[d, "f_U"] \\
p_0^{-1}(U) \arrow[r, "Q"] & q^{-1}(U)
\end{tikzcd}$$

It is easy to verify that f_U is a bijective map. Furthermore, applying this exact argument using h_U^{-1} we obtain an inverse f_U^{-1}. Thus, $P\xi$ is locally trivial. □

Observe that the total space of $P\xi$ consists of points of the form (b, L) where $b \in B$ and L is a one dimensional linear subspace inside the fiber $p^{-1}(b)$. Taking the projection $q : E' \to B$ of the projective bundle we can consider the pull-back $q^*\xi$ over E'. Consider the subpace E_{λ_ξ} of $E(q^*\xi)$ consisting of all points $((L, b), x)$ such that $x \in L$ and define $p_{\lambda_\xi} : E_{\lambda_\xi} \to B$ to be the restriction of the projection map $p_{q^*\xi}$ of the bundle $q^*\xi$ over E'. It is not hard, but rather cumbersome, to prove that λ_ξ is indeed a line bundle. Furthermore, we have the following result concerning the topological nature of $E(P(\xi))$.

Lemma 5.5. *If ξ is an $n-$vector bundle over paracompact space B then $E(P\xi)$ is a paracompact space.*

This lemma now enables us to use proposition 2.11 of chapter 2 to guarantee the existence of a complement σ for the line bundle λ_ξ. Thus, we have that $q^*\xi = \lambda_\xi \oplus \sigma$.

The following proposition, which states another property worth noting of the projective bundle, will enable us to apply the Leray-Hirch theorem in the context of projective bundles.

Proposition 5.6. *let $\xi = (E, B, p)$ be an $n-$vector bundle over a paracompact space B, let $P\xi = (E(P\xi), B, q)$ be its projective vector bundle, let $b \in B$ and let U be an open neighborhood of b for which there exists a local trivialization $f : U \times \mathbb{C}^n \to E_U$. Consider also the local trivialization $f' : U \times \mathbb{C}P^{n-1} \to q^{-1}(U)$ obtained from f and let $j_b : \mathbb{C}P^{n-1} \to E(P(\xi))$ be the composition of maps*

$$\mathbb{C}P^{n-1} \xrightarrow{i_b} b \times \mathbb{C}P^{n-1} \xrightarrow{f'} q^{-1}(b)$$

Then $j_b^*(\lambda_\xi)$ is isomorphic to the canonical line bundle γ_1^n.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}_0^n & \xrightarrow{fi_b} & p^{-1}(b)_0 \\ \downarrow & & \downarrow \\ \mathbb{C}P^{n-1} & \xrightarrow{j_b} & q^{-1}(b) \end{array}$$

where the vertical maps are the usual quotient maps. Consider now the map $F : \mathbb{C}P^{n-1} \times \mathbb{C}_0^n \to q^{-1}(b) \times p^{-1}(b)_0$ defined as $F = (j_b, fi_b)$. Then, by the commutativity of the previous diagram, we have that if h is the restriction of F on $E(\gamma_1^n)$ then h is a bundle map between γ_1^n and $(\lambda_\xi)_{q^{-1}(b)}$. In a completely similar fashion we can construct an inverse $h^{-1} : (\lambda_\xi)_{q^{-1}(b)} \to \gamma_1^n$ for h. Thus, by proposition 1.10 of chapter 1 we obtain that $j_b^*(\lambda_\xi) \cong \gamma_1^n$. $\square$

5.3 Definition of the Chern Classes

In this section we finally define the Chern classes of a complex vector bundle. Remember that in the previous chapter we proved that the cohomology ring $H^*(\mathbb{C}P^\infty; \mathbb{Z})$ is isomorphic to the polynomial algebra $\mathbb{Z}[e((\gamma_1)_R)]$. Through out the rest of this chapter we shall denote $e((\gamma_1)_R)$ by z and we shall assume that we are always working with the ring $\mathbb{Z}$.

Since the space $E(P(\xi))$ is a paracompact space given that the base space B of ξ is paracompact we have, by theorem 2.10, that there exists a map $f : E(P(\xi)) \to \mathbb{C}P^\infty$ such that $f^*(\gamma_1) \cong \lambda_\xi$. Denote by a_ξ the cohomology class $f^*(z)$ which is independent of the choice of f since this map is unique up to homotopy.

The following result, which is just a consequence of the Leray-Hirch theorem, is what will bring Chern classes to life.

Proposition 5.7. *The cohomology classes $1, a_\xi, \ldots, a_\xi^{n-1}$ conform a basis for the $H^*(B)-$module $H^*(E(P(\xi)))$. In particular, the map $q^* : H^*(B) \to H^*(E(P(\xi)))$ is a monomorphism.*

Proof. Observe first of all that that we have $a_\xi^k \in H^{2k}(E(P(\xi)))$ for all k. In order to prove this result we just need to verify that the conditions of the Leray-Hirch Theorem hold. Applying proposition 1.23 of chapter 1, we can find a countable open cover $\{U_i\}_i$ of B such that $P(\xi)$ is trivial on each U_i. Let U_i be one of these open sets, let b be an element in U_i and define $j_{b,i} : \mathbb{C}P^{n-1} \to q^{-1}(b)$ as it was defined in the Leray-Hirch theorem. By proposition 5.6 we have that $j_{b,i}^* f^*(\gamma_1) \cong \gamma_1^n$. But on the other hand we also have that the inclusion map $i : \mathbb{C}P^{n-1} \to \mathbb{C}P^\infty$ is such that $i^*\gamma_1 \cong \gamma_1^n$. Thus, by corollary 2.14 of chapter 2 we must have that $f j_{b,i}$ and i are homotopy equivalent. Since by proposition 4.18 of chapter 4 we have that $H^*(\mathbb{C}P^{n-1}; \mathbb{Z}) \cong \mathbb{Z}[e((\gamma_1^n))_R]/(e((\gamma_1^n)_R)^n)$ and since $j_{b,i}^*(a_\xi^k) = e((\gamma_1^n)_R))^k$ for all k we can conclude that $1, j_{b,i}^*(a_\xi), \ldots, j_{b,i}^*(a_\xi^{n-1})$ is a $\mathbb{Z}-$basis for $H^*(\mathbb{C}P^{n-1})$ and thus by the Leray-Hirsh theorem we obtain the desired result. $\square$

Since $1, a_\xi, \ldots, a_\xi^{n-1}$ conform a basis for the $H^*(B)-$module $H^*(E(P(\xi)))$ we can find cohomology classes $c_i(\xi) \in H^{2i}(B; \mathbb{Z})$ such that $c_0(\xi) = 1$, $c_i(\xi) = 0$ for $i > n$ and such that

$$a_\xi^n = \sum_{j=1}^{n} (-1)^{j+1} c_j(\xi) a_\xi^{n-j}$$

Definition 5.8. For a complex vector bundle ξ, the cohomology class $c_i(\xi) \in H^{2i}(B; \mathbb{Z})$ is the ***i-th Chern class*** of ξ and $1 + c_1(\xi) + \ldots + c_n(\xi)$ is called the total Chern class of ξ.

5.4 Properties of the Chern Classes

The focus of this section will be to discuss the axioms that characterize uniquely the Chern classes of complex vector bundles. In particular, we shall prove that the Chern Classes satisfy the following four axioms:

1. For each natural number i there is a Chern class $c_i \in H^{2i}(B; \mathbb{Z})$ with the condtion that $c_0(\xi) = 1$ and $c_i(\xi) = 0$ if $i > n$.
2. *Naturality* If $f : B' \to B$ is a continuous map then $f^*(c_i(\xi)) = c_i(f^*\xi)$ for all i.
3. *Whitney sum formula.* For two vector bundles ξ and δ we have that $c_m(\xi \oplus \delta) = \sum_{i+j=m} c_i(\xi) \cup c_j(\delta)$.
4. For the universal line bundle γ_1 over $\mathbb{C}P^\infty$ we have that $c_1(\gamma_1) = e((\gamma_1)_R)$.

Clearly the Chern Classes satisfy property (1). To verify the naturality condition observe that a map $f : B' \to B$ induces a fiber bundle map $u : E(P(f^*\xi)) \to E(P(\xi))$ which makes the following diagram commute

$$\begin{tikzcd} E(P(f^*\xi)) \arrow[r, "u"] \arrow[d, "q'"'] & E(P(\xi)) \arrow[d, "q"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

Since clearly we have that $\lambda_{f^*\xi} \cong u^*(\lambda_\xi)$ we also have then that $a_{f^*\xi} = u^*(a_\xi)$. Thus, when applying u^* to both sides of the equality $a_\xi^n = \sum_{i=1}^n (-1)^{i+1} c_i(\xi) a_\xi^{n-i}$ we obtain by the commutativity of the above diagram that

$$a_{f^*\xi}^n = \sum_{i=1}^n (-1)^{i+1} f^*(c_i(\xi)) a_{f^*\xi}^{n-i}$$

Thus, applying propostion 5.7 we obtain that $f^*(c_i(\xi)) = c_i(f^*\xi)$.

For the verification of axiom (4) we observe first that $E(P(\gamma_1)) = \mathbb{C}P^\infty$ and that $\lambda_{\gamma_1} = \gamma_1$. In this case we have thus $c_1(\gamma_1) = a_{\gamma_1} = Id^*(e((\gamma_1))_R) = e((\gamma_1)_R)$ and thus we also have that the Chern classes satisfy this axiom.

The proof of the third axiom is a little bit more elaborate and will be postponed until a later section.

5.5 Splitting Maps and Uniqueness of the Chern Classes

We have now all the technical machinery to introduce the following concept.

Definition 5.9. Let ξ be a vector bundle over B. A splitting map for ξ is a map $f_1 : B_1 \to B$ such that $f^*\xi$ is the whitney sum of line bundles and $f^* : H^*(B) \to H^*(B_1)$ is an injective map.

With the next proposition we establish the actual existence of splitting maps

Proposition 5.10. *Any $n-$vector bundle ξ over a space B has a splitting map.*

Proof. We prove this theorem by induction on the dimension of the fibers of ξ. If ξ is a $1-$bundle then clearly the map $Id : B \to B$ is a splitting map for ξ. Suppose now that the result holds for vector bundles of dimension $n-1$ and let ξ be an $n-$vector bundle. Remember that we have proved that the $n-$bundle $q^*\xi = (E(q^*\xi), E(P(\xi)), p')$ can be decomposed as the whitney sum $\lambda_\xi \oplus \sigma$ where λ_ξ is a line bundle and σ is a $n-1-$bundle. Then, if $f : B_1 \to E(P(\xi))$ is a splitting map for σ we have that $qf : B_1 \to B$ is a splitting map for ξ since $q^* : H^*(B) \to H^*(E(P(\xi)))$ is a monomorphism by proposition 5.7.

□

The previous result can be furhter refined in the following way.

Corollary 5.11. *If $\xi_1, \dots, \xi_n$ are vector bundles over a space B then there exists a map $f : B_1 \to B$ which is a splitting map for all the vector bundles ξ_i with $i = 1, \dots, n$.*

Proof. We shall prove this corollary by induction on the number of vector bundles defined on B. Clearly, by the previous proposition we have our result for $n = 1$. Suppose then that the result holds for vector bundles $\xi_1, \dots, \xi_{k-1}$ over B where each ξ_i has fibers of dimension $n(i)$. Then there exists a map $g : B' \to B$ such that $g^*\xi_i \cong \lambda_{i_1} \oplus \dots \oplus \lambda_{i_{n(i)}}$ for all i and such that $g^* : H^*(B) \to H^*(B')$ is a monomorphism. If ξ_k is an additional vector bundle over B we have that there exists a splitting map $f : B_1 \to B'$ for the vector bundle $g^*\xi_k$. Then clearly $gf : B_1 \to B$ is a splitting map for all the vector bundles $\xi_1, \dots, \xi_k$ and this concludes the proof.

□

We can now prove that the axioms stated in the previous section determine completely the Chern classes.

Theorem 5.12. *If c_i' is another sequence of characteristic classes which satisfies the axioms of the previous section then we must have that $c_i'(\xi) = c_i(\xi)$ for any $n-$vector bundle ξ.*

Proof. Since c_i and c_i' satisfy the same set of axioms we have that $c_1(\gamma_1) = c_1'(\gamma_1)$ and thus by naturality we have that $c_1(\delta) = c_1'(\delta)$ for any line bundle δ. Furthermore, suppose ξ is some $n-$bundle over a paracompact space B and let $f : B_1 \to B$ be a splitting map for ξ, i.e, f^* is injective and $f^*\xi \cong \delta_1 \oplus \dots \oplus \delta_n$ for some line bundles δ_i. Then, applying the Whitney sum formula and the fact that c and c' coincide for line bundles we have that $c_i(f^*\xi) = c_i'(f^*\xi)$ for all i and thus $f^*(c_i(\xi)) = f^*(c_i'(\xi))$. Since f^* is a monomorphism we obtain finally that $c_i(\xi) = c_i'(\xi)$.

□

5.6 The Whitney Sum Formula

This entire section will be devoted to the proof of the Whitney sum formula for Chern classes. We begin with the following proposition definition

Definition 5.13. Let $\xi = (E, B, p)$ and $\delta = (E_1, B_1, p_1)$ be two complex vector bundles. The ***exterior product*** $\xi \overline{\otimes} \delta$ of ξ and δ is defined to be the complex vector bundle over $B \times B_1$ such that the fiber on (b, b_1) is $p^{-1}(b) \otimes p_1^{-1}(b_1)$.

The following proposition allows us to compute the Chern class of the tensor product of line bundles.

Proposition 5.14. *If δ_1 and δ_2 are line bundles over a paracompact space B then we have the relation $c_1(\delta_1 \otimes \delta_2) = c_1(\delta_1) + c_1(\delta_2)$.*

Proof. Let γ_1 be the universal line bundle on $\mathbb{C}P^\infty$ and let $k : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ be the classifying map for $\gamma_1 \overline{\otimes} \gamma_1$. Let $i : \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ be the map given by $x \to (x, x_0)$ for some fixed x_0, let $p_i : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ be the projection onto the i-th component and denote by e_i the cohomology class $p_i^*(c_1(\gamma_1))$. By the Kunneth theorem we have that e_1 and e_2 conform a $\mathbb{Z}-$basis for $H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty; \mathbb{Z})$ and thus there exists unique $a_1, a_2 \in \mathbb{Z}$ such that $a_1 e_1 + a_2 e_2 = k^*(c_1(\gamma_1))$. Our first step is to prove that we have in fact $a_1 = a_2 = 1$. In order to do this, first observe that we have $p_1 i = Id$ and that $p_2 i$ is constant .Thus, it follows that $i^*(e_1) = c_1(\gamma_1)$ and $i^*(e_2) = 0$. Using these relations we compute the following:

$$\begin{aligned} a_1 c_1(\gamma_1) &= a_1 i^*(e_1) = a_1 i^*(e_1) + a_2 i^*(e_2) \\ &= i^*(a_1 e_1 + a_2 e_2) = i^*(k^*(c_1(\gamma_1))) \\ &= c_1(i^* k^* \gamma_1) = c_1(i^* \gamma_1 \overline{\otimes} \gamma_1) \\ &= c_1(i^*(p_1^* \gamma_1 \otimes p_2^* \gamma_1)) = c_1(i^* p_1^* \gamma_1 \otimes i^* p_2^* \gamma_1) = c_1(\gamma_1) \end{aligned} \tag{4}$$

For the last equality we applied the fact that $i^* p_2^* \gamma_1$ is a trivial line bundle. Since we have that $c_1(\gamma_1)$ constitutes a $\mathbb{Z}-$basis for $H^1(\mathbb{C}P^\infty; \mathbb{Z})$, by the previous calculations we obtain that $a_1 = 1$. In a totally analogous fashion it can be proven that $a_2 = 1$ and thus $k^*(c_1(\gamma_1)) = c_1(\gamma_1 \overline{\otimes} \gamma_1) = e_1 + e_2$.

Now let δ_1 and δ_2 be line bundles over a paracompact space B and let $k_1 : B \to \mathbb{C}P^\infty$ and $k_2 : B \to \mathbb{C}P^\infty$ be classifying maps for δ_1 and δ_2 respectively. Taking the diagonal map $d : B \to B \times B$ it is not hard to verify that we have $\delta_1 \otimes \delta_2 \cong d^*(\delta_1 \overline{\otimes} \delta_2) \cong d^*(k_1 \times k_2)^*(\gamma_1 \overline{\otimes} \gamma_1)$. These isomorphisms allows then to do the following calculations:

$$\begin{aligned} c_1(\delta_1 \otimes \delta_2) &= c_1(d^*(k_1 \times k_2)^*(\gamma_1 \overline{\otimes} \gamma_1)) = d^*(k_1 \times k_2)^*(e_1 + e_2) \\ &= d^*(k_1 \times k_2)^*(p_1^*(c_1(\gamma_1))) + d^*(k_1 \times k_2)^*(p_2^*(c_1(\gamma_1))) \\ &= k_1^*(c_1(\gamma_1)) + k_2^*(c_1(\gamma_1)) = c_1(\delta_1) + c_1(\delta_2) \end{aligned} \tag{5}$$

The second last equality follows from the fact that $k_i = p_i(k_1 \times k_2)d$ for $i = 1, 2$ and thus we have completed the theorem.

□

As a useful consequence of this proposition we obtain the following

Corollary 5.15. $c_1(\overline{\delta}) = -c_1(\delta)$ *for any line bundle* δ.

The following proposition is the essential step for proving the Whitney sum formula for Chern classes.

Proposition 5.16. *For line bundles $\delta_1, \dots, \delta_n$ over a paracompact space B we have the relation $c(\delta_1 \oplus \dots \oplus \delta_n) = (1 + c_1(\delta_1)) \dots (1 + c_1(\delta_n))$.*

Proof. Denote by η the $n-$vector bundle $\delta_1 \oplus \dots \oplus \delta_n$. Taking the pull-back of η on $q : E(P(\eta)) \to B$ we obtain the equality

$$q^*\delta_1 \oplus \dots \oplus q^*\delta_n = \lambda_\eta \oplus \sigma$$

Tensoring both sides with $\overline{\lambda_\eta}$ we thus obtain

$$(q^*\delta_1 \otimes \overline{\lambda_\eta}) \oplus \dots \oplus (q^*\delta_n \otimes \overline{\lambda_\eta}) = (\lambda_\eta \otimes \overline{\lambda_\eta})) \oplus (\sigma \otimes \overline{\lambda_\eta}))$$

By the previous equality we have that $q^*\eta \otimes \overline{\lambda_\eta}$ admits a section s which is nowhere zero and that projects to a section s_i on each $q^*\delta_i \otimes \overline{\lambda_\eta}$. For $i = 1, \dots, n$ let V_i be the open set of $E(P(\eta))$ such that $s_i \neq 0$. Observe that since s is nowhere zero we must have that $\bigcup_{1 \leq i \leq n} V_i = E(P(\eta))$. If for all $i = 1, \dots, n$ we denote by $j_i : V_i \to E(P(\eta))$ the usual inclusion map, then by the way we defined the open sets V_i we must have $j_i^*(c_1(q^*\delta_i \otimes \overline{\lambda_\eta})) = 0$. Thus, by taking the long exact sequence of the pair $(E(P(\eta)), V_i)$ we must have that $c_1(q^*\delta_i \otimes \overline{\lambda_\eta})$ can be represented by a cocycle $\tau_i \in C^1(E(P(\eta)), \mathbb{Z})$ which vanishes on all $1-$chains contained in V_i. However, the cup product $[\tau_1] \cup \dots \cup [\tau_n]$ is an element of $H^*(E(P(\eta)), \bigcup_{1 \leq i \leq n} V_i) = H^*(E(P(\eta)), E(P(\eta)) = 0$. Thus we obtain the following identity

$$c_1(q^*\delta_1 \otimes \overline{\lambda_\eta}) \dots c_1(q^*\delta_n \otimes \overline{\lambda_\eta}) = (q^*(c_1(\delta_1)) + c_1(\overline{\lambda_\eta})) \dots (q^*(c_1(\delta_n)) + c_1(\overline{\lambda_\eta}) = 0$$

Applying the fact that $c_1(\overline{\lambda_\eta}) = -c_1(\lambda_\eta)$ and that $a_\eta = c_1(\lambda_\eta)$, the previous identity turns into

$$(q^*(c_1(\delta_1)) - a_\eta) \dots (q^*(c_1(\delta_n)) - a_\eta) = 0$$

If we denote by α_k the element $\sum c_1(\delta_{i(1)}) \dots c_1(\delta_{i(k)})$ with $1 \leq i(1) < \dots < i(k) \leq n$ then the previous identitity can be rewritten as:

$$a_\eta^n = \textstyle\sum_{1 \leq i \leq n} (-1)^{i+1} q^*(\alpha_1) a_\eta^{n-i}$$

Observe that in order to obtain the last equality we relied heavily on the fact that the cohomology classes $q^*(c_1(\delta_j))$ and a_η are in $H^2(E(P(\eta)); \mathbb{Z})$ and thus there is no problem with signs when permuting any two of these elements. Since we also must have

$$a_\eta^n = \textstyle\sum_{1 \leq i \leq n} (-1)^{i+1} q^*(c_i(\eta)) a_\eta^{n-i}$$

it follows from theorem 5.7 that $c(\eta) = (1 + c_1(\delta_1)) \dots (1 + c_1(\delta_n))$.

∎

We can now proceed to prove the Whitney sum formula for Chern classes.

Theorem 5.17. *For two vector bundles η and δ over a paracompact space B we have that $c(\eta \oplus \delta) = c(\eta)c(\delta)$.*

Proof. By corollary 5.11 we have that there exists a map $f : B_1 \to B$ which is a splitting map for both η and δ. That is, we have that $f^* : H^*(B) \to H^*(B_1)$ is a monomorphism and that $f^*\eta = \lambda_1 \oplus \ldots \oplus \lambda_n$ and $f^*\delta = \lambda_{n+1} \oplus \ldots \lambda_{n+m}$ for some line bundles λ_i over B_1. Thus, applying the previous proposition we obtain

$$f^*(c(\eta \oplus \delta)) = (1 + c_1(\lambda_1)) \ldots (1 + c_1(\lambda_n))(1 + c_1(\lambda_{n+1})) \ldots (1 + c_1(\lambda_{n+m}))$$

But on the other hand it is plain that we also have

$$f^*(c(\eta)c(\delta)) = (1 + c_1(\lambda_1)) \ldots (1 + c_1(\lambda_n))(1 + c_1(\lambda_{n+1})) \ldots (1 + c_1(\lambda_{n+m}))$$

Then, since f^* is injective we must have that $c(\eta \oplus \delta) = c(\eta)c(\delta)$.

□

Remark: Absoloutely everything we have done in this chapter can be reproduced using real vector bundles, real projective spaces and the ring $\Lambda = \mathbb{Z}_2$ by making the necessary simplifications. For example, in the definition 5.8 the minus signs can be omitted since we would be working with $\mathbb{Z}_2$ coefficients. This will then yields us an alternate construction of the Stiefel-Whitney classes which is preferred by the author since it only relies on tools belonging to the theory of vector bundles (as opposed to the use of Steenrod squaring operations) and since it manifests that Stiefel-Whitney and Chern classes have a common origin.

5.7 The Cohomology Rings $H^*(G_n(\mathbb{C}^\infty)); \mathbb{Z})$ and $H^*(G_n(\mathbb{R}^\infty); \mathbb{Z}_2)$

This section is devoted to describe the cohomology rings $H^*(G_n(\mathbb{C}); \mathbb{Z})$ and $H^*(G_n(\mathbb{R}); \mathbb{Z}_2)$ and give an explicit description of the ring of characterisitc classes for comlex and real vector bundles when working respectively with the rings $\mathbb{Z}$ and $\mathbb{Z}_2$. Before jumping into the main core of this section we discuss the following algebraic technicalities.

Let Λ be a ring with unit 1, let $R[x_1, \ldots, x_n]$ be the ring of polynomials in n variables and let S_n be the symmetric group of order n. If $P \in R[x_1, \ldots, x_n]$ and if $\tau \in S_n$ then we define $P_\tau \in R[x_1, \ldots, x_n]$ by the relation $P_\tau(x_1, \ldots, x_n) = P(x_{\tau(1)}, \ldots, x_{\tau(n)})$.

Definition 5.18. A polynomial $P \in R[x_1, \ldots, x_n]$ is called ***symmetric*** if $P = P_\tau$ for all $\tau \in S_n$.

Examples of symmetric polynomials in $R[x_1, \ldots, x_n]$ (the most important ones as we shall soon see) are $\sigma_0^n = 1$, $\sigma_1^n(x_1, \ldots, x_n) = x_1 + \ldots + x_n$, $\sigma_n^n(x_1, \ldots, x_n) = x_1 \ldots x_n$ and $\sigma_k^n(x_1, \ldots, x_n) = \sum x_{i(1)} \ldots x_{i(k)}$ where $1 \leq i(1) < \ldots < i(k) \leq n$.

The following theorem, whose proof can be found in [Hu], is one of the key ingredients for the material of this section.

Theorem 5.19. *The subring $R[\sigma_1, \ldots, \sigma_n]$ of $R[x_1, \ldots, x_n]$ contains all the symmetric functions and the functions $\sigma_1, \ldots, \sigma_n$, called the elementary symmetric functions, are algebraically independent, that is, they don't satisfy any polynomial relations.*

Now we focus on proving the main result of this section and indeed of the whole project. Our first step will be to prove the following.

Proposition 5.20. *Let $h_n : \mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty \to G_n(\mathbb{C}^\infty)$ be a classifying map for $\gamma_1 \times \ldots \times \gamma_1$. Then h_n is a splitting map for the canonical bundle γ_n over $G_n(\mathbb{C}^\infty)$.*

Proof. we start off by observing that if $p_i : \mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is the projection map onto the i-th component then we have that

$$h_n^*\gamma_n \cong \gamma_1 \times \ldots \times \gamma_1 \cong p_1^*\gamma_1 \oplus \ldots \oplus p_n^*\gamma_1$$

Thus, the only thing we have to prove is that h_n^* is a monomorphism. Let $f : B \to G_n(\mathbb{C}^\infty)$ be a splitting map for γ_n with $f^*\gamma_n \cong \lambda_1 \oplus \ldots \oplus \lambda_n$ where λ_i is a line bundle over B. Let $g_i : B \to \mathbb{C}P^\infty$ be a classifying map for λ_i, i.e, a map g_i such that $\lambda_i \cong g_i^*\gamma_1$. Then, if we define $g : B \to \mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty$ as $g = (g_1, \ldots, g_n)$ we obtain

$$g^*(\gamma_1 \times \ldots \times \gamma_1) \cong g^*(p_1^*\gamma_1) \oplus \ldots \oplus g^*(p_n^*\gamma_1) \cong g_1^*(\gamma_1) \oplus \ldots \oplus g_n^*(\gamma_1) \cong \lambda_1 \oplus \ldots \oplus \lambda_n$$

Thus, by corollary 2.14 of chapter 2, we have that $f \cong h_n g$ and consequently we have $f^* = g^* h_n^*$. Since f^* is injective we must have that h_n^* is also injective. $\square$

The following lemmas will also be needed

Lemma 5.21. *Consider the n-fold product*

$$A = A_1 \times \ldots \times A_n = \mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty$$

Let $\tau \in S^n$ and let a_i denote the element $p_i^*(c_1(\gamma_1))$ where p_i is the projection onto the $i-$th component. If $f_\tau : A \to A$ is the map which maps A_i identically to $A_{\tau(i)}$ we have that $f_\tau^*(a_{\tau(i)}) = a_i$ and $(f_\tau^{-1})^*(a_i) = a_{\tau(i)}$

Proof. This result just follows from the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "f_\tau"] \arrow[d, "p_i"'] & A \arrow[d, "p_{\tau(i)}"] \\ A_i \arrow[r, "f_\tau"] & A_{\tau(i)} \end{tikzcd}$$

$\square$

Lemma 5.22. *With the notation of the previous lemma we have that $H^*(\mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty) = \mathbb{Z}[a_1, \ldots, a_n]$.*

Proof. This is just a direct application of the Kunneth theorem. $\square$

Now we can prove the main result

Theorem 5.23. *Let c_i denote $c_i(\gamma_n)$, where γ_n is the canonical universal $n-$bundle. Then the cohomology ring $H^*(G_n(\mathbb{C}^\infty);\mathbb{Z})$ is isomorphic to $\mathbb{Z}[c_1, \ldots, c_n]$ and the classes $c_1, \ldots, c_n$ are algerbaically independent.*

Proof. Consider a classifying map $h_n : \mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty \to G_n(\mathbb{C}^\infty)$ for $\xi = \gamma_1 \times \ldots \times \gamma_1$, which by the previous proposition is a splitting map for γ_n. As it was mentioned in the proof of 5.20, we have that $\xi \cong p_1^*\gamma_1 \oplus \ldots \oplus p_n^*\gamma_1$ and thus $c(\xi) = (1 + a_1) \ldots (1 + a_n)$. In particular we have that

- $c_1(\xi) = a_1 + \ldots + a_n$
- $c_n(\xi) = a_1 \ldots a_n$
- $c_k(\xi) = \sum a_{i(1)} \ldots a_{i(k)}$ with $1 \leq i(1) < \ldots < i(k) \leq n$.

that is, $c_i(\xi)$ is the $i-$th elementary symmetric function σ_i with the variables $a_1, \ldots, a_n$. If $\tau \in S_n$ and if $f_\tau : A \to A$ is the map which was defined in lemma 5.22 we have that both $h_n(f_\tau)^{-1}$ and h_n are classifying maps for the bundle ξ and thus $h_n(f_\tau)^{-1} \cong h_n$. Thus, from lemma 5.21 it follows that $\mathrm{Im}h_n^*$ lies in the subring $\mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ of $\mathbb{Z}[a_1, \ldots, a_n]$. Since $h_n^*(c_i) = \sigma_i$ and since h_n is injective we conclude that

$$H^*(G_n(\mathbb{C}^\infty);\mathbb{Z}) = \mathbb{Z}[c_1, \ldots, c_n]$$

and since the σ_i are algebraically independent then so are the Chern classes c_i of γ_n. $\square$

As an immediate corollary of this theorem we obtain the following result.

Corollary 5.24. *If ξ is an $n-$vector bundle over a paracompact space B then each characteristic class of ξ is of the form $\phi(c_1(\xi), \ldots, c_n(\xi))$ where ϕ is some polynomial of n indeterminates and with integer coefficients. Conversely, any element of the form $\phi(c_1(\xi), \ldots, c_n(\xi))$ is a characteristic class for ξ.*

Remark: The version of theorem 5.23 and corollary 5.24 for real vector bundles and coefficients in $\mathbb{Z}_2$ are proven in a totally analogous manner. In particular, we have given a proof of theorem 3.4, which describes all characteristic classes for real vector bundles when dealing with $\mathbb{Z}_2$.

5.8 Pontrjagin Classes... and Good Bye

We finish this project by introducing yet another variety of characteristic classes for *real* vector bundles: ***Pontrjagin Classes***. Eventhough our discussion of such classes will be brief it would be a shame to not mention them with all the machinery we have at our disposal.

Let then ξ be a real $n-$vector bundle. We begin with the following definition

Definition 5.25. The complex $n-$vector bundle $(\xi \otimes \mathbb{C})$, obtained by tensoring each fiber of ξ with $\mathbb{C}$, is called the ***complexification*** of ξ.

By means of continuous functors (see chapter 1) we have that $(\xi \otimes \mathbb{C})$ has a natural topology which makes it into a $n-$complex vector bundle. The following propositions are the key ingredients to define the Pontrjagin classes

Proposition 5.26. *For any real $n-$ vector bundle ξ we have that $(\xi \otimes \mathbb{C}) \cong (\xi \otimes \mathbb{C})^*$.*

Proof. Consider the function $F : E((\xi \otimes \mathbb{C})) \to E((\xi \otimes \mathbb{C})^*)$ defined fiberwise as $F(a \otimes z) = F(a \otimes \overline{z})$. Then, it is not hard to verify that this is will indeed give us a bundle isomorphism. □

Proposition 5.27. *Let $\xi = (E, B, p)$ be a complex $n-$vector bundle. Then, the Chern classes for ξ^* are given by $c_i(\xi^*) = (-1)^i c_i(\xi)$.*

Proof. By corollary 5.15 we have that $c_1(\delta^*) = -c_1(\delta)$ for complex line bundles. Let ξ be then any $n-$complex vector bundle and consider a splitting map $f : B_1 \to B$ for ξ such that $f^*\xi \cong \lambda_1 \oplus \dots \lambda_n$. In this case, it is not hard to verify that we have $f^*\xi^* \cong \lambda_1^* \oplus \dots \lambda_n^*$. With this relation at hand we obtain

$$c(f^*\xi^*) = (1 - c_1(\lambda_1)) \dots (1 - c_1(\lambda_n)) = \textstyle\sum_{0 \leq i \leq n} (-1)^i c_i(f^*(\xi))$$

Since f^* is a monomorphism we obtain the desired relation.

□

Corollary 5.28. *If a complex vector bundle ξ is isomorphic to ξ^* then $c_{2i+1}(\xi) = 0$ for all $i \geq 0$.*

Proof. This is just an immediate consequence of the naturality of the Chern classes and the previous propositions. □

In particular, we have that the previous proposition applies to the complexification $\xi \otimes \mathbb{C}$. We have reached thus the following definition

Definiton 5.29. Let ξ be a real $n-$vector bundle. The $i-$***th Pontrjagin class*** of ξ is defined to be $p_i(\xi) = c_{2_i}(\xi \otimes \mathbb{C})$, which is an element in $H^{4i}(B; \mathbb{Z})$. Naturality of these classes is an obvious consequence of the naturality of the Chern classes.

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