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A natural classifying space for cohomology with coefficients in a finite chain complex

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Introduction

This document is a master thesis written at the University of Copenhagen.

The purpose of the thesis and the research conducted for it was to find a natural classifying space for cohomology with coefficients in a finite chain complex. Cohomology with coefficients in a chain complex is a generalized cohomology theory which contains ordinary cohomology. It was, as far as I know first defined by Dold in [Dold], and it was further developed in the article [Brown64], in which it was also proved that cohomology with coefficients in a chain complex could be described in terms of ordinary cohomology up to functoriality in the coefficient variable. This description yields a classifying space for this cohomology theory. However, the fact that this description is not natural in the chain complex means that information is lost and that the classifying space obtained this way cannot contain the information given by functoriality of the cohomology theory in the coefficient variable.

In this thesis we describe a classifying space that does contain this information. However, we limit ourselves to the case of finite complexes, and complexes that are infinite in one direction.

The first chapter contains the definition of cohomology with coefficients in a chain complex, and the description in terms of ordinary cohomology. This leads up to a discussion of classifying spaces and the definition of a natural classifying space, which yields the formulation of the main task of the thesis. The material in this chapter is mainly taken from the article [Brown64].

The second chapter contains two results, namely a series of long exact sequences and equivalence of the singular cohomology functor and a cellular cohomology functor. This chapter consists of material developed for this thesis.

The third chapter contains background material on homotopy fibers. This material is described in many books on algebraic topology. However, since it is important that we are able to work effectively with homotopy fibers, the material is included here.

The last chapter contains the description of the natural classifying space and the proof that it is in fact a natural classifying space. The material of this chapter is the central part of the research developed for the thesis. The last section of this chapter also contains speculations of how to extend this result to the case of general complexes.

The reader is assumed to know basic algebraic topology such as ordinary cohomology and the fact that Eilenberg-MacLane spaces are classifying spaces for ordinary cohomology.

I would like to thank my advisor Jesper Michael Møller for suggesting the classifying space and outlining the strategy of the proof, and also for helping me along the way with many inspiring discussions. I would also like to thank Ronald Brown for his interest in this work and for drawing my attention to his article [Brown70].

Chapter 1

Cohomology with coefficients in a chain complex

In this section we aim to define cohomology with coefficients in a chain complex of abelian groups, and prove some basic results, particularly that we have defined a cohomology theory. Taking cohomology with coefficients in the complex having the group B in dimension 0 and 0's elsewhere we get cohomology with coefficients in the group B. So the concept of cohomology with coefficients in a chain complex generalizes that of cohomology with coefficients in a group. The main result in this section is that the groups $H^n(X, A)$ and $\prod_n H^n(X, H_{*-n}(A))$ are isomorphic. But the fact that this isomorphism is not natural in the complex A means that cohomology with coefficients in a complex contains more information than what is expressible in ordinary cohomology.

The last subsection of this chapter formulates the main problem of this thesis, namely that of constructing a natural classifying space for this cohomology theory. The definitions follow the article [Brown64].

1.1 The Hom functor

First we shall define some basic terminology:

Definition 1.1.1. *A chain complex of abelian groups is a sequence of abelian groups A_n with differentials $\partial_n : A_n \to A_{n-1}$, such that $\partial_{n-1}\partial_n = 0$. A cochain complex of abelian groups is a sequence of groups A_n and differentials $\delta : A_n \to A_{n+1}$ such that $\delta_{n+1}\delta_n = 0$.*

In what follows a chain complex of abelian groups will often just be denoted a chain complex.

We define cycles, boundaries, cocyles and coboundaries as usual, and this leads to the usual definition of homology of a chain complex and cohomology of a cochain complex.

Definition 1.1.2. *Let (A, ∂_A) and (B, ∂_B) be chain complexes of abelian groups. We define the cochain complex* $\mathrm{Hom}(A, B)$ *by setting* $\mathrm{Hom}(A, B)_p = \prod_n \mathrm{Hom}(A_n, B_{n-p})$*, with differential* $\delta : \mathrm{Hom}(A, B)_p \to \mathrm{Hom}(A, B)_{p+1}$ *given by* $\delta f = \partial_B f - (-1)^p f \partial_A$*.*

Let

$$B_0 \xrightarrow{h_0} \cdots \xrightarrow{h_{r-1}} B_r$$

be a finite chain complex. That is, $h_i h_{i-1} = 0$. *Let* B *be the chain complex with* B_0 *to* B_r *in dimensions* 0 *to* $-r$ *and zero in all other dimensions, and with the* h_i *maps. We define:*

$$\mathrm{Hom}(A; B_0 \to \ldots \to B_r) = \mathrm{Hom}(A; B)$$

Note that the definition of the differential is supposed to be understood the only way it makes sense. That is, an element in $\mathrm{Hom}(A, B)_p$ is a collection of maps $f : A_n \to B_{n-p}$ (one for each n). This means that $\partial_B f$ and $f\partial_A$ are collections of maps $A_n \to B_{n-(p+1)}$, and the minus sign means pointwise subtraction.

Note also that this definition could be immediately generalized to chain complexes of modules over some ring. For our purposes we are only interested in chain complexes of abelian groups.

A note on the finite complexes: When we write:

$$B_0 \overset{h_0}{\to} B_1 \to \ldots \overset{h_{r-1}}{\to} B_r$$

we mean the infinite chain complex with the groups B_0 to B_r in dimensions 0 to $-r$ and zeros elsewhere. This interpretation makes the class of finite complexes a subcategory of the category of chain complexes. We have chosen this notation in stead of the somewhat more logical notation

$$B_0 \overset{h_0}{\to} B_{-1} \to \ldots \overset{h_{-r+1}}{\to} B_{-r}$$

to avoid the many minussigns.

We define the length of a finite complex to be the number of groups in it. That is, the length of $B_0 \to \ldots \to B_r$ is $r + 1$.

We need to prove that Definition 1.1.2 does in fact define a cochain complex. That is, we need to prove that $\delta\delta$ is zero. This is easily verified:

$$\delta\delta f = \delta(\partial_B f - (-1)^p f\partial_A) =$$

$$\partial_B\partial_B f - (-1)^p\partial_B f\partial_A - (-1)^{p+1}(\partial_B f\partial_A - (-1)^p f\partial_A\partial_A) = 0$$

Definition 1.1.3. *Let* (A, ∂_A) *and* (B, ∂_B) *be chain complexes. A chain map of degree* p *is a sequence of maps* $f_n : A_n \to B_{n+p}$ *that commutes with the differential, that is* $f\partial_A = \partial_B f$. *A chain map of degree zero is often just called a chain map. If* (C, δ_C) *and* (D, δ_D) *are cochains, we define cochain maps of degree* p *to be sequences of maps* $A_n \to B_{n-p}$ *that commute with differentials. By cochain maps we simply mean cochain maps of degree* 0.

The next proposition shows that Hom is a functor of two variables that is contravariant in the first and covariant in the second.

Proposition 1.1.4. *Let* $f : A \to A'$ *and* $g : B \to B'$ *be chain maps. The maps* $f^* : \mathrm{Hom}(A', B) \to \mathrm{Hom}(A, B)$ *and* $g_* : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$ *defined by* $f^*(h) = hf$ *and* $g_*(h) = gh$ *are cochain maps.*

Clearly the maps $f \mapsto f^*$ and $g \mapsto g_*$ respects compositions, are associative and map identity to identity, so that Hom becomes a functor.

Proof. We only need to prove that f^* and g_* commutes with differentials. This is done by straight ahead computations:

$$\delta(f^*h) = \delta(hf) = \partial_B hf - (-1)^p hf\partial_A$$

and

$$f^*\delta(h) = f^*(\partial_B h - (-1)^p h\partial_{A'}) = \partial_B hf - (-1)^p h\partial_{A'} f$$

Since f commutes with derivatives we get the desired identity.

To prove that g_* commutes with derivatives we compute:

$$\delta g_*(h) = \delta gh = \partial_{B'} gh - (-1)^p gh\partial_A$$

and

$$g^*\delta(h) = g^*(\partial_B h - (-1)^p h\partial_A) = g\partial_B h - (-1)^p gh\partial_A$$

Again we use the fact that g commutes with differentials to get the desired equality.

□

Now we are ready for the main definition of this section:

Definition 1.1.5. *Let (A, ∂_A) and (B, ∂_B) be chain complexes. We define the cohomology of A with coefficients in B to be:*

$$H^*(A; B) = H^*(\mathrm{Hom}(A, B))$$

If

$$B_0 \xrightarrow{h_0} B_1 \longrightarrow \cdots \xrightarrow{h_{n-1}} B_n$$

is a finite chain complex (that is $h_i \circ h_{i-1} = 0$), we denote B the chain complex with the groups B_0 to B_{-n} in dimensions 0 to $-n$, and zeros elsewhere, and define:

$$H^*(A; B_0 \to \ldots \to B_{-n}) = H^*(A; B)$$

Since Hom is a functor of two variables from the category of chain complexes with chain maps to the same category, $H^*(-,-)$ also becomes a functor of two variables. It is contravariant in the first variable and covariant in the second.

Remark 1.1.6. If B_0 is an abelian group, we can view B_0 as a finite chain complex, and definition 1.1.5 gives us a new definition of cohomology with coefficients in B_0. Ordinary cohomology is obtained by applying the functor $\mathrm{Hom}(-, B_0)$ to the chain complex A and taking cohomology of the resulting chain complex $\mathrm{Hom}(A, B_0)$. If B denotes the chain complex with B_0 in dimension 0 and zeros elsewhere, we notice that $\mathrm{Hom}(A, B)_p = \mathrm{Hom}(A_p, B_0) = \mathrm{Hom}(A, B_0)_p$, and the derivative is just the derivative of ordinary cohomology up to a sign. Thus definition 1.1.5 generalizes the definition of cohomology with coefficients in a group.

Next we define the shift operator on a chain complex:

Definition 1.1.7. *Suppose B is a chain complex. We define the chain complex SB by:*

$$SB_n = B_{n-1}$$

and setting $\partial_{SB} = \partial_B$.

Proposition 1.1.8. *There exists an isomorphism:*

$$H^*(A;B) \simeq H^{*-1}(A;SB)$$

that is natural in both variables.

Proof. We define a cochain map of degree 1:

$$\phi : \mathrm{Hom}(A;B) \to \mathrm{Hom}(A;SB)$$

Suppose

$$f = (f_n)_{n\in\mathbb{Z}} \in \mathrm{Hom}(A;B)_p = \prod_{n\in\mathbb{Z}} \mathrm{Hom}(A_n;B_{n-p})$$

we define

$$\phi(f) = ((-1)^n f_n)_{n\in\mathbb{Z}} \in \mathrm{Hom}(A;SB)_{p-1} =$$
$$\prod_n \mathrm{Hom}(A_n;SB_{n-p+1}) = \prod_n \mathrm{Hom}(A_n;B_{n-p})$$

We need to prove that this map commutes with ∂. So we compute:

$$(\partial\phi(f))_n = (\partial((-1)^k f_k)_{k\in\mathbb{Z}})_n = (-1)^n \partial_B f_n - (-1)^{p-1}(-1)^{n+1} f_{n+1}\partial_A$$

and

$$(\phi\partial(f))_n = (\phi(\partial_B f_k - (-1)^p f_{k+1}\partial_A)_{k\in\mathbb{Z}})_n = (-1)^n(\partial_B f_n - (-1)^p f_{n+1}\partial_A)$$

Thus, ϕ is in fact a cochain map of degree 1, and since it is clearly bijective, it induces an isomorphism on the homology groups. Naturality is clear. □

We will particularly be interested in cohomology with coefficients in a finite chain complex.

Corollary 1.1.9. *There exists an isomorphism:*

$$H^*(A;B_0 \to \ldots \to B_k) \simeq H^{*+1}(A;0 \to B_0 \to \ldots \to B_k)$$

That is natural in both variables.

We next prove a useful lemma, but first we need a definition:

Definition 1.1.10. *Two chain maps $f, f' : A \to B$ are called homotopic if there exists a sequence of maps $S = \{S_n\}$, with $S_n : A_n \to B_{n+1}$ such that $f - f' = S\partial_A + \partial_B S$. Likewise we call two cochain maps $g, g' : C \to D$ homotopic if there exist a sequence of maps $T = \{T_n\}$ with $T_n : C_n \to D_{n-1}$, such that $g - g' = T\partial_C + \partial_D T$. The maps S and T are often called homotopies.*

The situation of the homotopic chain maps is illustrated in this diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial_A"] \arrow[d, "f-g"] & A_n \arrow[r, "\partial_A"] \arrow[d, "f-g"] \arrow[dl, "S"] & A_{n-1} \arrow[r] \arrow[d, "f-g"] \arrow[dl, "S"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial_B"] & B_n \arrow[r, "\partial_B"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

The reason that homotopic chain maps are interesting is that they induce the same map on homology, as is easily seen directly or in [Rotman] (thm. 6.8). Likewise homotopic cochain maps induce the same map on cohomology.

Lemma 1.1.11. *If $f, f' : A \to A'$ are homotopic chain maps, then f^*, f'^* induce the same map in cohomology. If $g, g' : B \to B'$ are homotopic chain maps then $g_*, g'_* :$ $\mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$ are homotopic as cochain maps, and therefore induce the same maps in cohomology.*

Proof. Let S be a homotopy from f to f'. This means that $(f - f')^* = (\partial_{A'} S + S\partial_A)^*$. We want to prove that

$$(f - f')^* = S^*\delta - \delta S^*$$

up to a sign. This will imply that f^* and f'^* induce the same map on cohomology. The rest is just computations: If $h \in \mathrm{Hom}(A', B)_p$ then

$$(S^*\delta - \delta S^*)h = S^*(\partial_B h - (-1)^p h\partial_{A'}) - \partial_B hS + (-1)^{p-1} hS\partial_A =$$

$$(-1)^{p+1}(h\partial_{A'} S + hS\partial_A) = (-1)^{p+1}(f - f')^* h$$

Now, let T be a homotopy from g to g'. We want to show that T_* is a homotopy from g_* to g'_*. Again, if $h \in \mathrm{Hom}(A', B)_p$ we get:

$$(T_*\delta + \delta T_*)h = T\partial_B h - (-1)^p Th\partial_A + \partial'_B Th - (-1)^{p+1} Th\partial_A =$$

$$T\partial_B h + \partial_{B'} Th = (g - g')_* h$$

Which proves the lemma. □

We are now ready to define cohomology of a space, and prove that this does in fact define a cohomology theory.

Definition 1.1.12. *If X is a topological space, let $\Delta_*(X)$ denote the singular chain complex. If B is a chain complex we define (singular) cohomology of X with coefficients in B to be*

$$H^n(X; B) = H^n(\Delta_*(X); B)$$

If (X, A) is a pair of spaces, we define the relative cohomology with coefficients in B to be:

$$H^n(X, A; B) = H^n(\Delta_*(X, A); B)$$

where $\Delta_*(X, A) = \Delta_*(X)/\Delta_*(A)$ is the singular chain complex of the pair.

Definition 1.1.13. *Let $\hat{\Delta}_*(X)$ denote the augmented singular complex of the space X, that is the complex:*

$$\cdots \xrightarrow{\partial} \Delta_1(X) \xrightarrow{\partial} \Delta_0(X) \xrightarrow{\epsilon} \mathbb{Z}$$

Where $\epsilon(\Sigma_i a_i \sigma_i) = \Sigma_i a_i$. We define reduced (singular) cohomology with coefficients in the chain complex B to be:

$$\tilde{H}^n(X; B) = H^n(\hat{\Delta}_*(X); B)$$

Relative reduced singular cohomology is defined as:

$$\tilde{H}^n(X, A; B) = H^n(X/A; B)$$

As usual, we get $H^n(X,\emptyset;B) = H^n(X;B)$, so the absolute definition reduces to the relative. Continuous maps between spaces (respectively pairs of spaces) induce chain maps between the singular complexes. Since $H^*(-,-)$ as defined on chain complexes is a functor of two variables, the topological definition of cohomology with coefficients in chain complexes becomes a functor that is contravariant in the first variable and covariant in the second. The same holds for reduced cohomology.

Definition 1.1.14. *A sequence of contravariant functors $(h^n(-))_{n\in\mathbb{Z}}$ from pairs of spaces in some category of topological spaces $\mathcal{C}$ to abelian groups is called a cohomology theory on $\mathcal{C}$ if it satisfies the following axioms:*

Homotopy invariance. *If $f,g : (X,A) \to (Y,B)$ are homotopic maps, then $f^* = g^* : h^*(Y,B) \to h^*(X,A)$.*

Long exact sequence of a pair. *For every pair (X,A) of spaces there exists natural maps $\delta : h^*(A) \to h^{*+1}(X,A)$, such that:*

$$\cdots \longrightarrow h^*(X,A) \xrightarrow{j^*} h^*(X) \xrightarrow{i^*}$$

$$h^*(A) \xrightarrow{\delta} h^{*+1}(X,A) \longrightarrow \cdots$$

is exact. Here $i : A \to X$ and $j : (X,\emptyset) \to (X,A)$ are the inclusions.

Excision. *Let A,B be subsets of a space X, such that $\mathrm{int}(A)\cup\mathrm{int}(B) = X$. If $i : (B, A\cap B) \to (X,A)$ denotes the inclusion, then $i^* : h^*(X,A) \to h^*(B,A\cap B)$ is an isomorphism.*

Addition. *Let $\coprod_\alpha X_\alpha$ denote the disjoint union of the spaces X_α, and $i_\alpha : X_\alpha \to \coprod_\alpha X_\alpha$ the inclusions. Then*

$$\prod_\alpha (i_\alpha)^* : h^*(\coprod_\alpha X_\alpha) \to \prod_\alpha h^*(X_\alpha)$$

is an isomorphism.

Dimension. *Let $*$ denote the one point space. $h^n(*) = 0$ for all $n \neq 0$.*

$h^n(-)$ is called a generalized cohomology theory, if it satisfies all above axioms, except perhaps the dimension axiom.

Theorem 1.1.15. *Cohomology with coefficients in a chain complex is a generalized cohomology theory on the category of topological spaces.*

One can easily prove the following consequence of the excision axiom:

Lemma 1.1.16. *Let h^n be a generalized cohomology theory, and let (X,A) be a pair of pointed CW-complexes, A a subcomplex of X. Then the map:*

$$h^n(X/A,*) = h^n(X/A,A/A) \to h^n(X,A)$$

induced by the map $(X,A) \to (X/A,A/A)$ that collapses A, is an isomorphism for all n.

This is proved as in ordinary cohomology, see for instance [Hatcher] Prop. 2.22.

Definition 1.1.17. *A sequence of contravariant functors $(\tilde{h}^n(-))_{n\in\mathbb{Z}}$ from pairs of spaces in some category of pointed topological spaces $\mathcal{C}$ to abelian groups is called a reduced cohomology theory on $\mathcal{C}$ if it satisfies the axioms of homotopy invariance, the axiom of the long exact sequence of the pair, and the following axiom:*

Wedge axiom. *Suppose (X_α) is a family of pointed spaces. Define $i_\alpha : X_\alpha \to \bigvee X_\alpha$ to be the inclusions. Then the map:*

$$\prod_\alpha : \tilde{h}^n(\bigvee_\alpha X_\alpha) \to \prod_\alpha \tilde{h}^n(X_\alpha)$$

is an isomorphism for all n.

Theorem 1.1.18. *Reduced cohomology with coefficients in a chain complex is a reduced cohomology theory on the category of pointed CW-complexes.*

In the category of pointed CW-complexes, a pair of spaces is a pair (X, A) where A is a subcomplex of X. All maps are pointed.

Once we have proved that these functors are cohomology theories, a lot of tools become available to us. One of the more important tools is the Mayer-Vietoris sequence, which will be available in both reduced and unreduced cohomology.

Before we prove the theorems, we need to establish a little vocabulary and a few lemmas from homological algebra. The first lemma is wellknown, and much used by algebraic topologists.

Definition 1.1.19. *A sequence of chain maps of chain complexes of abelian groups*

$$A \to B \to C$$

is called exact, if for each n the sequence of groups $A_n \to B_n \to C_n$ is exact.

Lemma 1.1.20 (Short exact sequence to long exact sequence). *If*

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \longrightarrow 0$$

is a short exact sequence of chain complexes of abelian groups then there exist a sequence of natural maps $\partial_n : H_n(C) \to H_{n-1}(A)$ such that the sequence

$$\cdots \longrightarrow H_{n+1}(C) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C) \longrightarrow \cdots$$

is exact.

Proof. This is Theorems 6.3 and 6.4 in [Rotman]. □

Remark 1.1.21. The fact that the connecting homomorphism is natural implies that the operation, that creates the long exact sequence from the short exact should be natural. This should be understood the following way: If

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "i"] \arrow[d, "f"] & A \arrow[r, "p"] \arrow[d, "g"] & A'' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & B' \arrow[r, "j"] & B \arrow[r, "q"] & B'' \arrow[r] & 0
\end{tikzcd}$$

is a commuting diagram of chain complexes of groups with exact rows, then the lemma produces long exact sequences, and assures that the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(A'') \arrow[r, "\delta"] \arrow[d, "f_*"] & H_n(A)' \arrow[r, "i_*"] \arrow[d, "f_*"] & H_n(A) \arrow[r, "p_*"] \arrow[d, "g_*"] & H_n(A'') \arrow[r] \arrow[d, "h_*"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(B'') \arrow[r, "\delta"] & H_n(B') \arrow[r, "j_*"] & H_n(B) \arrow[r, "q_*"] & H_n(B'') \arrow[r] & \cdots
\end{tikzcd}$$

By the duality of chain complexes and cochain complexes the same result holds in cohomology. In this case, if

$$0 \longrightarrow A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \longrightarrow 0$$

is a short exact sequence of cochain complexes of modules, then the connecting homomorphism δ_n raises dimension, that is $\delta_n : H^n(A) \to H^{n+1}(C)$, and the sequence:

$$\cdots \longrightarrow H^{n-1}(A) \xrightarrow{\delta} H^n(C) \xrightarrow{p^*} H^n(B) \xrightarrow{i^*} H^n(A) \longrightarrow \cdots$$

is exact. Of course, the same results on naturality holds in cohomology.

The next issue we need to discuss is exactness of functors.

Definition 1.1.22. *Let F be a functor between categories, in which we have defined exactness (that is the categories of abelian groups and chain complexes of abelian groups). We say that F is an exact covariant functor, if exactness of*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

implies exactness of

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$$

Exactness of contravariant functors is defined the same way. A covariant functor F is called left exact, if exactness of

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C$$

implies exactness of

$$0 \longrightarrow FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$$

A contravariant functor F is called left exact, if exactness of

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

implies exactness of

$$0 \longrightarrow FC \xrightarrow{Fg} FB \xrightarrow{Ff} FA$$

Definition 1.1.23. *Suppose A is a subcomplex of (B, ∂). That is $\partial(A) \subset A$. We can define the quotient*

$$(A/B)_n = A_n/B_n$$

with differential induced by ∂.

Lemma 1.1.24. *Let B be a chain complex of abelian groups. The contravariant functor* $\operatorname{Hom}(-, B)$ *is left exact. If*

$$0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$$

is an exact sequence of chain complexes of abelian groups, such that

$$0 \longrightarrow C_n \xrightarrow{i} D_n \xrightarrow{p} E_n \longrightarrow 0$$

is split exact for every n, then

$$0 \longrightarrow \operatorname{Hom}(E, B) \xrightarrow{p^*} \operatorname{Hom}(D, B) \xrightarrow{i^*} \operatorname{Hom}(C, B) \longrightarrow 0$$

is exact.

Proof. For good reasons, the proof of the first part resembles the proof of the fact that $\operatorname{Hom}(-, M)$ is a left exact functor, when M is a module. It is in fact the same proof (see [Rotman] thm. 2.9). We include this fairly easy argument for completeness.

Suppose

$$C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$$

is an exact sequence of chain complexes of abelian groups. We need to prove that

$$0 \longrightarrow \operatorname{Hom}(E, B) \xrightarrow{p^*} \operatorname{Hom}(D, B) \xrightarrow{i^*} \operatorname{Hom}(C, B)$$

is exact. Assume $f, g \in \operatorname{Hom}(E, B)$ are such that $fp = gp$. The fact that p is assumed to be surjective implies that $f = g$. Because $pi = 0$ we have that $i^*p^* = (pi)^* = 0$.

We now only need to prove that $\ker i^* \subseteq \operatorname{im} p^*$. Suppose $f \in \ker i^*$. This means exactly that f vanishes on $\operatorname{im} i$. Since i is a chain map $\operatorname{im} i$ is a subcomplex of D, and since for each n we have $E_n \simeq (D/\operatorname{im} i)_n$, we have $E \simeq D/\operatorname{im} i$. So we can define a map $g : E \to B$ such that the composition:

$$D \longrightarrow D/\operatorname{im} i \simeq E \xrightarrow{g} B$$

is f. This implies that $f \in \operatorname{im} p^*$.

To prove the last part of the lemma, we only need to verify that i^* is surjective. Suppose

$$f = (f_n)_{n \in \mathbb{Z}} \in \operatorname{Hom}(C, B)_p = \prod_n \operatorname{Hom}(C_n, B_{n-p})$$

For every n we have that D_n is on the isomorphic to $C_n \oplus D'_n$ for some D'_n. This means that each f_n can be extended to D_n by letting it be zero on D'_n. This way we extend f to an element $f' \in \operatorname{Hom}(D, B)_p$, and $i^*(f') = f$ as desired. $\square$

Proof of theorem 1.1.15. **Homotopy invariance.** In the proof of Theorem 2.10 in [Hatcher], it is proved that the prism operator P is a homotopy between the maps f_* and g_* considered as chain maps from $\Delta_*(X, A)$ and $\Delta_*(Y, B)$. Lemma 1.1.11 then tells us that $f^* = g^* : H^*(Y, B; C) \to H^*(X, A; C)$.

Long exact sequence of a pair. For each n the sequence:

$$0 \longrightarrow \Delta_n(A) \xrightarrow{i_*} \Delta_n(X) \xrightarrow{j_*} \Delta_n(X, A) \longrightarrow 0$$

is split exact, since if we let $D_n(X,A)$ be the free abelian group generated by all simplices whose image is not contained in A, then:

$$\Delta_n(X) \simeq \Delta_n(A) \oplus D_n(X,A)$$

By lemma 1.1.24 this implies that the sequence:

$$0 \longrightarrow \mathrm{Hom}(\Delta_*(X,A),B) \xrightarrow{j^*} \mathrm{Hom}(\Delta_*(X),B) \xrightarrow{i^*} \mathrm{Hom}(\Delta_*(A),B) \longrightarrow 0$$

is exact. Now lemma 1.1.20 gives the long exact sequence of the pair. Notice that the naturality of 1.1.20 implies naturality of the long exact sequence in (X,A) as well as naturality in the coefficient variable B.

Excision. In the proof of the excision property in ordinary homology as it is done in [Hatcher], it is proved that the map $i_* : \Delta_*(B, B\cap A) \to \Delta_*(X,A)$ induced by inclusion is an homotopy equivalence of chain complexes. That is, there is a map $j : \Delta_*(X,A) \to \Delta_*(B, B\cap A)$ such that the maps i_*j_* and j_*i_* are chain homotopic to the identity. Lemma 1.1.11 then implies that i^* on the level of cohomology is an isomorphism.

Additivity. Since the image of a simplex is connected, we know that the map:

$$\oplus_\alpha (i_\alpha)_* : \oplus_\alpha \Delta_*(X_\alpha) \to \Delta_*(\coprod_\alpha X_\alpha)$$

is a bijective chain map. Hence it induces isomorphism on cohomology. The cohomology with coefficients in B of the right side of this equation is $\mathrm{Hom}(\coprod_\alpha X_\alpha; B)$, so we only need to prove that the cohomology of the left side is $\prod_\alpha \mathrm{Hom}(X_\alpha, B)$.

By the same identity on modules ([Rotman] thm 2.4) we know that:

$$\mathrm{Hom}(\oplus_\alpha \Delta_*(X_\alpha); B) \simeq \prod_\alpha \mathrm{Hom}(\Delta_*(X_\alpha); B)$$

One might express this identity as 'a map from a direct sum to B is the same as a family of maps to B'. Since the cohomology of a product of chain complexes is the product of the cohomology of the chains, we have the identity:

$$H^*(\oplus_\alpha \Delta_*(X_\alpha); B) \simeq \prod_\alpha H^*(\Delta_*(X_\alpha); B) = \prod_\alpha H^*(X_\alpha; B)$$

□

Remark 1.1.25. Notice that the long exact sequence of the pair is natural in the coefficient variable. This is the reason we chose this construction. The fact that this sequence is natural in the coefficient variable implies that the Mayer-Vietoris sequence is natural in the coefficient variable.

Before we prove Theorem 1.1.18, we need to establish the relationship between reduced and unreduced cohomology with coefficients in a chain complex. There are no surprises here. It is exactly the same as in ordinary cohomology.

Proposition 1.1.26. *For $X \neq \emptyset$ there exist isomorphisms:*

$$\tilde{H}^n(X;B) \simeq H^n(X,*;B)$$

and

$$H^n(X;B) \simeq H^n(X,*;B) \oplus H^n(*;B)$$

which are natural in pointed maps in X and the coefficient complex B. There also exists a natural (in both variables) isomorphism:

$$\tilde{H}^n(X,A;B) \simeq H^n(X,A;B)$$

This isomorphism is only defined when (X,A) is a pair of CW-complexes and A is not empty.

To prove this proposition, we need another proposition:

Proposition 1.1.27. *Suppose $\phi : A \to A'$ induces an isomorphism $H(\phi) : H(A) \to H(A')$. Then ϕ also induces an isomorphism on cohomology with coefficients in a chain complex.*

We will prove this in the next section.

Proof of Proposition 1.1.26. Consider the chain map:

$$\phi : \hat{\Delta}_*(X) \to \Delta_*(X,*)$$

induced by the inclusion of X into $(X,*)$. Since this map clearly is natural (in pointed maps) in X and commutes with the boundary, it induces a natural map:

$$\phi_* : \tilde{H}_n(X) \to H_n(X,*)$$

It factorizes as:

$$\begin{tikzcd}
\hat{\Delta}_*(X) \arrow[d] \arrow[dr, "\phi"] & \\
\Delta_*(X) \arrow[r] & \Delta_*(X,*)
\end{tikzcd}$$

Since the two other maps in the diagram induce isomorphisms in ordinary homology groups in dimensions $n \geq 1$, so does ϕ. Using the fact, that $\tilde{H}^0(X)$ consists of a copy of $\mathbb{Z}$ for each component of X that does not contain the basepoint, and $H^0(X,*)$ consists of the same, it is easy clear that ϕ also induces an isomorphism on H_0.

Since ϕ induces an isomorphism on homology, it also induces an isomorphism

$$H^n(X,*;B) \to \tilde{H}^n(X;B)$$

This isomorphism is natural in B and in pointed maps in X.

Consider the long exact sequence of the pair $(X,*)$:

$$H^{n-1}(*;B) \longrightarrow H^n(X,*;B) \longrightarrow H^n(X;B) \xrightarrow{i^*} H^n(*;B) \longrightarrow H^{n+1}(X,*)$$

Let $c : X \to *$ denote the constant map. Then $i^*c^* = (ci)^* = id^* = id$ which implies that the sequence above splits, so that:

$$H^n(X;B) \simeq H^n(X,*;B) \oplus H^n(*;B)$$

Since this isomorphism is natural, this implies the first part of the proposition.

The last isomorphism is:

$$H^n(X,A;B) \simeq H^n(X/A,*;B) \simeq \tilde{H}^n(X/A;B) \simeq \tilde{H}^n(X,A;B)$$

□

Proof of Theorem 1.1.18. **Homotopy Invariance.** This follows from the naturality of the second isomorphism in Proposition 1.1.26 and homotopy invariance in the unreduced case.

Long exact sequence of a pair. Again we use the isomorphisms of Proposition 1.1.26. Replacing as we can in the long exact sequence of the pair from unreduced cohomology, we gain a long exact sequence:

$$\cdots \longrightarrow \tilde{H}^n(X,A;B) \xrightarrow{j^*} H^n(X,*;B) \oplus H^n(*;B) \xrightarrow{i^*\oplus(id)^*}$$

$$H^n(A,*;B) \oplus H^n(*;B) \xrightarrow{\delta} \tilde{H}^{n+1}(X,A;B) \longrightarrow \cdots$$

Notice that the map:

$$i^* \oplus (id)^* : H^n(X,*;B) \oplus H^n(*,B) \to H^n(A,*;B) \oplus H^n(*,B)$$

has the same kernel and image as i^*. We may thus remove $H^n(*;B)$ from the sequence and use the identity $\tilde{H}^n(X;B) \simeq H^n(X,*;B)$ to obtain the desired sequence.

Wedge axiom. Consider the commutative diagram:

$$\begin{tikzcd}
\tilde{H}^n(\bigvee_\alpha X_\alpha;B) \arrow[rr, "\prod_\alpha (i_\alpha)^*"] \arrow[d, "\simeq"] & & \prod_\alpha \tilde{H}^n(X_\alpha;B) \arrow[d, "\simeq"] \\
H^n(\bigvee_\alpha X_\alpha,*;B) \arrow[r] & H^n(\coprod_\alpha X_\alpha, \coprod_\alpha *_\alpha;B) \arrow[r, "\simeq"] & \prod_\alpha H^n(X_\alpha,*_\alpha;B)
\end{tikzcd}$$

The first map in the bottom line is induced by the map that collapses $\coprod *_\alpha$ to a point. This map induces an isomorphism by Lemma 1.1.16, since we are working in the category of pointed CW-complexes.

The fact that the map

$$H^n(\coprod_\alpha X_\alpha, \coprod_\alpha *_\alpha;B) \to \prod_\alpha H^n(X_\alpha,*_\alpha)$$

is an isomorphism follows from using the five lemma on the long exact sequence of the pair and using the the fact that unreduced cohomology satisfies the additivity axiom. $\square$

The next proposition is an easy corollary to a theorem of the next section. It could also be proved by direct computations.

Proposition 1.1.28. *Let $*$ denote the one point space. For each chain complex we have:*

$$H^n(*;B) \simeq H_{-n}(B)$$

and

$$\tilde{H}^n(*;B) \simeq 0$$

Thus cohomology with coefficients in a chain complex does not satisfy the dimension axiom.

1.2 Connection to ordinary cohomology

The aim of this section is first to prove that $H^*(X;B) \simeq \prod_n H^n(X, H_{n-*}(B))$ as groups, and second to discuss the limitations of this isomorphism. At first sight this isomorphism suggests that cohomology with coefficients in a chain complex is completely expressible in terms of cohomology with coefficients in a group. But we will show that this isomorphism loses extra information, that is available in cohomology with coefficients in a chain complex.

First we will quote two lemmas from [Brown64], but not prove them. The lemmas are lemma 1.7 and proposition 2.2 in the article. Notice that a graded group can be thought of as a chain complex with differential maps zero.

Lemma 1.2.1. *Let G be any graded group. There is a free chain complex F and an isomorphism $H(F) \to G$. For any chain complex A, free chain complex F, and map $\phi : H(F) \to H(A)$ there is a chain map $f : F \to A$, such that $H(f) = \phi$.*

Lemma 1.2.2. *Let K be a free chain complex and $f : A \to A'$ a chain map that induces an isomorphism on homology. Then $f_* : H^*(K, A) \to H^*(K, A')$ is an isomorphism.*

If B is a chain complex, let $H(B)$ denote the chain complex of homology groups of B with zero differential. From the definitions we see that $\mathrm{Hom}(A; H(B))_p = \prod_n \mathrm{Hom}(A_n, H_{n-p}(B))$, and the differential is just $\delta f = (-1)^{p+1} f \circ \partial_A$, so this proves that:

$$H^*(A; H(B)) \simeq \prod_n H^n(A, H_{n-*}(B))$$

Lemma 1.2.1 provides a free chain complex C and maps:

$$\begin{tikzcd} C \arrow[r, "\psi"] \arrow[d, "\phi"] & H(B) \\ B & \end{tikzcd}$$

with $H(\phi)$ and $H(\psi)$ isomorphisms. Lemma 1.2.2 now tells us that:

$$\psi_* \phi_*^{-1} : H^*(A; B) \to H^*(A; H(B))$$

is an isomorphism. We have just proved:

Theorem 1.2.3. *For all chain complexes A, B the groups $H^*(A;B)$ and $\prod_n H^n(A; H_{n-*}(B))$ are isomorphic. The isomorphism is natural in A.*

This proves Proposition 1.1.28, and Proposition 1.1.27 follows from the naturality of the isomorphism in Theorem 1.2.3 and the Universal Coefficient Theorem.

Remember that a chain map $f : B \to B'$ induces a map $f_* : H^*(A, B) \to H^*(A, B')$. It also induces a map $H(f) : H(B) \to H(B')$, which in turn induces a map $f_* : H^*(A, H(B)) \to H^*(A, H(B'))$. So it would be natural to ask whether the isomorphism of Theorem 1.2.3 is natural in B. It is a key observation that this is not the case, and this fact allows us to conclude that cohomology with coefficients in a chain complex contains information, that is lost when reducing it to terms of ordinary cohomology. The next example is taken from [Brown64] and shows that the isomorphism cannot be natural.

Example 1.2.4. Let A be a free chain complex with two generators a and b in dimensions 0 and 1 respectively, with $\partial b = 2a$. Let B be the free chain complex with generators c, d, d' in dimensions $0, 1, 1$ respectively and differential given by $\partial d = \partial d' = 2c$. Define $\tau : B \to B$ to be the chain map that exchanges d and d'. We aim to show that $\tau_* : H^*(A, B) \to H^*(A, B)$ is an isomorphism, that is not the identity, whereas $\tau_* : H^*(A, H(B)) \to H^*(A, H(B))$ is the identity. Thus, there can be no isomorphism $H^*(A; B) \to H^*(A; H(B))$ that commutes with τ.

Let us first compute $H(B)$. $H_1(B)$ is $\mathbb{Z}$ generated by $d - d'$, and $H_0(B)$ is $\mathbb{Z}/2\mathbb{Z}$. All other homologygroups are zero. The action on $H(B)$ induced by τ changes sign in $H_1(B)$ and is the identity on $H_0(B)$.

We have the following isomorphisms:

$$\begin{aligned} H^n(A, H(B)) &\simeq H^n(A, H_0(B)) \oplus H^{n+1}(A, H_1(B)) \\ &\simeq H^n(A, \mathbb{Z}/2\mathbb{Z}) \oplus H^{n+1}(A, \mathbb{Z}) \end{aligned}$$

Since τ_* is the identity on $H_0(B)$ we only need consider, what it does to the second summand.

First let us compute $H^n(A, \mathbb{Z})$ using the universal coefficient theorem ([Hatcher] thm 3.2). We easily get that $H_0(A) \simeq \mathbb{Z}/2\mathbb{Z}$ and $H_*(A) \simeq 0$ for $\star \neq 0$. The universal coefficient theorem then implies that $H^1(A, \mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $H^*(A, \mathbb{Z}) \simeq 0$ for $* \neq 1$. Thus, the isomorphism that τ induces on $H^*(A, \mathbb{Z})$ can only be the identity. We conclude that $\tau_* : H^*(A, H(B)) \to H^*(A, H(B))$ is the identity.

To prove that $\tau_* : H^*(A; B) \to H^*(A; B)$ is not the identity, we must first compute $H^*(A; B)$.

We have the following identities:

$$\begin{aligned} \mathrm{Hom}(A, B)_1 &\simeq \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z} \\ \mathrm{Hom}(A, B)_0 &\simeq \mathrm{Hom}(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \simeq (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z} \\ \mathrm{Hom}(A, B)_{-1} &\simeq \mathrm{Hom}(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

Here the identifications $\mathrm{Hom}(\mathbb{Z}, G) \simeq G$ are by the map $f \mapsto f(1)$. τ_* on $\mathrm{Hom}(A, B)_1$ is the identity. On $\mathrm{Hom}(A, B)_0$, we have $\tau_*((x, y), z) = ((y, x), z)$. In dimension -1 we get the map $\tau_*(x, y) = (y, x)$. To compute the cohomology of A with coefficients in B let us write out the chain complex $\mathrm{Hom}(A, B)$ with all differential maps:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(A,B)_{-1} \arrow[r, "\delta_{-1}"] \arrow[d, "\simeq"] & \mathrm{Hom}(A,B)_0 \arrow[r, "\delta_0"] \arrow[d, "\simeq"] & \mathrm{Hom}(A,B)_1 \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

Consider the diagram below, which illustrates an element of $\mathrm{Hom}(A, B)_{-1}$:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\cdot 2 \oplus \cdot 2"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

f is the only nonzero map, and the element in $\mathrm{Hom}(A, B)_{-1}$ is identified with the element $(x, y) = f(1) \in \mathbb{Z} \oplus \mathbb{Z}$. $\delta_{-1}(x, y)$ is a pair of maps (f', g') with $f' : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ and

$g' : \mathbb{Z} \to \mathbb{Z}$. This corresponds to an element $(g'(1), f'(1)) \in (\mathbb{Z}\oplus\mathbb{Z})\oplus\mathbb{Z}$, and we get $g'(1) = \cdot 2 \oplus \cdot 2(x, y) = 2x + 2y$ and $f'(1) = f(2) = (2x, 2y)$. Thus $\delta_{-1}(x, y) = ((2x, 2y), 2x + 2y)$.

Likewise, by considering the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\cdot 2 \oplus \cdot 2"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We see that the map δ_0 is given by $\delta_0((x, y), z) = 2(x + y - z)$.

Thus, we get that $H^{-1}(A, B) \simeq \mathbb{Z}$ generated by $(1, -1)$, and τ_* is the sign change map on $H^{-1}(A, B)$. We conclude that τ_* is not the identity on $H^*(A, B)$. For completeness, we get $H^0(A; B) \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $\xi = ((1, 0), -1)$ and $\eta = ((0, 1), -1)$. Here τ_* is the map that exchanges ξ and η. $H^1(A, B) \simeq \mathbb{Z}/2\mathbb{Z}$, and τ_* is the identity here.

Remark 1.2.5. The reader might ask why we didn't simply use Theorem 1.2.3 to prove that $H^n(-, B)$ is a cohomology theory. The reason is, that we are interested in the naturality in the coefficient variable. The proof we used includes a natural construction of the long exact sequence of the pair. This construction shows that the connecting homomorphism $\delta : H^n(A; B) \to H^{n+1}(X, A; B)$ is natural in B. Naturality of this map implies naturality of the Mayer-Vietoris sequence.

1.3 Classifying spaces

In this section we will define the main problem of this thesis. $[X, Y]$ denotes the homotopy classes of basepoint-preserving maps $X \to Y$. Two pointed maps $f, g : X \to Y$ represent the same element in $[X, Y]$ if there is a *basepoint-preserving* homotopy from f to g.

Definition 1.3.1. *Suppose F and G are functors from the category $\mathfrak{A}$ to the category $\mathfrak{B}$. A natural transformation T from F to G is a family of morphisms $T_X : FX \to GX$ such that the following diagram commutes for every morphism f:*

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "T_X"] & FY \arrow[d, "T_Y"] \\
GX \arrow[r, "Gf"] & GY
\end{tikzcd}$$

The natural transformation T is called an equivalence of functors if for every object X in $\mathfrak{A}$ T_X is an isomorphisms.

Definition 1.3.2. *Suppose $h^n(-)$ is a cohomology theory. The space K_n is called a classifying space for h^n if $[X, K_n]$ has a group structure for each space X and $h^n(-)$ and $[-, K_n]$ are equivalent as functors.*

Example 1.3.3. The Eilenberg-MacLane spaces $K(B, n)$ are classifying spaces for ordinary reduced cohomology with coefficients in the group B. That is, the functors $[-, K(B, n)]$ and $\tilde{H}^n(-, B)$ are naturally equivalent for every group B and every $n \geq 0$.

Let us determine the classifying space for reduced cohomology with coefficients in the chain complex B. We know from Theorem 1.2.3 that

$$\begin{aligned}\tilde{H}^*(X;B) &\simeq \prod_n \tilde{H}^n(X;H_{n-*}(B)) \\ &\simeq \prod_n [X, K(H_{n-*}(B),n)] \\ &\simeq [X, \prod_n K(H_{n-*}(B),n)]\end{aligned}$$

Note that all these isomorphisms are natural in X, so this proves that $\prod_n K(H_{n-*}(B),n)$ is a classifying space for cohomology with coefficients in B. The case we are particularly interested in is cohomology with coefficients in a homomorphism. In this case we get that $K(\ker h, *) \times K(\operatorname{coker} h, * - 1)$ is a classifying space for $H^*(-, A \xrightarrow{h} B)$.

Unfortunately this is only half the story. Before we discuss what is missing, lets review the construction of Eilenberg-MacLane spaces.

Suppose A is an abelian group and $n \geq 0$ is an integer. We aim to construct a space $K(A,n)$, such that $\pi_n(K(A,n)) \simeq A$ and $\pi_*(K(A,n)) \simeq 0$ for $* \neq n$. We would also like a canonical identification of $\pi_n(K(A,n))$ with A.

In the case $n = 0$ we set $K(A,0) = A$ with the discrete topology and the neutral element as base point. Then $K(A,0)$ becomes a topological group, and therefore a group-like H-space, such that $\pi_0(K(A,n))$ has a group structure. Clearly this construction satisfies the requirements above.

Now suppose $n \geq 1$. Choose an exact sequence

$$R \xrightarrow{r} F \xrightarrow{p} A \longrightarrow 0$$

where F and R are free abelian groups. This gives $A \simeq F/\operatorname{im} r$, so the generators of F are called the generators of A, and the elements in $\operatorname{im} r$ are called the relations.

We build $K(A,n)$ by setting $K(A,n)^{(n)} \simeq \bigvee_\alpha S^n_\alpha$, where α runs over the generators of F. We now have $\pi_n(K(A,n)^{(n)}) \simeq F$, with a natural identification. Create $K(A,n)^{(n+1)}$ by for each generator β of R attaching an $(n+1)$-cell along a map representing the element $r(\beta) \in K(A,n)^{(n)}$. This space has $\pi_n(K(A,n)^{(n+1)}) \simeq A$. The only thing left to do is to kill the higher homotopy groups. This is done by attaching $(n+2)$-cells along each nontrivial element of $\pi_{n+1}(K(A,n)^{(n+1)})$, thereby getting a space with π_{n+1} zero. Repeating this procedure, we attach $(n+3)$-cells to kill π_{n+2} of the resulting space, and continue to attach cells, thereby killing all higher homotopy groups. The details left out here can be found in section VII.11 of [Bredon].

Since Eilenberg-Maclane spaces are unique up to homotopy type we know that they are group-like H-spaces, since $K(A,n) \simeq \Omega K(A,n+1)$, thus $[X, K(A,n)]$ is a group for all spaces X.

Proposition 1.3.4. *Suppose A and B are abelian groups and $n \geq 1$ is a natural number. The map*

$$\phi : [K(A,n), K(B,n)] \to \operatorname{Hom}(\pi_n(K(A,n)), \pi_n(K(B,n))) = \operatorname{Hom}(A,B)$$

given by $\phi(f) = f_*$ is a group isomorphism.

Proof. We use the identifications $\pi_n(K(A,n)) \simeq A$ and $\pi_n(K(A,n)) \simeq A$ from the construction above.

We first show that ϕ is a homomorphism. Let $f, g \in [K(A,n), K(B,n)]$ and let $\sigma : S^n \to K(A,n)$ represent an element in A. We need to show that $(f+g)_*(\sigma) = f_*(\sigma) + g_*(\sigma)$, thus proving that $f_* + g_* = (f+g)_*$. $f_*(\sigma) + g_*(\sigma) = f \circ \sigma + g \circ \sigma$ where the $+$ is given by the H-cogroup structure on S^n. On the other hand $(f+g)_*\sigma$ is the sum of $f \circ \sigma$ and $g \circ \sigma$, where sum is defined by the H-space structure on $K(B,n)$. Theorem VII.3.3 of [Bredon] tells us that these sums are the same.

To see that ϕ is surjective, let $h : A \to B$ be a homomorphism. Suppose the spaces $K(A,n)$ and $K(B,n)$ have been constructed using the sequences:

$$R \xrightarrow{r} F \xrightarrow{p} A \longrightarrow 0$$

$$R' \longrightarrow F' \longrightarrow B \longrightarrow 0$$

and that the identifications $\pi_n(K(A,n)) \simeq A$ and $\pi_n(K(B,n)) \simeq B$ are made the standard way.

For each generator $\alpha \in F$, let $f_\alpha : S^n \to K(B,n)$ be a map representing the element $h(p(\alpha))$. Putting these maps together defines a map $f : K(A,n)^{(n)} = \bigvee_\alpha S^n_\alpha \to K(B,n)$. This map can be extended over the $(n+1)$-skeleton, since if e^{n+1} is an $(n+1)$-cell, then its attaching map is a map $\sigma : S^n \to \bigvee_\alpha S^n_\alpha$ corresponding to an element in F in the kernel of p. Thus $f \circ \sigma$ is nullhomotopic, which means that we can extend f over e^{n+1}. We can extend f over the rest of $K(A,n)$ since all the higher homotopy groups of $K(B,n)$ are zero. Clearly $f_* = h$, since it maps each generator in $\pi_n(K(A,n))$ to a map representing its image by h.

To prove that ϕ is injective, assume $f_* = g_*$. This means that for each generator $\alpha \in F$ if $\sigma : S^n \to \bigvee_\alpha S^n_\alpha$ denotes the identity onto the α factor, the compositions:

$$S^n \xrightarrow{\sigma} \bigvee_\alpha S^n_\alpha \xrightarrow{f} K(B,n)$$

and

$$S^n \xrightarrow{\sigma} \bigvee_\alpha S^n_\alpha \xrightarrow{g} K(B,n)$$

are homotopic. This implies that there exists a homotopy F from $f|_{K(A,n)^{(n)}}$ to $g|_{K(A,n)^{(n)}}$. We next prove that we can extend F over the $(n+1)$-skeleton of $K(A,n)$. Suppose e^{n+1} is an $(n+1)$-cell of $K(A,n)$. Since we already have defined F on the n-skeleton of $K(A,n)$ we are looking at a map defined on $e^{n+1} \times \partial I \cup \partial e^{n+1} \times I$ that we want to extend to $e^{n+1} \times I$. The pair $(D^{n+1} \times I, D^{n+1} \times \partial I \cup S^n \times I)$ is homeomorphic to the pair (D^{n+2}, S^{n+1}), so this extension problem is equivalent to the problem of extending a map $S^{n+1} \to K(B,n)$ to D^{n+2} which can be done since $\pi_{n+1}(K(B,n)) \simeq 0$. The same argument shows that F can be extended over all the higher skeletons of $K(A,n)$. We conclude that f is homotopic to g, and therefore ϕ is injective. $\square$

Unfortunately we need more than this construction for this thesis. For each $n \geq 0$, we need a functor $K(-,n)$ from the category of abelian groups to the category of pointed CW-complexes. A functor like this is described in [May65]. For any group A, $K(A,n)$ is a CW-complex with $(n-1)$-skeleton consisting only of the basepoint $*$. In the case $A = 0$ we have $K(0,n) \simeq *$. In the case $n = 0$ we take $K(-,0)$ to be the functor, that to A associates A as a discrete group.

$K(-,n)$ is also assumed to be a right inverse of the functor π_n. This should be understood the following way: $\pi_n K(-,n)$ is a functor from the category of abelian groups to the category of abelian groups. This functor is equivalent to the identity functor. This means that we are given identifications $\pi_n(K(A,n)) \simeq A$ for all groups A such that the following diagram commutes for all homomorphisms h:

$$\begin{tikzcd} \pi_n(K(A,n)) \arrow[r, "\pi_n(K(h,n))"] \arrow[d, "\simeq"] & \pi_n(K(B,n)) \arrow[d, "\simeq"] \\ A \arrow[r, "h"] & B \end{tikzcd}$$

In the following we will identify A with $\pi_n(K(A,n))$ using this natural transformation.

We also have another natural equivalence:

$$S : [\Sigma-,-] \to [-,\Omega-]$$

where Σ denotes the reduced suspension, given the following way. Suppose

$$f : \Sigma X = X \times I/(X \times \partial I \cup \{*\} \times I \to Y)$$

is a map, then

$$S(f) : X \to \Omega Y$$

is the map defined by $S(f)(x)(t) = f(x,t)$. Notice that the set $[\Sigma X, Y]$ has a group structure induced by the H-cospace structure on ΣX and $[X, \Omega Y]$ has a group structure induced by the H-space structure on ΩY. It is clear, that S induces a group homomorphism.

The natural equivalence S induces a natural equivalence:

$$S^l \pi_n \to \pi_{n-l}\Omega^l$$

for all $l \geq 0$. Using this equivalence, we get an identification:

$$\pi_{n-l}\Omega^l K(A,n) \simeq A$$

Lemma 1.3.5. *Suppose $n \geq l$. We can always choose a homotopy equivalence:*

$$f : \Omega^l K(A,n) \to K(A,n-l)$$

such that the following diagram commutes:

$$\begin{tikzcd} \pi_{n-l}(\Omega^l K(A,n)) \arrow[rr, "f_*"] \arrow[dr, "\simeq"] & & \pi_{n-l}(K(A,n-l)) \arrow[dl, "\simeq"] \\ & A & \end{tikzcd}$$

Proof. Suppose first that $n - l \geq 1$. The map f_* is an isomorphism, so we can always choose an isomorphism

$$\xi : \pi_{n-l}(K(A,n-l)) \to \pi_{n-l}(K(A,n-l))$$

such that ξf_* makes the diagram commute. Now, Proposition 1.3.4 tells us that ξ is realizable as a homotopy equivalence from $K(A,n-l)$ to itself. If we compose f with this map, we get the desired result.

In the case $n = l$ we can construct a homotopy equivalence:

$$f : \Omega^n K(A, n) \to K(A, 0)$$

by mapping each loop to the element in A represented by this loop. This map is continuous since it is constant on each component of $\Omega^n K(A, n)$, and it clearly becomes a weak homotopy equivalence. Since $\Omega^n K(A, n)$ is homotopic to $K(A, 0)$ which is a CW-complex, the weak homotopy equivalence is in fact a homotopy equivalence by Whiteheads Theorem. $\square$

In the rest of this thesis, whenever a homotopy equivalence $K(A, n - l)$ to $\Omega^l K(A, n)$ is mentioned, we mean a homotopy equivalence satisfying the requirements of Lemma 1.3.5

Let us briefly review how the natural transformation $T : [-, K(B, n)] \to \tilde{H}^n(-, B)$ is defined. If f represents an element of $[X, K(B, n)]$, then $T(f) = f^*(\iota)$, where $\iota \in \tilde{H}^n(K(B, n), B)$ is the image of the identity. We shall describe ι using cellular cohomology, since this is the simplest.

Let $(C_*(X), \partial)$ denote the cellular chain complex. ι is represented by the map

$$\iota : C_n(K(B, n)) \to B$$

defined as follows. If e^n is an n-cell of $K(B, n)$, let $\sigma : D^n \to K(B, n)$ denote its characteristic map. Since $K(B, n)^{(n-1)} = *$, σ maps the boundary of D^n to a point, and therefore represents an element $\iota(e^n) \in \pi_n(K(B, n)) \simeq B$. In books on algebraic topology ι is described in the text and almost never in theorems, but the details are written out in [Bredon] p. 491 in the text leading up to thm VII.12.1, and also in [Hatcher] in the text immediately following the proof of thm 4.57.

The fact that T is given by a map of the type $f \mapsto f^*(\iota)$ should come as no surprise, as this next lemma tells us:

Lemma 1.3.6. *Let F be a contravariant functor from the category of topological spaces to abelian groups, and suppose for some space Y, $R : [-, Y] \to F(-)$ is a natural transformation. Then R is on the form $R(f) = f^*(\alpha)$ for some element $\alpha \in F(Y)$.*

Proof. Let $\alpha = R([id_Y])$. Then $R([f]) = R(f^*([id])) = f^* R([id]) = f^*(\alpha)$. $\square$

Lemma 1.3.6 is stronger than it seems, as the proof of the next theorem suggests.

Theorem 1.3.7. *Let A and B be abelian groups, and let $f : A \to B$ be a homomorphism. The diagram:*

$$\begin{tikzcd}
{[X, K(A,n)]} \arrow[r, "K(f,n)_*"] \arrow[d, "\simeq"', "T_A"] & {[X, K(B,n)]} \arrow[d, "\simeq"', "T_B"] \\
\tilde{H}^n(X, A) \arrow[r, "f_*"] & \tilde{H}^n(X, B)
\end{tikzcd}$$

commutes for every space X.

Proof. Since the compositions $f_* \circ T_A$ and $T_B \circ K(f, n)_*$ are natural transformations, lemma 1.3.6 tells us that it suffices to prove that $f_* \circ T_A([id]) = T_B \circ K(f, n)_*([id])$. Both of these are elements in $H^n(K(A, n), B)$.

$T_A([id])$ is represented by the map $C_n(K(A,n)) \to A$ that maps an n-cell e^n in $K(A,n)$ to $\sigma \in \pi_n(K(A,n)) \simeq A$, where σ is the characteristic map of e^n. This means that $f_* \circ S_A([id])$ is the map $a : C_n(K(A,n)) \to B$ that to the n-cell e^n relates $K(f,n) \circ \sigma \in \pi_n(K(B,n)) \simeq B$.

$T_B \circ K(f,n)_*([id]) = T_B \circ K(f,n)^*([id]) = K(f,n)^* T_B([id])$. Now, $K(f,n)^* \circ T_B([id])$ is represented by the map $b : C_n(K(A,n)) \to B$ that maps e^n to the element $K(f,n) \circ \sigma \in \pi_n(K(B,n)) \simeq B$. This proves the theorem. □

Corollary 1.3.8. *The functors $[-, K(-,n)]$ and $\tilde{H}^n(-,-)$ taking one contravariant variable in the category of topological spaces and one covariant in the category of abelian groups, and having image in the category of abelian groups are naturally equivalent in both variables.*

We are now ready for the main definition of this section:

Definition 1.3.9. *Suppose $h^n(-,-)$ is a cohomology theory, with coefficients in some category $\mathfrak{A}$. A natural classifying space for $h^n(-,-)$ is a functor $K_n(-)$ such that there exists a natural equivalence from the functor $h^n(-,-)$ to the functor $[-, K_n(-)]$.*

Example 1.3.10. As we have just shown, $K(-,n)$ is a natural classifying space for cohomology with coefficients in the category of groups.

$\prod_n K(H_{*-n}(B), n)$ is not a natural classifying space for $\tilde{H}^*(-,B)$, since the isomorphism $\tilde{H}^*(X,B) \to [X, \prod_n K(H_{*-n}(B),n)]$, was constructed as the composition:

$$\tilde{H}^*(X,B) \xrightarrow{\phi} \prod_n \tilde{H}^n(X, H_{n-*}(B)) \longrightarrow [X, \prod_n H^n(X, H_{*-n}(B))]$$

and the map ϕ is not natural in the chain complex B. So there is no way that chain maps $B \to B'$ could correspond to maps $\prod_n K(H_{*-n}(B),n) \to \prod_n K(H_{*-n}(B'),n)$ in a natural way.

The main task of this thesis is to construct a natural classifying space

$$K(B_0 \to \ldots \to B_k, n)$$

for reduced cohomology with coefficients in the finite chain complex $B_0 \to \ldots \to B_k$. Keep in mind that we have defined a morphism in the category of finite chain complexes to be a set of maps $\phi = (\phi_0, \ldots, \phi_k)$ such that the following diagram commutes:

$$\begin{tikzcd}
B_0 \arrow[r, "h_0"] \arrow[d, "\phi_0"] & B_1 \arrow[r, "h_1"] \arrow[d, "\phi_1"] & \cdots \arrow[r, "h_{k-1}"] & B_k \arrow[d, "\phi_k"] \\
B'_0 \arrow[r, "h'_0"] & B'_1 \arrow[r, "h'_1"] & \cdots \arrow[r, "h_{k-1}"] & B_k
\end{tikzcd}$$

The functor $K(-,n)$ should expand the given functor defined on the category of abelian groups, which can be thought of as a subcategory of the category of finite chains of abelian groups.

Chapter 2

Two important results

This short chapter contains two results. The first result is a series of long exact sequences, that in a natural way relate cohomology with coefficients in a finite chain complex with cohomology with coefficients in shorter chain complexes, thus inductively relating cohomology with coefficients in a finite chain complex to ordinary cohomology. This result is more useful for our purpose than the result of Theorem 1.2.3 since it is natural in both variables.

The second result is equivalence of cellular and singular cohomology with coefficients in a chain complex. Cellular cohomology is obtained by replacing the singular chain complex with the cellular chain complex in definition 1.1.12.

Both results will be used to find the natural classifying space for cohomology with coefficients in a chain complex, and both results have been developed for this thesis.

2.1 The long exact sequences

Example 2.1.1. Lets look a little into our main example, namely cohomology with coefficients in a group. If $A \xrightarrow{h} B$ is a homomorphism of abelian groups, and K is a chain complex, we see that:

$$\mathrm{Hom}(K; A \xrightarrow{h} B)_p = \mathrm{Hom}(K_p, A) \times \mathrm{Hom}(K_{p-1}, B)$$

A good way to illustrate an element $(f_0, f_{-1}) \in \mathrm{Hom}(K; A \xrightarrow{h} B)_p$ is to draw the diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & K_{p+1} \arrow[r, "\partial"] \arrow[d, "0"] & K_p \arrow[r, "\partial"] \arrow[d, "f_0"] & K_{p-1} \arrow[r, "\partial"] \arrow[d, "f_{-1}"] & K_{p-1} \arrow[r, "\partial"] \arrow[d, "0"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & A \arrow[r, "h"] & B \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

This illustrates the derivation in $\mathrm{Hom}(K, A \xrightarrow{h} B)$. The first coordinate of $\delta(f_0, f_{-1})$ is the only map $K_{p+1} \to A$, that you can build from this diagram (with the right sign), and the second coordinate is the difference of the two maps $K_p \to B$ that you can build with the right sign. To be more precise we get:

$$\delta(f_0, f_{-1}) = ((-1)^{p+1} f_0 \partial, h f_0 - (-1)^p f_{-1} \partial)$$

Thus, a cycle in $\mathrm{Hom}(K, A \xrightarrow{h} B)_p$ is a pair (f_0, f_{-1}) of maps such that $f_0\partial = 0$, and this diagram commutes up to the right sign:

$$\begin{tikzcd} K_p \arrow[r, "\partial"] \arrow[d, "f_0"] & K_{p-1} \arrow[d, "f_{-1}"] \\ A \arrow[r, "h"] & B \end{tikzcd}$$

This means that f_0 represents an element in $H^p(K, A)$ and if g represents an element in $H^{p-1}(K, B)$, then $(0, g)$ represents an element in $H^p(K, A \xrightarrow{h} B)$. Thus we have maps:

$$\psi : H^n(K, A \xrightarrow{h} B) \to H^n(K, A)$$

and

$$\phi : H^{n-1}(K, B) \to H^n(K, A \xrightarrow{h} B)$$

induced by projection and inclusion respectively. If we combine these maps with the map $h_* : H^n(K, A) \to H^n(K, B)$ known from ordinary cohomology theory, we get an exact sequence:

$$\cdots \longrightarrow H^n(K, A \xrightarrow{h} B) \xrightarrow{\psi} H^n(K, A) \xrightarrow{h_*}$$

$$H^n(K, B) \xrightarrow{\phi} H^{n+1}(K, A \xrightarrow{h} B) \xrightarrow{\psi} \cdots$$

We will prove a generalization of this construction:

Theorem 2.1.2. *Suppose*

$$B_0 \stackrel{h_0}{\to} B_1 \to \ldots \stackrel{h_{k-1}}{\to} B_k$$

is a finite chain complex. For every $r \in \{0, \ldots, k-1\}$ there is a long exact sequence on the form:

$$\cdots \longrightarrow H^n(A, B_0 \to \ldots \to B_k) \xrightarrow{\psi} H^n(A, B_0 \to \ldots \to B_r) \xrightarrow{(h_r)_*}$$

$$H^{n-r}(A, B_{r+1} \to \ldots \to B_k) \xrightarrow{\phi} H^{n+1}(A, B_0 \to \ldots \to B_k) \longrightarrow \cdots$$

Here, the map ψ is given by projection and the map ϕ is given by inclusion. This sequence is natural in both A and the coefficient complex.

Remark 2.1.3. We need to explain what the map $(h_r)_*$ is. Suppose

$$(f_0, \ldots, f_r) \in \mathrm{Hom}(A_n, B_0) \times \ldots \times \mathrm{Hom}(A_{n-r}, B_r) = \mathrm{Hom}(A, B_0 \to \ldots \to B_r)_n$$

represents an element in $H^n(A, B_0 \to \ldots \to B_r)$. We define $(h_r)_*[(f_0, \ldots, f_r)]$ to be the element represented by

$$(h_r \circ f_r, 0, \ldots, 0) \in \mathrm{Hom}(A_{n-r}, B_{r+1}) \times \ldots \times \mathrm{Hom}(A_{n-k+1}, B_k) =$$

$$\mathrm{Hom}(A; B_{r+1} \to \ldots \to B_k)_{n-r}$$

Proof. The chain map i:

$$\begin{tikzcd}
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & B_{r+1} \arrow[r] \arrow[d, "id"] & \cdots \arrow[r] & B_k \arrow[d, "id"] \\
B_0 \arrow[r] \arrow[from=u] & \cdots \arrow[r] & B_r \arrow[r] & B_{r+1} \arrow[r] & \cdots \arrow[r] & B_k
\end{tikzcd}$$

induces the inclusion:

$$\phi : H^n(A; 0 \to \ldots \to 0 \to B_{r+1} \to \ldots \to B_k) \to H^n(A; B_0 \to \ldots \to B_k)$$

and the chain map p:

$$\begin{tikzcd}
B_0 \arrow[r] \arrow[d, "id"] & \cdots \arrow[r] & B_r \arrow[r] \arrow[d, "id"] & B_{r+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & B_k \arrow[d] \\
B_0 \arrow[r] & \cdots \arrow[r] & B_r \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0
\end{tikzcd}$$

induces the projection:

$$\psi : H^n(A; B_0 \to \ldots \to B_k) \to H^n(A; B_0 \to \ldots \to B_r)$$

Since clearly the sequence:

$$0 \longrightarrow \mathrm{Hom}(A; 0 \to \ldots \to 0 \to B_{r+1} \to \ldots \to B_k)_n \xrightarrow{i_*}$$

$$\mathrm{Hom}(A; B_0 \to \ldots \to B_k)_n \xrightarrow{p_*} \mathrm{Hom}(A; B_0 \to \ldots \to B_r)_n \longrightarrow 0$$

is short exact for every n, the long exact sequence follows from Lemma 1.1.20 and the identification:

$$H^n(A; \underbrace{0 \to \ldots \to 0}_{r+1} \to B_{r+1} \to \ldots \to B_k, n) \simeq H^{n-r-1}(A; B_{r+1} \to \ldots \to B_k)$$

of corollary 1.1.9.

The only thing left to prove is that the connecting homomorphism:

$$D : H^n(A, B_0 \to \ldots \to B_r) \to H^{n+1}(A, \underbrace{0 \to \ldots \to 0}_{r+1} \to B_{r+1} \to \ldots \to B_k)$$

is in fact the map $(h_r)_*$.

We can describe D by following the construction of the boundary map as on page 116 in [Hatcher]. Suppose $(f_0, \ldots, f_r)$ represents an element in $H^n(A, B_0 \to \ldots \to B_r)$. Then $\delta(f_0, \ldots, f_r) = 0$. The construction tells us to pick an element x such that $p_*(x) = (f_0, \ldots, f_r)$. Let us choose $(f_0, \ldots, f_r, 0, \ldots, 0)$. Using the fact that $\delta(f_0, \ldots, f_r) = 0$ we get

$$\delta(f_0, \ldots, f_r, 0, \ldots, 0) = (\underbrace{0, \ldots, 0}_{r}, h_r \circ f_r, 0, \ldots, 0)$$

Since $i_*(h_r \circ f_r, 0, \ldots, 0) = (0, \ldots, 0, h_r \circ f_r, 0, \ldots, 0)$ the construction tells us that

$$D[(f_0, \ldots, f_r)] = [(h_r \circ f_r, 0, \ldots, 0)]$$

as desired.

The naturality of the long exact sequence mentioned follows from the naturality in Lemma 1.1.20. Naturality in the coefficients variable is clear. $\square$

The version of the long exact sequence, that we will use the most is the case $r = k-1$.

Corollary 2.1.4. *The sequence:*

$$\cdots \longrightarrow H^n(A, B_0 \to \ldots \to B_k) \xrightarrow{\psi} H^n(A, B_0 \to \ldots \to B_{k-1}) \xrightarrow{(h_{k-1})_*}$$

$$H^{n-k+1}(A, B_k) \xrightarrow{\phi} H^{n+1}(A, B_0 \to \ldots \to B_k) \longrightarrow \cdots$$

is natural and exact.

Corollary 2.1.5. *Suppose $k > n+1$ and X is a space. Then the map:*

$$\psi : \tilde{H}^n(X, B_0 \to \ldots \to B_k) \to \tilde{H}^n(X, B_0 \to \ldots \to B_{n+1})$$

is an isomorphism. Notice that this map is natural in both variables.

Proof. Notice that for $n < 0$ we have by Theorem 1.2.3

$$\tilde{H}^n(X, B_0 \to \ldots \to B_k) \simeq \prod_l \tilde{H}^l(X, H_{l-n}(B_0 \to \ldots \to B_k)) \simeq 0$$

since $H_{l-n}(B_0 \to \ldots \to B_k)$ is only nontrivial for $l \in \{n, \ldots, n-k\}$.
Suppose $k > n+1$. Then Theorem 2.1.2 gives us a long exact sequence:

$$\tilde{H}^{-2}(X; B_{n+1} \to \ldots \to B_k) \longrightarrow \tilde{H}^n(X, B_0 \to \ldots \to B_k) \longrightarrow$$

$$\tilde{H}^n(X, B_0 \to \ldots \to B_n) \longrightarrow \tilde{H}^{-1}(X, B_{n+1} \to \ldots \to B_k)$$

Since the first and the last groups in this sequence are zero by the above analysis, this proves the corollary. $\square$

This means that the only interesting cases of cohomology with coefficients in a finite complex, are the ones with $k \leq n+1$, and we can restrict our attention to those.

Remark 2.1.6. Had we done the analysis for Theorem 2.1.2 in the case of infinite chains, the arguments from the corollary above would have showed, that for a general chain complex B, $\tilde{H}^n(X, B)$ is only affected by the groups in B in dimension greater than or equal to $-n-1$.

2.2 Cellular cohomology

Definition 2.2.1. *Let X be a CW-complex, and let $C_*(X)$ denote the cellular complex of X. We define the cellular cohomology of X with coefficients in the chain complex B as:*

$$H^n_{CW}(X; B) = H^n(C_*(X); B)$$

Let (X, A) be a CW-pair and let $C_*(X, A)$ denote the relative cellular complex of (X, A). We define relative cellular cohomology with coefficients in B to be:

$$H^n_{CW}(X, A; B) = H^n(C_*(X, A); B)$$

Let $\hat{C}_*(X)$ denote the augmented cellular complex of X. We define reduced cellular cohomology of X with coefficients in the chain complex B to be:

$$\tilde{H}^n_{CW}(X;B) = H^n(\hat{C}_*(X);B)$$

Reduced relative cellular homology is defined as:

$$\tilde{H}^n_{CW}(X,A;B) = \tilde{H}^n_{CW}(X/A;B)$$

Thus $H^n_{CW}(-,-)$ is a bifunctor taking one variable in the category of CW-complexes with cellular maps and a chain complex of abelian groups, with values in abelian groups. It is contravariant in the first variable and covariant in the second.

We would like to say that cellular cohomology is the same as singular cohomology. It is well-known that if X is a CW-complex and B is an abelian group, then $H^n_{CW}(X,B) \simeq H^n(X,B)$, if H^n_{CW} denotes ordinary cellular cohomology. This is sufficient for the purpose of computing cohomology groups, but since we are interested in the naturality of the cohomology functor, what we need is a natural equivalence of the functors $H^n(-,-)$ and $H^n_{CW}(-,-)$. That such an equivalence actually exists in the case of ordinary cohomology is proved in [May98] on pages 147-148.

The next theorem tells us that this natural equivalence generalizes to cohomology with coefficients in a chain complex.

Theorem 2.2.2. *Let $H^n(-,-)$ and $H^n_{CW}(-,-)$ denote singular and cellular cohomology of pairs, with coefficients in a chain complex. The functors $H^n(-,-)$ and $H^n_{CW}(-,-)$ are naturally equivalent.*

Proof. Let $T : H_n^{CW}(-) \to H_n(-)$ denote a natural equivalence of functors in ordinary homology. That this exists is proved in [May98] p. 117. Lemma 1.2.1 implies that for each pair (X,A) the isomorphism $T_{(X,A)} : H_n^{CW}(X,A) \to H_n(X,A)$ is induced by a chain map $f_{(X,A)} : C_*(X,A) \to \Delta_*(X,A)$. Before we define the transformation on cohomology with coefficients in a chain complex, lets define the transformation on ordinary cohomology $S : H^n(-,-) \to H^n_{CW}(-,-)$ by letting $S_{(X,A),B}$ be the map induced by $f_{(X,A)}$.

First, we need to prove that this is in fact a well defined transformation, since there might be several maps $f_{(X,A)}$ that induce the same map on homology. If $f_{(X,A)}$ and $f'_{(X,A)}$ are two such maps, then $(f_{(X,A)})_* = (f'_{(X,A)})_*$ on homology, and the universal coefficient theorem tells us that they induce the same map on cohomology.

The transformation S is clearly natural in the coefficient variable. To prove that it is natural in (X,A), let $g : (X,A) \to (Y,C)$ be a cellular map. We need to prove that this diagram commutes:

$$\begin{tikzcd}
H^n(\Delta_*(Y,C),B) \arrow[r, "(f_{(Y,C)})^*"] \arrow[d, "g^*"] & H^n(C_*(Y,C),B) \arrow[d, "g^*"] \\
H^n(\Delta_*(X,A),B) \arrow[r, "(f_{(X,A)})^*"] & H^n(C_*(X,A),B)
\end{tikzcd}$$

In other words we need to prove that $g^*(f_{(Y,C)})^* = (f_{(X,A)})^* g^*$ on cohomology. But naturality of T tells us that $g_*(f_{(X,A)})_* = (f_{(Y,C)})_* g_*$ on homology, and the result follows from the universal coefficient theorem. Since $f_{(X,A)}$ induces an isomorphism on homology it induces an isomorphism on cohomology, so we conclude that S is a natural equivalence.

We can now define the transformation $R : H^n(-,-) \to H^n_{CW}(-,-)$ on cohomology with coefficients in a chain complex, by letting $R_{(X,A),B}$ be the map induced by $f_{(X,A)}$ as in ordinary cohomology.

Again we need to check that this map is in fact well defined, since there might be several maps $f_{(X,A)}$ inducing the same map $T_{(X,A)}$. Consider the following commutative diagram:

$$\begin{tikzcd}
H^n(\Delta_*(X,A);B) \arrow[r, "\simeq"] \arrow[d, "(f_{(X,A)})^*"] & \prod_* H^*(\Delta_*(X,A), H_{n-*}(B)) \arrow[d, "\prod(f_{(X,A)})^*", "\simeq"'] \\
H^n(C_*(X,A);B) \arrow[r, "\simeq"] & \prod_* H^*(C_*(X,A), H_{n-*}(B))
\end{tikzcd}$$

Here the vertical isomorphism is the isomorphism of Theorem 1.2.3, and the diagram is commutative by the naturality of this isomorphism. Since the vertical map on the right is independent of the choice of $f_{(X,A)}$, R is independent of the choice of $f_{(X,A)}$.

This diagram also tells us two other things. First, since all other maps in the diagram are isomorphisms, $R_{(X,A),B}$ is an isomorphism for all (X,A) and B. Secondly, since all other maps in the diagram are natural in (X,A), R is natural in (X,A). Since clearly R is natural in B, we conclude that R is an natural equivalence of functors. □

Since we are really interested in reduced cohomology, we need the following corollary.

Corollary 2.2.3. *The functors $\tilde{H}^n_{CW}(-,-)$ and $\tilde{H}^n(-,-)$ defined on the category of* pointed *CW-complexes with pointed cellular maps and chain complexes of abelian groups, are naturally equivalent.*

Proof. Proposition 1.1.26 proves that the functors $\tilde{H}^n(-,-)$ and $H^n(-,*;-)$ are equivalent on the category of pointed CW-complexes. The same argument as in the proof of Proposition 1.1.26 proves that the functors $\tilde{H}^n_{CW}(-,-)$ and $H^n_{CW}(-,*;-)$ are equivalent. Since Theorem 2.2.2 tells us that $H^n(-,*;-)$ and $H^n_{CW}(-,*;-)$ are naturally equivalent, this proves the corollary. □

This result tells us that as long as we are looking at the category of pointed CW-complexes, the functors $\tilde{H}^n_{CW}(-,-)$ and $\tilde{H}^n(-,-)$ are essentially the same, and in the following chapters, we shall not distinguish between them.

Chapter 3

The homotopy fiber

This chapter contains some background material, that will be needed to determine the natural classifying space. Consider the long exact sequence of cohomology with coefficients in a homomorphism $A \xrightarrow{h} B$. Using classifying spaces this will correspond to a long exact sequence of functors $[, Y_n]$, which in term correspond to the sequence of space Y_n.

Our guess will be that this sequence of spaces is in fact the Puppe sequence for the map $K(A,n) \xrightarrow{K(h,n)} K(B,n)$. In the next chapter we shall prove that this guess is right, but before we do that we must introduce the Puppe sequence.

The material in this chapter can be found in most books on algebraic topology, and the main references for this presentation are [Hatcher] and [Whitehead]. We include this material here because it is central in the proof of the main theorem concerning the natural classifying space.

3.1 Fibrations

Definition 3.1.1. *A map $E \xrightarrow{p} B$ between topological spaces is called a fibration if the diagram:*

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\tilde{F}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"] \arrow[ur, dashed, "G"] & B \end{tikzcd}$$

has a lift G for all X and all maps $F, \tilde{F}$. Here $I = [0,1]$. We sometimes refer to the ability to lift all homotopies like this as the homotopy lifting property.

We define the fiber of the fibration $E \xrightarrow{p} B$ to be $F = p^{-1}(b)$ for $b \in B$, and write $F \longrightarrow E \xrightarrow{p} B$. The next proposition tells us that this is well defined.

Proposition 3.1.2. *Let $E \xrightarrow{p} B$ be a fibration. If $b, b' \in B$ belong to the same pathcomponent then $p^{-1}(b)$ is homotopy equivalent to $p^{-1}(b')$.*

Proof. Let $\gamma : I \to B$ be any path in B (no restriction on endpoints). We adopt the notation $F_b = p^{-1}(b)$. If we define maps $F_{\gamma(0)} \times \{0\} \to E$ and $F_{\gamma(0)} \times I \to B$, by letting the first map be the identity on $F_{\gamma(0)}$ and letting the second be given by the path γ, then this diagram commutes:

$$\begin{tikzcd}
F_{\gamma(0)} \times \{0\} \arrow[r, hook] \arrow[d, hook] & E \arrow[d, "p"] \\
F_{\gamma(0)} \times I \arrow[r] & B
\end{tikzcd}$$

The fibration property then gives maps $L_{\gamma,t} : F_{\gamma(0)} \to F_{\gamma(t)}$ for each t. This gives a map $L_\gamma = L_{\gamma,1} : F_{\gamma(0)} \to F_{\gamma(1)}$, and we consider the correspondence $\gamma \mapsto L_\gamma$. This correspondence has two key properties:

1. L_γ is well defined up to homotopy. That is, its homotopyclass is independent of the choice of lift. If the two paths γ, γ' are homotopic then the maps L_γ and $L_{\gamma'}$ are homotopic.

2. $\gamma \mapsto L_\gamma$ respects composition of paths. That is $L_{\gamma\sigma}$ is homotopic to $L_\gamma L_\sigma$. Note that our convention on path composition is that $\gamma\sigma$ means the path:

$$\gamma\sigma(t) = \begin{cases} \sigma(2t) & t \leq 1/2 \\ \gamma(2t-1) & t \geq 1/2 \end{cases}$$

Letting γ^{-1} denote the inverse path of γ, the two properties above combine to prove that $L_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$ and $L_{\gamma^{-1}} : F_{\gamma(1)} \to F_{\gamma(0)}$ are homotopy equivalences, thus proving the proposition.

To prove the first property let $\sigma : I \times I \to B$ be a homotopy between the paths γ and γ'. To be more precise, σ is a map such that $\sigma|_{I\times\{0\}}$ is the constant map to $\gamma(0)$, and $\sigma|_{I\times\{1\}}$ is constant, and $\sigma|_{\{1\}\times I}$ is γ, and $\sigma|_{\{0\}\times I}$ is γ'. We will show that this gives rise to a homotopy between L_γ and $L_{\gamma'}$. The case $\gamma = \gamma'$, where σ is the identity homotopy proves welldefinedness.

Define for all $(s,t) \in \partial I \times I$ $L_{\sigma,s,t} : F_{\gamma(0)} \to F_{\sigma(s,t)}$ using the lifts $L_{\gamma,t}$ and $L_{\gamma',t}$, and define $L_{\sigma,s,t}$ on $I \times \{0\}$ to be the identity. Then the following diagram commutes:

$$\begin{tikzcd}
F_{\gamma(0)} \times (I \times \{0\} \cup \partial I \times I) \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\
F_{\gamma(0)} \times I \times I \arrow[r] & B
\end{tikzcd}$$

This diagram has a lift, since the pair $(I \times I, \partial I \times I \cup I \times \{0\})$ is homeomorphic to the pair $I \times (I, \{0\})$. Thus we have maps $L_{\sigma,s,t} : F_{\gamma(0)} \to F_{\sigma(s,t)}$ for all $s,t \in I$. Setting $t = 1$ gives a homotopy from L_γ to $L_{\gamma'}$. Notice that this is a homotopy with image inside $F_{\gamma(1)}$ at all times since $p \circ L_{\sigma,s,1} = \sigma(s,1) = \gamma(1)$.

To prove the second property, suppose $L_{\gamma,s}$ and $L_{\sigma,s}$ are lifts for γ and σ respectively. Then the lift defined by $L_{\gamma,2s}$ for $s \leq 1/2$ and $L_{\sigma,2s-1}L_\gamma$ for $s \geq 1/2$ is a lift for $\sigma\gamma$. Thus $L_{\sigma\gamma}$ is homotopic to $L_\sigma L_\gamma$. □

We have one main example of a fibration:

Example 3.1.3. Let B be a pointed space with basepoint $*$. Let PB denote the path space of B, that is

$$PB = \{\gamma : I \to B | \gamma(0) = *\}$$

PB is given the compact open topology. The map $p : PB \to B$ given by $p(\gamma) = \gamma(1)$ is a fibration. To prove this suppose this diagram commutes:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\tilde{F}"] \arrow[d, hook] & PB \arrow[d, "p"] \\
X \times I \arrow[r, "F"] & B
\end{tikzcd}$$

Let for each $x \in X$, $\gamma_x = \tilde{F}(x, 0)$. Define $G(x, s)$ to be the path defined by:

$$G(x,s)(t) = \begin{cases} \gamma_x(\frac{t}{1-s}) & t \leq 1-s \\ F(x, t-1+s) & t \geq 1-s \end{cases}$$

To see that this defines a continuous path, notice that $t = 1-s$ we get $\gamma(\frac{t}{1-s}) = \gamma_x(1)$ and $F(x, 0)$. These are equal since the diagram commutes. The map G is continuous since the evaluation map $X \times I \times I \to B$ given by $(x, s, t) \mapsto G(x, s)(t)$ is continuous.

Thus we get the diagram:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\tilde{F}"] \arrow[d, hook] & PB \arrow[d, "p"] \\
X \times I \arrow[r, "F"] \arrow[ur, "G"] & B
\end{tikzcd}$$

The upper triangle in this diagram commutes since $G(x, 0) = \gamma_x$, and the lower triangle commutes since $G(x, s)(1) = F(x, s)$.

The fiber of the fibration $PB \xrightarrow{p} B$ is $p^{-1}(*) = \Omega B$, the loop space over B.

The example above will provide a whole series of fibrations using this next construction. Suppose $p : E \to B$ is a fibration, and $f : A \to B$ is a map. We construct the pullback of the fibration p as:

$$f^*(E) = \{(a, e) \in A \times E | f(a) = p(e)\}$$

This means that we have natural projections $\psi : f^*(E) \to A$ and $\xi : f^*(E) \to E$, such that the following diagram commutes:

$$\begin{tikzcd}
f^*(E) \arrow[r, "\psi"] \arrow[d, "\xi"] & E \arrow[d, "p"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

The next proposition justifies the construction of the pullback.

Proposition 3.1.4. *The map $f^*(E) \to A$ is a fibration.*

Proof. First notice that a map $\phi : W \to f^*(E)$ is the same as a pair of maps ϕ_1, ϕ_2 that makes this diagram commute:

$$\begin{tikzcd}
W \arrow[r, "\phi_1"] \arrow[d, "\phi_2"] & E \arrow[d, "p"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Now, suppose we have a commutative diagram:

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\tilde{F}"] \arrow[d, hook] & F^*(E) \arrow[r, "\psi"] \arrow[d, "\xi"] & E \arrow[d, "p"] \\
X \times I \arrow[r, "F"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

Since p is a fibration, there is a lift of the composition $f \circ F$. This provides the map $X \times I \to E$. If we let the map $X \times I \to A$ simply be F, these two maps constitute a map $X \times I \to F^*(E)$, which is a lift of $\tilde{F}$. $\square$

It is easy to see what the fiber of a pullback is. If $a \in A$, then $\xi^{-1}(a) = \{(a, e) \in F^*(E) | p(e) = f(a)\} = \{a\} \times p^{-1}(f(a))$. In other words, the fiber of the pullback fibration is a copy of the fiber of the original fibration. One usually writes

$$\begin{tikzcd}
F \arrow[r, equal] \arrow[d] & F \arrow[d] \\
f^*(E) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

There is one other important proposition that needs to be mentioned here. First we need a definition:

Definition 3.1.5. *Suppose $E \xrightarrow{p} B$ and $E' \xrightarrow{p'} B$ are fibrations over the same space B. A map $f : E \to E'$ is fiber preserving if the following diagram commutes:*

$$\begin{tikzcd}
E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\
& B &
\end{tikzcd}$$

Two fiber preserving maps $f, f' : E \to E'$ are fiber homotopy equivalent if there is a homotopy F from f to f' such that this diagram commutes:

$$\begin{tikzcd}
E \times I \arrow[r, "F"] \arrow[d] & E' \arrow[d, "p'"] \\
E \arrow[r, "p"] & B
\end{tikzcd}$$

In this situation F is called a fiber preserving homotopy. The fibrations $E \xrightarrow{p} B$ and $E' \xrightarrow{p'} B$ are fiber homotopy equivalent if there exist fiber preserving maps $f : E \to E'$ and $f' : E' \to E$ such that $f \circ f'$ and $f' \circ f$ are fiber homotopy equivalent to the identity through fiber preserving homotopies. One could view these maps f and f' as families of homotopy equivalences between the fibers.

Fiber homotopy equivalence is of course an equivalence relation. Notice that fiber homotopy equivalent fibrations have homotopic fibers.

Proposition 3.1.6. *Let $E \xrightarrow{p} B$ be a fibration. If $f, g : A \to B$ are homotopic maps then $f^*(E)$ and $g^*(E)$ are fiber homotopy equivalent. In particular, the fibers of these fibrations are homotopic.*

Lemma 3.1.7. *Let $E \xrightarrow{p} B$ be a fibration, and let $A \subseteq B$. Then the restriction $p : p^{-1}(A) \to A$ is a fibration.*

Proof. Obvious. ☐

Proof of Proposition 3.1.6. Suppose $F : A \times I \to B$ is a homotopy from f to g. The fibration $q : F^*(E) \to A \times I$ contains the fibrations $f^*(E)$ and $g^*(E)$, as $q : q^{-1}(A \times \{0\}) \to A \times \{0\} \simeq A$ and $q : q^{-1}(A \times \{1\}) \to A \times \{1\} \simeq A$ respectively. So it suffices to prove that if $p : E \to B \times I$ is a fibration, then all the fibrations $E_s = p^{-1}(B \times \{s\}) \to B$ are fiber homotopy equivalent.

Suppose $\gamma : I \to I$ is a path in I. We can define a homotopy $g_t : E_{\gamma(0)} \to B \times I$ by $g_t(x) = (p(x), \gamma(t))$, with initial lift $L_{\gamma,0} : E_{\gamma(0)} \to E$ defined by the inclusion. Then the homotopy lifting property enables us to lift g_t to a homotopy $L_{\gamma,t} : E_{\gamma(0)} \to E$ such that $\mathrm{im}(L_{\gamma,t}) \subseteq E_{\gamma(t)}$, and we define the homotopy $L_\gamma = L_{\gamma,1} : E_{\gamma(0)} \to E_{\gamma(1)}$.

Like in the proof of Proposition 3.1.2 we prove that the association $\gamma \mapsto L_\gamma$ is well defined up to homotopy, and that if to paths γ and γ' are homotopic (which in this case just means that they have the same endpoints since I is contractible), then L_γ is homotopic to $L_{\gamma'}$. We also prove that $L_\gamma L_\sigma$ is homotopic to $L_{\gamma\sigma}$. This implies that L_γ and $L_{\gamma^{-1}}$ are fiber homotopy equivalences that are inverses of each other. Notice that if γ and γ' are homotopic then the homotopy constructed from L_γ to $L_{\gamma'}$ preserves fibers since it is a lift of the map $E_{\gamma(0)} \times I \to B \times I$ defined by $(x, t) \mapsto (p(x), \gamma(1))$. ☐

3.2 Left exact sequences of spaces

The reason that fibrations are mentioned here is that they are left exact.

If A, X, Y are pointed spaces and $f : X \to Y$ is a pointed map, lets for a moment define the kernel of $f_* : [A, X] \to [A, Y]$ as $f_*^{-1}([*])$, where $[*]$ denotes the class represented by the constant map to basepoint.

Definition 3.2.1. *A sequence of pointed spaces and maps:*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called left exact if for all pointed CW-complexes A, the sequence:

$$[A, X] \xrightarrow{f_*} [A, Y] \xrightarrow{g_*} [A, Z]$$

is exact. That is if the kernel of g_* is exactly the image of f_*.

Lemma 3.2.2. *All fibrations $F \longrightarrow E \xrightarrow{p} B$ of pointed spaces are left exact.*

Before we prove this, we need another lemma, which is taken from [Bredon]:

Lemma 3.2.3. *Let $F \longrightarrow E \xrightarrow{p} B$ be a fibration of pointed spaces. Suppose (X, A) is a pair of pointed CW-complexes, then the lifting problem:*

$$\begin{tikzcd}
X \times \{0\} \cup A \times I \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "f_t"] & B
\end{tikzcd}$$

always has a solution $G : X \times I \to E$.

Proof. We do this by induction over the skeletons of $X \setminus A$. To expand G over a new cell of $X \setminus A$ is equivalent to solving the lifting problem:

$$\begin{tikzcd}
D^n \times \{0\} \cup S^{n-1} \times I \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\
D^n \times I \arrow[r, "f_t"] \arrow[ur, dashed] & B
\end{tikzcd}$$

Which can be solved since the pairs $(D^n \times I, D^n \times \{0\} \cup S^{n-1} \times I)$ and $(D^n \times I, D^n \times \{0\})$ are homeomorphic, and p is a fibration. □

Proof of Lemma 3.2.2. Let i denote the inclusion of F into E. Since $F = p^{-1}(*)$ it is clear that the image of i_* is a subset of the kernel of p_*.

Now, suppose $f : X \to E$ is a map such that $p \circ f$ is homotopic to the constant map to $*$. We need to show that f is homotopic to a map whose image is contained in F. Consider the diagram:

$$\begin{tikzcd}
X \times \{0\} \cup \{*\} \times I \arrow[r, "f \cup *"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times I \arrow[r, "G"] & B
\end{tikzcd}$$

Here G is a homotopy from $p \circ f$ to the constant map to basepoint. Since p is a fibration Lemma 3.2.3 tells us that G lifts to a homotopy $\tilde{G} : X \times I \to E$ from f to a map whose image is contained in $p^{-1}(*) = F$. The fact that we restricted $\tilde{G}$ on $\{*\} \times I$ to be the constant map to $*$ ensures that this homotopy is basepoint-preserving. □

At this point we need to discuss an important construction. Suppose $f : X \to Y$ is a pointed map, that is not necessarily a fibration. We can write f as a homotopy equivalence followed by a fibration in the following way.

We define FY to be the set of all continuous maps $I \to Y$ with no restriction on the endpoints, and give this set the compact open topology. Let F^f denote the space:

$$F^f = \{(x, \gamma) \in X \times FY | f(x) = \gamma(1)\}$$

There is a natural projection $\pi : F^f \to X$ and a map $q : F^f \to Y$ given by $q(x, \gamma) = \gamma(0)$. There is also an inclusion $j : X \to F^f$ given by $j(x) = (x, e_{f(x)})$ where $e_{f(x)}$ denotes the constant path to $f(x)$. We claim that j and π are homotopy equivalences and q is a fibration.

It is clear that $\pi j : X \to F^f \to X$ is the identity. The composition $j\pi$ maps (x, γ) to $(x, e_{f(x)})$. This map is homotopy equivalent to the identity on F^f using the homotopy given by contraction of paths.

q is a fibration by the same argument that proves that $PB \to B$ is a fibration. We have

$$qj(x) = q(x, e_{f(x)}) = f(x)$$

So we have written f as a homotopy equivalence followed by a fibration.

The fiber of q is the space:

$$T^f = \{(x, \gamma) \in X \times FY | f(x) = \gamma(1), \gamma(0) = *\} =$$

$$\{(x, \gamma) \in X \times PY | f(x) = \gamma(1)\}$$

called the homotopy fiber of f. The projection $p : T^f \to X$ is a fibration since it is the pullback of the fibration $PX \to X$ via the map f. The fiber of this fibration is ΩY included into T^f by the map $i : \gamma \mapsto (*, \gamma)$.

Thus we have this diagram:

$$\begin{tikzcd} T^f \arrow[r, hook] \arrow[dr, "p"'] & F^f \arrow[d, "\simeq"] \arrow[dr, "q"] & \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

The diagram at the left commutes. To see that the diagram at the right commutes up to homotopy, notice that q is the map $(x, \gamma) \mapsto \gamma(0)$ and the map given by going down and then right is $(x, \gamma) \mapsto f(x) = \gamma(1)$. Now, the map $h_t : (x, \gamma) \mapsto \gamma(t)$ is a homotopy between these two maps.

Since $T^f \longrightarrow F^f \xrightarrow{q} X$ is a fibration the sequence:

$$T^f \xrightarrow{p} X \xrightarrow{f} Y$$

is left exact. Notice that all maps here are basepoint-preserving. Since we also have that $\Omega Y \to T^f \to X$ is a fibration we get a left exact sequence:

$$\Omega Y \xrightarrow{i} T^f \xrightarrow{p} X \xrightarrow{f} Y$$

The next theorem tells us that we can expand this sequence infinitely to the left.

Theorem 3.2.4. *The sequence*

$$\cdots \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{\Omega i} \Omega T^f \xrightarrow{\Omega p} \Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{i} T^f \xrightarrow{p} X \xrightarrow{f} Y$$

is left exact.

To prove this theorem we need two lemmas.

Lemma 3.2.5. *If $f : X \to Y$ is a fibration then the inclusion $j : X \to F^f$ is a fiber homotopy equivalence. In particular, the inclusion of the fiber of f into T^f is a homotopy equivalence.*

Proof. Define $g : I \times F^f \to Y$ by $g(t,(x,\gamma)) = \gamma(t)$. On $\{1\} \times F^f$ we can define the lift $\tilde{g} : \{1\} \times F^f \to X$ by $\tilde{g}(1,(x,\gamma)) = x$. Now, apply the homotopy lifting property to the diagram:

$$\begin{tikzcd}
\{1\} \times F^f \arrow[r, "\tilde{g}|_{\{0\}\times F^f}"] \arrow[d, hook] & X \arrow[d, "f"] \\
I \times F^f \arrow[r, "g"] \arrow[ur, dashed, "\tilde{g}"] & Y
\end{tikzcd}$$

To get $\tilde{g}$. We can now define a homotopy $h_t : F^f \to F^f$ by

$$h_t(x,\gamma) = (\tilde{g}(t,(x,\gamma)), \gamma|_{[0,t]})$$

Here $\gamma|_{[0,t]}$ denotes the restriction of the path to the first part of the path, that is $\gamma|_{[0,t]}(s) = \gamma(st)$. Notice that $h_1 = id_{F^f}$, and that $h_0(F^f) \subseteq X$. Thus we can view h_0 as a map from F^f to X. Also notice that h_t is a fiber preserving homotopy.

Now, $h_0 \circ j \simeq h_1 \circ j = id_X$ and $j \circ h_0 = h_0 \simeq h_1 = id_{F_f}$, so j is a fiber homotopy equivalence with inverse h_0. □

Lemma 3.2.6. *$T^{\Omega f}$ is homeomorphic to ΩT^f. The homeomorphism makes this diagram commute:*

$$\begin{tikzcd}
 & \Omega T^f \arrow[d, "\simeq"] \arrow[dr, "\Omega p"] & & & \\
\Omega^2 Y \arrow[r] \arrow[ur, "\Omega i"] & T^{\Omega f} \arrow[r] & \Omega X \arrow[r, "\Omega f"] & \Omega Y \arrow[r, "i"] & T^f
\end{tikzcd}$$

Proof. From the definition we have:

$$T^{\Omega f} = \{(\sigma,\gamma) \in \Omega X \times P\Omega Y | f \circ \sigma = \gamma(1)\}$$

Notice here that there is a canonical identification of $P\Omega Y$ with $\Omega P Y$. We have two projections:

$$\begin{tikzcd}
 & T^f \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X & & PY
\end{tikzcd}$$

From definitions we also have:

$$\Omega T^f = \{\phi : I \to T^f | \phi(0) = \phi(1) = (*, e_*)\} =$$

$$\{\phi : I \to X \times PY | \phi(0) = \phi(1) = (*, e_*), f \circ p(\phi(t)) = (q \circ \phi(t))(1)\}$$

So we can define maps

$$\Phi : \Omega T^f \to T^{\Omega f}$$

and

$$\Psi : T^{\Omega f} \to \Omega T^f$$

by $\Phi(\phi) = (p \circ \phi, q \circ \phi)$ and $\Psi(\sigma,\gamma)(t) = (\sigma(t), \gamma(t))$. These maps are clearly each others inverses. To see that Ψ is continuous it suffices to prove that the map $T^{\Omega f} \times I \to X \times PY$ given by Ψ and evaluation is continuous. But this is clearly the case.

By the definition of subset topology, Φ is continuous iff the composition

$$\Omega T^f \xrightarrow{\Phi} T^{\Omega f} \longrightarrow \Omega X \times P\Omega Y$$

is continuous. By the definition of the compact open topology, this composition is continuous iff the composition:

$$\Omega T^f \times I \xrightarrow{\Phi \times id} T^{\Omega f} \times I \longrightarrow \Omega X \times P\Omega Y \times I \longrightarrow X \times PY$$

is continuous. The last map here is given by evaluation. But this is clearly the case, since this composition is the evaluation map on ΩT^f. This proves that the maps Φ and Ψ are homeomorphisms.

At this point the only thing left to prove is commutativity of the diagram. Consider the diagram:

$$\begin{tikzcd}
\Omega T^f \arrow[d, "\Phi", "\simeq"'] \arrow[dr, "\Omega p"] & \\
T^{\Omega f} \arrow[r] & \Omega X
\end{tikzcd}$$

$\Omega p(\phi) = p \circ \phi$, and the composition down and right in the diagram is the map:

$$\phi \mapsto (p \circ \phi, q \circ \phi) \mapsto p \circ \phi$$

So this diagram commutes. Consider the other diagram in question:

$$\begin{tikzcd}
 & \Omega T^f \arrow[d, "\Phi", "\simeq"'] \\
\Omega^2 Y \arrow[ur, "\Omega i"] \arrow[r] & T^{\Omega f}
\end{tikzcd}$$

The bottom map is $\gamma \mapsto (e_*, \gamma)$. On the other hand $\Omega i(\gamma)$ is the loop $t \mapsto (*, \gamma(t)) \in X \times \Omega Y$, and Φ evaluated on this loop gives $(e_*, \gamma) \in \Omega X \times \Omega^2 Y$. This proves that the last diagram commutes. □

Proof of Theorem 3.2.4. We need to prove that the sequence

$$\Omega X \xrightarrow{\Omega f} \Omega Y \longrightarrow T^f$$

is left exact. If we can prove that, we can apply the construction again to the map Ωf and Lemma 3.2.6 will give us the rest.

The idea will be to construct the homotopy fiber of the map $p : T^f \to X$. We will call this T^p. Using the fact that the projection $T^p \to T^f$ is a fibration we will gain a left exact sequence of spaces, that we will prove is essentially the same as the sequence we were trying to prove was left exact.

So, consider

$$T^p = \{(x, \gamma) \in T^f \times PX | \gamma(1) = p(x)\} =$$

$$\{(x, \sigma, \gamma) \in X \times PY \times PX | f(x) = \sigma(1), \gamma(1) = x\}$$

The inclusion of the fiber of p into the homotopy fiber of p is given by

$$\gamma \in \Omega Y \mapsto (*, \gamma, e_*) \in T^p \subset X \times PY \times PX$$

By Lemma 3.2.5 this is a homotopy equivalence. As usual the projection map $T^p \to T^f$ is a fibration since it is the pullback of a fibration. Its fiber is ΩX and the inclusion of this fiber into T^p is given by the map

$$\gamma \in \Omega X \mapsto (*, e_*, \gamma) \in T^p$$

Thus we have a commutative diagram:

$$\begin{tikzcd} & \Omega Y \arrow[d, "\simeq"] \arrow[dr, "i"] & & \\ \Omega X \arrow[r] & T^p \arrow[r] & T^f \arrow[r, "p"] & X \end{tikzcd}$$

Where the bottom line is left exact.

Now, define $-\Omega f : \Omega X \to \Omega Y$ by $(-\Omega f)(\gamma) = f \circ \gamma^{-1}$, where γ^{-1} is the inverse path of γ. We claim now is that the following diagram commutes up to homotopy.

$$\begin{tikzcd} & \Omega Y \arrow[d, "\simeq"] \\ \Omega X \arrow[ur, "-\Omega f"] \arrow[r] & T^p \end{tikzcd}$$

The bottom arrow here is the map $\gamma \mapsto (*, e_*, \gamma)$ and the map given by going diagonal and down is $\gamma \mapsto (*, f \circ \gamma^{-1}, e_*)$. We can define a homotopy between these maps by

$$h_t(\gamma) = (\gamma(t), \sigma_t, \tau_t)$$

where

$$\sigma_t(s) = \begin{cases} * & \text{for } s \leq t \\ f \circ \gamma(t + 1 - s) & \text{for } s \geq t \end{cases}$$

and

$$\tau_t(s) = \begin{cases} * & \text{for } s + t \leq 1 \\ \gamma(t + s - 1) & \text{for } s + t \geq 1 \end{cases}$$

We easily check that for all t we have $\sigma_t(1) = f \circ \gamma(t)$ and $\tau_t(1) = \gamma(t)$, such that h_t maps into T^p for all t. Also $h_1(\gamma) = (*, e_*, \gamma)$ and $h_0(\gamma) = (*, f \circ \gamma^{-1}, e_*)$.

Thus the sequence

$$\Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{i} T^f$$

is left exact. Clearly this implies that

$$\Omega X \xrightarrow{\Omega f} \Omega Y \xrightarrow{i} T^f$$

is left exact, which proves the theorem. □

There is an interesting corollary to the theorem:

Corollary 3.2.7. *If $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration, we have an exact sequence of the form:*

$$\cdots \longrightarrow \pi_{n+1}(B) \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*}$$

$$\pi_n(B) \longrightarrow \cdots \longrightarrow \pi_0(E) \xrightarrow{p_*} \pi_0(B)$$

Where the maps involved are homomorphisms when this makes sense, and just maps, when the sets involved have no group structure (there is no group structure on π_0).

Proof. Apply $[S^0, -]$ to the long exact sequence of Theorem 3.2.4 and use the isomorphism $[X, \Omega Y] \simeq [\Sigma X, Y]$, where Σ denotes the reduced suspension. The last thing to notice is that for a fibration the homotopy fiber is homotopy equivalent to the actual fiber as we proved in Lemma 3.2.5. $\square$

The last thing we need to mention in this chapter is a naturality in the construction of the homotopy fiber, that will imply naturality of the classifying spaces once we have proved that these are the homotopy fibers of certain maps.

Proposition 3.2.8. *Suppose we have a commutative diagram of pointed maps*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi_X"] & Y \arrow[d, "\phi_Y"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

Then the map ϕ mapping $(x, \gamma) \in X \times PY$ to $(\phi_X(x), \phi_Y \circ \gamma) \in X' \times PY'$ induces a map between the homotopy fibers T^f and $T^{f'}$.

Proof. The only thing we need to prove is that if $\gamma(1) = f(x)$, then $\phi_Y \circ \gamma(1) = f' \circ \phi_X(x)$, which follows from the commutativity of the diagram. $\square$

For later, we need a lemma which is proved in [Brown70].

Lemma 3.2.9. *Suppose we have a commutative diagram of pointed maps:*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\simeq"] & B \arrow[d, "\simeq"] \\ A' \arrow[r, "h'"] & B' \end{tikzcd}$$

Where the vertical maps are homotopy equivalences. Then the induced map between the homotopy fibers is also a homotopy equivalence.

Chapter 4

The natural classifying space

In this chapter we define the natural classifying space for cohomology with coefficients in a *finite* chain complex, and prove that it is in fact a classifying space for this cohomology theory. The classifying space will be constructed inductively, and the fact that it is a classifying space will be proved by induction on the length of the chain complex. Thus the restriction to finite chains.

The basis will be the functor $K(-,n)$ from abelian groups to topological spaces and the transformation $[-,K(-,n)] \to H^n(-,-)$ on ordinary cohomology as described in section 1.3. Everything in this chapter is theory developed for this thesis.

Remember that Corollary 2.1.5 showed that we need only take care of $H^n(-,B_0 \to \ldots \to B_k)$ in the case of $k \leq n+1$. We need a little more than that, which is provided by the next lemma:

Lemma 4.0.10. *The functors $\tilde{H}^n(-,-)$ and $\tilde{H}^{n+1}(\Sigma-,-)$ are naturally equivalent.*

Proof. If we write ΣX as the union of two cones:

$$\Sigma X = C_+(X) \cup C_-(X)$$

with $C_+(X) \cap C_-(X) = X$ and use the Mayer-Vietoris sequence, we get for every chain complex B:

$$\tilde{H}^n(C_+(X);B) \oplus \tilde{H}^n(C_-(X);B) \longrightarrow \tilde{H}^n(X;B) \longrightarrow$$

$$\tilde{H}^{n+1}(\Sigma X;B) \longrightarrow \tilde{H}^{n+1}(C_+(X);B) \oplus \tilde{H}^{n+1}(C_-(X);B)$$

Since

$$\tilde{H}^*(C_+(X);B) \oplus \tilde{H}^*(C_-(X);B) \simeq 0$$

for all $*$, we have proved that there is a natural equivalence of functors

$$\tilde{H}^n(-;-) \to \tilde{H}^{n+1}(\Sigma-;-)$$

□

Remember that we defined the length of the finite chain complex $B_0 \to \ldots \to B_k$ to be $k+1$.

The lemma shows that if we can prove that the functors $\tilde{H}^n(-,-)$ and $[-,K(-,n)]$ taking coefficient variable in the category of complexes with length at most $n+1$ are equivalent for all n, then we have an equivalence of functors:

$$\tilde{H}^n(-,-) \to \tilde{H}^{n+1}(\Sigma-,-) \to [\Sigma-,K(-,n)] \to [-,\Omega K(-,n)]$$

defined on the category of chain complexes of length at most $n+2$. So in this case we have that $\Omega K(-,n)$ is a natural classifying space for cohomology with coefficients in a chain with length at most $n+2$.

This argument shows, that it is sufficient to consider the case $n \geq k$.

4.1 The functor K(-,n)

The aim of this section is to expand the functor $K(-,n) : Ab \to Top$ to the category of finite chain complexes. We will begin with the case of a bicomplex $A \xrightarrow{h} B$. Consider the long exact sequence for cohomology with coefficients in h:

$$\cdots \longrightarrow \tilde{H}^{n-1}(X,A) \xrightarrow{h_*} \tilde{H}^{n-1}(X,B) \longrightarrow \tilde{H}^n(X, A \xrightarrow{h} B) \longrightarrow$$

$$\tilde{H}^n(X,A) \xrightarrow{h_*} \tilde{H}^n(X,B) \longrightarrow \cdots$$

Suppose $K(A \xrightarrow{h} B, n)$ is a classifying space for $H^n(-, A \xrightarrow{h} B)$. Expressing this sequence in terms of classifying spaces we get:

$$\cdots \longrightarrow [X, \Omega K(A,n)] \xrightarrow{\Omega K(B,n)} [X, \Omega K(B,n)] \longrightarrow [X, K(A \xrightarrow{h} B, n)] \longrightarrow \cdots$$

$$[X, K(A,n)] \xrightarrow{K(h,n)_*} [X, K(B,n)] \longrightarrow \cdots$$

Here, all the maps involved are natural in X. We know the first and the last map here from Proposition 1.3.7. This next lemma is very much in the spirit of lemma 1.3.6

Lemma 4.1.1. *Let $T : [-,X] \to [-,Y]$ be a natural transformation. Then T is of the form f_* for some map $f : X \to Y$.*

Proof. Let $[f] = T([id])$, and let $g : A \to Y$ be any map. Then

$$T([g]) = T(g^*[id]) = g^*T([id]) = g^*[f] = [fg] = f_*[g]$$

□

Thus the long exact sequence above corresponds to a right exact sequence of spaces:

$$\cdots \longrightarrow \Omega K(A,n) \xrightarrow{\Omega K(h,n)} \Omega K(B,n) \longrightarrow K(A \xrightarrow{h} B, n) \longrightarrow$$

$$K(A,n) \xrightarrow{K(h,n)} K(B,n) \longrightarrow \cdots$$

The key observation here is that this resembles the Puppe sequence for the map $K(h,n) : K(A,n) \to K(B,n)$. Our guess will be that this is in fact the Puppe-sequence, such that

$K(A \xrightarrow{h} B, n)$ is the homotopy fiber for $K(h, n)$. We will for now define $K(A \xrightarrow{h} B, n)$ to be the homotopy fiber, and later on we will prove that this is in fact a classifying space for $H^n(-, A \xrightarrow{h} B)$.

The next thing to prove is that this defines a functor from the category of bicomplexes to the category of topological spaces. The morphisms in the category of bicomplexes of abelian groups are pairs of homomorphisms $\phi = (\phi_0, \phi_1)$ that make the diagram below commute:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "\phi_0"] & B \arrow[d, "\phi_1"] \\ A' \arrow[r, "h'"] & B' \end{tikzcd}$$

We need this pair to define a map $K(\phi, n) : K(A \xrightarrow{h} B, n) \to K(A' \xrightarrow{h'} B', n)$. Since $K(-, n)$ is a functor defined on the category of abelian groups, the commutative diagram above induces the commutative diagram:

$$\begin{tikzcd} K(A, n) \arrow[r, "K(h,n)"] \arrow[d, "K(\phi_0,n)"'] & K(B, n) \arrow[d, "K(\phi_1,n)"] \\ K(A', n) \arrow[r, "K(h',n)"] & K(B', n) \end{tikzcd}$$

By proposition 3.2.8 $(K(\phi_0, n), K(\phi_1, n))$ induces a map

$$K(\phi, n) : K(A \xrightarrow{h} B, n) \to K(A' \xrightarrow{h'} B', n)$$

as desired. Clearly, if ϕ is the identity, then $K(\phi, n)$ is the identity. It is also clear that $K(\psi \circ \phi, n) = K(\psi, n) \circ K(\phi, n)$. Thus $K(-, n)$ is a functor from the category of bicomplexes to the category of topological spaces.

We note two important obvious consequences of this definition: $K(A \longrightarrow 0, n) = K(A, n)$ and $K(0 \longrightarrow B, n) = \Omega K(B, n)$.

Let us define $K(B_0 \overset{h_0}{\to} B_1 \overset{h_0}{\to} B_2, n)$. Consider the commutative diagram:

$$\begin{tikzcd} B_0 \arrow[r, "h_0"] \arrow[d] & B_1 \arrow[d, "h_1"] \\ 0 \arrow[r] & B_2 \end{tikzcd}$$

Taking $K(-, n)$ on this diagram we get:

$$\begin{tikzcd} K(B_0, n) \arrow[r, "K(h_0,n)"] \arrow[d] & K(B_1, n) \arrow[d, "K(h_1,n)"] \\ \star \arrow[r] & K(B_2, n) \end{tikzcd}$$

The pair $(0, K(h_1, n))$ induces a map

$$K(h_1, n) : K(B_0 \to B_1, n) \to K(0 \to B_2, n) = \Omega K(B_2, n)$$

We define $K(B_0 \to B_1 \to B_2, n)$ to be the homotopy fiber of $K(h_1, n)$. There is an abuse of notation here since $K(h_1, n)$ might be one of two maps, either $K(B_1, n) \to K(B_2, n)$ or

$K(B_0 \to B_1, n) \to \Omega K(B_2, n)$. In general however, it will be clear from context, what is meant.

To describe the functoriality of this definition, suppose we have a set of maps $\phi = (\phi_0, \phi_1, \phi_2)$ such that the following diagram commutes:

$$\begin{tikzcd}
B_0 \arrow[r, "h_0"] \arrow[d, "\phi_0"] & B_1 \arrow[r, "h_1"] \arrow[d, "\phi_1"] & B_2 \arrow[d, "\phi_2"] \\
B_0' \arrow[r, "h_0'"] & B_1' \arrow[r, "h_1'"] & B_2'
\end{tikzcd}$$

By functoriality in the case of complexes of length two we get a commutative diagram:

$$\begin{tikzcd}
K(B_0 \to B_1, n) \arrow[r, "{K(h_1,n)}"] \arrow[d, "{K((\phi_0,\phi_1),n)}"'] & \Omega K(B_2, n) \arrow[d, "{\Omega K(\phi_2,n)}"] \\
K(B_0' \to B_1', n) \arrow[r, "{K(h_1',n)}"] & \Omega K(B_2', n)
\end{tikzcd}$$

Such that we get an induced map $K(\phi, n) : K(B_0 \to B_1 \to B_2, n) \to K(B_0' \to B_1' \to B_2', n)$ on the homotopy fibers. It is clear that this induces a functor on the category of chain complexes of length at most three.

Notice what we get in the case $B_0 = 0$. In this case $K(0 \to B_1 \to B_2, n)$ becomes the homotopy fiber of the map

$$K(h_1, n) : K(0 \to B_1, n) \to \Omega K(B_2, n)$$

which is really just the map:

$$\Omega K(h_1, n) : \Omega K(B_1, n) \to K(B_2, n)$$

and by Lemma 3.2.6 we get the homeomorphism:

$$K(0 \to B_1 \to B_2, n) = \Omega K(B_1 \to B_2, n)$$

In general, we define $K(B_0 \to \ldots \to B_{k+1}, n)$ from $K(B_0 \to \ldots \to B_k, n)$ the following way. From functoriality in the case of k complexes, using the following diagram:

$$\begin{tikzcd}
B_0 \arrow[r, "h_0"] \arrow[d] & \cdots \arrow[r] & B_{k-1} \arrow[r, "h_{k-1}"] \arrow[d] & B_k \arrow[d, "h_k"] \\
0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & B_{k+1}
\end{tikzcd}$$

we obtain a commutative diagram:

$$\begin{tikzcd}
K(B_0 \to \ldots \to B_{k-1}, n) \arrow[r, "{K(h_{k-1},n)}"] \arrow[d] & K(\underbrace{0 \to \ldots \to 0}_{k-1} \to B_k, n) \arrow[d, "{K(h_k,n)}"] \\
\star \arrow[r] & K(\underbrace{0 \to \ldots \to 0}_{k-1} \to B_{k+1}, n)
\end{tikzcd}$$

This induces a map

$$K(h_k, n) : K(B_0 \to \ldots \to B_k, n) \to \Omega K(\underbrace{0 \to \ldots \to 0}_{k-1} \to B_{k+1}, n)$$

We define $K(B_0 \to \ldots \to B_{k+1}, n)$ to be the homotopy fiber of this map. As in the case above we get functoriality by induction on the length of the complex.

We need one important lemma:

Lemma 4.1.2. *$K(0 \to B_1 \to \ldots \to B_k, n)$ is homeomorphic to $\Omega K(B_1 \to \ldots \to B_k, n)$. The homeomorphism is natural in the finite complex.*

Proof. The proof is by induction on k. The cases $k = 1, 2$ have been proved in the text above. $K(0 \to B_1 \to \ldots \to B_k, n)$ is by definition and using the induction hypothesis the homotopy fiber of the map

$$K(h_{k-1}, n) : K(0 \to B_1 \to \ldots \to B_{k-1}, n) \to \Omega^k K(B_k, n)$$

Using the induction hypothesis and the identity of Lemma 3.2.6 we see that this map is in fact the map

$$\Omega K(h_{k-1}, n) : \Omega K(B_1 \to \ldots \to B_{k-1}, n) \to \Omega^k K(B_k, n)$$

Using Lemma 3.2.6 again we get the desired result. Naturality is clear. $\square$

In the case of $k > n+1$ Corollary 2.1.5 shows that

$$\tilde{H}^n(X, B_0 \to \ldots \to B_k) \simeq \tilde{H}^n(X, B_0 \to \ldots \to B_{n+1})$$

An equation like that should be reflected in the classifying space, and the next lemma shows that this is the case.

Lemma 4.1.3. *Suppose $k > n+1$. Then there exists a natural homotopy equivalence:*

$$K(B_0 \to \ldots \to B_k, n) \to K(B_0 \to \ldots \to B_{n+1}, n)$$

Naturality means that it commutes with chain maps.

Proof. It suffices to show that there exists a natural homotopy equivalence:

$$K(B_0 \to \ldots \to B_k, n) \to K(B_0 \to \ldots \to B_{k-1}, n)$$

If $k > n+1$ then $\Omega^{k-1} K(B_k, n)$ is null homotopic. So we can construct a commutative diagram:

$$\begin{tikzcd} K(B_0 \to \ldots \to B_{k-1}, n) \arrow[r] \arrow[d, equal] & \Omega^{k-1} K(B_k, n) \arrow[d, "\simeq"] \\ K(B_0 \to \ldots \to B_{k-1}, n) \arrow[r] & \star \end{tikzcd}$$

Since both vertical maps here are homotopy equivalences, this map induces a homotopy equivalence between the fibers (Lemma 3.2.9) which is the homotopy equivalence we were looking for. Naturality is clear. $\square$

To motivate this construction consider the long exact sequence of cohomology groups:

$$\cdots \longrightarrow \tilde{H}^{n-k}(X, B_k) \longrightarrow \tilde{H}^n(X, B_0 \to \ldots \to B_k) \longrightarrow$$

$$\tilde{H}^n(X, B_0 \to \ldots \to B_{k-1}) \xrightarrow{(h_{k-1})^*} \tilde{H}^n(X, B_k) \longrightarrow \cdots$$

As above, this defines a right exact sequence of chain complexes:

$$\cdots \longrightarrow \Omega^k K(B_k, n) \longrightarrow K(B_0 \to \ldots \to B_k, n) \longrightarrow$$

$$K(B_0 \to \ldots \to B_{k-1}, n) \longrightarrow K(B_k, n) \longrightarrow \cdots$$

Our guess is that this is in fact the Puppe sequence, and that the map from $K(B_0 \to \ldots \to B_{k-1}, n)$ to $K(B_k, n)$ is $K(h_{k-1}, n)$.

To prove that $K(A \xrightarrow{h} B, n)$ is a classifying space the idea will be to construct a natural transformation $T : [-, K(-, n)] \to \tilde{H}^n(-, -)$ such that this diagram commutes for all spaces and morphisms:

$$\begin{tikzcd}
{[X, \Omega K(B, n)]} \arrow[r] \arrow[d, "T_B"] & {[X, K(A \xrightarrow{h} B, n)]} \arrow[r] \arrow[d, "T"] & {[X, K(A, n)]} \arrow[d, "T_A"] \\
H^{n-1}(X, B) \arrow[r] & H^n(X, A \xrightarrow{h} B) \arrow[r] & H^n(X, A)
\end{tikzcd} \tag{4.1}$$

The five lemma will then imply that T is an isomorphism. Of course, for this we need $[X, K(A \xrightarrow{h} B, n)]$ to be a group, and the maps involved to be morphisms. The question of group structure will be postponed a while. For now, we shall concentrate on defining a natural transformation, that makes the diagram commute.

In the general case of finite complexes, the proof will be on induction on the length of the complex. The argument will be the argument sketched above, where the long exact sequence of the homomorphism, will be replaced be the sequence:

$$\cdots \longrightarrow \tilde{H}^{n-k}(X, B_k) \longrightarrow \tilde{H}^n(X, B_0 \to \ldots \to B_k) \longrightarrow$$

$$\tilde{H}^n(X, B_0 \to \ldots \to B_{k-1}) \xrightarrow{(h_{k-1})^*} \tilde{H}^n(X, B_k) \longrightarrow$$

This argument of course generalizes the argument for homomorphisms and will therefore be the only one presented.

4.2 A CW-structure on $K(B_0 \to \ldots \to B_k, n)$

Lemma 1.3.6 tells us that a natural transformation

$$T : [-, K(B_0 \to \ldots \to B_k, n)] \to \tilde{H}^n(-, B_0 \to \ldots \to B_k)$$

must be on the form $[f] \mapsto f^*(\xi)$, with

$$\xi \in \tilde{H}^n(K(B_0 \to \ldots \to B_k, n), B_0 \to \ldots \to B_k)$$

To construct the transformation, we must define the element ξ. In the construction of this element we shall use cellular cohomology. This means that we shall need a CW-structure on $K(B_0 \to \ldots \to B_k, n)$. Such a structure exists up to homotopy by the next lemma:

Lemma 4.2.1. *The space $K(B_0 \to \ldots \to B_k, n)$ is homotopy equivalent to a CW-complex if $n \geq k$.*

Remark 4.2.2. Remember that Corollary 2.1.5 and Lemma 4.0.10 showed that we could restrict our attention to the case $n \geq k$.

Proof of Lemma 4.2.1. We prove this by induction on k. The lemma is clearly true in the case $k = 0$. For the inductions step we need to prove that $K(B_0 \to \ldots \to B_k, n)$ is homotopy equivalent to a CW-complex, under the assumption that $K(B_0 \to \ldots \to B_{k-1}, n)$ is homotopy equivalent to a CW-complex. Consider the diagram:

$$\begin{tikzcd} K(B_0 \to \ldots \to B_{k-1}, n) \arrow[r, "K(h_{k-1},n)"] \arrow[d, "\phi_0", "\simeq"'] & \Omega^k K(B_k, n) \arrow[d, "\phi_1", "\simeq"'] \\ X \arrow[r, "\phi_1 K(h_{k-1},n)\phi_0^{-1}"] & K(B_k, n-k) \end{tikzcd}$$

Here X is a CW-complex, that is homotopy equivalent to $K(B_0 \to \ldots \to B_{k-1}, n)$ and the vertical maps are homotopy equivalences. ϕ_0^{-1} is a homotopy inverse to ϕ_0 so the diagram commutes up to homotopy.

The homotopy fiber of the map in the top row is $K(B_0 \to \ldots \to B_k, n)$ and the homotopy fiber in the bottom row is homotopy equivalent to a CW-complex by a famous theorem of Milnor [Milnor]. So the aim is to prove that these two homotopy fibers are homotopy equivalent.

Consider the diagram:

$$\begin{tikzcd} K(B_0 \to \ldots \to B_{k-1}, n) \arrow[r, "K(h_{k-1},n)"] \arrow[d, equal] & \Omega^k K(B_k, n) \arrow[d, "\phi_1"] \\ K(B_0 \to \ldots \to B_{k-1}, n) \arrow[r, "\phi_1 \circ K(h_{k-1},n)"] & K(B_k, n-k) \end{tikzcd}$$

Since this diagram commutes, by lemma 3.2.9 it induces a homotopy equivalence between the homotopy fibers of the two horizontal maps in the diagram.

Likewise consider the diagram:

$$\begin{tikzcd} K(B_0 \to \ldots \to B_{k-1}, n) \arrow[r, "\phi_1 \circ K(h_{k-1},n)\phi_0^{-1}\phi_0"] \arrow[d, "\phi_0", "\simeq"'] & K(B_k, n-k) \arrow[d, equal] \\ X \arrow[r, "\phi_1 \circ K(h_{k-1},n)\phi_0^{-1}"] & K(B_k, n-k) \end{tikzcd}$$

Since this diagram commutes, by Lemma 3.2.9 it induces a homotopy equivalence between the homotopy fibers.

Since the maps $\phi_1 K(h_{k-1}, n)\phi_0^{-1}\phi_0$ and $\phi_1 K(h_{k-1}, n)$ are homotopy equivalent, by Proposition 3.1.6 the corresponding homotopy fibers are homotopy equivalent. This proves the lemma. □

For our purpose however, we need a special CW-structure on $K(B_0 \to \ldots \to B_k, n)$. To get this, we need a couple of theorems from homotopy theory.

Theorem 4.2.3 (Cellular Approximation Theorem). *Suppose $f : (X, A) \to (Y, B)$ is a map between CW-complexes such that $f|_A$ is cellular. Then there exists a cellular map f' homotopic to f through a homotopy that is stationary on A.*

Proof. This is Theorem 4.8 in [Hatcher]. □

Theorem 4.2.4. *Suppose (X, A) is a pair of spaces, such that A is a non-empty CW-complex. Then there exists a CW-complex Y with A as a subcomplex, and a weak homotopy equivalence $f : X \to Y$ that is the identity on A. If (X, A) is n-connected, that is if $\pi_k(X, A) = 0$ for all $k \leq n$ then we may assume that $Y^{(n)} \subseteq A$. If X is homotopy equivalent to a CW-complex, then f is a homotopy equivalence.*

Proof. Everything except the last line is Prop. 4.13 in [Hatcher]. The last line follows from Whiteheads theorem. □

Theorem 4.2.5. *Suppose (X, A) is a pair of spaces and $\gamma : A \to A'$ is a weak homotopy equivalence with A' a CW-complex, then there exists a CW-complex X' with A' as a subcomplex, and a weak homotopy equivalence $\sigma : X \to X'$ such that $\sigma|_A = \gamma$. If γ is a homotopy equivalence, and X is homotopy equivalent to a CW-complex, then σ is a homotopy equivalence.*

Proof. This is a theorem on page 76 in [May98]. The last part is Whiteheads theorem. □

From the construction we have a natural inclusion:

$$i : \Omega^k K(B_k, n) \to K(B_0 \to \ldots \to B_k, n)$$

given by $i(\gamma) = (*, \gamma)$.

We would like $\Omega^k K(B_k, n)$ to be a subcomplex of $K(B_0 \to \ldots \to B_k, n)$, and we would like $K(B_0 \to \ldots \to B_k, n)$ to be constructed from $\Omega^k K(B_k, n)$ by attaching cells of dimension greater than $n - k$. For this we need the theorems above and the next two lemmas.

Lemma 4.2.6. *$K(B_0 \to \ldots \to B_k, n)$ is $n - k - 1$ connected.*

Proof. This is done by induction on k. For $k = 0$ we need to prove that $\pi_l(K(B_0, n)) = 0$ for $l \leq n - 1$, which is clearly true. For the induction step, consider the fibration

$$K(B_0 \to \ldots \to B_k, n) \longrightarrow K(B_0 \to \ldots \to B_{k-1}, n) \overset{K(h_{k-1},n)}{\longrightarrow} \Omega^{k-1} K(B_k, n)$$

and the following part of the long exact sequence of homotopy groups associated to it:

$$\pi_{l+1}(\Omega^{k-1} K(B_k, n)) \longrightarrow \pi_l(K(B_0 \to \ldots \to B_k, n)) \longrightarrow \pi_l(K(B_0 \to \ldots \to B_{k-1}, n))$$

Induction tells us that $\pi_l(K(B_0 \to \ldots \to B_{k-1}, n))$ is zero for $l \leq n-k$, and $\pi_{l+1}(\Omega^{k-1} K(B_k, n))$ is zero for $l + 1 + k - 1 < n$ that is for $l \leq n - k - 1$. Thus, for $l \leq n - k - 1$ we have $\pi_l(K(B_0 \to \ldots \to B_k, n)) = 0$ as desired. □

Lemma 4.2.7. *The pair $(K(B_0 \to \ldots \to B_k, n), \Omega^k K(B_k, n))$ is $(n - k)$-connected.*

Proof. We need to prove that the map:

$$i_* : \pi_l(\Omega^k K(B_k, n)) \to \pi_l(K(B_0 \to \ldots \to B_k, n))$$

is an isomorphism for $l < n - k$ and an epimorphism for $l = n - k$. For $l < n - k$ both groups are zero, so the map is clearly an isomorphism.

For $l = n - k$ consider the following part of the long exact sequence:

$$\pi_{n-k}(\Omega^k K(B_k, n)) \xrightarrow{i_*} \pi_{n-k}(K(B_0 \to \ldots \to B_k, n)) \longrightarrow \pi_{n-k}(K(B_0 \to \ldots \to B_{k-1}, n))$$

Lemma 4.2.6 tells us that $(K(B_0 \to \ldots \to B_{k-1}, n))$ is $(n - k)$-connected, so i_* is an epimorphism as desired. □

Proposition 4.2.8. *There exists a CW-complex X with $K(B_k, n - k)$ as a subcomplex, and a homotopy equivalence*

$$f : (K(B_0 \to \ldots \to B_k, n), \Omega^k K(B_k, n)) \to (X, K(B_k, n - k))$$

of pairs, such that $X^{(n-k)} \subseteq K(B_k, n - k)$.

Proof. Since all spaces involved are homotopic to CW-complexes all weak homotopy equivalences are automatically homotopy equivalences.

Since we know that $\Omega^k K(B_k, n)$ and $K(B_k, n-k)$ are homotopy equivalent, Theorem 4.2.5 tells us that there exists a CW-complex Y such that $(Y, K(B_k, n-k))$ and $(K(B_0 \to \ldots \to B_k, n), \Omega^k K(B_k, n))$ are homotopy equivalent. Now, Theorem 4.2.4 and Lemma 4.2.7 tells us that $(Y, K(B_k, n - k))$ is homotopy equivalent to another CW-pair $(X, K(B_k, n - k))$ with $X^{(n-k)} \subseteq K(B_k, n - k)$ as desired. □

Since $K(B_0 \to \ldots \to B_k, n)$ is the homotopy fiber of the map:

$$K(h_{k-1}, n) : K(B_0 \to \ldots \to B_{k-1}, n) \to \Omega^{k-1} K(B_k, n)$$

we have a commutative diagram of maps:

$$\begin{tikzcd}
\Omega^k K(B_k, n) \arrow[r, "i"] & K(B_0 \to \ldots \to B_k, n) \arrow[r, "p_k"] \arrow[d, "q_k"] & K(B_0 \to \ldots \to B_{k-1}) \arrow[d, "K(h_{k-1}, n)"] \\
& P\Omega^{k-1} K(B_k, n) \arrow[r, "\eta"] & \Omega^{k-1} K(B_k, n)
\end{tikzcd}$$

where p and q are the projections, and η is the map $\gamma \mapsto \gamma(1)$. Notice that $p_k i$ is the constant map to $*$.

If we replace the spaces in the upper line of this diagram with CW approximations as given in Proposition 4.2.8 we get a diagram that only commutes up to homotopy, but still has $p_k i = *$. According to The Cellular Approximation Theorem (Thm 4.2.3), we may also replace p_k with a cellular map that is homotopic to p_k through a homotopy that maps $\Omega^k K(B_k, n)$ to $*$ at all times. Having done this replacement, we know that $K(h_{k-1}, n) \circ p_k$ is homotopic to $\eta \circ q$ through a homotopy, that maps $\Omega^k K(B_k, n)$ to $*$ at all times.

4.3 The natural transformation

Corollary 2.1.5 and Lemma 4.0.10 shows that it is sufficient to define T in the case $n \geq k$, so in the following we shall always assume that $n \geq k$.

To define the natural transformation

$$T : [-, K(B_0 \to \ldots \to B_k, n)] \to \tilde{H}^n(-, B_0 \to \ldots \to B_k)$$

we must choose an element

$$\xi \in H^n(K(B_0 \to \ldots \to B_k, n), B_0 \to \ldots \to B_k)$$

such that $T([f]) = f^*(\xi)$. An element in $\tilde{H}^n(X, B_0 \to \ldots \to B_k)$ is represented by a set of maps $(\xi_0, \ldots, \xi_k)$ such that the following diagram commutes up to sign:

$$\begin{tikzcd}
C_{n+1} \arrow[r, "\partial"] \arrow[d] & C_n(X) \arrow[r, "\partial"] \arrow[d, "\xi_0"] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d, "\xi_1"] & \cdots \arrow[r] & C_{n-k}(X) \arrow[d, "\xi_k"] \\
0 \arrow[r] & B_0 \arrow[r, "h_0"] & B_1 \arrow[r] & \cdots \arrow[r] & B_k
\end{tikzcd}$$

To be more precise we must have $\xi_0 \circ \partial = 0$ and $\xi_l \circ \partial = (-1)^n h_{l-1}\xi_{l-1}$. The diagram is meant as a way to think of elements in $\tilde{H}^n(X, B_0 \to \ldots \to B_k)$ and the equations presented are exactly the equations, that tell us that $(\xi_0, \ldots, \xi_k)$ is a cycle.

In the case $k = 0$ we already have a definition that works. As described in section 1.3, the map $\xi_0 : C_n(K(B, n)) \to B$ is the map that to an n-cell σ associates $\sigma \in \pi_n(K(B, n)) \simeq B$, using the identification $\pi_n(K(B, n)) \simeq B$ given by the natural transformation of $\pi_n K(-, n)$ to the identity functor. This works, since the $(n-1)$-skeleton of $K(B, n)$ is a point. Notice that this works for all $n \geq 0$. The reason that $\pi_0(K(B, 0))$ is a group is that $K(B, n)$ is a group-like H-space.

In the case $k = 1$ we must define (ξ_0, ξ_1). Let us first define the map:

$$\xi_0 : C_n(K(B_0 \xrightarrow{h_0} B_1, n)) \to B_0$$

Let $\xi_0' : C_n(K(B_0, n)) \to B_0$ be defined as above. Since the map

$$p_1 : K(B_0 \xrightarrow{h_0} B_1, n) \to K(B_0, n)$$

is cellular, we can define ξ_0 to be $p_0^*(\xi_0')$. Secondly, we must define the map:

$$\xi_1 : C_{n-1}(K(B_0 \xrightarrow{h_0} B_1), n) \to B_1$$

Suppose $\sigma : D^{n-1} \to K(B_0 \xrightarrow{h_0} B_1, n)$ is the attaching map of an $(n-1)$-cell. Since the $(n-1)$-skeleton of $K(B_0 \xrightarrow{h_0} B_1, n)$ is contained in $\Omega K(B_1, n)$ whose $(n-2)$-skeleton is a point, σ defines a map $\sigma : S^{n-1} \to \Omega K(B_1, n)$. This map defines an element $\xi_1(\sigma) \in \pi_{n-1}(\Omega K(B_1, n)) \simeq B_1$. For the signs to work out right however, we will consider the pair $(\xi_0, (-1)^n \xi_1)$, with ξ_0 and ξ_1 defined as above.

Rename the two maps defined above to (ξ_0', ξ_1'). In the case $k = 2$ we aim to define (ξ_0, ξ_1, ξ_2). Since the map:

$$p_2 : K(B_0 \to B_1 \to B_2, n) \to K(B_0 \to B_1, n)$$

is cellular, we set $(\xi_0, \xi_1) := (p_2^*(\xi_0'), (-1)^n p_2^*(\xi_1'))$. If $\sigma : D^{n-2} \to K(B_0 \to B_1 \to B_2, n)$ is the attaching map of an $(n-2)$-cell, then since

$$K(B_0 \to B_1 \to B_2, n)^{(n-2)} \subseteq \Omega^2 K(B_2, n)$$

σ defines an element in $\pi_{n-2}(\Omega^2 K(B_2, n)) \simeq B_2$, and we define $\xi_2(\sigma)$ to be this element.

In general, we define inductively $(\xi_0, \dots, \xi_k)$ from the set $(\xi_0', \dots, \xi_{k-1}')$ defined in the case $k-1$, to be

$$(p_k^*(\xi_0'), \dots, (-1)^{(k-1)n} p_k^*(\xi_{k-1}'), (-1)^{kn} \xi_k)$$

where ξ_k is defined to be the map that to an $(n-k)$-cell associates, the attaching map considered as an element in $\pi_{n-k}(\Omega^k K(B_k, n))$.

Another more direct way to express the map:

$$\xi_l : C_{n-l}(K(B_0 \to \dots \to B_k, n)) \to B_l$$

is the following: Let σ be an $(n-l)$-cell of $K(B_0 \to \dots \to B_k, n)$ represented by its attaching map. Then the map:

$$p_{l+1} \dots p_k \circ \sigma : D^{n-l} \to K(B_0 \to \dots \to B_l, n)$$

is a cellular map, and therefore has image contained in $\Omega^l K(B_l, n)$. Since the $(n-l)$-skeleton of $\Omega^l K(B_l, n)$ is a point (the CW-structure on it is given by $K(B_l, n-l)$), this map represents an element in $\pi_{n-l}(\Omega^l K(B_l, n))$, that we call $\xi_l(\sigma)$.

Notice that in the case $k = 0$ the transformation T reduces to the transformation known from ordinary cohomology. Even more, since the CW-approximation used for

$$K(\underbrace{0 \to \dots \to 0}_{l} \to B, n) = \Omega^l K(B, n)$$

is in fact $K(B, n-l)$ in this case:

$$T : [-, \Omega^l K(B, n)] \to H^n(-, \underbrace{0 \to \dots \to 0}_{l} \to B) = H^{n-l}(-, B)$$

also reduces to the case known from ordinary cohomology. Thus, we have:

Proposition 4.3.1. *For all groups B and all CW-complexes X and for $n \geq l$ the map:*

$$T : [X, \Omega^l K(B, n)] \to \tilde{H}^{n-l}(X, B)$$

is an isomorphism.

In the following we will prove a couple of things about the definition of T. First of all we need to prove that $\xi = (\xi_0, \dots, \xi_k)$ is a cycle, so that T will be welldefined. This is done by proving that it satisfies the equations from the beginning of this section.

The second thing we need to prove is that we in this way have defined a transformation, that is natural in the coefficient variable. This is the requirement that will prove that $K(B_0 \to \dots \to B_k, n)$ is a *natural* classifying space.

The last thing we need to prove is that the transformation T makes the diagram 4.1 commute.

So lets get started:

Proposition 4.3.2. *The set* $(\xi_0, \ldots, \xi_k)$ *as defined above is a cycle.*

Proof. The proof will be by induction on k. In the case $k = 0$ we need to prove that $\xi_0 \partial = 0$, which is true, since if the map $\sigma : S^n \to K(B, n)$ can be extended to D^{n+1} it represents the zero element in $\pi_n(K(B, n))$.

For the induction step, suppose

$$(\xi'_0, (-1)^n \xi'_1, \ldots, (-1)^{(k-1)n} \xi'_{k-1})$$

is a cycle, where

$$\xi'_l : C_{n-l}(K(B_0 \to \ldots \to B_{k-1}, n)) \to B_l$$

That is, it satisfies the equations:

$$\xi'_0 \partial = 0$$

and

$$\xi'_l \partial = h_{l-1} \xi'_{l-1}$$

for all l. Notice here, that the signs $(-1)^n$ cancel. We need to prove that the element

$$(p_k^*(\xi'_0), (-1)^n p_k^*(\xi'_1), \ldots, (-1)^{(k-1)n} p_k^*(\xi'_{k-1}), (-1)^{nk} \xi_k)$$

satisfies the same equations. Most of the equations follow trivially from the induction hypothesis. The only thing left to prove is:

$$\xi_k \partial = h_{k-1} p_k^*(\xi'_{k-1})$$

Suppose σ is an $(n-k+1)$-cycle of $K(B_0 \to \ldots \to B_k, n)$ represented by its characteristic map $\sigma : D^{n-k+1} \to K(B_0 \to \ldots \to B_k, n)$. $p_k^*(\xi'_{k-1})(\sigma)$ is by definition the element

$$p_k \circ \sigma \in \pi_{n-k+1}(\Omega^{k-1} K(B_{k-1}, n)) \simeq B_{k-1}$$

Since the identification $\pi_{n-k+1}(\Omega^{k-1} K(B_{k-1}, n)) \simeq B_{k-1}$ comes from the natural transformation from $\pi_{n-k+1}(\Omega^{k-1} K(-, n))$ to the identity functor, $h_{k-1} \circ p_k^*(\xi'_{k-1})(\sigma)$ is the element

$$\Omega^{k-1} K(h_{k-1}, n) \circ p_k \circ \sigma \in \pi_{n-k+1}(\Omega^{k-1} K(B_k, n)) \simeq B_k$$

It is clear from definitions that the map:

$$K(h_{k-1}, n) : K(B_0 \to \ldots \to B_{k-1}, n) \to \Omega^{k-1} K(B_k, n)$$

restricted to $\Omega^{k-1} K(B_{k-1}, n)$ is $\Omega^{k-1} K(h_{k-1}, n)$, thus $h_{k-1} \circ p_k^*(\xi'_{k-1})(\sigma)$ is the element:

$$K(h_{k-1}, n) \circ p_k \circ \sigma \in \pi_{n-k+1}(\Omega^{k-1} K(B_k, n)) \simeq B_k$$

At this point, we would like to remind the reader of this diagram:

$$\begin{tikzcd}
\Omega^k K(B_k, n) \arrow[r, "i"] & K(B_0 \to \ldots \to B_k, n) \arrow[r, "p_k"] \arrow[d, "q_k"] & K(B_0 \to \ldots \to B_{k-1}) \arrow[d, "K(h_{k-1}, n)"] \\
& P\Omega^{k-1} K(B_k, n) \arrow[r, "\eta"] & \Omega^{k-1} K(B_k, n)
\end{tikzcd}$$

which commuted up to a homotopy, that was constant $*$ on

$$\Omega^k(K(B_k, n)) = K(B_0 \to \ldots \to B_k, n)^{(n-k)}$$

This means that $h_{k-1} \circ p_k^*(\xi'_{k-1})(\sigma)$ is also represented by the map:

$$\eta \circ q_k \circ \sigma \in \pi_{n-k+1}(\Omega^{k-1}K(B_k, n)) \simeq B_k$$

On the other side of the equation we have $\xi_k \partial\sigma$ which is the element:

$$\partial\sigma \in \pi_{n-k}(\Omega^k K(B_k, n)) \simeq B_k$$

represented by the element $\partial\sigma = \sigma|_{S^{n-k}} : S^{n-k} \to \Omega^k K(B_k, n)$. Using the natural transformation:

$$S^{-1} : \pi_{n-k}\Omega \to \pi_{n-k+1}$$

we get that the same element is represented by:

$$S^{-1}(\partial\sigma) \in \pi_{n-k+1}(\Omega^{k-1}K(B_k, n))$$

Now, all we need to complete the proof is a homotopy from $S^{-1}(\partial\sigma)$ to $\eta \circ q_k \circ \sigma$ inside $\Omega^k K(B_k, n)$. Let

$$F : D^{n-k+1} \times I / D^{n-k+1} \times \{0\} \to \Omega^{k-1}K(B_k, n)$$

be defined as follows: $F(x, t) : (q_k \circ \sigma(x))(t)$. Now,

$$F|_{D^{n-k+1} \times \{1\}} = \eta \circ q_k \circ \sigma$$

and

$$F|_{S^{n-k} \times I} = S^{-1}(\partial\sigma)$$

So F provides the desired homotopy. $\square$

Thus if $\xi = (\xi_0, \ldots, \xi_k)$ then we can define a natural transformation

$$T : [-, K(B_0 \to \ldots \to B_k, n)] \to \tilde{H}^n(-, B_0 \to \ldots \to B_k)$$

by setting $T([f]) = f^*(\xi)$. Since our goal was to define a natural classifying space, the next step is to prove that this transformation is natural in the coefficient variable. Let us first make it clear what is meant by that.

A morphism in $\phi : (B_0 \to \ldots \to B_k) \longrightarrow (B'_0 \to \ldots \to B'_k)$ in the category of finite chains, is a set of maps $(\phi_0, \ldots, \phi_k)$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
B_0 & \xrightarrow{h_0} & B_1 & \xrightarrow{h_1} & \cdots & \longrightarrow & B_k \\
\downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & & & \downarrow{\scriptstyle \phi_k} \\
B'_0 & \xrightarrow{h'_0} & B'_1 & \xrightarrow{h'_1} & \cdots & \longrightarrow & B'_k
\end{array}$$

Since $K(-, n)$ is a functor, ϕ defines a map

$$K(\phi, n) : K(B_0 \to \ldots \to B_k, n) \to K(B'_0 \to \ldots \to B'_k, n)$$

We need to prove that the following diagram commutes for all X:

$$\begin{tikzcd}
{[X, K(B_0 \to \ldots \to B_k, n)]} \arrow[r, "K(\phi,n)_*"] \arrow[d, "T"] & {[X, K(B'_0 \to \ldots \to B'_k, n)]} \arrow[d, "T'"] \\
\tilde{H}^n(X, B_0 \to \ldots \to B_k) \arrow[r, "\phi_*"] & \tilde{H}^n(X, B'_0 \to \ldots \to B'_k)
\end{tikzcd}$$

Since all maps in this diagram are natural transformations, by Lemma 1.3.6 it suffices to prove that:

$$\phi_* T([id]) = TK(\phi, n)_*([id])$$

Proposition 4.3.3. *The transformation T is natural in the coefficient variable.*

Proof. The proof will be by induction on k. The case $k = 0$ is Theorem 1.3.7. In this case we actually proved that the diagram commuted in a much more precise way. If

$$\xi : C_n(K(A, n)) \to A$$

and

$$\eta : C_n(K(B, n)) \to B$$

were the maps such that T_A and T_B were given as $T_A([f]) = f^*([\xi])$ and $T_B([f]) = f^*([\eta])$, and $h : A \to B$ was a morphism, then

$$h \circ \xi = \eta \circ K(h, n)_*$$

This is stronger than what is needed, since we only needed that $h\xi - \eta K(h, n)_*$ was a boundary.

We shall prove that this stronger identity holds in general by induction. The induction step is as follows: Suppose

$$\xi' = (\xi'_0, \ldots, (-1)^{n(k-1)}\xi'_{k-1})$$

and

$$\eta' = (\eta'_0, \ldots, (-1)^{n(k-1)}\eta'_{k-1})$$

are defined as above, such that

$$T : [-, K(B_0 \to \ldots \to B_{k-1}, n)] \to H^n(-, B_0 \to \ldots \to B_{k-1})$$

is given by $T([f]) = f^*([\xi])$ and

$$T' : [-, K(B'_0 \to \ldots \to B'_{k-1}, n)] \to H^n(-, B'_0 \to \ldots \to B'_k)$$

is given by $T'([f]) = f^*([\eta])$. We shall use the names $\phi = (\phi_0, \ldots, \phi_{k-1})$ and $\bar{\phi} = (\phi_0, \ldots, \phi_k)$. The induction hypothesis will be that

$$\phi \circ \xi' = \eta' \circ K(\phi, n)_*$$

Now we define

$$\xi = (p_k^*(\xi'_0), \ldots, (-1)^{n(k-1)}p_k^*(\xi'_{k-1}), (-1)^{nk}\xi_k)$$

and

$$\eta = (p_k^*(\eta'_0), \ldots, (-1)^{n(k-1)}p_k^*(\eta'_{k-1}), (-1)^{nk}\eta_k)$$

as usual. We need to prove that

$$\bar{\phi} \circ \xi = \eta \circ K(\bar{\phi}, n)_*$$

Let us first concentrate on the first $k-1$ places above. Suppose $l \in \{1, \ldots, k-1\}$. We need to prove that:

$$\phi_l \circ \xi' \circ (p_k)_* = \eta \circ (p_k)_* K(\bar{\phi}, n)_*$$

If we can prove that

$$p_k \circ K(\bar{\phi}, n) = K(\phi, n) \circ p_k$$

then the equation will follow from the induction hypothesis. But remember that the functoriality of $K(-, n)$ was defined inductively, by defining $K(\bar{\phi}, n)$ to be the map, induced on the homotopy fibers by $K(\phi, n)$ and $\Omega^{k-1}K(\phi_k, n)$. This means that it made this diagram commute:

$$\begin{tikzcd}
K(B_0 \to \ldots \to B_k, n) \arrow[r, "p_k"] \arrow[d, "K(\bar{\phi},n)"] & K(B_0 \to \ldots \to B_{k-1}, n) \arrow[r, "K(h_k,n)"] \arrow[d, "K(\phi,n)"] & \Omega^{k-1}K(B_k, n) \arrow[d, "\Omega^{k-1}K(\phi_k,n)"] \\
K(B_0' \to \ldots \to B_k', n) \arrow[r, "p_k"] & K(B_0' \to \ldots \to B_{k-1}', n) \arrow[r, "K(h_k',n)"] & \Omega^{k-1}K(B_k', n)
\end{tikzcd}$$

So clearly $p_k \circ K(\bar{\phi}, n) = K(\phi, n) \circ p_k$.

The last thing needed to prove is that

$$\phi_k \circ \xi_k = \eta_k \circ K(\bar{\phi}, n)_*$$

Notice that on $K(B_0 \to \ldots \to B_k, n)^{(n-k)} = \Omega^k K(B_k, n)$ we have that $K(\bar{\phi}, n)$ is just $\Omega^k K(\phi_k, n)$. Remember that ξ_k is the map, that to an $(n-k)$-cell associates the element in $\pi_{n-k}(\Omega^k K(B_k, n))$ represented by its characteristic map, and $\eta \circ \Omega^k K(\phi_k, n)$ is the map that to an $(n-k)$-cell σ associates the element in $\pi_{n-k}(\Omega^k K(B_k', n))$ represented by $\Omega^k K(\phi_k, n) \circ \sigma$. Now, since the natural equivalence from $\pi_{n-k}\Omega^k K(-, n)$ gave the commutative diagram:

$$\begin{tikzcd}
\pi_{n-k}(\Omega K(B_k, n)) \arrow[r, "\Omega^k K(\phi_k,n)_*"] \arrow[d, "\simeq"] & \pi_{n-k}(\Omega^k K(B_k', n)) \arrow[d, "\simeq"] \\
B_k \arrow[r, "\phi_k"] & B_k'
\end{tikzcd}$$

we have that $\Omega^k K(\phi_k, n) \circ \sigma$ represents $\phi_k \circ \xi(\sigma)$, which proves the proposition. □

The last thing we need to prove is that diagram 4.1 commutes. Actually, diagram 4.1 is the diagram we need for the case $k = 1$, in general we need the following diagram to commute:

$$
\begin{tikzcd}
{[X,\Omega K(B_0 \to \ldots \to B_{k-1},n)]} \arrow[r, "T"] \arrow[d, "\Omega K(h_{k-1},n)"] & \tilde{H}^{n-1}(X,B_0 \to \ldots \to B_{k-1}) \arrow[d, "(h_{k-1})_*"] \\
{[X,\Omega^k K(B_k,n)]} \arrow[r, "T"] \arrow[d, "i_*"] & \tilde{H}^{n-k}(X,B_k) \arrow[d, "\phi"] \\
{[X,K(B_0 \to \ldots \to B_k,n)]} \arrow[r, "T"] \arrow[d, "p_*"] & \tilde{H}^{n}(X,B_0 \to \ldots \to B_k) \arrow[d, "\psi"] \\
{[X,K(B_0 \to \ldots \to B_{k-1},n)]} \arrow[r, "T"] \arrow[d, "K(h_{k-1},n)"] & \tilde{H}^{n}(X,B_0 \to \ldots \to B_{k-1}) \arrow[d, "(h_{k-1})_*"] \\
{[X,\Omega^{k-1} K(B_k,n)]} \arrow[r, "T"] & \tilde{H}^{n-k+1}(X,B_k)
\end{tikzcd}
\tag{4.2}
$$

The transformation

$$T : [X, \Omega^k K(B_k, n)] \to \tilde{H}^{n-k}(X, B_k)$$

Should be interpreted as follows: There is an identification:

$$\Omega^k K(B_k, n) = K(\underbrace{0 \to \ldots \to 0}_{k} \to B_k, n)$$

and another identification:

$$H^{n-k}(X, B_k) = \tilde{H}^n(X, \underbrace{0 \to \ldots \to 0}_{k} \to B_k)$$

Thus the transformation T in question becomes the transformation:

$$[-, K(\underbrace{0 \to \ldots \to 0}_{k} \to B_k, n)] \to \tilde{H}^n(-, \underbrace{0 \to \ldots \to 0}_{k} \to B_k)$$

We make the same identifications in the top and bottom horizontal line in the diagram.

Proposition 4.3.4. *Diagram 4.2 commutes.*

Proof. The top square and the bottom square commute by naturality of T in the coefficient variable.

The map

$$\phi : \tilde{H}^{n-k}(X, B_k) \to \tilde{H}^n(X, B_0 \to \ldots \to B_k)$$

is actually the map induced by the chain map:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & B_k \arrow[d, "id"] \\
B_0 \arrow[r, "h_0"] & B_1 \arrow[r, "h_1"] & \cdots \arrow[r] & B_{k-1} \arrow[r, "k_{k-1}"] & B_k
\end{tikzcd}
$$

Likewise the map ψ is induced by the chain map:

$$\begin{tikzcd} B_0 \arrow[r, "h_0"] \arrow[d, "id"] & B_1 \arrow[r, "h_1"] \arrow[d, "id"] & \cdots \arrow[r] & B_{k-1} \arrow[r, "h_{k-1}"] \arrow[d] & B_k \arrow[d] \\ B_0 \arrow[r, "h_0"] & B_1 \arrow[r, "h_1"] & \cdots \arrow[r] & B_{k-1} \arrow[r] & 0 \end{tikzcd}$$

This means that the two squares in the middle of diagram 4.2 commute by naturality of T in the coefficient variable. ☐

Our aim was to prove that T was an isomorphism for all spaces X and all finite complexes. The idea of the proof was to prove that diagram 4.2 commuted, that all the maps involved were homomorphisms, and then use induction on k and the Five Lemma to prove that T was an isomorphism. So the next thing that is needed is a group structure on $[X, K(B_0 \to \ldots \to B_k, n)]$. We can get that group structure if we replace X by ΣX and use the H-cogroup structure on ΣX. This way $[\Sigma X, K(B_0 \to \ldots \to B_k, n)]$ becomes a group the same way $\pi_n(X)$ becomes a group.

Now, for any pointed map:

$$f : Y \to Z$$

the map

$$f_* : [\Sigma X, Y] \to [\Sigma X, Z]$$

becomes a group homomorphism. Thus all the vertical maps in the diagram 4.2 become group homomorphisms. The fact that the horizontal maps become homomorphisms is due to the next lemma, which is taken from [Hatcher](Lemma 4.60).

Lemma 4.3.5. *Suppose h is a contravariant functor from some category of pointed spaces to abelian groups satisfying the homotopy invariance axiom and the wedge axiom. Then $(f+g)^* = f^* + g^*$ for maps $f, g : \Sigma X \to Y$.*

Proof. Let

$$c : \Sigma X \to \Sigma X \vee \Sigma X$$

denote the map that collapses $X \times \{\frac{1}{2}\}$. Then by definition $f + g = (f \vee g) \circ c$. Consider the following commutative diagram:

$$\begin{tikzcd} h(K) \arrow[r, "(f\vee g)^*"] & h(\Sigma X \vee \Sigma X) \arrow[r, "c^*"] \arrow[d, shift left, "i_1^*\oplus i_2^*"] & h(\Sigma X) \\ & h(\Sigma X) \oplus h(\Sigma X) \arrow[u, shift left, "q_1^*\oplus q_2^*"] & \end{tikzcd}$$

Here $q_1 : \Sigma X \vee \Sigma X \to \Sigma X$ is the identity on the first factor, and collapses the second factor. Likewise q_2 is the identity on the second and a collapse on the first. i_1 and i_2 denote the inclusions onto the first and second factor respectively. This means that $q_1 \circ i_1$ is the identity, as well as $q_2 \circ i_2$ is the identity. So the two vertical maps are inverses of each other, $i_1^* \oplus i_2^*$ being an isomorphism by assumption.

Since $(f \vee g) \circ i_1 = f$, we have

$$i_1^*(f \vee g)^*(x) = f^*(x)$$

Likewise:

$$i_2^*(f \vee g)^*(x) = g^*(x)$$

Since $q_1 \circ c$ is homotopic to the identity, we have

$$c^* q_1^*(x) = x$$

Likewise:

$$c^* q_2^*(x) = x$$

If we sum these equations up, we see that:

$$c^* \circ (f \vee g)^*(x) = f^*(x) + g^*(x)$$

But by definition we have

$$c^* \circ (f \vee g)^*(x) = (f + g)^*(x)$$

which proves the lemma. □

Thus we have proved that all the maps in diagram 4.2 are homomorphisms, so we can apply the Five Lemma. We are now ready for the main theorem:

Theorem 4.3.6. *The functors $[-, \Omega K(-, n+1)]$ and $\tilde{H}^n(-,-)$ taking the first variable in the category of pointed CW-complexes, and the second in the category of finite complexes of abelian groups are naturally equivalent.*

Proof. We will first prove that the maps:

$$T : [\Sigma X, \Omega^l K(B_0 \to \ldots \to B_k, n)] \to \tilde{H}^{n-l}(\Sigma X, B_0 \to \ldots \to B_k)$$

are isomorphisms as long as $n - k - l \geq 0$ (l can be zero). This is done by induction on k. The case $k = 0$ is known from ordinary cohomology (Proposition 4.3.1). The induction step is just the five lemma used on diagram 4.2. Notice here that the diagram also takes care of the case $l \neq 0$ since

$$\Omega^l K(B_0 \to \ldots \to B_k, n) = K(\underbrace{0 \to \ldots \to 0}_{l} \to B_0 \to \ldots \to B_k, n)$$

This proves that the functors $[\Sigma -, K(-, n)]$ and $\tilde{H}^n(\Sigma -, -)$ are naturally equivalent functors defined on chains of length at most $n + 1$. Using equivalence of functors:

$$[-, \Omega K(-, n)] \simeq [\Sigma -, K(-, n)]$$

and

$$\tilde{H}^n(\Sigma -, -) \simeq \tilde{H}^{n-1}(-, -)$$

we get a natural equivalence of functors

$$[-, \Omega K(-, n+1)] \simeq \tilde{H}^n(-, -)$$

defined on chains of length at most $n + 2$. Now, suppose $k > n + 2$ then we have natural homotopy equivalences:

$$\Omega K(B_0 \to \ldots \to B_k, n+1) \simeq K(0 \to B_0 \to \ldots \to B_k, n+1) \simeq$$

$$K(0 \to B_0 \to \ldots \to B_{n+1}, n+1) \simeq \Omega K(B_0 \to \ldots \to B_{n+1}, n+1)$$

by lemma 4.1.3 and since Corollary 2.1.5 gives a natural equivalence:

$$\tilde{H}^n(-, B_0 \to \ldots \to B_k) \simeq \tilde{H}^n(-, B_0 \to \ldots \to B_{n+1})$$

the equivalence of functors holds in the case of arbitrary finite chains. This concludes the proof. $\square$

4.4 The case of infinite chains

If we allow the chains to be infinite in one direction, that is if we allow chains on the form:

$$B_0 \to B_1 \to \ldots$$

The arguments from Remark 2.1.6 show that:

$$\tilde{H}^n(X, B_0 \to B_1 \to \ldots) \simeq \tilde{H}^n(X, B_0 \to \ldots \to B_{n+1})$$

Thus $\Omega K(B_0 \to \ldots \to B_{n+1}, n+1)$ is a natural classifying space for $\tilde{H}^n(-, B_0 \to B_1 \to \ldots)$. So we define (as a matter of notation):

$$\Omega K(B_0 \to B_1 \to \ldots, n+1)$$

to be

$$\Omega K(B_0 \to \ldots \to B_{n+1}, n+1)$$

The following is an idea of how to deal with the case of infinite chains.

Suppose we have a chain complex B_*, that is infinite in both directions:

$$\cdots \longrightarrow B_{-1} \longrightarrow B_0 \longrightarrow B_1 \longrightarrow \cdots$$

The idea in the following will be to approximate cohomology with coefficients in B_* with coefficients in the complex:

$$B_0 \longrightarrow B_{-1} \longrightarrow B_{-2} \longrightarrow \cdots$$

and then improve this approximation by approximating with

$$B_1 \longrightarrow B_0 \longrightarrow B_{-1} \longrightarrow \cdots$$

and go to the limit in some way.

So consider the chain map:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & B_{-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r] & B_{-1} \arrow[r] & \cdots
\end{tikzcd}$$

This map induces a map:

$$\tilde{H}^n(-, \ldots \to 0 \to B_0 \to B_{-1} \to \ldots) \to \tilde{H}^n(-, \ldots \to 0 \to B_1 \to B_0 \to B_{-1} \to \ldots)$$

Since

$$\tilde{H}^n(-,\ldots \to 0 \to B_1 \to B_0 \to \ldots) \simeq \tilde{H}^{n+1}(-,B_1 \to \ldots) \simeq [-,\Omega K(B_1 \to \ldots, n+2)]$$

We have that this map corresponds to a map:

$$[-,\Omega K(B_0 \to B_{-1} \to \ldots, n+1)] \to [-,\Omega K(B_1 \to B_0 \to \ldots, n+2)]$$

Which corresponds to a homotopy class of maps:

$$[\phi] \in [\Omega K(B_0 \to B_{-1} \to \ldots, n+1), \Omega K(B_1 \to B_0 \to \ldots, n+2)]$$

Using Theorem 1.2.3 we see that:

$$\tilde{H}^n(S^i;B) \simeq \tilde{H}^i(S^i;H_{i-n}(B))$$

So that

$$\pi_i(\Omega K(B_0 \to \ldots, n+1)) = 0$$

for $i > n$. Arguments as in the proof of Theorem 2.1.2 give us an exact sequence of the form:

$$\tilde{H}^n(X,B_1) \longrightarrow \tilde{H}^n(X,\ldots \to 0 \to B_0 \to B_{-1} \to \ldots) \longrightarrow$$
$$\tilde{H}^n(X,\ldots \to 0 \to B_1 \to B_0 \to \ldots) \longrightarrow \tilde{H}^{n+1}(X,B_1)$$

Which shows that the map ϕ induces isomorphisms on π_i for $i < n$ and an epimorphism for $i = n$.

Continuing this way we obtain a sequence of maps defined up to homotopy:

$$\Omega K(B_0 \to \ldots, n+1) \to \Omega K(B_1 \to \ldots, n+2) \to \Omega K(B_2 \to \ldots, n+3) \to \ldots$$

corresponding to the sequence:

$$\tilde{H}^n(-,\ldots \to 0 \to B_0 \to B_{-1} \to \ldots) \to \tilde{H}^n(-,\ldots \to 0 \to B_1 \to B_0 \to \ldots) \to$$
$$\tilde{H}^n(-,\ldots \to 0 \to B_2 \to B_1 \to \ldots) \to \ldots$$

where

$$\pi_i(\Omega K(B_r \to \ldots, n+r+1)) = 0$$

for $i > n+r$ and the map:

$$\phi_r : \Omega K(B_r \to \ldots, n+r+1) \to \Omega K(B_{r+1} \to \ldots, n+r+2)$$

induces an isomorphism on π_i for $i < n+r$ and an epimorphism on π_{n+r}.

The maps ϕ_r are only defined up to homotopy. If we choose specific maps we can construct the mapping telescope:

$$T(B_*, n)$$

as

$$\coprod_r \Omega K(B_r \to \ldots, n+r+1) \times [r, r+1]/\sim$$

where $\sim$ is the relation that identifies

$$(x, r+1) \in \Omega K(B_r \to \ldots, n+r+1) \times [r, r+1]$$

with

$$(\phi_r(x), r+1) \in \Omega K(B_{r+1} \to \ldots, n+r+2) \times [r+1, r+2]$$

and that identifies all points on the form $(*, t)$. This last identification collapses $\{*\} \times I$ to a point, that we set to be the base point of the mapping telescope.

There is an inclusion:

$$\kappa_r : \Omega K(B_r \to \ldots, n+r+1) \to T(B_*, n)$$

given by identifying

$$\Omega K(B_r \to \ldots, n+r+1)$$

with

$$\Omega K(B_r \to \ldots, n+r+1) \times \{r\}$$

Suppose B_* is only infinite in one direction, that is suppose $B_r = 0$ for r sufficiently large. Then the sequence:

$$\Omega K(B_0 \to \ldots, n+1) \to \Omega K(B_1 \to \ldots, n+2) \to \ldots$$

stabilizes. That is, all maps are homotopy equivalences from a certain point. In this case the mapping telescope is homotopic to (or at least weakly homotopic to) the space that the sequence stabilizes to. This means that, in a way the mapping telescope extends the definition of the classifying space from the finite case.

Conjecture 4.4.1. *The space:*

$$T(B_*, n)$$

is welldefined up to homotopy. That is, changing the maps involved to homotopy equivalent maps does not change the homotopy type of the space. Further more, the space is a natural classifying space for cohomology with coefficients in a chain complex.

In the following we shall restrict ourselves to finite CW-complexes and prove part of the conjecture in this case.

Proposition 4.4.2. *Suppose*

$$T(B_*, n)$$

is welldefined and has a group-like H-space structure such that κ_r is a H-space map for all r. Then the functors $[-, T(B_*, n)]$ and $\tilde{H}^n(-, B_*)$ are equivalent on the category of finite dimensional *CW-complexes.*

The proof of this will be split up into a series of lemmas. First we remind the reader of the definition of a direct limit of a sequence of groups.

Suppose we have a sequence of groups and homomorphisms:

$$B_0 \xrightarrow{h_0} B_1 \xrightarrow{h_1} B_2 \xrightarrow{h_2} \cdots$$

We define the direct limit of this sequence to be:

$$\varinjlim B_i = (\bigoplus_i B_i)/S$$

where S is the subgroup generated by elements of the form $x_i - h_i(x_i)$. It is an easy consequence of this definition that if the sequence is stable from some point, that is if all B_i are isomorphic to some B and all h_i are isomorphisms for i greater than some n then:

$$\varinjlim B_i \simeq B$$

The most interesting feature of this construction is that if we have a commutative diagram of morphisms:

$$\begin{tikzcd} \cdots \arrow[r] & B_i \arrow[r, "h_i"] \arrow[d, "\phi_i"] & B_{i+1} \arrow[r, "h_{i+1}"] \arrow[dl, "\phi_{i+1}"] & \cdots \\ & C & & \end{tikzcd}$$

Then there exists a unique induced map:

$$\phi : \varinjlim B_i \to C$$

Such that this diagram commutes for all i:

$$\begin{tikzcd} B_i \arrow[rr] \arrow[dr, "\phi_i"] & & \varinjlim B_i \arrow[dl, "\phi"] \\ & C & \end{tikzcd}$$

Lemma 4.4.3. *Suppose X is a finite dimensional CW-complex. Then the sequence:*

$$\tilde{H}^n(X; \ldots \to 0 \to B_0 \to B_{-1} \to \ldots) \to \tilde{H}^n(X; \ldots \to 0 \to B_1 \to B_0 \to B_{-1} \to \ldots) \to$$

$$\tilde{H}^n(X; \ldots \to 0 \to B_2 \to B_1 \to B_0 \to B_{-1} \to \ldots) \to \ldots$$

is stable from some point and

$$\varinjlim \tilde{H}^n(X; \ldots \to 0 \to B_r \to B_{r-1} \to \ldots) \simeq \tilde{H}^n(X; B_*)$$

Proof. Suppose X has dimension i. For $r = i - n + 1$ $\mathrm{Hom}(C_*(X), \ldots \to 0 \to B_r \to B_{r-1} \to \ldots)_n$ looks like this:

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & C_i(X) \arrow[r] \arrow[d] & C_{i-1}(X) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & B_r \arrow[r] & B_{r-1} \arrow[r] & B_{r-2} \arrow[r] & \cdots \end{tikzcd}$$

This means that the groups of B in dimensions greater than r have no influence on the n'th cohomology group of X. Thus the map:

$$\tilde{H}^n(X, \ldots \to 0 \to B_r \to B_{r-1} \to \ldots) \longrightarrow \tilde{H}^n(X, \ldots \to 0 \to B_{r+1} \to B_r \to \ldots)$$

is an isomorphism for $r \geq i - n + 1$. And for $r \geq i - n + 1$ we have:

$$\tilde{H}^n(X, \ldots \to 0 \to B_r \to B_{r-1} \to \ldots) \simeq \tilde{H}^n(X, B_*)$$

$\square$

Lemma 4.4.4. *The canonical map:*

$$\kappa_r : \Omega K(B_r \to \ldots, n+r+1) \to T(B_*, n)$$

induces an isomorphism on π_i for $i < n+r$, and an epimorphism on π_{n+r}.

Proof. Since this diagram commutes up to homotopy:

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega K(B_r \to \ldots, n+r+1) \arrow[r, "\phi_r"] \arrow[d, "\kappa_r"] & \Omega K(B_{r+1} \to \ldots, n+r+2) \arrow[r, "\phi_{r+1}"] \arrow[dl, "\kappa_{r+1}"] & \cdots \\
& T(B_*, n) & &
\end{tikzcd}$$

we obtain a commutative diagram of homotopy groups:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_i(\Omega K(B_r \to \ldots, n+r+1)) \arrow[r, "(\phi_r)_*"] \arrow[d, "(\kappa_r)_*"] & \pi_i(\Omega K(B_{r+1} \to \ldots, n+r+2)) \arrow[r, "(\phi_{r+1})_*"] \arrow[dl, "(\kappa_{r+1})_*"] & \cdots \\
& \pi_i(T(B_*, n)) & &
\end{tikzcd}$$

Since the map:

$$\pi_i(\Omega K(B_r \to \ldots, n+r+1)) \to \varinjlim \pi_i(\Omega K(B_l \to \ldots, n+l+1))$$

is an isomorphism for $i < n+r$ and an epimorphism for $i = n+r$, it suffices to show that the induced map:

$$\phi : \varinjlim \pi_i(\Omega K(B_r \to \ldots, n+r+1)) \to \pi_i(T(B_*, n))$$

is an isomorphism for all i.

Let us first prove that ϕ is surjective. Suppose

$$f : S^i \to T(B_*, n)$$

represents an element in

$$\pi_i(T(B_*, n))$$

Since the set

$$\{\Omega K(B_0 \to \ldots, n+1) \times [0,1] \cup \ldots \cup \Omega K(B_r \to \ldots, n+r+1) \times [r, r+1[\}_{r \in \mathbb{N}}$$

is an open covering of $T(B_*, n)$, and S^i is compact f must have image contained in some

$$\Omega K(B_0 \to \ldots, n+1) \times [0,1] \cup \ldots \cup \Omega K(B_r \to \ldots, n+r+1) \times [r, r+1[$$

Since this set is contractible to

$$\Omega K(B_r \to \ldots, n+r+1) \times \{r\}$$

This implies that $[f]$ is in the image of κ_r

To prove injectivity of ϕ suppose we have a null homotopy of some map $f : S^i \to T(B_*, n)$. This is the same as a map:

$$F : S^i \times I/\{*\} \times I \cup S^i \times \{1\} \to T(B_*, n)$$

By the same arguments as before, we can lift F to some $\Omega K(B_r \to \ldots, n+r+1)$ up to homotopy, which proves that f represents 0 in some $\pi_i(\Omega K(B_r \to \ldots, n+r+1))$. $\square$

Lemma 4.4.5. *Suppose X and Y are connected (0-connected) spaces and $f : X \to Y$ is a pointed map that induces an isomorphism on all π_i for $i < n$ and an epimorphism for $i = n$. If K is a CW-complex of dimension less than n then the map:*

$$f_* : [K, X] \to [K, Y]$$

is a bijection.

Proof. This is Theorem IV.7.16 and Lemma IV.7.12 of [Whitehead]. □

Lemma 4.4.6. *Suppose X is a finite dimensional CW-complex. Then the sequence:*

$$[X, \Omega K(B_0 \to \ldots, n+1)] \to [X, \Omega K(B_1 \to \ldots, n+2)] \to \ldots$$

stabilizes and

$$\varinjlim [X, \Omega K(B_r \to \ldots, n+r+1)] \simeq [X, T(B_*, n)]$$

Proof. The isomorphism is the map:

$$\varinjlim [X, \Omega K(B_r \to \ldots, n+r+1)] \simeq [X, T(B_*, n)]$$

induced by the diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & {[X, \Omega K(B_r \to \ldots, n+r+1)]} \arrow[r, "(\phi_r)_*"] \arrow[d, "(\kappa_r)_*"] & {[X, \Omega K(B_{r+1} \to \ldots, n+r+2)]} \arrow[r, "(\phi_{r+1})_*"] \arrow[dl, "(\kappa_{r+1})_*"] & \cdots \\
& {[X, T(B_*, n)]} & &
\end{tikzcd}$$

obtained as in the proof of Lemma 4.4.4.

Lemma 4.4.5 tells us that the sequence:

$$[X, \Omega K(B_0 \to \ldots, n+1)] \to [X, \Omega K(B_1 \to \ldots, n+2)] \to \ldots$$

is stable from a certain point. It also tells us that the map:

$$[X, \Omega K(B_r \to \ldots, n+r+1)] \to [X, T(B_*, n)]$$

is bijective from a certain point. This proves the lemma. □

Proof of Proposition 4.4.2. For a finite dimensional CW-complex X we get isomorphisms:

$$\tilde{H}^n(X, B_*) \simeq \varinjlim \tilde{H}^n(X; \ldots \to 0 \to B_r \to B_{r-1} \to \ldots) \simeq$$

$$\varinjlim [X, K(B_r \to \ldots, n+r+1)] \simeq [X, T(B_*, n)]$$

All maps involved are clearly natural in X. □

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