Relaxed vertex colorings of simplicial complexes

Natalia Dobrinskaya[1] Jesper M. Møller[2] Dietrich Notbohm[1]

[1]Department of Mathematics
Vrije Universiteit, Amsterdam

[2]Department of Mathematics
University of Copenhagen

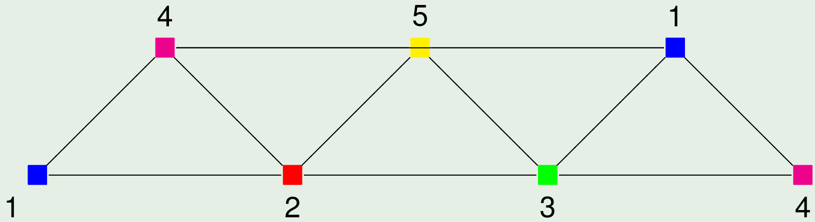
Bergen Topology Symposium, June 10 – 11, 2010

Outline

- Introduction
  - Standard Vertex Colorings and Relaxed Vertex Colorings of simplicial complexes: Examples
  - Formal definitions
- Colorings and vector bundles
  - Davis–Januszkiewicz spaces
  - Canonical vector bundles over Davis–Januszkiewicz spaces
  - Colorings = Splittings of canonical vector bundles
- Colorings and Stanley–Reisner rings
  - Stanley–Reisner rings
  - Colorings = Factorizations in Stanley–Reisner rings
- Conclusion
  - Summary
  - Questions to think about

Example (Standard coloring of the Möbius band MB)

Standard coloring of 5-vertex complex MB using 5 colors

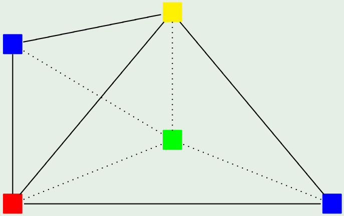


Example (Sudoku)

SUDOKU is an 8-dimensional simplicial complex with $9 + 9 + 9$ maximal simplices. A sudoku problem consists in completing a given partial standard coloring to a full standard coloring of SUDOKU using 9 colors.

Standard Colorings

Example



Standard coloring of K
= Standard coloring of
1-skeleton of K
$\Longrightarrow$ graph theory

Theorem (Standard colorings live on the 1-skeleton)

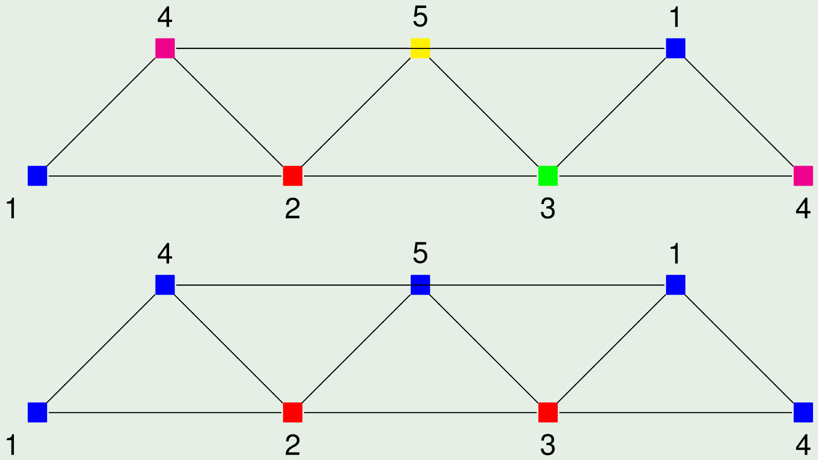
Standard coloring of K = Standard coloring of $\mathrm{sk}_1(K)$

Proof.

A coloring of the vertices is a coloring of K if and only if K contains no monochrome 1-simplices. □

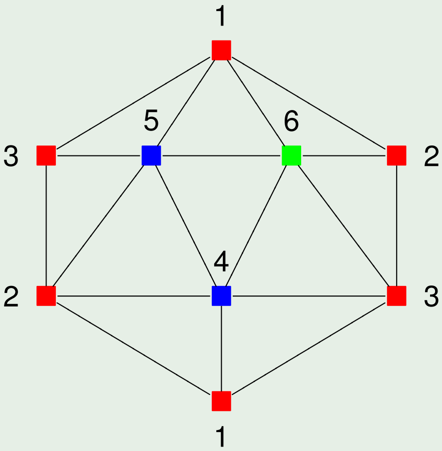
Standard and Relaxed Colorings

Example (Standard and Relaxed coloring of Möbius band MB)



Relaxed colorings

Example (Relaxed Coloring of projective plane P^2)

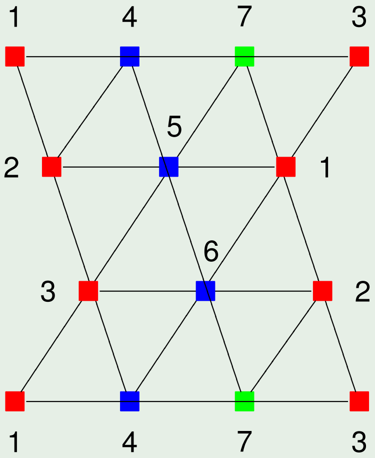


A $(3, 2)$-coloring of a triangulation P^2 of the projective plane.

No monochrome 2-dimensional simplices

A standard coloring of P^2 needs 6 colors.

Example (Relaxed coloring of the torus T^2)



A $(3, 2)$-coloring of Möbius' minimal triangulation of the torus.

A standard coloring needs 7 colors.

Abstract Simplicial Complexes

Definition (Simplex)

A simplex is the set $D[\sigma]$ of all subsets of a finite set σ.

Definition (ASC)

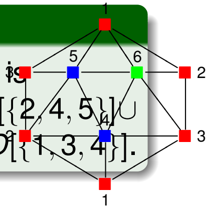
An Abstract Simplicial Complex is a union of simplices:

$$K = \bigcup_{\sigma} D[\sigma]$$

$n(K)$: number of vertices in maximal simplex of K
$m(K)$: number of vertices in K ($|V|$)

Example

The 2-dimensional ASC P^2 with vertex set $\{1, \ldots, 6\}$ is
$D[\{1,3,5\}] \cup D[\{1,5,6\}] \cup D[\{1,2,6\}] \cup D[\{2,3,5\}] \cup D[\{2,4,5\}] \cup$
$D[\{4,5,6\}] \cup D[\{3,4,6\}] \cup D[\{2,3,6\}] \cup D[\{1,2,4\}] \cup D[\{1,3,4\}]$.



Colorings of Simplicial Complexes

Let K be an ASC on vertex set V and P a finite palette of r colors.

Definition ((r, s)-coloring of an ASC)

A (P, s)-coloring (or (r, s)-coloring) of K is a map $f\colon V \to P$ that is at most s-to-1 on all simplices of K.

$f\colon V \to P$ is an (r, s)-coloring if and only if K contains no monochrome s-simplices.

Remark

An (r, s) coloring with

$s = 1$ is a standard coloring using r colors

$s > 1$ is a relaxed coloring using r colors

Theorem ((r, s)-colorings live on the s-skeleton)

(r, s)-colorings of K = (r, s)-colorings of $\mathrm{sk}_s(K)$

Coloring the Poincaré homology 3-sphere

A $(4, 2)$-coloring of a 16-vertex triangulation of the Poincaré homology 3-sphere:

$\{1, 2, 4, 9\}, \{1, 2, 4, 15\}, \{1, 2, 6, 14\}, \{1, 2, 6, 15\}, \{1, 2, 9, 14\}, \{1, 3, 4, 12\}, \{1, 3, 4, 15\}, \{1, 3, 7, 10\},$
$\{1, 3, 7, 12\}, \{1, 3, 10, 15\}, \{1, 4, 9, 12\}, \{1, 5, 6, 13\}, \{1, 5, 6, 14\}, \{1, 5, 8, 11\}, \{1, 5, 8, 13\}, \{1, 5, 11, 14\},$
$\{1, 6, 13, 15\}, \{1, 7, 8, 10\}, \{1, 7, 8, 11\}, \{1, 7, 11, 12\}, \{1, 8, 10, 13\}, \{1, 9, 11, 12\}, \{1, 9, 11, 14\}, \{1, 10, 13, 15\},$
$\{2, 3, 5, 10\}, \{2, 3, 5, 11\}, \{2, 3, 7, 10\}, \{2, 3, 7, 13\}, \{2, 3, 11, 13\}, \{2, 4, 9, 13\}, \{2, 4, 11, 13\}, \{2, 4, 11, 15\},$
$\{2, 5, 8, 11\}, \{2, 5, 8, 12\}, \{2, 5, 10, 12\}, \{2, 6, 10, 12\}, \{2, 6, 10, 14\}, \{2, 6, 12, 15\}, \{2, 7, 9, 13\}, \{2, 7, 9, 14\},$
$\{2, 7, 10, 14\}, \{2, 8, 11, 15\}, \{2, 8, 12, 15\}, \{3, 4, 5, 14\}, \{3, 4, 5, 15\}, \{3, 4, 12, 14\}, \{3, 5, 10, 15\}, \{3, 5, 11, 14\},$
$\{3, 7, 12, 13\}, \{3, 11, 13, 14\}, \{3, 12, 13, 14\}, \{4, 5, 6, 7\}, \{4, 5, 6, 14\}, \{4, 5, 7, 15\}, \{4, 6, 7, 11\}, \{4, 6, 10, 11\},$
$\{4, 6, 10, 14\}, \{4, 7, 11, 15\}, \{4, 8, 9, 12\}, \{4, 8, 9, 13\}, \{4, 8, 10, 13\}, \{4, 8, 10, 14\}, \{4, 8, 12, 14\}, \{4, 10, 11, 13\},$
$\{5, 6, 7, 13\}, \{5, 7, 9, 13\}, \{5, 7, 9, 15\}, \{5, 8, 9, 12\}, \{5, 8, 9, 13\}, \{5, 9, 10, 12\}, \{5, 9, 10, 15\}, \{6, 7, 11, 12\},$
$\{6, 7, 12, 13\}, \{6, 10, 11, 12\}, \{6, 12, 13, 15\}, \{7, 8, 10, 14\}, \{7, 8, 11, 15\}, \{7, 8, 14, 15\}, \{7, 9, 14, 15\},$
$\{8, 12, 14, 15\}, \{9, 10, 11, 12\}, \{9, 10, 11, 16\}, \{9, 10, 15, 16\}, \{9, 11, 14, 16\}, \{9, 14, 15, 16\}, \{10, 11, 13, 16\},$
$\{10, 13, 15, 16\}, \{11, 13, 14, 16\}, \{12, 13, 14, 15\}, \{13, 14, 15, 16\}$

Run the magma program demo.prg from /home/moller/projects/simplicial/version04/presentation.

Chromatic Numbers of ASCs

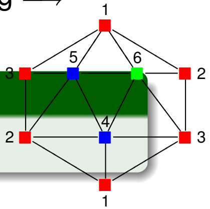
Definition (Chromatic numbers of ASCs)

The s-chromatic number, $\mathrm{chr}^s(K)$, is the least r such that K admits an (r,s) coloring.

- $|V| \geq \mathrm{chr}^1(K) \geq \mathrm{chr}^2(K) \geq \cdots \geq \mathrm{chr}^{1+\dim K}(K) = 1$
- $\mathrm{chr}^s(D[V]) = \left\lceil \frac{|V|}{s} \right\rceil$
- $K \subset K' \implies \mathrm{chr}^s(K) \leq \mathrm{chr}^s(K')$
- $\left\lceil \frac{n(K)}{s} \right\rceil \leq \mathrm{chr}^s(K) \leq \left\lceil \frac{m(K)}{s} \right\rceil$
- K admits a (r,s)-coloring $\implies$
  any maximal simplex of K admits a (r,s)-coloring $\implies$
  $n(K) \leq rs$

Example (The chromatic numbers of P^2)

$\mathrm{chr}^1(P^2) = 6$, $\mathrm{chr}^2(P^2) = 3$, and $\mathrm{chr}^3(P^2) = 1$



Chromatic numbers of manifolds

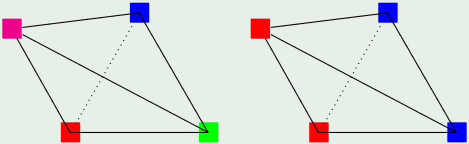
Definition (Chromatic numbers of manifolds)

The s-chromatic number of the manifold M is the maximum

$$\mathrm{chr}^s(M) = \max\{\mathrm{chr}^s(K) \mid |K| \simeq M\}$$

Example (Chromatic numbers of S^2)

Colorings of the tetrahedron $\partial D[3_+]$ shows that $\mathrm{chr}^1(S^2) \geq 4$ and $\mathrm{chr}^2(S^2) \geq 2$.



Theorem (4-color theorem)

$\mathrm{chr}^1(S^2) = 4$, $\mathrm{chr}^2(S^2) = 2$, $\mathrm{chr}^3(S^2) = 1$.

Chromatic numbers of $S^3, S^4, \ldots$

Proposition

$\mathrm{chr}^1(S^3) = \infty$ *and* $\mathrm{chr}^2(S^3) \geq 4$*.*

Proof.

For any finite set V there exist triangulations of S^3 with vertex set V such that any two vertices are connected by an edge.
There are triangulations K of S^3 with $\mathrm{chr}^2(K) = 4$. $\square$

The first interesting chromatic numbers for spheres are

$$\mathrm{chr}^n(S^{2n-1}) \qquad \text{and} \qquad \mathrm{chr}^n(S^{2n})$$

as $\mathrm{chr}^s(S^{2n-1}) = \infty = \mathrm{chr}^s(S^{2n})$ for all $n \geq 1$ and all $s < n$.

Speculations

- Is $\mathrm{chr}^2(S^3)$ finite?
- Is $\mathrm{chr}^n(S^{2n}) = 4$ $(n \geq 1)$ and $\mathrm{chr}^n(S^{2n-1}) = 4$ $(n \geq 2)$?

Davis–Januszkiewicz spaces

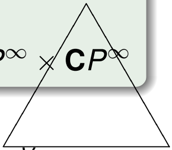
Definition (The Davis–Januszkiewicz space of K)

- Let $\mathrm{DJ}(D[V]) = \mathrm{map}(V, \mathbf{C}P^\infty) = \overbrace{\mathbf{C}P^\infty \times \cdots \times \mathbf{C}P^\infty}^{m(K)}$
- For $\sigma \subset V$ consider $\mathrm{DJ}(D[\sigma]) = \mathrm{map}(V, V - \sigma; \mathbf{C}P^\infty, *)$ as the subspace of the σ-axes of $\mathrm{DJ}(D[V]) = \mathrm{map}(V, \mathbf{C}P^\infty)$
- $\mathrm{DJ}(K) = \bigcup_{\sigma \in K} \mathrm{DJ}(D[\sigma])$

Example

If $K = D[\{1,2\}] \cup D[\{1,3\}] \cup D[\{2,3\}] \subset D[\{1,2,3\}]$ then $\mathrm{DJ}(K)$ is

$$\mathbf{C}P^\infty \times \mathbf{C}P^\infty \times \{*\} \cup \mathbf{C}P^\infty \times \{*\} \times \mathbf{C}P^\infty \cup \{*\} \times \mathbf{C}P^\infty \times \mathbf{C}P^\infty$$



- $K \subset K' \implies \mathrm{DJ}(K) \subset \mathrm{DJ}(K')$
- $\bigvee_V \mathbf{C}P^\infty = \mathrm{DJ}(V) \subset \mathrm{DJ}(K) \subset \mathrm{DJ}(D[V]) = (\mathbf{C}P^\infty)^V$

Vector bundles over Davis–Januszkiewicz spaces

Definition (The canonical vector bundle λ_K)

The canonical vector bundle λ_K over $\mathrm{DJ}(K)$ is the restriction

$$\begin{array}{ccc} \lambda_K & \longrightarrow & \lambda \times \cdots \times \lambda \\ \downarrow & & \downarrow \\ \mathrm{DJ}(K) & \hookrightarrow & \mathbf{C}P^\infty \times \cdots \times \mathbf{C}P^\infty \end{array} \qquad \dim \lambda_K = m(K)$$

to $\mathrm{DJ}(K)$ of the product of the tautological complex line bundles.

Theorem (The canonical vector bundle ξ_K)

There exists a short exact sequence of vector bundles

$$0 \to \xi_K \to \lambda_K \to \mathbf{C}^{m(K)-n(K)} \to 0$$

where $\dim \xi_K = n(K)$.

Colorings and splittings of vector bundles

Theorem (Colorings = Splittings of vector bundles)

Assume that $n(K) \leq rs$. Then K admits an (r, s)-coloring if and only if there exists a lift in either of the diagrams

$$\begin{tikzcd} & BU(s)^r \arrow[d, "\oplus"] \\ \mathrm{DJ}(K) \arrow[r, "\lambda_K"'] \arrow[ur, dotted, "?"] & BU \end{tikzcd} \qquad \begin{tikzcd} & BU(s)^r \arrow[d, "\oplus"] \\ \mathrm{DJ}(K) \arrow[r, "\xi_K"'] \arrow[ur, dotted, "?"] & BU(rs) \end{tikzcd}$$

The combinatorial problem of existence of an (r, s)-coloring of K has been translated to a topological problem of stably splitting the canonical vector bundle λ_K into r vector bundles of dimension s.

A failed proof of the 4-color theorem

Theorem (The 4-color theorem)

$\mathrm{chr}^1(K) \leq 4$ *for all triangulations* K *of* S^2.

Failed Proof.

$$\begin{tikzcd} & BU(1)^4 \arrow[d, "\oplus"] \\ \mathrm{DJ}(K) \arrow[ur, dotted, "?"] \arrow[r, "\lambda_K"'] & BU \end{tikzcd} \qquad \begin{tikzcd} & BU(1)^4 \arrow[d, "\oplus"] \\ \mathrm{DJ}(K) \arrow[ur, dotted, "?"] \arrow[r, "\xi_K"'] & BU(4) \end{tikzcd}$$

$\square$

Colorings of other compact surfaces? $\left\lfloor \frac{7+\sqrt{49-24\chi}}{2} \right\rfloor$. The 5-color theorem.

The Stanley–Reisner algebra

Definition (The Stanley–Reisner algebra of K)

$\mathrm{SR}(K;R) = R[V]/\left(\prod \tau \mid \tau \in D[V] - K\right)$ is the quotient of the polynomial algebra on V (in degree 2) by the monomial ideal generated by the (minimal) non-simplices of K.

Theorem (Davis–Januszkiewicz)

$\mathrm{SR}(K;R) = H^*(\mathrm{DJ}(K);R)$

- If $V = \{v_1, v_2, v_3\}$ then
  - $\mathrm{SR}(D[V];R) = R[v_1, v_2, v_3]$
  - $\mathrm{SR}(\partial D[V];R) = R[v_1, v_2, v_3]/\langle v_1 v_2 v_3 \rangle$
- $K \subset K' \implies \mathrm{SR}(K;R) \twoheadleftarrow \mathrm{SR}(K';R)$
- $R[V] \twoheadrightarrow \mathrm{SR}(K) = \lim(P(K)^{\mathrm{op}}; \mathrm{SR}(D[\sigma])) \subset \prod_{\sigma \in K} R[\sigma]$

$$\begin{tikzcd} & \mathrm{DJ}(K) \arrow[d, "H^*(-;R)"] \\ K \arrow[ur, "\mathrm{DJ}(-)"] \arrow[r, "\mathrm{SR}(-;R)"'] & \mathrm{SR}(K;R) \end{tikzcd}$$

Colorings and the Stanley–Reisner ring

Theorem (Stanley–Reisner recognition of colorings)

The partition $V = V_1 \cup \cdots \cup V_r$ is an (r,s)-coloring of K if and only if

$$\prod_{v \in V}(1+v) = \prod_{1 \leq j \leq r} c_{\leq s}(V_j)$$

in $\mathrm{SR}(K;\mathbf{Z})$.

Theorem (Colorings = Factorizations of symmetric polynomials)

K admits an (r,s)-coloring iff there exist r elements $c_1,\ldots,c_r$ of $\mathrm{SR}(K;\mathbf{Z})$ *such that* $\deg(c_j) \leq 2s$ *and*

$$\prod_{v \in V}(1+v) = \prod_{1 \leq j \leq r} c_j$$

in $\mathrm{SR}(K;\mathbf{Z})$.

The Stanley–Reisner ring of $\mathbb{P}^2$ and C_5

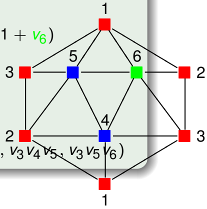
Example

Since $[1,2,3,4,5,6] \to [{\color{red}\blacksquare}, {\color{red}\blacksquare}, {\color{red}\blacksquare}, {\color{blue}\blacksquare}, {\color{blue}\blacksquare}, {\color{green}\blacksquare}]$ is a $(3,2)$-coloring, the identity

$$\prod_{1 \le i \le 6} (1 + v_i) = (1 + v_1 + v_2 + v_3 + v_2 v_3 + v_1 v_3 + v_1 v_2)(1 + v_4 + v_5 + v_4 v_5)(1 + v_6)$$

holds in the Stanley–Reisner ring for $P2$

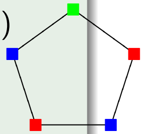
$\mathrm{SR}(\mathbb{P}^2; \mathbf{Z}) = \mathbf{Z}[v_1, \ldots, v_6]/(v_1 v_2 v_3, v_1 v_2 v_5, v_1 v_3 v_6, v_1 v_4 v_5, v_1 v_4 v_6, v_2 v_3 v_4, v_2 v_4 v_6, v_2 v_5 v_6, v_3 v_4 v_5, v_3 v_5 v_6)$



Example

$$\mathrm{SR}(C_5; \mathbf{Z}) = \mathbf{Z}[v_1, \ldots, v_5]/(v_1 v_3, v_1 v_4, v_2 v_4, v_2 v_5, v_3 v_5)$$

$$\prod_{1 \le i \le 5} (1 + v_i) = (1 + v_1 + v_3)(1 + v_2 + v_4)(1 + v_5)$$



Another failed proof of the 4-color theorem

Theorem (The 4-color theorem)

$\mathrm{chr}^1(K) \leq 4$ *for all triangulations K of S^2.*

Failed Proof.

Let K be a triangulation of S^2 with vertex set V. There exist 4 elements $c_1, c_2, c_3, c_4 \in \mathrm{SR}(K; \mathbf{Z})$ of degree ≤ 2 so that

$$\prod_{v \in V} (1 + v) = c_1 c_2 c_3 c_4$$

in $\mathrm{SR}(K; \mathbf{Z})$. □

Summary

What we learned today

- An (r, s)-coloring is a coloring of the vertices by r colors so that at most s vertices of any simplex has the same color
- (r, s)-colorings depend only on the s-skeleton
- (r, s)-coloring is equivalent to splitting the canonical vector bundle over the Davis–Januszkiewicz space
- (r, s)-coloring is equivalent to factorizing the total Chern class of the canonical vector bundle in the Stanley–Reisner ring

What we didn't learn today

- (L, s)-colorings of the vertices of K where L is some ASC. (Example: $(3, 2)$-colorings of $P2$ with 2-colors in every simplex.)
- (L, s)-colorings of the t-simplices of K

Questions to think about

Questions

- Is $\mathrm{chr}^2(S^3) = 4$?
- Is $\mathrm{chr}^n(S^{2n-1}) = 4$ for all $n \geq 2$?
- Is $\mathrm{chr}^n(S^{2n}) = 4$ for all $n \geq 1$?
- With these tools, is it possible to find a topological proof of the 4-color theorem?
- With these tools, is it possible to compute the chromatic numbers of the compact surfaces?
- Is there a connection between the face numbers and the chromatic numbers (as in the 6-color theorem)?