Poisson boundaries of groups:
- Entropy, growth and amenability

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Abstract

The main interest of this thesis is the study of the Poisson boundary of a group G and its applications. The concept, originally introduced by Furstenberg in 1963, is based on the idea of extending the boundary representation of harmonic functions from the classical Dirichlet problem to a general group-theoretic setting. The Poisson boundary itself is an abstractly defined probability space, on which any harmonic function on G can be represented (using the Poisson transform, appropriately defined) by a *boundary* function on it. The Poisson boundary is associated to a random walk on the group, and the construction is done both in a topological and a measurable setting, where in the latter case two explicit realisations are provided.

We then investigate conditions ensuring triviality of the Poisson boundary and show that this can, in many cases, be characterised by means of vanishing of the Shannon entropy, a criteria due to Kaimanovich and Vershik. Further, we study applications of the Poisson boundary to analytic properties of the group, such as amenability, and discuss a related conjecture by Furstenberg. Finally, we tie in the notion of amenable actions, as defined by Zimmer. While an amenable group always acts amenably on any G-space, it is an important feature that any group acts amenably on its Poisson boundary. This is a result of Zimmer, which we discuss.

As an additional treat, we mention very recent surprising applications of the Furstenberg boundary (which we also introduce) to the settling of a number of long-standing open problems regarding group C*-algebras.

Contents

1 Introduction

In 1963 Harry Furstenberg published his paper [10], wherein he introduced the Poisson boundary for semi-simple Lie groups. The purpose of doing so was to extend the boundary representation of harmonic functions known from the Dirichlet problem to a more general setup. In order to do so he also defined both harmonic functions on a group and the Poisson transform, the latter which serves as the group theoretic analogue of the Poisson integral from the classical case. In the years following [10], all of the above ideas was extended to locally compact groups, and in 1973 Furstenberg published [11], wherein he defined the so-called μ-boundaries of a group G, a class of G-spaces extending the ideas behind the Poisson boundary. The μ-boundaries can be obtained as quotients of the Poisson boundary, and will also allow a boundary representation of some of the harmonic functions on the group, whereas only the Poisson boundary allows a boundary representation of all harmonic functions on G.

The Poisson boundary itself is a compact G-space equipped associated to a random walk on the group, a random walk which also serves as a crucial component in the construction of the Poisson boundary. Here, the random walk is used to equip the space of bounded, left uniformly continuous, harmonic functions on the group with the structure of a commutative, unital C^*-algebra, and the Poisson boundary is defined to be the spectrum of this C^*-algebra. This gives an isomorphism between the harmonic functions on the group and the continuous functions on the Poisson boundary, an isomorphism which can be realised as a Poisson transform for a appropriately defined probability measure on the Poisson boundary. The construction can also be carried out for measurable harmonic functions on the group, where the measurable setup also allows some more explicit constructions of the Poisson boundary. We will in this thesis present the constructions of the so-called *Stationary boundary* and *Exit boundary.*

An important question in the context of μ-boundaries is whether the Poisson boundary is trivial or not. If G is an abelian group the Poisson boundary is trivial, whenever the semigroup generated by the support of μ is all of G, as any μ-harmonic function in this cases is constant due to the Choquet-Deny theorem [5]. This result was later extended to the class of nilpotent groups as seen in [22]. For general groups, the question of triviality of the Poisson boundaries is more difficult to answer, but in 1983 (see [16]) Vadim Kaimanovich and Anatoly Vershik provided a characterisation of boundary triviality for a big class of measures, using the information theoretic concept of entropy, originally introduced by Claude Shannon [26]. Indeed, they proved that under the assumption of μ having finite entropy, then boundary triviality is equivalent to entropy of the pair (G, μ) being zero. This is called the entropy criteria for triviality of the Poisson boundary. In order to determine whether this entropy is zero or not, we apply the group theoretic concept of growth to characterise a class of groups G and Borel probability measures μ on G, for which the pair (G, μ) has zero entropy. In particular, any group with subexponential growth and measure with either finite support or finite first moment, will give zero entropy of the pair (G, μ). An examination of the above results can be found in Section 4. The main results therein are found in the above mentioned paper of Kaimanovich and Vershik, but for auxiliary results and additional details the Master's thesis of Cağri Sert [25] has been of much use for the author.

Another application of the Poisson boundary arises in the theory of amenable groups and amenable actions. The first of these relates to a conjecture by Harry Furstenberg from his 1973 paper [11]. In here, he proved that any non-amenable group admits a probability measure with full support and non-trivial Poisson boundary. He then conjectured that the converse implication was true, as well, or in other words that *G possesses a measure μ whose support is all of G and for which no nontrivial μ-boundaries exists if and only if G is amenable.* The second implication was later proved independently by Joseph Rosenblatt in [23] and Kaimanovich and Vershik [16], and we will in Section 5 provide a short survey of the proof provided by Rosenblatt. A more detailed examination can be found in the author's Master's project [21]. The second major result is that a group G acts amenably on its Poisson boundary whenever the corresponding measure is absolutely continuous with respect to the Haar measure on the group. Amenable actions have been defined in two different ways, but we will rely on the concept introduced by Robert Zimmer in [28], the paper in which he also proved the above statement in the case of a discrete group. The general result was proved in [1]. However, in this thesis, we will present the proof from Chapter 7 in [2].

Finally, we take a quick look at another type of boundary of a group, the so-called G-boundaries, which was also introduced by Furstenberg. The G-boundary is defined in a way that somewhat resembles the μ-boundaries, but without the probabilistic nature of the μ-boundary. The class of G-boundaries also admits a universal object, in the sense of before, an object which is called the Furstenberg boundary. Very recently history, the Furstenberg boundary has found surprising applications in the theory of group C^*-algebras, where it has been used by Matthew Kennedy and Mehrdad Kalantar first in [19], and later in collaboration with Emmanuel Breuillard and Narutaka Ozawa in [18]. The Furstenberg boundary was used to provide new characterisations of several classes of groups, most importantly the C^*-simple groups and the groups with the unique trace property. This characterisations also led to the proof that any C^*-simple group has the unique trace property, an open question for many years. We will not provide any details regarding the Furstenberg boundary and its applications, but Section 6 will include a list of important definitions and results. The sections is only meant as an appetiser for the reader with an interest in the theory of C^*-algebras and thus, the reader will need deeper knowledge of C^*-algebras to understand the statements provided therein.

Notation used in the thesis

- $\mathbb{N}_0$: the positive integers $0, 1, 2, \ldots$.
- $B(x, r)$: the open ball around x with radius $r > 0$.
- $\partial B(x, r)$: the boundary of $B(x, r)$.
- $C^n(\Omega)$: complex-valued functions on Ω, which are n times continuously differentiable.

- $C^n(\Omega, \mathbb{R})$: real-valued functions on Ω, which are n times continuously differentiable.
- G is always a group and the identity will be denoted by e.
- $C(X)$: the continuous functions from a Hausdorff topological space X to $\mathbb{C}$.
- $C_c(X)$: the compactly supported, continuous functions from a locally compact, Hausdorff topological space X to $\mathbb{C}$.
- $L^p(X, \mu)$: the p-integrable functions from a measure space (X, μ) to $\mathbb{C}$.
- $\mathcal{P}(X)$: the set of Borel probability measures on a measurable space X.

1.1 The classical setup

Before delving into the main topics of the thesis and the world of groups, we start out with a short survey on classical harmonic analysis. Herein we prove a couple of results that will act as a motivation for constructions and terminology in the abstract case. Given a simply connected domain $\Omega \subset \mathbb{R}^n$ and $u \in C^2(\Omega)$, we say that u is harmonic on Ω if

$$\Delta u = \frac{\partial^2}{\partial x_1^2} u + \cdots + \frac{\partial^2}{\partial x_n^2} u = 0.$$

Here Δ is the usual Laplace operator, so in other words $u \in C^2(\Omega)$ is harmonic if and only if it belongs to the kernel of Δ. We will now turn our attention to the two-dimensional case, i.e., $\Omega \subset \mathbb{R}^2$ where we identify $\mathbb{R}^2$ with $\mathbb{C}$, whenever beneficial. In this context, it is worth noticing that the Cauchy-Riemann equations ensure that all holomorphic functions are harmonic, but also that the class of harmonic functions exceeds the one of holomorphic functions. The latter statement follows from the fact that both the real and imaginary part of a holomorphic function is harmonic, but it is holomorphic only in the case where the original function is constant.

This latter connection between real-valued harmonic functions and holomorphic functions turns out to be rather strong, as seen in the following proposition:

Proposition 1.1. *Any real-valued harmonic function u on Ω is the real part of some holomorphic function F on Ω.*

Proof. It is well-known fact from classical two-dimensional analysis that the identity $\frac{\partial^2}{\partial x} u = -\frac{\partial^2}{\partial y} u$ gives rise to a function $v \colon \Omega \to \mathbb{R}$ such that

$$\frac{\partial}{\partial x} v = -\frac{\partial}{\partial y} u \quad \text{and} \quad \frac{\partial}{\partial y} v = \frac{\partial}{\partial x} u.$$

From here we can consider the function $F \colon \Omega \to \mathbb{C}$ given by $F = u + iv$, which is holomorphic due to the above identities, and clearly $\mathrm{Re}(F) = u$. $\square$

Such harmonic functions have a variety of interesting properties, but there are two properties of certain appeal for our journey onwards. We start out by considering a real-valued harmonic functions f on $B(0,1)$ and later extend the results to complex-valued case.

1. **The mean value property:** For any $x \in B(0,1)$ and $r > 0$ such that $\overline{B(x,r)} \subset B(0,1)$, the following holds

$$f(x) = \int_{\partial B(x,r)} f(z) \, \mathrm{d}\sigma_r(z).$$

   Here σ_r is the normalised Lebesgue measure on $\partial B(x,r)$.

2. **Boundary representation:** There exists a unique $\hat{f} \in C(\partial B(0,1))$, such that

$$f(re^{i\theta}) = \int_{\partial B(0,1)} P(r, \theta - t) \hat{f}(t) \, \mathrm{d}\sigma(t),$$

   for all $r \in [0,1)$ and $\theta \in [-\pi, \pi]$, where σ is the Lebesgue measure on $\partial B(0,1)$. Here P is the Poisson kernel, which we will describe in detail later.

The above properties will be our motivation for the group theoretic approach to harmonic functions. The mean value property will give rise to the very definition of harmonic functions, while the boundary representation will inspire the study of boundaries for probability measures on a group. The concrete translations will be saved for the introductory parts of Section 2 and 3, respectively.

Finally, we finish this section by going through the proofs of the above claims. Let us start out with the mean value property, which also holds true in more general cases, i.e., when the domain need not be the unit disc.

Theorem 1.2 (The mean-value property)**.** *Let f be a real-valued harmonic function on a simply connected domain $\Omega \subset \mathbb{R}^2$. For any open ball $B(x,r)$ such that $\overline{B(x,r)}$ is contained in Ω, we have the integral representation*

$$f(x) = \int_{\partial B(x,r)} f(z) \, \mathrm{d}\sigma_r(z) = \int_{\partial B(0,r)} f(x+y) \, \mathrm{d}\sigma_r(y).$$

Proof. The equality between the integrals is due to translation invariance of the Lebesgue measure, so let us focus on the equality between $f(x)$ and the latter integral. For this, let $R = \sup\{r \geq 0 \mid \overline{B(x,r)} \subset \Omega\}$ and consider the map $\varphi \colon [0,\infty) \to \mathbb{R}$, defined by

$$\varphi(r) = \int_{\partial B(0,1)} f(x + ry) \, \mathrm{d}\sigma(y), \quad r > 0,$$

where σ denotes the Lebesgue measure on $\partial B(0,1)$. Then φ is continuous on $[0,R]$ and differentiable in $(0,R)$ with

$$\varphi'(r) = \int_{\partial B(0,1)} \nabla f(x+ry) \cdot y \, \mathrm{d}\sigma(y), \quad 0 < r < R.$$

From here on, we can apply Green's formula to obtain the equality

$$\varphi'(r) = \int_{\partial B(0,1)} \nabla f(x+ry) \cdot y \, \mathrm{d}\sigma(y) = \int_{B(x,r)} \Delta f(z) \, \mathrm{d}z = 0,$$

for all $r \in (0, R)$, since $\Delta f = 0$ by assumption. With a derivative equal to zero, we conclude that φ is constant, so by continuity of f we obtain

$$f(x) = \varphi(0) = \varphi(r) = \int_{\partial B(0,r)} f(x+y)\,\mathrm{d}\sigma_r(y),$$

which was the desired result. $\square$

With the mean value property settled, we turn to the boundary representation, for which we start out by considering harmonic functions on an area strictly larger than $B(0,1)$. Let $u\colon B(0,R) \to \mathbb{R}$ be a harmonic function, for some $R > 1$, and let $F\colon B(0,R) \to \mathbb{C}$ be the associated holomorphic function, in the sense of Proposition 1.1. This allows us to express F in terms of its Taylor series, i.e.,

$$F(z) = \sum_{n=0}^{\infty} c_n z^n, \qquad |z| < R. \tag{1}$$

As $u = \mathrm{Re}(F)$, we know that $u = (F + \overline{F})/2$ and hence

$$u(re^{i\theta}) = \frac{1}{2}\left(\sum_{n=0}^{\infty} c_k r^k e^{ik\theta} + \sum_{k=0}^{\infty} c_k r^k e^{-ik\theta}\right) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta},$$

whenever $r < R$ and $-\pi \leq \theta \leq \pi$. Taking a closer look at the coefficients above we notice that $a_0 = \mathrm{Re}(c_0), a_k = c_k/2$ for $k > 0$ and $a_k = c_{-k}/2$ for $k < 0$. In particular,

$$u(e^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}, \quad \theta \in [-\pi, \pi],$$

and so the a_k's must be the Fourier coefficients for the map $\theta \mapsto u(e^{i\theta})$, i.e.,

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it})e^{-ikt}\,\mathrm{d}t.$$

As the convergence in (1) is uniform on compact subsets of $B(0,R)$, we can interchange the sum and the integral to obtain

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it})\left(\sum_{n=-\infty}^{\infty} r^{|k|}e^{ik(\theta-t)}\right)\mathrm{d}t, \quad \theta \in [-\pi,\pi], 0 \leq r < R.$$

The infinite series appearing in this latter integral will be uniformly convergent for $r \in [0,1)$, with limit

$$\sum_{n=-\infty}^{\infty} r^{|k|}e^{ik(\theta-t)} = \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} = \frac{1-r^2}{|1-re^{i(\theta-t)}|^2}.$$

To ease notation further on, we define $P\colon [0,1) \times [-\pi,\pi] \to \mathbb{R}$ by

$$P(r,t) = \frac{1-r^2}{|1-re^{it}|^2},$$

such that the harmonic function u can be expressed as

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta - t)\cdot u(e^{it})\,\mathrm{d}t, \tag{2}$$

for $r \in [0,1)$ and $\theta \in [-\pi,\pi]$, a formula that will serve as inspiration for the general group theoretic definition. Note by the way that the above integral can be viewed as the convolution of $t \mapsto u(e^{it})$ and $P(r,\cdot)$, when considered as integrable functions on $[-\pi,\pi]$ equipped with the normalised Lebesgue measure. If we let f denote the map $t \mapsto u(e^{it})$, we say that u is the Poisson integral of f and denote it by $u = P(f)$. Notice finally that even though we only considered real-valued functions so far, the results also apply to complex-valued harmonic functions through linearity of the integral. From here on the harmonic functions will be complex-valued.

In order to recognise the Fourier series above, the construction of this Poisson integral relies heavily on R being strictly greater than 1, but we can use this idea to obtain a similar result in the case, when $R = 1$. This is the solution of the Dirichlet problem for the unit disc in $\mathbb{R}^2$.

Theorem 1.3. *Let u be a harmonic function on $B(0,1)$ satisfying*

$$\sup_{0\le r<1}\int_{-\pi}^{\pi} |u(re^{it})|^p\,\mathrm{d}t < \infty,$$

for some $p > 1$. Then there is a function $f \in L^p([-\pi,\pi])$ such that

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)f(t)\,\mathrm{d}t, \quad r < 1, \theta \in [-\pi,\pi].$$

In other words, $u = P(f)$.

Proof. In order to use the original construction in this case of $R = 1$, let (r_n) be an increasing sequence in $[0,1)$ converging to 1. Let furthermore (f_n) be a sequence in $L^p([-\pi,\pi])$ given by

$$f_n(t) = u(r_n e^{it}), \qquad t \in [-\pi,\pi]$$

By hypothesis on u, the sequence $(f_n)_{n\ge1}$ is bounded in $L^p([-\pi,\pi])$, which we identify with $L^q([-\pi,\pi])^*$, where q is the dual exponent of p. As the closed balls in this dual space are weak*-compact, there is a subnet $(f_{n_\alpha})_{\alpha\in A}$ converging to $f \in L^q([-\pi,\pi])^*$ in the weak*-topology, that is

$$\int g(t)f_{n_\alpha}(t)\,\mathrm{d}t \to \int g(t)f(t)\,\mathrm{d}t,$$

for all $g \in L^q([-\pi,\pi])$, and in particular for $t \mapsto P(r,\theta-t)$, which is a continuous map on a compact interval, whenever $r < 1$.

Next up, we notice that for each $n \ge 1$, the map $z \mapsto u(r_n z)$ is harmonic on $B(0,r_n^{-1})$. By our choice of $(r_n)_{n\ge1}$, the numbers r_n^{-1} are all strictly greater than 1, so by the previous discussion

$$u(r_n re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)f_n(t)\,\mathrm{d}t,$$

for $r < 1$ and $\theta \in [-\pi, \pi]$. Combining this with the weak*-convergence, we get

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \theta - t) f(t)\,\mathrm{d}t, \quad r < 1, \theta \in [-\pi, \pi],$$

as we had set out to prove. □

This result can also be extended to the case $p = \infty$, or, in other words, where $\sup_{z\in B(0,1)} |u(z)| < \infty$. To see this, we identify $L^\infty([-\pi, \pi])$ with the dual of $L^1([-\pi, \pi])$ and repeat the proof above. The case when $r = 1$ does cause some problems though, since $L^1([-\pi, \pi])$ can **not** be identified with the dual of any such L^p-space. However, we can embed $L^1([-\pi, \pi])$ into the regular bounded Borel measures on $[-\pi, \pi]$, which can be identified with the dual of $C([-\pi, \pi])$. Repeating the previous proof once more in this new setup, will then give us the following corollary:

Corollary 1.4. *Let u be a harmonic function on $B(0, 1)$ such that*

$$\sup_{0\le r<1} \int_{-\pi}^{\pi} |u(re^{it})|\,\mathrm{d}t < \infty.$$

Then there is a bounded regular Borel measure on μ on $[-\pi, \pi]$ such that,

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \theta - t)\,\mathrm{d}\mu(t).$$

We will let $P(\mu)$ denote this function and call it the Poisson integral of μ.

Through these results we have seen that for most harmonic functions on $B(0, 1)$, we can associate either some L^p-function or a Borel measure via the Poisson integral. What we will see now is that this association also works the other way around.

Theorem 1.5. *Let $f \in L^p([-\pi, \pi])$ for $1 \le p \le \infty$, and let u be the Poisson integral of f, i.e.,*

$$u_r(\theta) := u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \theta - t) f(t)\,\mathrm{d}t, \quad r < 1, -\pi \le \theta \le \pi.$$

Then u is harmonic on $B(0, 1)$, and $\sup_{0\le r<1} \|u_r\|_p \le \|f\|_p$.

Proof. Assume first that f is real-valued and let $f(t) = \sum_{k=-\infty}^{\infty} a_k e^{ikt}, t \in [-\pi, \pi]$, be the Fourier series of f. By arguments seen earlier, we have

$$u(re^{i\theta}) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta}, \qquad r < 1, -\pi \le \theta \le \pi.$$

By the assumption of f being real-valued, we get $a_{-k} = \overline{a_k}$, for all $k \in \mathbb{Z}$, and hence

$$\begin{aligned} u(re^{i\theta}) &= \sum_{k=-\infty}^{\infty} a_k r^{|k|} e^{ik\theta} = a_0 + \sum_{k=1}^{\infty} r^k (a_k e^{ik\theta} + a_{-k} e^{-ik\theta}) \\ &= a_0 + \sum_{k=1}^{\infty} r^k \operatorname{Re}(a_k e^{ik\theta}) = \operatorname{Re}\left(a_0 + \sum_{k=1} a_k (re^{i\theta})^k\right). \end{aligned}$$

This is the real part of a holomorphic function on $B(0,1)$, so u is harmonic on $B(0,1)$. To obtain the inequality of the norms, one should notice that for $r < 1$, the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |P(r,\theta-t)|\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,\theta-t)\,\mathrm{d}t = \sum_{k=-\infty}^{\infty} r^{|k|}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}\,\mathrm{d}t,$$

equals 1, using classical theory of Fourier series to see that the one, nonzero term in the series corresponds to $k = 0$. To finish this part, we notice that u_r can be written as the convolution of $P(r,\cdot)$ and f, and thus

$$\|u_r\|_p \leq \|P(r,\cdot)\|_1 \cdot \|f\|_p = \|f\|_p$$

for $1 \leq p \leq \infty$. This latter fact can be found in Appendix A, and is generally known as Young's inequality for convolutions. For the general case, where f is complex-valued, we can handle the real and imaginary parts individually and reach the conclusion by linearity. □

This result can also be extended to Borel measures on $[-\pi,\pi]$, i.e., the Poisson integral of such a measure is a harmonic function on $B(0,1)$. By identifying the unit sphere $\partial B(0,1)$ as a quotient of $[-\pi,\pi]$, we obtain a one-to-one correspondence between bounded harmonic functions on $B(0,1)$ and bounded functions on $\partial B(0,1)$. With $f \in L^\infty(\partial B(0,1))$ and $u = P(f)$, we will in some sense view f as a boundary description of the behaviour of u. This connection becomes even more clear, as

$$u(re^{it}) \to f(t), \text{ as } r \to \infty,$$

for almost every $t \in [-\pi,\pi]$. This result can be found as Theorem 11.23 in [24]. With this in mind, we can turn our attention back to Theorem 1.5 in the case of $p = \infty$ and realise that $\|f\|_\infty = \|P(f)\|_\infty$, thus giving us an isometric identification. For a final remark of this section, let us view a harmonic function f and its boundary description as one function $\tilde{f}$, i.e., given $f \in L^\infty(\partial B(0,1))$, we define $\tilde{f} : \overline{D} \to \mathbb{C}$ by

$$\tilde{f}(re^{it}) = \begin{cases} f(re^{it}), & r = 1 \\ P(f)(re^{it}), & r < 1 \end{cases}$$

The integral representation from before can then be rewritten as

$$\tilde{f}(re^{it}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r,t-\theta)\tilde{f}(\theta)\,\mathrm{d}\theta = \int_{\partial B(0,1)} P(r,t-\theta)\tilde{f}(\theta)\,\mathrm{d}\sigma(\theta).$$

Note finally that since $P(0,t) = 1$, for all $t \in [-\pi,\pi]$, it is clear that these two integral representations coincide, whenever f is a harmonic function on a simply connected domain Ω which contains the closed unit ball.

2 Harmonic functions

The purpose of this first section is to define the group theoretic notion of a harmonic function, and also develop a series of tools needed in the construction of the boundaries to come. We will follow the first chapter of [2], and all results will be found in there unless otherwise mentioned.

Let G be a locally compact group with countable basis. The group G will be equipped with a right Haar measure λ. We will also let Δ denote the modular function corresponding to λ, i.e., $\lambda(xA) = \Delta(x)\lambda(A)$, for all Borel sets $A \subset G$ and $x \in G$. For more details on the Haar measure and the modular function, the reader is referred to the first part of Appendix A. We will also use the notion of convolutions from time to time, for which the reader may read the second part of Appendix A.

Definition 2.1. Let μ be a Borel probability measure on G be given. A μ-harmonic function on G is a function $f\colon G \to \mathbb{C}$ such that

$$f(g) = \int_G f(gx)\,\mathrm{d}\mu(x), \qquad \text{for all } g \in G. \tag{3}$$

The space of bounded μ-harmonic functions on G will be denoted by $H^\infty(G,\mu)$. In order for (3) to make sense, we implicitly require each translate ${}_{x^{-1}}f$, $x \in G$, given by $g \mapsto f(xg)$, to be an element of $L^1(G,\mu)$.

We will also let $H^\infty_{\mathrm{luc}}(G,\mu)$ denote the space of left uniformly continuous functions in $H^\infty(G,\mu)$, where left uniform continuity of a function $f\colon G \to \mathbb{C}$ is defined as follows. For every $\varepsilon > 0$, there exists an open neighbourhood U of the identity in G, such that for all $u \in U$,

$$\sup_{g\in G} |f(ug) - f(g)| < \varepsilon.$$

Note that left uniformly continuity in particular implies continuity.

In order to describe μ-harmonic functions in a different way, we introduce the following notation.

Definition 2.2. For a Borel measure μ on G and $f\colon G \to \mathbb{C}$ for which the integral below makes sense, we define $f \star \mu \in L^\infty(G)$ by

$$(f \star \mu)(g) = \int f(gx)\,\mathrm{d}\mu(x), \qquad \text{for all } g \in G.$$

In this way $f \star \mu = f * \tilde{\mu}$, where $\tilde{\mu}$ denotes the Borel probability measure on G, defined as $\tilde{\mu}(A) = \mu(A^{-1})$ for $A \subset G$ Borel, and furthermore, $f\colon G \to \mathbb{C}$ is μ-harmonic if and only if $f = f \star \mu$. It is also not difficult to see that $\|f \star \mu\|_\infty \leq \|\mu\| \cdot \|f\|_\infty$, and similarly, if f is in $L^1(G,\lambda)$, then $\|f \star \mu\|_1 \leq \|\mu\| \cdot \|f\|_1$.

2.1 Motivation

In order to understand the group theoretic description of harmonic functions, let us consider the group

$$G = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \;\middle|\; \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

In resemblance with the usual Möbius transformations on the Riemann sphere, we will identify each $g \in G$ with a holomorphic bijection of the complex plane onto itself. The identification goes as follows. The matrix

$$g = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} \in G,$$

is identified with holomorphic map $g(z) = (\alpha z + \beta)/(\overline{\beta} z + \overline{\alpha}), z \in \mathbb{C}$. This gives rise to an action of G on $\mathbb{C}$, which furthermore preserves both $B(0,1)$ and $\partial B(0,1)$ individually. We now investigate how this group action can be used to describe the mean value property from the classical setup. Let $f\colon B(0,1) \to \mathbb{C}$ be a harmonic function and $r < 1$ be given. The first thing to notice is that every $z \in B(0,1)$ can be written as $g(0)$, for some $g \in G$. For example, one could pick $\beta = \lambda z$ and $\alpha = \lambda$, with $\lambda = (\sqrt{1 - |z|^2})^{-1}$.

Through this identification, we define $\overline{f}\colon G \to \mathbb{C}$ by $\overline{f}(g) = f(g(0))$ for $g \in G$. From here on, we will once more let σ_r denote the normalised Lebesgue measure on $\partial B(0,r)$ and $\overline{\sigma_r}$ denote the lift of σ_r to G using the identification above. In other words

$$\overline{\sigma_r}(B) = \sigma_r(\{g(0) \in \partial B(0,r) \mid g \in B\})$$

for any measurable $B \subset G$. The mean-value property of f applied at zero can then be expressed as follows

$$f(0) = \int_{\partial B(0,r)} f(y) \, \mathrm{d}\sigma_r(y) = \int_G \overline{f}(h) \, \mathrm{d}\overline{\sigma_r}(h).$$

This is not the exact description we are looking for, but let us look at the composition $f \circ g$, for some $g \in G$. As the composition of holomorphic functions is holomorphic itself, we can apply Proposition 1.1 to conclude that $f \circ g$ is also harmonic on $B(0,1)$. As this holds true for all $g \in G$, we obtain

$$\overline{f}(g) = f(g(0)) = \int_{\partial B(0,r)} f(g(y)) \, \mathrm{d}\sigma_r(y) = \int_G \overline{f}(gh) \, \mathrm{d}\overline{\sigma_r}(h),$$

and thus, $\overline{f}$ is $\overline{\sigma_r}$-harmonic, whenever f is harmonic on $B(0,1)$.

2.2 Basic properties

In the classical case we assumed the harmonic functions to be C^2 and thus in particular, continuous. We did not make any such assumptions in the group theoretic definition, but as the following two propositions show, continuity occurs rather often.

Proposition 2.3. *Let μ be a Borel probability measure on G. If μ is absolutely continuous with respect to λ, then $f \star \mu$ is right uniformly continuous (and hence continuous), whenever f is bounded.*

Proof. By analogy with the definition of left uniform continuity, we say that a function $\varphi\colon G \to \mathbb{C}$ is right uniformly continuous, if for every $\varepsilon > 0$ there exists an open neighbourhood U of the identity e of G, such that

$$\sup_{g \in G} |\varphi(gt) - \varphi(g)| < \varepsilon, \qquad t \in U.$$

Let $\varphi \in L^1(G,\lambda)$ be the density of μ with respect to λ, let K be a compact neighbourhood of the identity and let $\varepsilon > 0$ be given. Then for $g \in G, t \in K$,

$$\begin{aligned}|(f \star \mu)(gt) - (f \star \mu)(g)| &= \left| \int_G f(gth)\varphi(h)\,\mathrm{d}\lambda(h) - \int_G f(gh)\varphi(h)\,\mathrm{d}\lambda(h) \right| \\ &= \left| \int_G f(gh)\varphi(t^{-1}h)\Delta(t)^{-1}\,\mathrm{d}\lambda(h) - \int_G f(gh)\varphi(h)\,\mathrm{d}\lambda(h) \right| \\ &\leq \|f\|_\infty \int_G |\varphi(t^{-1}h)\Delta(t)^{-1} - \varphi(h)|\,\mathrm{d}\lambda(h).\end{aligned}$$

These calculations show that it suffices to prove that the final integral above can be made arbitrarily small. For this, pick $\psi \in C_c(G)$ such that

$$\|\varphi - \psi\|_1 < \varepsilon/(3\|f\|_\infty).$$

By double use of the triangle inequality, we obtain

$$\int_G |\varphi(t^{-1}h)\Delta(t)^{-1} - \varphi(h)|\,\mathrm{d}\lambda(h) \leq \frac{2}{3\|f\|_\infty} + \int_G |\psi(t^{-1}h)\Delta(t)^{-1} - \psi(h)|\,\mathrm{d}\lambda(h).$$

The latter integrand will, for all $t \in K$, vanish outside some compact set, and hence it suffices to make the integrand small in a uniform manner on this compact set. First of, we insert yet another auxiliary term, and estimate for $t \in K, h \in G$,

$$\begin{aligned}|\psi(t^{-1}h)\Delta(t)^{-1} - \psi(t)| &\leq |\psi(t^{-1}h)\Delta(t)^{-1} - \psi(h)\Delta(t)| + |\psi(h)\Delta(t)^{-1} - \psi(h)| \\ &\leq \Delta(t)^{-1}|\psi(t^{-1}h) - \psi(h)| + \|\psi\|_\infty \cdot |\Delta(t)^{-1} - 1|.\end{aligned}$$

Looking at the first term in the sum above, we notice that continuity of Δ gives us an upper bound for $\Delta(t)^{-1}$. Furthermore any compactly supported continuous function is uniformly continuous, so we can make this term arbitrarily small. For the second term, we recall that $\Delta(e) = 1$ and hence $|\Delta(t)^{-1} - 1| \to 0$, as t tends to $e \in G$. Combining these considerations, we can pick an open neighbourhood U of $e \in G$, such that

$$\sup_{g\in G} |(f \star \mu)(gt) - (f \star \mu)(g)| < \varepsilon, \qquad t \in U,$$

and thus conclude that $f \star \mu$ is right uniformly continuous. □

We should note that if $f : G \to \mathbb{C}$ is μ-harmonic for some absolutely continuous measure μ, then f must be right uniformly continuous by the above proposition, as $f = f \star \mu$. It turns out that this fact holds true more generally, where we can weaken the assumption on μ, to the so-called spread-out measures. A Borel probability measure μ on G is said to be *spread-out* if there exists some positive integer p such that the measures μ^{*p} and λ are not mutually singular.

Proposition 2.4. *Let μ be a spread-out Borel probability measure on G. Then any bounded μ-harmonic function on G is right uniformly continuous (and hence continuous).*

Proof. Let $p > 1$ be such that μ^{*p} and λ are not mutually singular. For any $n \geq 1$, we let $\mu^{*np} = \alpha_n \cdot \lambda + \beta_n$, with $\alpha_n \in L^1(G, \lambda)$, be the Lebesgue decomposition. The choice of p ensures that none of the α_n's are constantly zero. By properties of the convolution of measures, the total variation of β_n tends to zero as n goes to infinity. Thus, for any bounded μ-harmonic function f, we have

$$\|f - f \star (\alpha_n \cdot \lambda)\|_\infty = \|f \star \beta_n\|_\infty \leq \|f\|_\infty \|\beta_n\|$$

As the latter term above tends to zero, when n tends to infinitity we conclude that the same is true for the first term above, and since uniform limits of continuous functions are continuous, the above estimates conclude the proof. The uniform convergence also implies that f is right uniformly continuous. □

2.3 Random walks

One of the main tools used in this thesis is that of a random walk on a group G. The reader may be familiar with this concept from classical probability theory, and will meet no surprises in the translation into the group theoretic version. Given a Borel probability measure μ on G, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = G^{\mathbb{N}_0}$ and $\mathcal{F}, \mathbb{P}$ are the product σ-algebra and product measure corresponding to the Borel sets in G and μ, respectively. If we let $(X_n)_{n\geq 0}$ denote the coordinate projections from Ω to G, we obtain a sequence of independent random variables with common distribution μ. With these projections we define the right random walk corresponding to μ as the sequence $(R_n)_{n\geq 0}$, where $R_0 \equiv e$ and $R_n = X_0 X_1 \cdots X_{n-1}$, for $n \geq 1$.

In the above construction we defined R_0 to be constantly equal to e, but this concept can easily be generalised. If now $\pi \in \mathcal{P}(G)$ is some Borel probability measure on G, we could consider R_0 as a random variable independent of each X_n and with distribution π, i.e., $R_0(\mathbb{P}) = \pi$. This would allow us to construct the *right random walk associated to μ with initial distribution* π denoted by $(R_n^\pi)_{n\geq 0}$, given as

$$R_0^\pi = R_0, \qquad R_n = R_0 X_0 X_1 \cdots X_{n-1}, \quad n \geq 1.$$

In this way the original random walk $(R_n)_{n\geq 0}$ was simply the random walk with initial distribution δ_e. Note also that the random walk with initial law δ_g, $g \in G$ can be described as $(R_n^{\delta_g})_{n\geq 0} = (gR_n)_{n\geq 0}$. We now turn to the connection between random walks and harmonic functions, where the theory of martingales plays a crucial role. For more information on martingales, see Chapter 7 in [17].

Proposition 2.5. *Let f be a bounded μ-harmonic function on G and $(R_n^\pi)_{n\geq 0}$ be the right random walk associated to μ with initial law π, for some $\pi \in \mathcal{P}(G)$. Then $(f(R_n^\pi))_{n\geq 0}$ is a bounded martingale with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$, where $\mathcal{F}_0 = \sigma(R_0)$ and $\mathcal{F}_n = \sigma(R_0, X_0, \ldots, X_{n-1})$, $n \geq 1$.*

Proof. That the process $(f(R_n^\pi))_{n\geq 0}$ is bounded follows from the boundedness of f, so let us focus on showing that it is indeed a martingale. For this, let $n \geq 1$ be given along with $Y \in \{R_0, X_0, \dots, X_{n-1}\}$ and a Borel subset of $B \subset \mathbb{C}$. We need to show that

$$\int_\Omega 1_{f^{-1}(B)}(Y) f(R_{n+1}^\pi)\, \mathrm{d}\mathbb{P} = \int_\Omega 1_{f^{-1}(B)}(Y) f(R_n^\pi)\, \mathrm{d}\mathbb{P}.$$

For this, we apply the abstract change of variable formula along with the independence of X_n from R_n^π and Y, respectively. Then

$$\begin{aligned}\int_\Omega 1_{f^{-1}(B)}(Y) f(R_n^\pi X_n)\, \mathrm{d}\mathbb{P} &= \int_\Omega 1_{f^{-1}(B)}(Y) \int_G f(R_n^\pi \cdot g)\, \mathrm{d}X_n(\mathbb{P})(g)\, \mathrm{d}\mathbb{P} \\ &= \int_\Omega 1_{f^{-1}(B)}(Y) \int_G f(R_n^\pi \cdot g)\, \mathrm{d}\mu(g)\, \mathrm{d}\mathbb{P} \\ &= \int_\Omega 1_{f^{-1}(B)}(Y) f(R_n^\pi)\, \mathrm{d}\mathbb{P},\end{aligned}$$

which is the desired result. □

By the martingale convergence theorem and the above proposition, we obtain an almost sure limit of the process $(f(R_n^\pi))_{n\geq 0}$, regardless of the initial distribution. We will use these limits to define a map $Z_f\colon G \to L^\infty(\Omega, \mathbb{P})$ by

$$Z_f(g) = \lim_{n\to\infty} f(gR_n), \quad g \in G,$$

which also gives us $f(g) = E(Z_f(g))$. Here $E(Z_f(g))$ denotes the expectation of $Z_f(g)$ with respect to $\mathbb{P}$, i.e., $E(Z_f(g)) = \int_\Omega Z_f(g)(\omega)\, \mathrm{d}\mathbb{P}(\omega)$.

If we let T denote the shift on Ω, i.e., $T(\omega_0, \omega_1, \omega_2, \dots) = (\omega_1, \omega_2, \omega_3, \dots)$, or equivalently, $T(\omega) = (X_1, X_2, \dots)(\omega)$, $\omega \in \Omega$, then for $g \in G$,

$$Z_f(g)(\omega) = \lim_{n\to\infty} f(gR_n)(\omega) = \lim_{n\to\infty} f(gX_0(\omega)R_n(T(\omega))),$$

and hence $Z_f(g) = Z_f(gX_0) \circ T$ $\mathbb{P}$-almost surely. Now, this type of invariance will be central in the following, so we let

$$I^\infty = \big\{Z\colon G \to L^\infty(\Omega, \mathbb{P}) \mid Z \text{ is bounded and } Z(g) = Z(gX_0) \circ T \ \mathbb{P}\text{-a.s.}\big\}$$

Combining the fact that $f(g) = E(Z_f(g))$ for $g \in G$, with the definition of Z_f, we see that the map $f \mapsto Z_f$ is an isometry from $H^\infty(G, \mu)$ to I^∞, when both spaces are equipped with the supremum norm. The relation between $H^\infty(G, \mu)$ and I^∞ becomes even more clear with the following result.

Proposition 2.6. *For any $Z \in I^\infty$, we obtain a bounded μ-harmonic function f on G by setting $f(g) = E(Z(g))$, for $g \in G$.*

Proof. The proof relies on the i.i.d. nature of the sequence of projections $(X_n)_{n\geq 0}$, which tells us that $(X_1, X_2, \dots)$ has distribution $\mathbb{P}$ as well. From here on it is once again a matter of applying the abstract change of variable formula along with certain independence properties. Namely, for any $g \in G$,

$$\begin{aligned} f(g) &= \int_\Omega Z(g)(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \int_\Omega Z(gX_0(\omega))(T(\omega))\,\mathrm{d}\mathbb{P}(\omega) \\ &= \int_G \int_\Omega Z(gx)(\omega)\,\mathrm{d}(X_1, X_2, \dots)(\mathbb{P})(\omega)\,\mathrm{d}X_0(\mathbb{P})(x) \\ &= \int_G \int_\Omega Z(gx)(\omega)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}\mu(x) \\ &= \int_G E(Z(gx))\,\mathrm{d}\mu(x) = \int_G f(gx)\,\mathrm{d}\mu(x). \end{aligned}$$

In other words, the function f is μ-harmonic. □

If we, in addition, assume left uniformly continuity of f, then Z_f is left uniformly continuous, as well, and this also holds true the other way around. Here left uniform continuity of a map $Z\colon G \to L^\infty(\Omega, \mathbb{P})$ is to be interpreted naturally as follows. Given $\varepsilon > 0$ there exists an open neighbourhood U of the identity in G, such that

$$\sup_{g \in G} \|Z(ug) - Z(g)\|_\infty < \varepsilon$$

for any $u \in U$. In other words $H^\infty_{\mathrm{luc}}(G, \mu)$ can be identified with the left uniformly continuous maps in I^∞, which we will denote by I^∞_{luc}. This identification of μ-harmonic functions will become important later on, when we introduce the boundary theory. Another application of this approach by random walks arises when we consider abelian or to a greater extend, nilpotent groups. We will only prove the proposition below for abelian groups, while a proof for the nilpotent case can be found in [22].

Proposition 2.7 ([27])**.** *Let G be an abelian group, and let μ be a Borel probability measure on G. If the semigroup generated by the support of μ is all of G, then every function $f \in H^\infty_{\mathrm{luc}}(G, \mu)$ is constant.*

Proof. With the notation introduced above, we have $Z_f(g) = \lim_{n\to\infty} f(g + R_n)$, for $g \in G$. With the filtration $(\mathcal{F}_n)_{n \geq 0}$ as earlier, we obtain

$$E(Z_f(g) \mid \mathcal{F}_n) = \lim_{m\to\infty} E(f(g + R_m) \mid \mathcal{F}_n) = f(g + R_n)$$

for all $n \geq 1$. Since G is abelian, we can permute the ordering of the terms in R_n, and thus $Z_f(g)$ is invariant under finite permutation of $(X_n)_{n\geq 0}$. By the Hewitt-Savage 0-1 law (see Theorem B.3), we conclude that $Z_f(g)$ is constant $\mathbb{P}$-a.s. and hence for all $g \in G$,

$$Z_f(g) = E(E(Z_f(g) \mid \mathcal{F}_n)) = E(f(g + R_n)) = f(g) \quad \mathbb{P}\text{- a. s.},$$

where the last equality follows from μ-harmonicity of f. From here on, we conclude that for all $g \in G$,

$$f(g + X_0) = f(g + R_1) = E(Z_f(g) \mid \mathcal{F}_1) = Z_f(g) = f(g) \quad \mathbb{P}\text{- a. s.}$$

As X_0 has distribution μ, we have $f(g + h) = f(g)$ for μ-a.e $h \in G$ and $g \in G$. By continuity of f, we also have $f(g + h) = f(g)$ for all $h \in \operatorname{supp}(\mu)$ and $g \in G$, so by assumption on μ, the function f is constant. □

The assumption on μ in the above proposition will appear many times later in the thesis, and the strengt of this property in combination with harmonic functions can be seen in this above proof.

2.4 Different degrees of harmonicity

In the study of measurable functions on measure spaces it is common not to distinguish between functions that only differ on a null-set, and for this reason we would like to extend our definition of harmonicity to equivalence classes of functions. To do this, we introduce the Poisson transform P_μ associated to some Borel probability measure μ on the group G. The basic version of the Poisson transform is a map $P_\mu \colon L^\infty(G,\lambda) \to L^\infty(G,\lambda)$, defined by

$$P_\mu(f)(g) = \int_G f(gx)\,\mathrm{d}\mu(x).$$

This map is well-defined, i.e., if $f = f'$ λ-a.s. then $P_\mu(f) = P_\mu(f')$ λ-a.s. This holds true since the Haar measure λ is equivalent to any translate of itself.

A natural generalisation is to consider the map $P_\alpha \colon L^\infty(G,\nu) \to L^\infty(G,\mu)$, for three Borel probability measures μ, ν and α on G. This map need not be well-defined, but there are certain assumptions on the measures which will make it all work. A sufficient condition for P_α to preserve equivalence classes is that for all Borel sets A in G and μ-a.e. $g \in G$, we have $\alpha(gA) = 0$, whenever $\nu(A) = 0$. This is equivalent to the fact that $\mu * \alpha$ is absolutely continuous with respect ν. To prove this, we observe that

$$\mu * \alpha(A) = \int_G \int_G 1_A(gh)\,\mathrm{d}\alpha(h)\,\mathrm{d}\mu(g) = \int_G \alpha(g^{-1}A)\,\mathrm{d}\mu(g),$$

which proves the statement.

Almost sure harmonicity

The first basic case above, i.e., the case where μ and ν are both equal to the Haar measure λ is particularly simple as we do not need any assumptions on the last measure. This allows us to define the map

$$P_\mu \colon L^\infty(G,\lambda) \to L^\infty(G,\lambda)$$

for any Borel probability measure μ on G, and inspired by the similarity of $P_\mu(f)$ and $f \star \mu$, we introduce the following definition.

Definition 2.8. We say that an equivalence class $[f] \in L^\infty(G,\lambda)$ is an almost-sure bounded μ-harmonic function if $[P_\mu(f)] = [f]$. The space of such equivalence classes of functions will be denoted $H^\infty_{\text{a.s.}}(G,\mu)$.

As the choice of representative in an equivalence class of $H^\infty_{\text{a.s.}}(G,\mu)$ does not matter, we will henceforth view elements of $H^\infty_{\text{a.s.}}(G,\mu)$ as functions, just as we usually do for measurable functions.

Proposition 2.9. *Any function $f \in H^\infty_{\text{a.s.}}(G,\mu)$ is the almost sure limit of a sequence $(f_n)_{n\geq 0}$ in $H^\infty_{\text{luc}}(G,\mu)$.*

Proof. Let $f \in H^\infty_{\text{a.s.}}(G,\mu)$ be given. Let $(\alpha_n)_{n\geq 0}$ be a sequnce of positive functions in $C_c(G)$, such that $\alpha_n \cdot \lambda$ converges to δ_e as $n \to \infty$. To see that such a sequence exists, we can use the countable basis of G to pick a sequence of open sets $(A_n)_{n\geq 0}$, satisfying the following conditions:

- $A_0 \subset K$, for some compact set $K \subset G$,
- $0 < \lambda(A_n) < \infty$, for all $n \in \mathbb{N}$,
- $A_0 \supset A_1 \supset \dots,$
- $\bigcap_{n=0}^\infty A_n = \{e\}$.

From here we define a sequence $(\alpha'_n)_{n\geq 0}$ of positive functions in $C_c(G)$ by

$$\alpha'_n = 1_{A_n} * 1_{A_n}.$$

Each such map is continuous, has support in A_nA_n, and hence is integrable with respect to λ. Finally, we define

$$\alpha_n = \frac{1}{\|\alpha'_n\|_1}\alpha'_n,$$

where the 1-norm is calculated with respect to the Haar measure λ. For each $n \geq 0$, define $f_n \colon G \to \mathbb{C}$ by

$$f_n(x) = \int \alpha_n(g) f(gx)\,\mathrm{d}\lambda(g), \quad x \in G$$

Then each f_n is continuous, and the sequence is converging to f almost surely. It remains to show that each f_n is in $H^\infty_{\text{luc}}(G,\mu)$. Harmonicity of f_n follows, as any $x \in G$ yields

$$\begin{aligned}
f_n(x) &= \int_G \alpha_n(g) f(gx)\,\mathrm{d}\lambda(g) = \int_G \alpha_n(g) P_\mu(f)(gx)\,\mathrm{d}\lambda(g) \\
&= \int_G\int_G \alpha_n(g) f(gxh)\,\mathrm{d}\mu(h)\,\mathrm{d}\lambda(g) \\
&= \int_G\int_G \alpha_n(g) f(gxh)\,\mathrm{d}\lambda(g)\,\mathrm{d}\mu(h) = \int_G f_n(xh)\,\mathrm{d}\mu(h).
\end{aligned}$$

To see that each f_n is left uniformly continuous, consider for $u, x \in G$ the estimates,

$$\begin{aligned}
|f_n(ux) - f_n(x)| &= \left| \int_G \alpha_n(g) f(gux)\,\mathrm{d}\lambda(g) - \int_G \alpha_n(g) f(gx)\,\mathrm{d}\lambda(g) \right| \\
&= \left| \int_G \alpha_n(gu^{-1}) f(gx)\,\mathrm{d}\lambda(g) - \int_G \alpha_n(g) f(gx)\,\mathrm{d}\lambda(g) \right| \\
&\leq \|f\|_\infty \int_G |\alpha_n(gu^{-1}) - \alpha_n(g)|\,\mathrm{d}\lambda(g).
\end{aligned}$$

As α_n is compactly supported, it is also right uniformly continuous and thus, the above integral can be made arbitrarily small. This tells us that f_n is left uniformly continuous, thus completing the proof. □

Weak harmonicity

Another case of interest is the one where all three Borel probability measures are equal to the same measure μ. In order for the Poisson transform to be well-defined, we would need μ^{*2} to be absolutely continuous with respect to μ. In general this is not true, but let us look at an interesting construction. Given a Borel probability measure μ, we define a new probability measure $\tilde{\mu}$ by

$$\tilde{\mu} = \sum_{n=0}^{\infty} 2^{-(n+1)} \mu^{*n}.$$

Here, the zeroth power of μ is the Dirac measure at the identity e. The construction of this new measure $\tilde{\mu}$ may potentially give us an entirely new class of harmonic functions, but as the following proposition shows, this is not the case.

Proposition 2.10. *Let μ be a Borel probability measure on G, and let $\tilde{\mu}$ be defined as above. Then a function $f\colon G \to \mathbb{C}$ is μ-harmonic if and only if f is $\tilde{\mu}$-harmonic.*

Proof. Assume first that f is μ-harmonic. By applying harmonicity twice we obtain

$$\int_G f(xy)\,\mathrm{d}\mu^{*2}(y) = \int_G \int_G f(xgh)\,\mathrm{d}\mu(h)\,\mathrm{d}\mu(g) = \int_G f(xg)\,\mathrm{d}\mu(g) = f(x),$$

for any $x \in G$ and hence f is μ^{*2}-harmonic. Similarly f is μ^{*n}-harmonic, for any $n \geq 1$, and thus, for any $x \in G$.

$$\int_G f(xy)\,\mathrm{d}\tilde{\mu}(y) = \sum_{n=0}^{\infty} 2^{-(n+1)} \int_G f(xy)\,\mathrm{d}\mu^{*n}(y) = f(x) \sum_{n=0}^{\infty} 2^{-(n+1)} = f(x).$$

In conclusion f is $\tilde{\mu}$-harmonic. For the other implication assume that f is $\tilde{\mu}$-harmonic and note that for any $x \in G$,

$$2f(x) = 2\sum_{n=0}^{\infty} 2^{-(n+1)} \int_G f(xy)\,\mathrm{d}\mu^{*n}(y) = f(x) + \sum_{n=1}^{\infty} 2^{-n} \int_G f(xy)\,\mathrm{d}\mu^{*n}(y).$$

For this latter infinite series, we can apply linearity of the convolution along with a reindexing of the terms, and we end up with

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{-n} \int_G f(xy)\,\mathrm{d}\mu^{*n}(y) &= \int_G \left(\sum_{n=0}^{\infty} 2^{-(n+1)} \int_G f(xgh)\,\mathrm{d}\mu^{*n}(h) \right) \mathrm{d}\mu(g) \\ &= \int_G \int_G f(xgh)\,\mathrm{d}\tilde{\mu}(h)\,\mathrm{d}\mu(g) = \int_G f(xg)\,\mathrm{d}\mu(g).\end{aligned}$$

The combination of these calculations yields the desired μ-harmonicity of f. $\square$

Note that the measure $\tilde{\mu}$ constructed above has the important property that $\tilde{\mu}^{*2}$ is absolutely continuous with respect to $\tilde{\mu}$, so we may consider the Poisson transform

$$P_{\tilde{\mu}}\colon L^{\infty}(G, \tilde{\mu}) \to L^{\infty}(G, \tilde{\mu}).$$

We will use this map to define a new class of μ-harmonic functions, whose relevance will become apparent, when we start the explicit constructions of the Poisson boundary.

Definition 2.11. We say that $[f] \in L^\infty(G, \tilde{\mu})$ is weakly μ-harmonic if

$$[f] = [P_{\tilde{\mu}}(f)].$$

The space of these functions will be denoted by $H^\infty_{\mathrm{w}}(G, \mu)$.

2.5 Topological G-spaces

For yet another generalisation of the Poisson transform, we add an action of G to the mix. A topological G-space is a topological space B along with a continuous action of G on B. Recall that an action of G on B is a map from $G \times B$ into B denoted by $(g, b) \mapsto g.b$, such that $e.b = b$ and for all $g, h \in G$ and $b \in B$,

$$g.(h.b) = (gh).b.$$

Given a compact G-space B, we have a natural way of extending the action of G on B to $C(B)$ and $\mathcal{P}(B)$. For $f \in C(B)$ and $g, x \in G$, we set $g.f(x) = f(g^{-1}x)$, and for $\nu \in \mathcal{P}(B)$ we set $g.\nu(f) = \nu(g^{-1}.f)$.

Given a Borel probability measure μ on G and a Borel probability measure ν on B, we define the convolution of μ and ν to be the Borel probability measure $\mu * \nu$ on B, given by

$$\mu * \nu(\varphi) = \int_G \int_B \varphi(g.b)\, \mathrm{d}\nu(b)\, \mathrm{d}\mu(g), \quad \varphi \in C(B).$$

Note that this definition coincides with the usual convolutions of measures, when G itself is viewed as a G-space by left multiplication.

Definition 2.12. Let μ be a Borel probability measure on G and let B be a G-space. A Borel probability measure ν on B is said to be μ-stationary if $\mu * \nu = \nu$. A G-space B equipped with a μ-stationary measure ν will be called a (G, μ)-space.

Remark 2.13. If B is a compact space, then the set of Borel probability measures on B will be a compact, convex subset of $C(B)^*$. By the Markov-Kakutani fixed point theorem, the linear map $\nu \mapsto \mu * \nu$ will have a fixed point, i.e., a μ-stationary measure on B. In other words, any compact G-space can be viewed as a (G, μ)-space.

In the following, we fix a Borel probability measure μ on G. Now, consider a μ-stationary measure ν on a G-space B. We define the Poisson transform P_ν from the bounded functions B to the bounded functions on G, by

$$P_\nu(f)(g) = \int_B f(g.b)\, \mathrm{d}\nu(b), \quad g \in G.$$

When f is a bounded function on B, then $P_\nu(f)$ is a bounded μ-harmonic function on G, as seen by the following calculations, where $g \in G$,

$$\begin{aligned}
P_\nu(f)(g) &= \int_B f(g.b)\,\mathrm{d}\nu(b) = \int_B f(g.b)\,\mathrm{d}\mu * \nu(b) \\
&= \int_B \int_G f(g.(h.x))\,\mathrm{d}\mu(h)\,\mathrm{d}\nu(b) \\
&= \int_G \int_B f((gh).x)\,\mathrm{d}\nu(b)\,\mathrm{d}\mu(h) \\
&= \int_G P_\nu(f)(gh)\,\mathrm{d}\mu(h).
\end{aligned}$$

Our interest concerning bounded harmonic functions will shortly turn to the subspace $H^\infty_{\mathrm{luc}}(G,\mu)$, and for this reason we could ask ourselves when $P_\nu(f) \in H^\infty_{\mathrm{luc}}(G,\mu)$. This boils down to studying the behaviour of $\sup_{b\in B}|f(g.b) - f(b)|$, for every $g \in G$, as

$$\begin{aligned}
\sup_{h\in G}|P_\nu(f)(gh) - P_\nu(f)(h)| &= \sup_{h\in G}\left|\int_B f(g.(h.b)) - f(h.b)\,\mathrm{d}\nu(b)\right| \\
&\leq \sup_{b\in B}|f(g.b) - f(b)|.
\end{aligned}$$

Thus, if the latter term tends to zero as g approaches the identity $e \in G$, then $P_\nu(f) \in H^\infty_{\mathrm{luc}}(G,\mu)$. If B is a compact space, then any $f \in C(B)$ will have this property and thus $P_\nu \colon C(B) \to H^\infty_{\mathrm{luc}}(G,\mu)$. In general this need not be the case, so we are forced to look at the subspace

$$C_u(B) = \left\{ f \in C(B) \;\middle|\; \sup_{b\in B}|f(g.b) - f(b)| \to 0, \text{ as } g \to e \right\}.$$

Then $P_\nu \colon C_u(B) \to H^\infty_{\mathrm{luc}}(G,\mu)$. Fortunately, most of the spaces we will encounter are compact and hence, this latter complication is not that important. Extending the Poisson transform to equivalence classes of functions, we recall the discussion in Section 2.4, which allows us to define $P_\nu \colon L^\infty(B,\nu) \to L^\infty(G,\mu)$. The stationarity of ν ensures that P_ν only takes values in $H^\infty_{\mathrm{w}}(G,\mu)$, giving us a new meaning to the notion of weak harmonicity.

2.6 Measurable G-spaces

A natural generalisation of a topological G-space is to loosen the restriction on the action of G. Instead of requiring continuity of the action, we will only require measurability. This leads to the following definition.

Definition 2.14. A measurable G-space B is a topological space on which G acts in a measurable way. Here, both B and G are equipped with their standard Borel structure. Let μ be a Borel probability measure on G. If ν is a μ-stationary Borel probability measure on B, we say that (B,ν) is a weak measurable (G,μ)-space.

Sacrificing continuity of the action on a weak measurable G-space (B,ν) does not alter any of the algebraic, nor measure theoretic considerations in Section 2.5, and hence most of the results can be applied to this case, as well. For instance, we are able to define the Poisson transform P_ν from $L^\infty(B,\nu)$ to $H^\infty_{\mathrm{w}}(G,\mu)$. However, it is not the case that the Poisson transform automatically maps into $H^\infty_{\text{a.s.}}(G,\mu)$. For this, we need some additional restrictions.

Now, instead of finding an answer to this very specific problem, we will first broaden the problem a bit. Recall that $H^\infty_{\text{a.s.}}(G,\mu)$ denotes the subspace of $L^\infty(G,\lambda)$, where each equivalence class satisfies $[f] = [P_\mu(f)]$. This equality is well-defined due to our opening discussion of Section 2.4. We also saw that this equality could make sense in more general cases, e.g., if ρ is a Borel probability measure on G such that $\rho * \mu$ is absolutely continuous with respect to ρ. In such case, we let

$$H^\infty_\rho(G,\mu) = \big\{[f] \in L^\infty(G,\rho) \mid [f] = [P_\mu(f)]\big\}.$$

By combining the discussion above with the opening discussion of Section 2.4, we obtain the following lemma as an immediate consequence.

Lemma 2.15. *Let (B,ν) be a weak measurable (G,μ)-space. For any Borel probability measure ρ on G such that $\rho * \mu$ is absolutely continuous with respect to ρ, the Poisson transform P_ν extends to a map from $L^\infty(B,\rho*\nu)$ into $H^\infty_\rho(G,\mu)$. If ρ is equivalent to λ, then P_ν extends to a map from $L^\infty(B,\rho*\nu)$ into $H^\infty_{\text{a.s.}}(G,\mu)$.*

Another setup we consider in this measurable setting is that of a strong measurable space. Before doing so, we say that a Borel probability measure α on B is G-quasi invariant, if α is equivalent to $g.\alpha$, for all $g \in G$.

Definition 2.16. Let (B,ν) be a weak measurable (G,μ)-space. If α is a G-quasi-invariant Borel probability measure on B such that ν is absolutely continuous with respect to α, then (B,ν,α) is said to be a strong measurable (G,μ)-space.

The importance of strong measurable spaces may seem unclear at this point, but they will reappear once we turn to the discussion of measurable boundaries in Section 3. For this reason, we will introduce a way of constructing a strong measurable (G,μ)-space from a weak measurable (G,μ)-space under certain assumptions.

Proposition 2.17. *Let μ be a spread-out Borel probability measure on G and (B,ν) be a weak measurable (G,μ)-space. If ρ is Borel probability measure on G and ρ is equivalent to λ, then $(B,\nu,\rho*\nu)$ is a strong measurable (G,μ)-space.*

Proof. Notice that if ρ is equivalent to λ, then ρ is also equivalent to $g.\rho$ for any $g \in G$. To see that $\rho*\nu$ is G-quasi-invariant, a simple calculation shows that $g.(\rho*\nu) = (g.\rho)*\nu$, for all $g \in G$, and hence the G-quasi-invariance of $\rho*\nu$ is inherited from that of ρ. What remains to show is that ν is absolutely continuous with respect to $\rho*\nu$.

As ν is spread-out there exists $p \geq 0$, such that μ^{*p} and λ are not mutually singular. From here we consider the Lebesgue decomposition of each μ^{*np} as $\alpha_n+\beta_n$, where α_n is absolutely continuous with respect to λ, while β_n and λ are mutually singular. An immediate consequence hereof is that $\|\beta_n\| \to 0$, as $n \to \infty$. By μ-stationarity of ν, we now see that for $n \geq 1$,

$$\|\nu - \alpha_n * \nu\| = \|\mu^{*np} * \nu - \alpha_n * \nu\| = \|\beta_n * \nu\| \leq \|\beta_n\|,$$

and hence $\alpha_n * \nu$ converges to ν in total variation. By the assumption on ρ, the measures α_n will also be absolutely continuous with respect to ρ, and hence $\alpha_n * \nu$ is absolutely continuous with respect to $\rho * \nu$, for all $n \geq 1$. By the above established convergence, we conclude that ν is absolutely continuous with respect to $\rho * \nu$, thus concluding the proof. $\square$

3 Boundaries

The next important topic of this thesis is that of boundaries associated to a group G. In the classical setup we saw that there is a one-to-one correspondence between harmonic functions on $B(0,1)$ and continuous functions on $\partial B(0,1)$. This correspondence is defined through the boundary representation, which we call the Poisson integral. In Section 2.5, we defined the Poisson transform in relation to a topological (G,μ)-space and saw how the Poisson transform gives rise to μ-harmonic functions. Let us try to make a parallel between these ideas, as we did with harmonic functions.

We let G be the group defined in Section 2.1 along with the associated action of G on $\mathbb{C}$. Let once more σ denote the Lebesgue measure on $\partial B(0,1)$, and consider for $\varphi \in L^1(\partial B(0,1))$ and $g \in G$, the integral

$$\begin{aligned}\int_{\partial B(0,1)} \varphi(g(\theta))\,\mathrm{d}\sigma(\theta) &= \int_{\partial B(0,1)} \varphi(\theta)|(g^{-1})'(\theta)|\,\mathrm{d}\sigma(\theta) \\ &= \int_{\partial B(0,1)} \varphi(\theta)P(r,t-\theta)\,\mathrm{d}\sigma(\theta),\end{aligned}$$

where $g(0)=re^{it}$. This gives a connection between the natural action of G on $\mathbb{C}$ and the classical Poisson integral. For $r \in (0,1)$, we equip G with the measure $\overline{\sigma_r}$ as defined in Section 2.1, and let f be a harmonic function on $B(0,1)$. Then $\overline{f}\colon G \to \mathbb{C}$, defined by $\overline{f}(g) = f(g(0))$ is $\overline{\sigma_r}$-harmonic. If we let $\tilde{f}\colon \overline{B(0,1)} \to \mathbb{C}$ denote the continuous extension of f as defined in the very end of Section 1.1, then for $g \in G$ with $g(0) = re^{it}$,

$$\begin{aligned}\overline{f}(g) = f(g(0)) = f(re^{it}) &= \int_{\partial B(0,1)} P(r,t-\theta)\tilde{f}(\theta)\,\mathrm{d}\sigma(\theta) \\ &= \int_{\partial B(0,1)} \tilde{f}(g(\theta))\,\mathrm{d}\sigma(\theta) = P_\sigma(\tilde{f}).\end{aligned}$$

By the bijective correspondence between harmonic functions on $B(0,1)$ and continuous functions on $\partial B(0,1)$, we see that P_σ defines a map from $C(\partial B(0,1))$ into the $\overline{\sigma_r}$-harmonic functions on G.

The motivating idea behind defining an appropriate notion of boundaries of a group G is to find compact (G,μ)-spaces for which the Poisson transform has certain properties such as: being an isometric isomorphism, an isometry or being multiplicative. The remainder of this section will follow Chapter 2 and the beginning of Chapter 3 in [2].

3.1 Poisson boundary

The idea behind the Poisson boundary is to create a topogical space B, such that every function $H^\infty_{\mathrm{luc}}(G,\mu)$ can be uniquely represented by a continuous function on B. This will mimic the integral representation from the classical case, with B representing the *boundary* of G, as seen in the example above. We will do this by equipping $H^\infty_{\mathrm{luc}}(G,\mu)$ with a multiplication and involution, such that it becomes a unital commutative C^*-algebra. Then the space of left uniformly continuous μ-harmonic functions on G will be isometrically isomorphic to the continuous functions on the spectrum Π_μ of this C^*-algebra.

The involution of choice is simply pointwise complex conjugation of the given function $f \in H^\infty_{\text{luc}}(G, \mu)$. The multiplication becomes a bit more tricky, as the pointwise product of μ-harmonic functions need not be μ-harmonic itself. Instead, we recall the isometric identification $f \mapsto Z_f$ of $H^\infty_{\text{luc}}(G, \mu)$ with I^∞_{luc}. The product $\odot$ is then be defined as follows

$$(f \odot g)(x) = E(Z_f(x)Z_g(x)), \quad f, g \in H^\infty_{\text{luc}}(G, \mu), x \in G.$$

The fact that this product is even well-defined and actually gives rise to a unital commutative C^*-algebra is seen in the following proposition.

Proposition 3.1. *Let μ be a Borel probability measure on G. The operation $\odot$ set forth above defines a multiplication on $H^\infty_{\text{luc}}(G, \mu)$, and pointwise complex conjugation defines an involution on $H^\infty_{\text{luc}}(G, \mu)$. Furthermore, the resulting $*$-algebra is a unital commutative C^*-algebra with respect to the supremum norm.*

Proof. To see that this product is even well-defined it suffices by Proposition 2.6 to show that I^∞_{luc} is closed under pointwise products. The invariance property of functions in I^∞_{luc} is clearly inherited through products, and it is not difficult to see that the product remains left uniformly continuous. For the latter statement, let $Z, Z' \in I^\infty_{\text{luc}}$ be given. Then for $u, g \in G$,

$$\begin{aligned}
\|ZZ'(ug) - ZZ'(g)\|_\infty &= \|Z(ug)Z'(ug) - Z(g)Z'(g)\|_\infty \\
&\leq \|Z(ug)Z'(ug) - Z(ug)Z'(g)\|_\infty + \|Z(ug)Z'(g) - Z(g)Z'(g)\|_\infty \\
&\leq \|Z\|\|Z'(ug) - Z'(g)\|_\infty + \|Z'\|\|Z(ug) - Z(g)\|_\infty,
\end{aligned}$$

which can be made arbitrarily small by applying left uniform continuity of both Z and Z'. That the product is distributive with respect to the pointwise addition follows directly from the pointwise distributivity of summation. Hovever, the associativity of this product is non-trivial, but the above construction is a particularly nice version of the Choi-Effros product, which is associative (see [4]). We are now left with showing that $\|f \odot g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$ and $\|\overline{f} \odot f\|_\infty = \|f\|^2_\infty$ for any $f, g \in H^\infty_{\text{luc}}(G, \mu)$. The first inequality is not a big problem, as

$$\begin{aligned}
\|f \odot g\|_\infty &= \sup_{x \in G} \left|(f \odot g)(x)\right| = \sup_{x \in G} \left|E(Z_f(x)Z_g(x))\right| \\
&\leq \sup_{x \in G} \left(\|Z_f(x)\|_\infty \cdot \|Z_g(x)\|_\infty\right) \\
&\leq \sup_{x \in G} \|Z_f(x)\|_\infty \cdot \sup_{x \in G} \|Z_g(x)\|_\infty = \|f\|_\infty \cdot \|g\|_\infty.
\end{aligned}$$

To see that the C^*-identity is satisfied, we will introduce an alternative approach to the product. Combining the convolution powers of μ and the random walk $(R^g_n)_{n \geq 0}$ it is evident that

$$(f \star \mu^{*n})(g) = Ef(gR_n),$$

for all $n \in \mathbb{N}$ and $g \in G$. In this way, we can characterise the product $f \odot g$ as the pointwise limit of $(f \cdot g) \star \mu^{*n}$, as $n \to \infty$. With this in mind it suffices to show that $\||f|^2 \star \mu^{*n}\| = \|f\|^2$, for all $n \in \mathbb{N}$. It is quite obvious that $|f|^2 \star \mu^{*n}$ is uniformly bounded by $\|f\|^2$, but for the reverse inequality we consider the case $n = 1$. We have

$$|f|^2 = \left|f \star \mu\right|^2 \leq \left(|f| \star \mu\right)^2 \leq |f|^2 \star \mu,$$

where the last bound is due to the Cauchy-Schwartz inequality. Applying supremum on both sides gives us the desired inequality, and a standard induction argument gives the general result.

Finally the commutativity of $\odot$ is inherited through pointwise multiplication of complex-valued functions, which all in all yields the desired conclusion. ☐

Definition 3.2. Let μ be a Borel probability measure on G. The spectrum Π_μ of $H^\infty_{\mathrm{luc}}(G, \mu)$ is called the Poisson boundary of the pair (G, μ).

In the context of Section 2.5, we could ask ourselves if the above established isomorphism could be realised as the Poisson transform of some Borel probability measure on Π_μ. This question leads to the following theorem.

Theorem 3.3. *Let μ be a Borel probability measure on G and let Π_μ be the corresponding Poisson boundary. Then there exists a stationary measure ν on Π_μ such that that P_ν is an isometric isomorphism from $C(\Pi_\mu)$ to $H^\infty_{\mathrm{luc}}(G, \mu)$. This measure will henceforth be denoted by $\overline{\nu}$.*

Proof. Note first of all that the canonical $*$-isomorphism $\Phi\colon C(\Pi_\mu) \to H^\infty_{\mathrm{luc}}(G, \mu)$ is a G-equivariant map, with the natural actions as described in Section 2.5. From here on, let L be a linear functional on $C(\Pi_\mu)$ defined by

$$L(\varphi) = \Phi(\varphi)(e), \quad \varphi \in C(\Pi_\mu).$$

By the Riesz Representation Theorem there exists a probability measure ν on Π_μ, such that $L(\varphi) = \int_{\Pi_\mu} \varphi \, \mathrm{d}\nu$, for all $\varphi \in C(\Pi_\mu)$. Then ν is μ-stationary as

$$\begin{aligned} \mu * \nu(\varphi) &= \int_G \int_{\Pi_\mu} \varphi(g.b) \, \mathrm{d}\nu(b) \, \mathrm{d}\mu(g) = \int_G \int_{\Pi_\mu} {}_{g^{-1}}\varphi(b) \, \mathrm{d}\nu(b) \, \mathrm{d}\mu(g) \\ &= \int_G L({}_{g^{-1}}\varphi) \, \mathrm{d}\mu(g) = \int \Phi(\varphi)(g) \, \mathrm{d}\mu(g) = \Phi(\varphi)(e) = \int_{\Pi_\mu} \varphi \, \mathrm{d}\nu, \end{aligned}$$

for all $\varphi \in C(\Pi_\mu)$. Furthermore, the Poisson transform P_ν is equal to Φ. To see this, let $\varphi \in C(\Pi_\mu)$ be given. Then G-equivariance yields

$$P_\nu(\varphi)(g) = \int_{\Pi_\mu} \varphi(g.b) \, \mathrm{d}\nu(b) = \int_{\Pi_\mu} {}_{g^{-1}}\varphi(b) \, \mathrm{d}\nu(b) = L({}_{g^{-1}}\varphi) = \Phi(\varphi)(g),$$

for all $g \in G$. ☐

Proposition 3.4. *Assume that the semigroup generated by* $\mathrm{supp}(\mu)$ *is all of G. Then $\overline{\nu}$ has full support in $\mathbb{P}_\mu$.*

Proof. Assume in order to reach a contradiction that $\operatorname{supp}(\overline{\nu})$ is a proper subset of Π_μ. Let $f \in C(\Pi_\mu)$ be given, with $\operatorname{supp}(f) \subset \Pi_\mu \setminus \operatorname{supp}(\overline{\nu})$ and $\|f\|_\infty = 1$. Then

$$0 = \int_{\Pi_\mu} f(x)\,\mathrm{d}\overline{\nu}(x) = \int_G \int_{\Pi_\mu} f(g.x)\,\mathrm{d}\overline{\nu}(x)\,\mathrm{d}\mu(g) = \int_G P_{\overline{\nu}}(f)(g)\,\mathrm{d}\mu(g),$$

and hence $P_{\overline{\nu}}(f)(g) = 0$ for μ-a.e. $g \in G$. Applying μ-stationarity of $\overline{\nu}$ multiple times along with our assumptions on μ, gives us $P_{\overline{\nu}}(f) = 0$ on a dense subset of G. As Π_μ is compact, $P_{\overline{\nu}}(f)$ is continuous and hence constantly equal to 0. But now $P_{\overline{\nu}}$ is no longer an isometry and hence we have reached a contradiction. In conclusion $\overline{\nu}$ must have full support. □

The assumption of $\operatorname{supp}(\mu)$ generating all of G as a semigroup is another way of saying that the random walk $(R_n)_{n\geq 0}$ will hit any element of G almost surely, and will appear later as an assumption in many of the coming results. Another notable consequence of this criteria is that, whenever G is abelian, then Π_μ is a singleton by Proposition 2.7.

3.2 An example

Let us as an example consider $\mathbb{F}_2$, the free group of two generators a, b, and let $A \subset \mathbb{F}_2$ denote the set $\{a, b, a^{-1}, b^{-1}\}$. As a probability measure on $\mathbb{F}_2$, we let μ denote the equidistribution on A, i.e.,

$$\mu = \frac{1}{4}\big(\delta_a + \delta_b + \delta_{a^{-1}} + \delta_{b^{-1}}\big).$$

In this way, a bounded function $f\colon \mathbb{F}_2 \to \mathbb{C}$ belongs to $H^\infty_{\mathrm{luc}}(\mathbb{F}_2, \mu)$ if and only if

$$f(\gamma) = \frac{1}{4}\big(f(\gamma a) + f(\gamma b) + f(\gamma a^{-1}) + f(\gamma b^{-1})\big), \quad \text{for } \gamma \in \mathbb{F}_2.$$

Our goal is now to identify the Poisson-boundary $(\Pi_\mu, \overline{\nu})$. With the classical case and the name *boundary* in mind, a naive and intuitive approach would be to guess that $\Pi_\mu = B$, where B denotes the space of infinite reduced words with letters from A. Fortunately this turns out to be the case, so let us study this space.

First of all, B is a closed subset of $A^{\mathbb{N}}$, and hence compact. Secondly $\mathbb{F}_2$ acts on B by adding $\gamma \in \mathbb{F}_2$ to the front of the infinite word and then reducing the obtained word. For an infinite word $\gamma \in B$, we let $\gamma|_n \in \mathbb{F}_2$ be the word consisting of the first n letters of γ. Finally, for $\gamma \in \mathbb{F}_2$, we let $[\gamma] \subset B$ be the set consisting of all infinite words α, such that $\alpha|_{|\gamma|} = \gamma$, where $|\gamma|$ is the length of γ. That is, $[\gamma]$ consists of all infinite continuations of γ. We will now define a measure ν on B by

$$\nu([\gamma]) = \frac{1}{4} \cdot \left(\frac{1}{3}\right)^{|\gamma|-1}, \quad \gamma \in \mathbb{F}_2.$$

Note that ν is actually uniquely determined by its values on $[\gamma]$ for $\gamma \in \mathbb{F}_2$ as these sets comprise an intersection stable generator of the Borel sets in B. Our claim is now that (B, ν) is the Poisson boundary of (G, μ).

To prove this claim, we start out by adding B to $\mathbb{F}_2$ and thus compactifying $\mathbb{F}_2$. The compactification is to be understood as follows. A sequence $(\gamma_n)_{n\geq 0}$ in $\mathbb{F}_2$ converges to $\gamma \in B$ if for any $k \in \mathbb{N}$ there exists $N_k \in \mathbb{N}$ such that $n \geq N_k$ implies $\gamma_n \in [\gamma|_k]$. The role of our proposed *boundary* can now be described through the theory of random walks.

Consider a sequence $(X_n)_{n\geq 0}$ of i.i.d. random variables with values in A and common distribution μ. As in Section 2.3, these random variables will be defined as coordinate projections from the sequence space $\Omega = A^{\mathbb{N}_0}$ equipped with the countable product measure $\mathbb{P} = \bigotimes_{n\geq 0} \mu$. From here we define the random walk $(R_n)_{n\geq 0}$ by $R_0 = e$ and $R_n = X_0 X_1 \cdots X_{n-1}$ for $n \geq 1$. The property of $(R_n)_{n\geq 0}$ which interests us the most is the average length of the words at each step, so let us define a sequence of numbers $(u_n)_{n\geq 0}$ by

$$u_n = E|R_n|, \quad n \geq 0,$$

where $|R_n|$ denotes the length of the word R_n. It is clear that $u_0 = 0$ and $u_1 = 1$, but from there on it gets a bit more complicated due to the possibility of cancellations. However, dividing Ω into two disjoint subsets representing cancellation and no cancellation, respectively, simplifies this problem. More precisely, for $n \geq 1$,

$$\begin{aligned} u_{n+1} = E|R_{n+1}| &= E1_{\{X_n = X_{n-1}^{-1}\}}|R_{n+1}| + E1_{\{X_n \neq X_{n-1}^{-1}\}}|R_{n+1}| \\ &= E1_{\{X_n = X_{n-1}^{-1}\}}|R_n - 1| + E1_{\{X_n \neq X_{n-1}^{-1}\}}|R_n + 1| \\ &= E|R_n| - \mathbb{P}(X_n = X_{n-1}^{-1}) + \mathbb{P}(X_n \neq X_{n-1}^{-1}) = u_n + 1/2 \end{aligned}$$

In conclusion, $u_0 = 0$ and $u_n = (n+1)/2$ for $n \geq 1$. In particular $(u_n/n)_{n\geq 1}$ will be a decreasing sequence converging to $1/2$. In order to use this convergence in mean, we notice that the shift operator T on Ω preserves $\mathbb{P}$ and

$$|R_{n+m}(\omega)| = |R_n(\omega)R_m(T^n(\omega))| \leq |R_n(\omega)| + |R_m(T^n(\omega))|.$$

Thus, $(|R_n|)_{n\geq 0}$ is a subadditive process, so the limit $\lim_{n\to\infty} |R_n|/n$ exists $\mathbb{P}$-almost surely and in mean by Kingman's Subadditive Ergodic Theorem (see Theorem B.1). Since $\lim_{n\to\infty} u_n/n > 0$, we conclude that $|R_n| \to \infty$ $\mathbb{P}$-a.s., as $n \to \infty$, and hence there exists $z\colon \Omega \to B$, such that $R_n(\omega) \to z(\omega)$, $\mathbb{P}$-almost surely. To determine the distribution of ν, let $\gamma \in \mathbb{F}_2$ be given and observe that

$$z(\mathbb{P})([\gamma]) = \mathbb{P}(z \in [\gamma|) = \mathbb{P}(z|_{|\gamma|} = \gamma) = \mathbb{P}(\lim_{n\to\infty} R_n|_{|\gamma|} = \gamma) = \mathbb{P}(R_{|\gamma|} = \gamma).$$

As the X_n's are independent and equidistributed on A, we see that $z(\mathbb{P})$ is actually equal to our previously defined measure ν. By stationarity of the process $(X_n)_{n\geq 0}$, we can also conclude that z and $z \circ T$ have the same distribution. Combining this fact with the $\mathbb{P}$-almost sure equality $z(\omega) = X_0(\omega).z(T(\omega))$, we conclude that the measure ν is μ-stationary. This is true, since for any $\varphi \in C(B)$,

$$\begin{aligned}\nu(\varphi) &= \int_B \varphi(\gamma)\,\mathrm{d}\nu(\gamma) = \int_\Omega \varphi(z(\omega))\,\mathrm{d}\mathbb{P}(\omega)\\ &= \int_\Omega \varphi(X_0(\omega).z(T(\omega)))\,\mathrm{d}\mathbb{P}(\omega)\\ &= \int_B\int_A \varphi(\alpha.\gamma)\,\mathrm{d}X_0(\mathbb{P})(\alpha)\,\mathrm{d}(z\circ T)(\mathbb{P})(\gamma)\\ &= \int_B\int_A \varphi(\alpha.\gamma)\,\mathrm{d}\mu(\alpha)\,\mathrm{d}\nu(\gamma) = \mu * \nu(\varphi).\end{aligned}$$

This shows that (B,ν) is actually an $(\mathbb{F}_2,\mu)$-space. What remains to show is that the Poisson transform $P_\nu\colon C(B)\to H^\infty_{\mathrm{luc}}(\mathbb{F}_2,\mu)$ is well-defined and a $*$-isomorphism. From our previous discussion on harmonic functions applied to random walks, we know that there exists an isometric, bijective correspondence between $H^\infty_{\mathrm{luc}}(\mathbb{F}_2,\mu)$ and I^∞_{luc}, which for $f\in H^\infty_{\mathrm{luc}}(\mathbb{F}_2,\mu)$ and $\gamma\in\mathbb{F}_2$ is given by

- $f(\gamma R_n(\omega))\to Z_f(\gamma)(\omega)$, $\mathbb{P}$-almost surely,
- $E(Z_f(\gamma)) = f(\gamma)$.

In order to translate this into a correspondence between $C(B)$ and $H^\infty_{\mathrm{luc}}(\mathbb{F}_2,\mu)$, we consider the behaviour of $Z_f(\gamma)$ for a given $f\in H^\infty_{\mathrm{luc}}(\mathbb{F}_2,\mu)$. For this, let $\omega,\omega'\in\Omega$ be given such that R_n converges in both ω and ω', but also satisfying $z(\omega)=z(\omega')$. Then there exist subsequences $(n_k)_{k\geq 0}$ and $(m_{n_k})_{k\geq 0}$ in $\mathbb{N}_0$, such that $R_{n_k}(\omega) = R_{m_{n_k}}(\omega')$, for all $k\geq 0$. This tells us that $Z_f(\gamma)(\omega)$ only depends on the value of $\gamma.z(\omega)$, so there exists a map $\varphi\in C(B)$ such that $\varphi(\gamma.z(\omega)) = Z_f(\gamma)(\omega)$.

The isometric correspondence between f and Z_f will then extend to an isometric correspondence between f and φ, so it remains to show that f arises as the Poisson transformation of φ. That this is true is a consequence of our construction, as

$$\begin{aligned}f(\gamma) = EZ_f(\gamma) &= \int_\Omega Z_f(\gamma)(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \int_\Omega \varphi(\gamma.z(\omega))\,\mathrm{d}\mathbb{P}(\omega)\\ &= \int_B \varphi(\gamma.\alpha)\,\mathrm{d}z(\mathbb{P})(\alpha) = \int_B \varphi(\gamma.\alpha)\,\mathrm{d}\nu(\alpha) = P_\nu(\varphi)(\gamma),\end{aligned}$$

for every $\gamma\in\mathbb{F}_2$. In conclusion, (B,ν) is the Poisson boundary of $(\mathbb{F}_2,\mu)$.

3.3 General μ-boundaries

At this point we have shown that given a compact (G,μ)-space (B,ν), the Poisson transform P_ν is a map from $C(B)$ into $H^\infty_{\mathrm{luc}}(G,\mu)$, and it is not difficult to see that it is also norm decreasing. Furthermore, if the (G,μ)-space is the Poisson boundary Π_μ with the measure $\overline{\nu}$ from Theorem 3.3, then $P_{\overline{\nu}}$ is an isometric isomorphism.

By the Banach-Stone theorem, no other compact (G,μ)-space (B,ν), up to homeomorphism, will induce an isometric isomorphism, but there is no reason why the Poisson transform could not be isometric nor a $*$-homomorphism. The question regarding when this happens for compact (G,μ)-spaces is fully answered by the following proposition.

Proposition 3.5. *Let (B, ν) be a compact (G, μ)-space. The Poisson transform P_ν is an isometric embedding of $C(B)$ into $H^\infty_{\text{luc}}(G, \mu)$ if and only if*

$$\delta_B \subset \overline{\{\delta_g * \nu \mid g \in G\}}^{w^*}.$$

Proof. Assume first that $\delta_B \subset \overline{\{\delta_g * \nu \mid g \in G\}}^{w^*}$, and let us show that P_ν is an isometry. As the Poisson transform P_ν is always norm decreasing it suffices to show that $\|f\|_\infty \leq \|P_\nu(f)\|_\infty$, for any $f \in C(B)$. Let $b \in B$ be given and use the hypothesis to pick a net $(g_i)_{i \in I}$ in G such that $\delta_{g_i} * \nu \xrightarrow{w^*} \delta_b$, i.e.,

$$\int_B f(x) \, \mathrm{d}(\delta_{g_i} * \nu)(x) \to \int_B f(x) \, \mathrm{d}\delta_b(x) = f(b),$$

for all $f \in C(B)$. For any $f \in C(B)$, we notice that

$$\int_B f(x) \, \mathrm{d}(\delta_{g_i} * \nu)(x) = P_\nu(f)(g_i) \leq \|P_\nu(f)\|_\infty,$$

and hence $|f(b)| \leq \|P_\nu(f)\|_\infty$. As $b \in B$ was arbitrary we see that P_ν is an isometry.

For the converse inclusion, assume that P_ν is an isometry and let $b \in B$ be given. By Urysohn's lemma we can consider the non-empty set

$$F_b = \{f \in C(B) \mid 0 \leq f \leq 1, \; f(b) = 1\}.$$

As ν is a probability measure, P_ν preserves positive functions, and hence for any $f \in F_b$ and $n \in \mathbb{N}$, the set

$$A_{f,n} = \{g \in G \mid P_\nu(f)(g) \geq 1 - \tfrac{1}{n}\},$$

is non-empty. Furthermore, if $f, f' \in F_b$ then $\inf(f, f') \in F_b$, so the inclusion

$$A_{f,n} \cap A_{f',n} \supset A_{\inf(f,f'),\sup(n,n')},$$

shows that the family $(A_{f,n})_{f \in F_b, n \in \mathbb{N}}$ has the finite intersection property. Consider now the family $(B_{f,n})_{f \in F_b, n \in N}$ of subsets in $\mathcal{P}(B)$ defined by

$$B_{f,n} = \{\delta_g * \nu \mid g \in A_{f,n}\}.$$

This family will inherit the finite intersection property and since $\mathcal{P}(B)$ inherits compactness from B, we can conclude that there exists some

$$\tau \in \bigcap_{f \in F_b, n \in \mathbb{N}} B_{f,n}.$$

By construction of the family $(B_{f,n})_{f \in F_b, n \in \mathbb{N}}$ this τ will satisfy $\int_B f \, \mathrm{d}\tau = 1$, for all $f \in F_b$. As τ is a probability measure and f is bounded by one, this implies that $f = 1$ on the support of τ. An important part of Urysohn's lemma is that the functions in F_b separate b from all other elements of B, and hence the support of τ must be the singleton $\{b\}$. In conclusion, $\delta_b = \tau \in \overline{\{\delta_g * \nu \mid g \in G\}}^{w^*}$. □

Definition 3.6. Let (B,ν) be a (G,μ)-space. If $\delta_B \subset \overline{\{\delta_g * \nu \mid g \in G\}}^{w^*}$, we say that (B,ν) is a contractible (G,μ)-space.

The contractible (G,μ)-spaces will, due to the above proposition, induce an isometric Poisson transform, but they are also related to the Poisson boundary, as seen in the following theorem.

Theorem 3.7. *Let (B,ν) be a compact contractible (G,μ)-space. Then there exists a G-equivariant map $q\colon (\Pi_\mu, \overline{\nu}) \to \mathcal{P}(B)$ such that $\delta_B \subset q(\Pi_\mu)$.*

Proof. We start out by looking at the map $\varphi\colon C(B) \to C(\Pi_\mu)$ given by $\varphi = P_{\overline{\nu}}^{-1} \circ P_\nu$. As both $P_{\overline{\nu}}^{-1}$ and P_ν are positive isometries, so is φ, and hence the adjoint map $\varphi^*\colon C(\Pi_\mu)^* \to C(B)^*$ is a positive isometry and maps $P(\Pi_\mu)$ to $P(B)$. We define $q\colon \Pi_\mu \to P(B)$ by $q(x) = \varphi^*(\delta_x)$, for $x \in G$, giving a G-equivariant map. The proof of the final claim will follow the same pattern as the second part of the proof of Proposition 3.5. For $b \in B$, we consider the set

$$F_b = \{f \in C(B) \mid 0 \leq f \leq 1, f(b) = 1\}.$$

As φ is a positive isometry, the sets $A_{f,n}$, for $f \in F_b$ and $n \in \mathbb{N}$, given by

$$A_{f,n} = \left\{x \in \Pi_\mu \mid \varphi(f)(x) \geq 1 - \tfrac{1}{n}\right\},$$

are all non-empty and the family $(A_{f,n})_{f\in F_b, n\in\mathbb{N}}$ has the finite intersection property. We can then consider the family $(B_{f,n})_{f\in F_b, n\in\mathbb{N}}$ of subsets in $\mathcal{P}(B)$ given by

$$B_{f,n} = \{\varphi^*(\delta_x) \mid x \in A_{f,n}\}.$$

This family inherits the finite intersection property and thus, there exists τ in $\bigcap_{f\in F_b, n\in\mathbb{N}} B_{f,n}$. By construction of $(B_{f,n})_{f\in F_b, n\in\mathbb{N}}$, the measure τ must be equal to δ_b, and hence $\delta_b \in q(\Pi_\mu)$. $\square$

We will now see how the random walk from Section 2.3 can be used to describe other types of (G,μ)-spaces.

Lemma 3.8. *Let (B,ν) be a compact (G,μ)-space with countable basis. Then there exists a map $\mathcal{V}\colon \Omega \to \mathcal{P}(B)$, such that*

$$gR_n(\omega).\nu \to g.\mathcal{V}(\omega), \qquad \mathbb{P}\text{-a.s.}$$

for all $g \in G$, where $(\Omega, \mathbb{P})$ is the probability space from Section 2.3. Furthermore the map $g \mapsto g.\mathcal{V}$ is left uniformly continuous for $\mathbb{P}$-a.s., and $\mathrm{E}\,\mathcal{V} = \nu$. Here $\mathrm{E}\,\mathcal{V}$ is defined as the probability measure on B, given as

$$\mathrm{E}\,\mathcal{V}(\varphi) = \int_\Omega \mathcal{V}(\omega)(\varphi)\,\mathrm{d}\mathbb{P}(\omega), \quad \varphi \in C(B).$$

Proof. The action of G on ν is the action induced from B, when viewing ν as a linear functional on $C(B)$. Let $\varphi \in C(B)$ be given and let $f = \mathcal{P}_\nu(\varphi) \in H^\infty_{\text{luc}}(G,\mu)$. Then $gR_n(\omega).\nu(\varphi) = f(gR_n(\omega))$, for $\omega \in \Omega, g \in G$ and thus $gR_n(\omega).\nu(\varphi)$ converges $\mathbb{P}$-almost surely to $Z_f(g)(\omega)$, as seen in Section 2.3. The countable basis of B ensures separability of $C(B)$, and thus $gR_n(\omega).\nu$ converges $\mathbb{P}$-almost surely.

Defining $\mathcal{V}\colon \Omega \to \mathcal{P}(B)$ to be this almost sure limit, will then give us the desired result. Left uniform continuity is inherited from that of $gR_n(\omega).\nu$ and the expectation formula follows from the Lebesgue dominated convergence theorem, as

$$E(g.\mathcal{V}(\varphi)) = \lim_{n\to\infty} E(gR_n.\nu(\varphi)) = E(g.\nu(\varphi)) = g.\nu(\varphi), \quad \varphi \in C(B).$$

The second equality above is a consequence of the stationarity of ν. □

It is worth pointing out that the above construction implies that $Z_{P_\nu(\varphi)}(g)$ is equal to $g.\mathcal{V}(\varphi)$ $\mathbb{P}$-almost surely. This identification will come in handy in the following proposition, in which we apply the above theorem to the Poisson boundary.

Proposition 3.9. *Let μ be a probability measure on G, and assume that the Poisson boundary $(\Pi_\mu, \overline{\nu})$ is second countable. Then $\mathcal{V}$ takes its values in δ_{Π_μ}, i.e., there exists a measurable map $z\colon \Omega \to \Pi_\mu$ such that*

$$\mathcal{V}(\omega) = \delta_{z(\omega)}, \qquad \mathbb{P}\text{- a. s.}$$

Proof. As $\mathcal{P}_{\overline{\nu}}\colon C(\Pi_\mu) \to H^\infty_{\text{luc}}(G,\mu)$ is a $*$-homomorphism, we see by the remark above that any $\varphi \in C(\Pi_\mu)$ satisfies

$$g.\mathcal{V}(\omega)(\varphi^2) = Z_{P_{\overline{\nu}}(\varphi^2)}(g)(\omega) = (Z_{P_{\overline{\nu}}(\varphi)}(G,\omega))^2 = (g.\mathcal{V}(\omega)(\varphi))^2,$$

where $\omega \in \Omega, g \in G$. By Jensen's inequality this only happens if φ is constant on the support of $g.\mathcal{V}(\omega)$, and since it holds true for all $\varphi \in C(\Pi_\mu)$, we conclude that $g.\mathcal{V}(\omega)$ and hence $\mathcal{V}(\omega)$ is a one-point measure $\mathbb{P}$-almost surely. From there on it is a matter of defining $z\colon \Omega \to \Pi_\mu$ to represent these points. □

We know that $(\Pi_\mu, \overline{\nu})$ is the only (G,μ)-space for which the Poisson transform $P_{\overline{\nu}}$ is a $*$-isomorphism, but the above proof only required a $*$-homomorphism. With this in mind it is natural to ask which (G,μ)-spaces give rise to a $*$-homomorphism. A necessary condition, by the above proposition, is that $\mathcal{V}$ takes values in the one-point measures, but a simple computation shows that this is also a sufficient condition. In other words, for any second countable (G,μ)-space (B,ν), the Poisson transformation $P_\nu\colon C(B) \to H^\infty_{\text{luc}}(G,\mu)$ is a $*$-homomorphism if and only if

$$gR_n(\omega).\nu \to g.\delta_{z(\omega)} \qquad \mathbb{P}\text{- a. s.},$$

for some measurable map $z\colon \Omega \to B$. This leads to the definition of a μ-boundary.

Definition 3.10. A second countable (G,μ)-space (B,ν) is said to be a μ-boundary if there exists a measurable map $z\colon \Omega \to B$, such that $R_n(\omega).\nu$ converges to $\delta_{z(\omega)}$, for $\mathbb{P}$-almost any $\omega \in \Omega$.

As the properties of a μ-boundary (B,ν) solely rely on the measure ν and the action of G, we will always assume that B has no proper G-invariant subset containing $\operatorname{supp}(\nu)$. This precaution gives us control of the set B, which otherwise could be made arbitrarily large and wild outside the reach of $\operatorname{supp}(\nu)$ and its G-translates. The following proposition shows that any compact, second countable G-space gives rise to a μ-boundary, a result for which we omit the proof.

Proposition 3.11 (2.12, [9]). *Let B be a second countable, compact G space, and let μ be a Borel probability measure on G, for which the semigroup generated by* $\operatorname{supp}(\mu)$ *is all of G. Then $\mathcal{P}(B)$ contains a μ-boundary.*

We will also introduce a small proposition regarding the map $z\colon \Omega \to B$ corresponding to the μ-boundary (B,ν).

Proposition 3.12. *Let (B,ν) be a μ-boundary, with $z\colon \Omega \to B$, such that $R_n(\omega).\nu$ converges to $\delta_{z(\omega)}$, for $\omega \in \Omega$. Then ν is the distribution of the random variable z.*

Proof. Let $f \in C(B)$ be given. Then an application of the abstract change of variable formula yields

$$\begin{aligned}
z(\mathbb{P})(f) &= \int_\Omega f(z(\omega))\,\mathrm{d}\mathbb{P}(\omega) = \int_\Omega \int_B f(x)\,\mathrm{d}\delta_{z(\omega)}(x)\,\mathrm{d}\mathbb{P}(\omega) \\
&= \lim_{n\to\infty} \int_\Omega \int_B f(R_n(\omega).x)\,\mathrm{d}\nu(x)\,\mathrm{d}\mathbb{P}(\omega) \\
&= \lim_{n\to\infty} \int_\Omega \int_B f(g.x)\,\mathrm{d}\nu(x)\,\mathrm{d}R_n(\mathbb{P})(g) \\
&= \lim_{n\to\infty} \int_\Omega \int_B f(g.x)\,\mathrm{d}\nu(x)\,\mathrm{d}\mu^{*n}(g) \\
&= \lim_{n\to\infty} \mu^{*n} * \nu(f) = \nu(f),
\end{aligned}$$

which proves the claim. □

A consequence of this proposition is that any μ-boundary (B,ν) for which ν has full support will also be contractible, as $z(\Omega)$ will be dense in B.

3.4 Universality of Π_μ.

Besides being an example of a μ-boundary, the Poisson boundary also turns out to be the largest amongst μ-boundaries, whenever the support of μ is sufficiently large. By this, we mean that any μ-boundary can be obtained as an equivariant image of Π_μ. Throughout this section, we assume that the semigroup generated by μ is all of G. The interest in this assumption is a consequence of the following proposition.

Proposition 3.13 ([9]). *Let (B,ν) be a compact (G,μ)-space. Then* $\operatorname{supp}(\nu)$ *is an invariant set under the action of G.*

Proof. As each $g \in G$ gives rise to an automorphism of B it suffices to show that $A = B \setminus \operatorname{supp}(\nu)$ is G-invariant. Let $x \in A$ be given and pick an open set U containing x and with $\nu(U) = 0$. By μ-stationarity of ν we see that

$$0 = \nu(U) = \mu * \nu(U) = \int_G \nu(g^{-1}U)\, \mathrm{d}\mu(g).$$

Then $\nu(g^{-1}U) = 0$ for any $g \in \operatorname{supp}(\mu)$, which again implies that $g^{-1}x \in A$ for all $g \in \operatorname{supp}(\mu)$. The first conclusion is that A is invariant under the action of $\operatorname{supp}(\mu)^{-1}$. For a general $g \in G$, we use our assumption on μ to pick $g_1, \dots, g_n \in \operatorname{supp}(\mu)$ such that $g^{-1} = g_1 \cdots g_n$. Applying our first conclusion n times to an element $x \in A$, we see that

$$gx = g_n^{-1}.g_{n-1}^{-1} \dots g_1^{-1}.x \in A.$$

In conclusion A and hence $\operatorname{supp}(\nu)$ is G-invariant. □

Thus, when the semigroup generated by $\operatorname{supp}(\mu)$ is all of G, we will only consider μ-boundaries where the measure has full support. Note that due to Proposition 3.4 this does not change the Poisson boundary Π_μ. We are now ready for the main result in the theory of the topological Poisson boundary

Theorem 3.14 ([9])**.** *Assume that the semigroup generated by the support of μ is all of G, and let (B, ν) be a μ-boundary. Then there exists a continuous, surjective, G-equivariant map $\varphi\colon \Pi_\mu \to B$, such that $\varphi(\overline{\nu}) = \nu$.*

Proof. As $\operatorname{supp}(\nu) = B$, the μ-boundary (B, ν) is also contractive and hence, the Poisson transform $P_\nu\colon C(B) \to H^\infty_{\mathrm{luc}}(G, \mu)$ is an isometric, multiplicative embedding. Then $\Phi\colon C(B) \to C(\Pi_\mu)$, defined by $\Phi = P_{\overline{\nu}}^{-1} \circ P_\nu$ is an injective, multiplicative map. With this in mind, consider $\Psi\colon C(\Pi_\mu)^* \to C(B)^*$ defined by

$$\Psi(\nu)(f) = \nu(\Phi(f)), \quad \nu \in C(\Pi_\mu)^*, f \in C(B).$$

For any $x \in \Pi_\mu$, the multiplicative linear functional δ_x is mapped to a multiplicative linear functional on $C(B)$, since for $x \in \Pi_\mu$ and $f, g \in C(B)$,

$$\Psi(\delta_x)(fg) = \Phi(fg)(x) = (\Phi(f)\Phi(g))(x) = \Phi(f)(x)\Phi(g)(x) = \Psi(\delta_x)(f)\Psi(\delta_x)(g).$$

The C^*-algebra $C(B)$ is commutative, so we know that for each $x \in \Pi_\mu$ there exists a unique $y \in B$ such that $\Psi(\delta_x) = \delta_y$. By this argument, we define $\varphi\colon \Pi_\mu \to B$, such that $\Psi(\delta_x) = \delta_{\varphi(x)}$. To see that φ is continuous, let $(x_\alpha)_{\alpha \in A}$ be a net in Π_μ converging to some $x \in \Pi_\mu$. Then for all $f \in C(B)$,

$$f(\varphi(x_\alpha)) = \Psi(\delta_{x_\alpha})(f) = \Phi(f)(x_\alpha) \to \Phi(f)(x) = \Psi(\delta_x)(f) = f(\varphi(x)),$$

and hence $\varphi(x_\alpha) \to \varphi(x)$. To show that φ is onto, assume in order to reach a contradiction that is it not, and pick $y_0 \in B \setminus \varphi(\Pi_\mu)$. By Urysohn's Lemma, there exists $f \in C(B)$ such that $f(y_0) = 1$ and $f(y) = 0$, for all $y \in \varphi(\Pi_\mu)$ and hence

$$\Phi(f)(x) = \Psi(\delta_x)(f) = \delta_{\varphi(x)}(f) = f(\varphi(x)) = 0, \qquad x \in \Pi_\mu,$$

i.e., $\Phi(f) = 0$. As f by construction is non-zero and Φ is an isometry, this is a contradiction, and thus, φ is onto.

Finally, it remains to prove that $\varphi(\overline{\nu}) = \nu$, so let $f \in C(B)$ be given. Then

$$\begin{aligned}\varphi(\overline{\nu})(f) &= \int_{\Pi_\mu} f(\varphi(x))\,\mathrm{d}\overline{\nu}(x) = \int_{\Pi_\mu} \delta_{\varphi(x)}(f)\,\mathrm{d}\overline{\nu}(x)\\ &= \int_{\Pi_\mu} \Psi(\delta_x)(f)\,\mathrm{d}\overline{\nu}(x) = \int_{\Pi_\mu} \Phi(f)(x)\,\mathrm{d}\overline{\nu}(x)\\ &= P_{\overline{\nu}}(\Phi(f))(e) = P_\nu(f)(e) = \nu(f).\end{aligned}$$

As $f \in C(B)$ was arbitrary, we conclude that $\varphi(\overline{\nu}) = \nu$. □

3.5 Measurable boundaries

In this section, we focus on measurable G-spaces and discuss constructions hereon, which are analogous to the ones from the topological setting. Without a continuous action from G on the space B, it no longer makes sense to consider the space $H^\infty_{\mathrm{luc}}(G,\mu)$, but we will instead focus on $H^\infty_\rho(G,\mu)$, for some Borel probability measure ρ on G equivalent to the Haar measure λ. This ensures that $\rho, \rho * \tau$ and $\tau * \rho$ are equivalent for all $\tau \in \mathcal{P}(G)$. Thus, for the remainder of this section, ρ will be such a measure on G. Note that this also implies $H^\infty_\rho(G,\mu) \cong H^\infty_{\mathrm{a.s.}}(G,\mu)$.

Just as for $f \in H^\infty_{\mathrm{luc}}(G,\mu)$ in the topological case, every $f \in H^\infty_\rho(G,\mu)$ gives rise to a bounded martingale $(f(R_n))_{n\geq 1}$ with respect to the filtration $(\mathcal{F}_n)_{n\geq 1}$, where $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_{n-1}), n \geq 1$. In fact, there is an isometric bijection from $H^\infty_\rho(G,\mu)$ to the set

$$I^\infty_\rho := \{Z \in L^\infty(G\times\Omega, \rho\otimes\mathbb{P}) \mid Z(g,\omega) = Z(gX_0(\omega), T(\omega)),\ \rho\otimes\mathbb{P} - a.s.\}.$$

The argument here is identical to that of Section 2.3, but we can no longer be sure to identify μ-harmonic functions pointwise, only ρ-almost surely.

The first approach to defining a measurable Poisson boundary mimics the one in the topological case. Here it is known that $L^\infty(G\times\Omega, \rho\otimes\mathbb{P})$ is a commutative von Neumann algebra in the weak*-topology, when viewed as the dual of $L^1(G\times\Omega, \rho\otimes\mathbb{P})$. The next step is to show that I^∞_ρ is a weak*-closed subset of $L^\infty(G\times\Omega, \rho\otimes\mathbb{P})$, such that itself becomes a von Neumann algebra. For this let $(Z_\alpha)_{\alpha\in A}$ be a net in I^∞_ρ converging to some Z in $L^\infty(G\times\Omega, \rho\otimes\mathbb{P})$, with respect to the weak*-topology, i.e.,

$$\int_{G\times\Omega} |Z_\alpha - Z|\cdot|f|\,\mathrm{d}\rho\otimes\mathbb{P} \to 0,$$

for all $f \in L^1(G\times\Omega, \rho\otimes\mathbb{P})$. To ease notation let $\tilde{\theta}\colon G\times\Omega \to G\times\Omega$ denote the map $\tilde{\theta}(g,\omega) = (gX_0(\omega), T(\omega))$, with $(g,\omega) \in G\times\Omega$. We are left with showing that $Z = Z\circ\tilde{\theta}$, $\rho\otimes\mathbb{P}$-a.s. For any $\alpha \in A$, we have

$$\begin{aligned}\int |Z - Z\circ\tilde{\theta}|\,\mathrm{d}\rho\otimes\mathbb{P} &\leq \int |Z - Z_\alpha|\,\mathrm{d}\rho\otimes\mathbb{P} + \int |(Z - Z_\alpha)\circ\tilde{\theta}|\,\mathrm{d}\rho\otimes\mathbb{P}\\ &= \int |Z - Z_\alpha|\,\mathrm{d}\rho\otimes\mathbb{P} + \int |Z - Z_\alpha|\,\mathrm{d}\tilde{\theta}(\rho\otimes\mathbb{P}).\end{aligned}$$

The first of these integrals converges to zero, since the constant function 1 lies in $L^1(G \otimes \Omega, \rho \otimes \mathbb{P})$. For the second integral, we notice that $\tilde{\theta}(\rho \otimes \mathbb{P}) = (\rho * \mu) \otimes \mathbb{P}$, which by assumption on ρ must be absolutely continuous with respect to $\rho \otimes \mathbb{P}$. This gives us a map $\varphi \in L^1(G \times \Omega, \rho \otimes \mathbb{P})$, such that φ is the density of $\tilde{\theta}(\rho \otimes \mathbb{P})$ with respect to $\rho \otimes \mathbb{P}$. Applying the weak*-convergence of $(Z_\alpha)_{\alpha \in A}$ to the second integral above, this will also converge to zero and hence $Z \in I_\rho^\infty$.

In particular, I_ρ^∞ is a commutative von Neumann algebra, and hence it is isomorphic to $L^\infty(\Pi_\mu^\rho, \nu_\rho)$ for some probability space (Π_μ^ρ, ν_ρ). The space (Π_μ^ρ, ν_ρ) is called the measurable Poisson boundary. Not also that Proposition 2.7 also applies to $H^\infty_{\text{a.s.}}(G, \mu)$, and hence if G is abelian and the semigroup generated by $\operatorname{supp}(\mu)$ is all of G, then Π_μ^ρ is a singleton.

Even though we have proved the existence of the measurable Poisson boundary (Π_μ^ρ, ν_ρ), we would prefer a more explicit realisation of this abstractly constructed probability space. All along, the requirement is that the bounded measurable functions on it coincides with I_ρ^∞. In order to compare such spaces of bounded measurable functions on different probability spaces, we will introduce a series of lemmas, the first of which is a basic measure theoretic tool to characterise functions that are almost surely equal.

Lemma 3.15. *Let $(X, \mathcal{F}, \mu)$ be a probability space and let $f, g\colon X \to \mathbb{C}$ be two measurable functions. Then $f = g$ μ-a.e. if and only if*

$$\mu(f^{-1}(A) \Delta g^{-1}(A)) = 0,$$

for all Borel sets $A \subset \mathbb{C}$.

Proof. Assume first that $f = g$ μ-a.e. and consider $A \subset \mathbb{C}$ Borel. Then

$$\begin{aligned}\mu(f^{-1}(A) \Delta g^{-1}(A)) &= \int_X |1_A(f(x)) - 1_A(g(x))| \,\mathrm{d}\mu(x) \\ &= \int_{\{f=g\}} |1_A(f(x)) - 1_A(g(x))| \,\mathrm{d}\mu(x) = 0.\end{aligned}$$

For the converse, assume that f and g are not equal μ-a.e., and pick $k \in \mathbb{N}$ so that

$$N_k = \{x \in X \mid |f(x) - g(x)| > 1/k\}$$

has strictly positive measure. As $\mathbb{C}$ is a separable metric space, we can construct a sequence $(A_j)_{j \geq 1}$ of pairwise disjoint Borel sets in $\mathbb{C}$, with $\operatorname{diam}(A_j) < 1/k$ for all $j \in \mathbb{N}$ and whose union is all of $\mathbb{C}$. Simply pick the balls of radius $1/(2k)$ around each point in a countable dense subset and make them disjoint in your favorite measure theoretic manner. Then $(N_k \cap f^{-1}(A_j))_{j \geq 1}$ constitutes a countable partition of N_k, so pick j_0 such that $\mu(N_k \cap f^{-1}(A_{j_0})) > 0$. As $|f(x) - g(x)| > 1/k$ for every $x \in N_k$, it is easy to see that $x \in N_k \cap f^{-1}(A_{j_0})$ implies $x \notin g^{-1}(A_{j_0})$ and hence

$$N_k \cap f^{-1}(A_{j_0}) = N_k \cap (f^{-1}(A_{j_0}) \backslash g^{-1}(A_{j_0})).$$

From here on it is a matter of applying monotonicity of μ, to obtain

$$\begin{aligned} 0 < \mu(N_k \cap f^{-1}(A_{j_0})) &= \mu(N_k \cap (f^{-1}(A_{j_0}) \backslash g^{-1}(A_{j_0}))) \\ &\leq \mu(N_k \cap (f^{-1}(A_{j_0}) \Delta g^{-1}(A_{j_0}))) \leq \mu(f^{-1}(A_{j_0}) \Delta g^{-1}(A_{j_0})), \end{aligned}$$

which, by contraposition proves the claim. □

The next result revolves around the question of when two sub-σ-algebras on a probability space generate the same bounded measurable functions. We start out by considering a probability space $(X, \mathcal{F}, \mu)$, with $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$. We say that $\mathcal{G}$ and $\mathcal{H}$ are equal μ-a.e. if for every $A \in \mathcal{G}$ there exists $B \in \mathcal{H}$ with $\mu(A \Delta B) = 0$ and vice versa.

Lemma 3.16. *Let $(X, \mathcal{F}, \mu)$ be a probability space with $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$. If $\mathcal{G} = \mathcal{H}$ μ-a.e. then*

$$L^\infty(X, \mathcal{G}, \mu) = L^\infty(X, \mathcal{H}, \mu).$$

Proof. This proof follows the classical three step strategy of going through characteristic functions, simple functions and finally measurable functions in general. For $A \in \mathcal{G}$ and $B \in \mathcal{H}$ with $\mu(A \Delta B) = 0$, it is evident by the previous lemma that 1_A and 1_B are equal μ-a.s. In addition, if $B, B' \in \mathcal{H}$ are given such that $\mu(A \Delta B) = \mu(A \Delta B') = 0$, then

$$0 \leq \mu(B \Delta B') = \mu((B \Delta A) \Delta (A \Delta B')) \leq \mu(B \Delta A) + \mu(A \Delta B') = 0,$$

and hence, $1_B = 1_{B'}$ μ-a.s. These considerations give rise to a bijective correspondence between the characteristic functions in $L^\infty(X, \mathcal{G}, \mu)$ and $L^\infty(X, \mathcal{H}, \mu)$ respectively. From here we obtain a natural isometric isomorphism between the simple functions in these two spaces, which extends to an isometric isomorphism from $L^\infty(X, \mathcal{G}, \mu)$ to $L^\infty(X, \mathcal{H}, \mu)$ as desired. □

Realisations of the measurable Poisson boundary

Going back to the definition of I_ρ^∞, we start out by considering the space $G \times \Omega$ equipped with the natural Borel product σ-algebra $\mathcal{A}$ and the probability measure $\rho \otimes \mathbb{P}$, where $\mathbb{P} = \bigotimes_{i=0}^\infty \mu$. This measure represents an initial distribution of ρ and the i.i.d. distributions for the construction of the random walk. We can now consider the completion of $\mathcal{A}$, denoted by $\overline{\mathcal{A}}$, instead of $\mathcal{A}$, as this, due to Proposition 3.16, will not change the measurable functions. Consider now the map $S \colon G \times \Omega \to G \times \Omega$ given by

$$S(g, \omega) = (g, g\omega_0, g\omega_0\omega_1, \dots), \qquad g \in G, \omega = (\omega_0, \omega_1, \dots) \in \Omega,$$

describing the paths of the random walk. Furthermore, we let $\mathbb{P}_\rho$ denote the image measure $S(\rho \otimes \mathbb{P})$, i.e., $\mathbb{P}_\rho$ is the distribution of the random walk $(R_n^\rho)_{n \geq 0}$. Similarly we let $\mathbb{P}_g$ denote the measure $S(\delta_g \otimes \mathbb{P})$. Before moving on to the specific construction, one should note that S is a bijection with inverse function

$$S^{-1}(h, \eta) = (h, h^{-1}\eta_0, \eta_0^{-1}\eta_1, \eta_1^{-1}\eta_2, \dots), \qquad h \in G, \eta = (\eta_0, \eta_1, \dots) \in \Omega.$$

Stationary boundary

The first goal is to identify I_ρ^∞ with the measurable functions on $G\times\Omega$ with respect to a certain σ-algebra. For this we introduce the transformations $T,\tilde{\theta}\colon G\times\Omega\to G\times\Omega$, given by $T(g,\omega_0,\omega_1,\dots) = (\omega_0,\omega_1,\dots)$ and $\tilde{\theta}(g,\omega_0,\omega_1,\dots) = (g\omega_0,\omega_1,\omega_2,\dots)$, respectively. In this way, $Z\in I_\rho^\infty$ if it belongs to $L^\infty(G\times\Omega,\rho\otimes\mathbb{P})$ and satisfies $Z = Z\circ\tilde{\theta}$. Notice also that $T = S\circ\tilde{\theta}\circ S^{-1}$, so if $Z\in I_\rho^\infty$, then the map $\tilde{Z} = Z\circ S^{-1}$ satisfies $\tilde{Z} = \tilde{Z}\circ T$. Indeed,

$$\tilde{Z}\circ T = Z\circ S^{-1}\circ T = Z\circ\tilde{\theta}\circ S^{-1} = Z\circ S^{-1} = \tilde{Z}.$$

Finally, notice that $T(\mathbb{P}_\rho) = \mathbb{P}_{\rho*\mu}$, so $\mathbb{P}_\rho$ is T-quasi invariant and hence I_ρ^∞ is isomorphic to the function space $L^\infty(G\times\Omega,\mathcal{A}_T,\mathbb{P}_\rho)$, where $\mathcal{A}_T$ consists of the sets in $A\in\overline{\mathcal{A}}$, such that $\mathbb{P}_\rho(A\Delta T^{-1}(A))$. Equivalently, we could use the σ-algebra

$$\mathcal{I}_T = \{A\in\overline{\mathcal{A}}\mid A = T^{-1}(A)\},$$

as these σ-algebras are equal $\mathbb{P}_\rho$-a.e. We could stop the construction at this point, but the σ-algebra $\mathcal{A}_T$ is somewhat unnaturally small for the set $G\times\Omega$, in the sense that it will most likely not even separate points of $G\times\Omega$. We can not hope to find a larger σ-algebra equal to $\mathcal{A}_T$ modulo $\mathbb{P}_\rho$, but we can find a smaller space instead.

We start out by defining an equivalence relation $\sim$ on $G\times\Omega$ by $x\sim x'$ if and only if there exists $n,n'\geq 0$ such that $T^n x = T^{n'}x'$. This allows us to construct the quotient space $\mathcal{S} = (G\times\Omega)/\sim$, which will be our space of interest. As G acts on $G\times\Omega$ by multiplying from the left on each coordinate, it is clear that $x\sim x'$ if and only if $g.x\sim g.x'$, for all $g\in G$, so it induces an action of G on $\mathcal{S}$ by $g.[x]_\sim = [g.x]_\sim$, where $[x]_\sim$ denotes the equivalence class containing x and g is an element of G.

We also want to equip $\mathcal{S}$ with the structure of a probability space, so let π denote the projection from $G\times\Omega$ to $\mathcal{S}$, i.e., the map $\pi(x) = [x]_\sim$. We define a σ-algebra $\mathcal{F}$ on $\mathcal{S}$ by $A\in\mathcal{F}$ if and only if $\pi^{-1}(A)\in\mathcal{I}_T$. Note that each equivalence class $[a]_\sim\in\mathcal{S}$ belongs to $\mathcal{I}_T$, as $Tx\sim x$ for all $x\in G\times\Omega$, so

$$x\in[a]_\sim \iff x\sim a \iff Tx\sim a \iff Tx\in[a] \iff x\in T^{-1}([a]_\sim),$$

and hence $\mathcal{F}$ separates points in $\mathcal{S}$. On the other hand, let $x\sim x'$ in $G\times\Omega$ be given and let $n,m\geq 0$ be such that $T^n x = T^m x'$. If $x\neq x'$, we can without loss of generality assume that $n = m+k$ for some $k\geq 1$ and hence $T^n x' = T^k(T^m(x))$. From here it follows that if $x\in A$ for $A\in\mathcal{I}_T$, then $x'\in A$. Then any $A\in\mathcal{I}_T$ can be written as a disjoint union of equivalence classes of $\sim$, so there is a one-to-one correspondence between $\mathcal{I}_T$ and $\mathcal{F}$, through π. Equipping $(\mathcal{S},\mathcal{F})$ with the image measure $\alpha_\rho = \pi(\mathbb{P}_\rho)$, we obtain an isomorphism

$$L^\infty(G\times\Omega,\mathcal{I}_T,\mathbb{P}_\rho)\cong L^\infty(\mathcal{S},\mathcal{F},\alpha_\rho).$$

Indeed, we showed that the sets in $\mathcal{I}_T$ do not separate equivalence classes of $\sim$, so every measurable functions must be constant on each equivalence class $\mathbb{P}_\rho$-a.s. and thus induces a unique map in $L^\infty(\mathcal{S},\mathcal{F},\alpha_\rho)$.

Note that the space $(\mathcal{S},\mathcal{F},\alpha_\rho)$ has the same properties fulfilled by (Π_μ^ρ,ν_ρ). We will call $(\mathcal{S},\alpha_\rho)$ the stationary boundary.

Exit boundary

A similar approach is to consider the σ-algebra $\mathcal{A}_\infty$ on $G \times \Omega$, given by

$$\mathcal{A}_\infty = \bigcap_{n=1}^{\infty} \sigma(R_n, R_{n+1}, \dots).$$

This σ-algebra is sometimes referred to as the *tail σ-algebra* of the random walk $(R_n)_{n\geq 0}$. Let us investigate the sets in here, so let $A \in \mathcal{A}_\infty$. By definition of $\mathbb{P}_\rho$, we know that $\mathbb{P}_\rho$-a.e. $x \in A$ is on the form

$$x = (g, gR_0(\omega), gR_1(\omega), gR_2(\omega), \dots),$$

for some $g \in G$ and $\omega \in \Omega$. The fact that $A \in \sigma(R_n, R_{n+1}, \dots)$ will then ensure the existence of Borel sets $A_1, A_2, \dots$ in G, such that

$$A = G \times \cdots G \times A_1 \times A_2 \times \cdots$$

with $n+1$ copies of G in the beginning of the above product. This is the same as the existence of a $B \in \overline{\mathcal{A}}$ with $T^{-(n+1)}(B) = A$. Thus, the σ-algebra

$$\left\{A \subset G \times \Omega \text{ Borel set} \;\middle|\; \exists (A_n)_{n\geq 1} \text{ in } \overline{\mathcal{A}} \text{ such that } T^{-n}(A_n) = A\right\}$$

will coincide with $\mathcal{A}_\infty$ modulo $\mathbb{P}_\rho$, i.e., they generate the same bounded measurable functions. The following proposition shows that $\mathcal{A}_\infty$ also coincides with $\mathcal{I}_T$ modulo $\mathbb{P}_\rho$, and hence we have the isomorphism

$$L^\infty(G \times \Omega, \mathcal{I}_T, \mathbb{P}_\rho) \cong L^\infty(G \times \Omega, \mathcal{A}_\infty, \mathbb{P}_\rho).$$

Proposition 3.17. *The σ-algebras $\mathcal{A}_\infty$ and $\mathcal{I}_T$ are equal $\mathbb{P}_\rho$-a.e.*

Proof. It is clear that $\mathcal{I}_T$ is contained in $\mathcal{A}_\infty$, as any $A \in \mathcal{I}_T$ satisfy $A = T^{-n}(A)$ for all $n \in \mathbb{N}$, so we may just pick the sequence $(A_n)_{n\geq 1}$ to be constantly equal to A. Let us now focus on the other implication. As these constructions solely depend on whether certain functions are μ-harmonic or not, we can without loss of generality assume that μ and μ^{*2} are not mutually singular due to Proposition 2.10, or in other words that

$$\|\mu - \mu^{*2}\| < 2.$$

Using the zero-two law (Theorem B.2), we are allowed to conclude that

$$\|\mu^{*p} - \mu^{*(p+1)}\| \to 0,$$

when $p \to \infty$, a fact that will come in handy later. Let now $A \in \mathcal{A}_\infty$ be given, with $(A_n)_{n\geq 1}$ in $\overline{\mathcal{A}}$ such that $A = T^{-n}(A_n)$ for all $n \in \mathbb{N}$. Note that this also gives us $T^{-1}(A_{n+1}) = A_n$ for all $n \in \mathbb{N}$. Next, we define a family of maps $(f_n)_{n\geq 1}$, such that $f_n \colon G \to \mathbb{R}$ is given by

$$f_n(g) = E_g(1_{A_n}),$$

where E_g denotes the expectation with respect to the measure $\mathbb{P}_g$. We could instead have written $f_n(g) = \mathbb{P}_g(A_n)$, but the approach using expectations will be more useful in a moment. It is not difficult to see, that $|f_n(g)| \le 1$ for all $n \in \mathbb{N}, g \in G$, but what is more important, each f_n turns out to be μ-harmonic. To prove this, we will first of all show that $f_n = P_\mu(f_{n+1})$, so let $g \in G$ be given. Then

$$f_n(g) = E_g(1_{A_n}) = E_g(1_{A_{n+1}} \circ T) = \int_{G\times\Omega} 1_{A_{n+1}}\,\mathrm{d}T(\mathbb{P}_g).$$

As seen earlier $T(\mathbb{P}_g) = \mathbb{P}_{\delta_g * \mu}$, which may also be written as $g.(\mu * \mathbb{P}_e)$ and hence,

$$\begin{aligned}
f_n(g) &= \int_{G\times\Omega} 1_{A_{n+1}}(gh, g\omega)\,\mathrm{d}\mu * \mathbb{P}_e(h,\omega)\\
&= \int_{G\times\Omega}\int_{G\times\Omega} 1_{A_{n+1}}(g\omega_0, g\omega_0\omega')\,\mathrm{d}\mathbb{P}_e(h,\omega')\,\mathrm{d}\mu(\omega_0)\\
&= \int_{G\times\Omega}\int_{G\times\Omega} 1_{A_{n+1}}(g\omega_0 h, g\omega_0\omega')\,\mathrm{d}\mathbb{P}_e(h,\omega')\,\mathrm{d}\mu(\omega_0)\\
&= \int_{G\times\Omega}\int_{G\times\Omega} 1_{A_{n+1}}(g\omega_0 h, g\omega_0\omega')\,\mathrm{d}\mathbb{P}_e(h,\omega')\,\mathrm{d}\mathbb{P}_e(h',\omega)\\
&= \int_{G\times\Omega}\int_{G\times\Omega} 1_{A_{n+1}}(gR_1(h',\omega)h, gR_1(h',\omega)\omega')\,\mathrm{d}\mathbb{P}_e(h,\omega')\,\mathrm{d}\mathbb{P}_e(h',\omega)\\
&= \int_{G\times\Omega}\int_{G\times\Omega} 1_{A_{n+1}}(h,\omega')\,\mathrm{d}\mathbb{P}_{R_1(g,\omega)}(h,\omega')\,\mathrm{d}\mathbb{P}_g(h',\omega)\\
&= E_g(E_{R_1}(1_{A_{n+1}})) = E_g(f_{n+1}(R_1))\\
&= \int_{G\times\Omega} f_{n+1}(gR_1(\omega))\,\mathrm{d}\mathbb{P}_e(\omega) = \int_G f_{n+1}(gx)\,\mathrm{d}\mu(x) = P_\mu(f_{n+1})(g)
\end{aligned}$$

This is not quite enough, but taking our application of the zero-two law into play, we will for each $g \in G$ have

$$|f_{n+1}(g) - f_n(g)| \le \|P_\mu^p(f_{n+p+1}) - P_\mu^{p+1}(f_{n+p+1})\| \le \|\mu^{*p} - \mu^{*(p+1)}\|.$$

As the latter norm above tends to zero as $p \to \infty$, we conclude that $f_n = f_{n+1}$ for each $n \in \mathbb{N}$. If we let f_0 denote this common function, then $f_0(g) = E_g(1_A)$. Furthermore f_0 will be an element of $H_\rho^\infty(G,\mu)$ and thus give rise to a unique $Z \in I_\rho^\infty$ such that $f_0(g) = E(Z(g,\cdot))$ ρ-almost surely. We may then consider $\tilde{Z} \in L^\infty(G\times\Omega, \mathcal{A}_T)$, defined as $\tilde{Z} = Z \circ S^{-1}$, and see that

$$\begin{aligned}
f_0(g) = E(Z(g,\cdot)) &= \int_\Omega Z(g,\omega)\,\mathrm{d}\mathbb{P}(\omega)\\
&= \int_{G\times\Omega} Z(h,\omega)\,\mathrm{d}\delta_g\otimes\mathbb{P}(h,\omega)\\
&= \int_{G\times\Omega} \tilde{Z}(h,\omega)\,\mathrm{d}S(\delta_g\otimes\mathbb{P})(h,\omega)\\
&= \int_{G\times\Omega} \tilde{Z}(h,\omega)\,\mathrm{d}\mathbb{P}_g(h,\omega) = \mathrm{E}_g(\tilde{Z}).
\end{aligned}$$

By uniqueness of Z, we must then have $1_A = \tilde{Z}$, $\mathbb{P}_\rho$-almost surely or in other words $1_A \in L^\infty(G\times\Omega, \mathcal{A}_T)$. This happens if and only if $\mathbb{P}_\rho(A\Delta T^{-1}(A))$, thus completing the proof. □

Once again, $\mathcal{A}_\infty$ is a very small σ-algebra on $G \times \Omega$, so we seek to construct a more useful quotient space generating the same bounded measurable functions. For this we consider the equivalence relation $\approx$ on $G \times \Omega$ given by $x \approx x'$ if and only if there exists $n \geq 0$, such that $T^n x = T^n x'$. Let $\mathcal{E}$ denote the quotient $(G \times \Omega)/\approx$ along with a projection $\pi_{\mathcal{E}}\colon G \times \Omega \to \mathcal{E}$.

As with the stationary boundary, we equip $\mathcal{E}$ with the σ-algebra $\mathcal{F}_{\mathcal{E}}$ of sets with preimages in $\mathcal{A}_\infty$ through $\pi_{\mathcal{E}}$ and the image measure $\alpha_{\rho,\mathcal{E}} = \pi_{\mathcal{E}}(\mathbb{P}_\rho)$. Our aim in this setting is to show that $[x]_\approx \in \mathcal{A}_\infty$ for any $x \in G \times \Omega$ and that every $A \in \mathcal{A}_\infty$ is a disjoint union of equivalence classes under $\approx$. This will give us a probability space with a σ-algebra that separates points and whose bounded measurable functions coincide with the ones of the measurable Poisson boundary. For the first property let $x \in G \times \Omega$ be given. Then

$$[x]_\approx = \{x' \in G \times \Omega \mid \exists n \geq 0 : T^n x = T^n x'\},$$

and thus, $[x]_\approx = T^{-n}(A_n)$ for any $n \geq 0$, with

$$A_n = \{x' \in G \times \Omega \mid \exists k \geq n : T^k x = T^k x'\}.$$

For the second part, i.e., to prove that $A \in \mathcal{A}_\infty$ is a disjoint union of equivalence classes, it suffices to show that $x \in A$ implies $[x]_\approx \subset A$. For this consider $x \in A$ and $x' \in [x]_\approx$, with $n \geq 1$ such that $T^n x = T^n x'$. With $A = T^{-n}(A_n)$ for some Borel set A_n, we see that $T^n x' \in A_n$, i.e. $x' \in T^{-n}(A_n) = A$. This gives a one-to-one correspondence between $\mathcal{A}_\infty$ and $\mathcal{F}_{\mathcal{E}}$ and hence

$$L^\infty(\mathcal{E}, \mathcal{F}_{\mathcal{E}}, \alpha_{\rho,\mathcal{E}}) \cong L^\infty(G \times \Omega, \mathcal{A}_\infty, \mathbb{P}_\rho).$$

The probability space $(\mathcal{E}, \alpha_{\rho,\mathcal{E}})$ is called the exit boundary of (G, μ).

Strongly approximatively transitive measures

After studying the measurable Poisson boundary, we could also try to extend the notion of a contractible (G, μ)-space and μ-boundaries to the measurable setting. In the topological case, a contractible (G, μ)-space (B, ν) was characterised by the Poisson transform $P_\nu\colon C(B) \to H^\infty_{\mathrm{luc}}(G, \mu)$ being an isometry. It is at this point that the strong measurable spaces, a notion developed in Section 2.6, comes into play

Proposition 3.18. *Let (B, ν, α) be a strong measurable space (G, μ). Then the following are equivalent*

1. *The convex hull of $G.\nu$ is dense in the space of the Borel probability measures on B, who are absolutely continuous with respect to α.*
2. *For each Borel set $A \subset B$ with $\alpha(A) > 0$ and each $\varepsilon > 0$, there exists $g \in G$, such that $\nu(g.A) > 1 - \varepsilon$.*
3. *P_ν is an isometry from $L^\infty(B, \alpha)$ into $H^\infty_{\mathrm{a.s.}}(G, \mu)$.*

Proof. Let us start out by assuming that the convex hull of $G.\nu$ is dense in the set of Borel probability measures on B which are absolutely continuous with respect to α. We wish to show that *(2)* is satisfied, so let A be a Borel subset of B with $\alpha(A) > 0$, and let $\varepsilon > 0$ be given. Let β denote the normalised restriction of α to A, that is the measure with density $\varphi \in L(B, \alpha)$ given by

$$\varphi(b) = \alpha(A)^{-1} 1_A(b), \qquad b \in B.$$

By assumption there exists a convex combination $\sum_{i=1}^n p_i g_i.\nu$, i.e., $g_1, \dots, g_n \in G$ and $p_1, \dots, p_n \in (0,1)$ satisfying $\sum_{i=1}^n p_i = 1$, such that

$$\left\| \beta - \sum_{i=1}^n p_i g_i.\nu \right\| < \varepsilon.$$

By definition of β we conclude that

$$1 - \sum_{i=1}^n p_i \nu(g_i.A) = \beta(A) - \sum_{i=1}^n p_i g_i.\nu(A) < \varepsilon,$$

and hence $\nu(g_i.A) > 1 - \varepsilon$ for some $i \in \{1, \dots, n\}$. For the next implication assume that *2.* holds true, and let us show that $P_\nu \colon L^\infty(B, \alpha) \to L^\infty(G, \mu)$ is an isometry. This argument will follow the traditional measure theoretic form, by considering characteristic functions, then simple functions and finally extending to all measurable functions.

For a Borel set A in B with $\alpha(A) > 0$, we know that $\|1_A\|_\infty = 1$, when 1_A is viewed as function in $L^\infty(B, \alpha)$. By *2.*, we see that

$$\|P_\nu(1_A)\|_\infty = \sup_{g \in G} \int_B 1_A(g.b) \, \mathrm{d}\nu(b) = \sup_{g \in G} \nu(g^{-1}.A) = 1,$$

and hence P_ν acts as an isometry on the characteristic functions. To continue from here let $\varphi = \sum_{i=1}^n \alpha_i 1_{A_i}$ be a simple function in $L^\infty(B, \alpha)$. We can without loss of generality assume that $A_1, \dots, A_n$ are disjoint sets with $\alpha(A_i) > 0$, $\alpha_1 = 1$ and $\max_{1 \leq i \leq n} |\alpha_i| \leq 1$. Note that this also implies $\|\varphi\| = 1$.

Let $\varepsilon > 0$ be given and pick $g \in G$, such that $\nu(g.A_1) > 1 - \varepsilon$ and hence also $\sum_{i=2}^n \nu(g.A_i) \leq \varepsilon$. Then

$$P_\nu(\varphi)(g^{-1}) = \nu(g.A_1) + \sum_{i=2}^n \alpha_i \nu(g.A_i) \geq \nu(g.A_1) + \sum_{i=2}^n \nu(g.A_i) > 1 - 2\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we see that $\|P_\nu(\varphi)\|_\infty = 1$, so P_ν acts as an isometry on the simple functions in $L^\infty(B, \alpha)$. By density of the simple functions in $L^\infty(B, \alpha)$, we conclude that P_ν is an isometry from $L^\infty(B, \alpha)$ to $L^\infty(G, \lambda)$.

In order to prove that *3.* implies *1.* we start out by looking at the convex hull of $G.\nu$. First of all, the quasi-invariance of α ensures that $g.\nu$ is absolutely continuous with respect to α, so *1.* makes sense. To ease notation, we let $L^1_1(B, \alpha)$ denote the set of such measures. For a $g \in G$, the measure $g.\nu$ can also be written as $\delta_g * \nu$, so the convex hull of $G.\nu$ can be written as the set

$$\{\beta * \nu \mid \beta \text{ is a convex combination of Dirac measures in } G\},$$

which we will denote by M. In order to reach a contradiction, we will now assume that M is not dense in and pick $\rho \in L^1_1(B,\alpha)\backslash\overline{M}$. By the Hahn-Banach theorem, there exists $f \in L^\infty(B,\alpha)$ and $\varepsilon > 0$ such that

$$\beta * \nu(f) \leq \rho(f) - \varepsilon \leq \|f\|_\infty - \varepsilon,$$

for all β in the convex hull of δ_G. In particular, $\delta_g * \nu(f) \leq \|f\|_\infty - \varepsilon$ for all $g \in G$, and since $\delta_g * \nu(f) = g.\nu(f) = P_\nu(f)(g)$, we conclude that $\|P_\nu(f)\|_\infty < \|f\|_\infty$, which contradicts P_ν being an isometry. □

We say that a measure ν on a measurable G-space is *strongly approximately transitive, abbreviated as SAT*, if it satisfies one of the equivalent conditions in Proposition 3.18. In this way a strongly measurable space (B,ν,α) gives rise to an isometric Poisson transform $P_\nu \colon L^\infty(B,\alpha) \to L^\infty(G,\mu)$ if and only if ν is SAT.

Measurable μ-boundaries

The motivation for μ-boundaries in the topological case were to ensure that the Poisson transform was a $*$-homomorphism. Namely, we showed that this was the case if and only if there exists a map $z \colon \Omega \to B$, such that $R_n(\omega).\nu \to \delta_{z(\omega)}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. It is this latter property that we will imitate in the measurable case. As we have moved to the measurable case, we will have to account for the measure ρ, which is still a probability measure on G equivalent to λ.

Definition 3.19. Let (B,ν) be a weak measurable (G,μ) space. Then (B,μ) is a measurable μ-boundary if there exists a measurable map $z \colon (\Omega,\mathbb{P}) \to (B,\nu)$ such that $gR_n(\omega).\nu(\varphi) \to \varphi(g.z(\omega))$, for $\rho \otimes \mathbb{P}$-a.e $(g,\omega) \in G \times \Omega$ and all $\varphi \in (B,\nu)$.

There are two results regarding measurable μ-boundaries that we want to show in this section. First of all, we want to show that the Poisson boundary is a μ-boundary, thus justifying the generalisation, but also that any other measurable μ-boundary is a G-equivariant image of the Poisson boundary. That the first assertion is true follows analogous from the topological case and in addition, by considering the stationary boundary $(\mathcal{S},\alpha_\rho)$ as a realisation of the Poisson boundary, the only choice of $z \colon (\Omega,\mathbb{P}) \to (\mathcal{S},\alpha_\rho)$ is $z(\omega) = [(e,(R_n(\omega)_{n\geq 0}))]_\sim$. Secondly, the measurable Poisson boundary is universal among measurable μ-boundaries, as seen below.

Proposition 3.20. *Let (B,ν) be a measurable μ-boundary. Then there exists a G-equivariant map from the stationary Poisson boundary $(\mathcal{S},\alpha_\rho)$ to B, which maps α_ρ to ν.*

Proof. Let $z \colon (\Omega,\mathbb{P}) \to B$ be the map corresponding to (B,ν). Define $Z \colon G\times\Omega \to B$, by $Z(g,\omega) = g.z(\omega)$. Then Z is $\tilde{\theta}$-invariant, where $\tilde{\theta} \colon G \times \Omega \to G \times \Omega$ is the map defined earlier by $\tilde{\theta}(g, \omega) = (g\omega_0, T(\omega))$. Indeed for any φ in $L^\infty(B, \nu)$, we may used the assumed convergence to obtain

$$\begin{aligned}\varphi(Z \circ \tilde{\theta}(g, \omega)) &= \varphi(Z(g\omega_0, T(\omega))) = \varphi(g\omega_0.z(T(\omega))) \\ &= \lim_{n\to\infty} \int_B \varphi(g\omega_0 R_n(T(\omega)).x)\, \mathrm{d}\nu(x) \\ &= \lim_{n\to\infty} \int_B \varphi(gR_n(\omega).x)\, \mathrm{d}\nu(x) = \varphi(g.z(\omega)) = \varphi(Z(g, \omega)),\end{aligned}$$

and hence $Z \circ \tilde{\theta}(g, \omega) = Z(g, \omega)$ almost surely. From here on, we use the construction of the stationary boundary, to obtain an induced map $\tilde{Z} \colon (\mathcal{S}, \alpha_\rho) \to (B, \nu)$. More precisely, $\tilde{Z}$ is defined by $\tilde{Z}([(h, y)]_\sim) = Z \circ S^{-1}(h, y)$ for $(h, y) \in G \times \Omega$. Since $\mathbb{P}_\rho$-a.e. $(h, y) \in G \times \Omega$ is on the form $(h, h\omega_0, h\omega_0\omega_1, \dots)$, for some $\omega \in \Omega$, we get the desired G-equivariance, as

$$\begin{aligned}g.\tilde{Z}([(h, y)]_\sim) &= g.(Z \circ S^{-1})(h, y) = g.Z((h, \omega)) = gh.z(\omega) \\ &= Z(gh, \omega) = Z \circ S^{-1}((gh, gy)) = \tilde{Z}(g.[(h, y)]_\sim),\end{aligned}$$

holds true for any $g \in G$. Analogous to the topological case, the distribution of the map z is exactly ν, so for the final part, it suffices to show that $z = \tilde{Z}(z')$, where $z' \colon (\Omega, \mathbb{P}) \to (\mathcal{S}, \alpha_\rho)$ is defined as $z'(\omega) = [(e, (R_n(\omega))_{n\geq 0})]_\sim$. Fortunately this is not very difficult, as

$$\tilde{Z}(z'(\omega)) = \tilde{Z}([(e, (R_n(\omega)))]_\sim) = Z((e, \omega)) = z(\omega),$$

for any $\omega \in \Omega$, thus proving the claim. $\square$

4 Entropy

The purpose of this section is to apply the information theoretic concept of entropy both to probability measures, and to partitions of a probability space, and see how this relate to the behaviour of the measurable Poisson boundary. This type of entropy is sometimes referred to as the Shannon entropy, in order to distinguish it from the thermodynamical notion of entropy. The theory covered here can be found in [16], and in more details in [25]. We will throughout this section assume that our group G is discrete and countable. In order to ease notation, given a probability measure μ on G and $x \in G$, we let $\mu(x)$ denote the measure of the singleton $\{x\}$.

In order to understand the concept of entropy, we first need to discuss the notion of information. The idea here is to play a game of two persons. Let $(X, \mathcal{F}, \mu)$ be a probability space and assume that the first player knows the probability of each set in $\mathcal{F}$. The second player will now pick an element $x \in X$ and then inform the first player of a set $A \in \mathcal{F}$ which contains x. The question at hand is how much the first player learns about the element x through the mentioning of A. The following requirements seems justifiable:

- The information should always be positive.
- If the surrounding event A has full measure, we do not get any information about the element.
- If the element is contained in two independent events, the information corresponding to this intersection should correspond to the the sum of the respective amounts of information.

In order to turn these ideas into a formal setting, we aim to define a map $I\colon [0,1] \to \mathbb{R}$, such that $I(\mu(A))$ represents the information obtained from the set A. The properties above can then be describes as follows:

- $I \geq 0$, i.e., $I(\mu(A)) \geq 0$ for all $A \in \mathcal{F}$.
- $I(1) = 0$, i.e., $I(\mu(A)) = 0$ for $A \in \mathcal{F}$ with $\mu(A) = 1$.
- $I(p_1 p_2) = I(p_1) + I(p_2)$ for $p_1, p_2 \in [0,1]$, i.e., for every pair of independent events A and B, we have $I(\mu(A \cap B)) = I(\mu(A)) + I(\mu(B))$.

The customary choice of this function is $I(t) = -\log(t)$ for $t \in (0,1]$ and the value zero in $t = 0$. As alternative choices one could have multiplied I with any positive number, but we stick with the customary choice. We are now ready to define the entropy of a measure μ, denoted by $H(\mu)$.

Definition 4.1. The entropy of a probability measure μ on a measurable space $(X, \mathcal{F})$ is defined as the average amount of information, where the average is to be thought of as the expected value of information given by the measure μ.

In the case when $\mu \in \mathcal{P}(G)$, the average of information can be expressed as follows $H(\mu) = -\sum_{g \in G} \mu(g) \log(\mu(g))$, with the convention that $0 \cdot \log(0) = 0$. This quantity might be finite, but it could just as well be infinite.

Remark 4.2. The existence of a measure on G with infinite entropy is a non-trivial fact. If we go through our favorite discrete distributions on $\mathbb{N}$, they will probably all have finite entropy, but let us look at an example of a measure with infinite entropy. Following [3], we let $g_1, g_2, \dots$ be an enumeration of our countable, discrete group G, and consider the Borel probability measure μ on G, defined by

$$\mu(g_n) = \frac{1}{\log_2(n+1)} - \frac{1}{\log_2(n+2)}, \quad n \geq 1,$$

where $\log_2$ denotes the logarithm with base 2. This is indeed a probability measure, as the infinite series $\sum_{n=1}^{\infty} \mu(g_n)$ is a telescoping sum leaving only the term $1/\log_2(2) = 1$. Then $H(\mu) = \infty$.

The first result regarding the entropy is the fact that $H(\mu \otimes \mu') = H(\mu) + H(\mu')$, for $\mu, \mu' \in \mathcal{P}(G)$. Indeed,

$$\begin{aligned}
H(\mu \otimes \mu') &= - \sum_{(x,x') \in G \times G} \mu \otimes \mu'(x,x') \log(\mu \otimes \mu'(x,x')) \\
&= - \sum_{x \in G} \sum_{x' \in G} \mu(x)\mu'(x') \log(\mu(x)\mu'(x')) \\
&= - \sum_{x' \in G} \sum_{x \in G} \mu'(x')\mu(x) \log(\mu(x)) - \sum_{x \in G} \sum_{x' \in G} \mu(x)\mu'(x') \log(\mu'(x')) \\
&= - \sum_{x \in G} \mu(x) \log(\mu(x)) - \sum_{x' \in G} \mu'(x') \log(\mu'(x')) = H(\mu) + H(\mu').
\end{aligned}$$

After multiplying two elements of G, we are unable to retrieve the original elements and thus, we expect that the information of a convolution is less than the information of the product measure. This is indeed the case.

Proposition 4.3. *Let μ, μ' be probability measures on G with finite entropy. Then $H(\mu * \mu')$ is less than or equal to $H(\mu) + H(\mu')$.*

Proof. This proof is not particularly deep, but merely a series of calculations using that μ, μ' are probability measures and that the logarithm is an increasing function. More precisely,

$$\begin{aligned}
H(\mu * \mu') - H(\mu) &= - \sum_{g \in G} \mu * \mu'(g) \log(\mu * \mu'(g)) - \sum_{g \in G} \mu'(g) H(\mu) \\
&= - \sum_{g \in G} \sum_{h \in G} \mu(h)\mu'(h^{-1}g) \log\left(\sum_{h' \in G} \mu(h')\mu'(h'^{-1}g) \right) \\
&\quad + \sum_{g \in G} \sum_{h \in G} \mu'(g) \log(\mu(h))\mu(h) \\
&\leq - \sum_{g \in G} \sum_{h \in G} \mu(h)\mu'(h^{-1}g) \log(\mu(h)\mu'(h^{-1}g)) \\
&\quad + \sum_{g \in G} \sum_{h \in G} \mu'(g) \log(\mu(h))\mu(h)
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{g\in G}\sum_{h\in G}\mu(h)\mu'(h^{-1}g)\big[\log(\mu(h)) + \log(\mu'(h^{-1}g))\big] \\
&\quad + \sum_{g\in G}\sum_{h\in G}\mu'(g)\log(\mu(h))\mu(h) \\
&= -\sum_{g\in G}\sum_{h\in G}\mu(h)\log(\mu(h))\big[\mu'(h^{-1}g) - \mu'(g)\big] \\
&\quad + \sum_{h\in G}\sum_{g\in G}\mu(h)\mu'(h^{-1}g)\log(\mu'(h^{-1}g)) \\
&= -\sum_{g\in G}\sum_{h\in G}\mu(h)\log(\mu(h))\Big[\sum_{g\in G}\mu'(h^{-1}g) - \sum_{g\in G}\mu'(g)\Big] \\
&\quad - \sum_{h\in G}\sum_{g\in G}\mu(h)\mu'(g)\log(\mu'(g)).
\end{aligned}
$$

As μ' is a probability measure the first term in the final line above equals zero, and since μ is a probability measure the second term is simply equal to $H(\mu')$, thus completing the proof. □

With this result, the sequence $(h_n)_{n\geq 1}$ given by $h_n = H(\mu^{*n})$ is subadditive, i.e., $h_{n+m} \leq h_n + h_m$ for $n, m \geq 1$, and hence the limit

$$h(G,\mu) := \lim_{n\to\infty}\frac{h_n}{n}$$

exists. The quantity $h(G,\mu)$ is called the entropy of the pair (G,μ). The main goal of this section is to prove that under the assumption that $\mu \in \mathcal{P}(G)$ has finite entropy, the condition $h(G,\mu) = 0$ is equivalent to triviality of the measurable Poisson boundary. This is called the entropy criteria for triviality of the Poisson boundary. The proof of this uses the realisation of the Poisson boundary as the exit boundary, through the study of partitions and entropy.

4.1 Entropy of partitions

In the following, we will perform a lot of work with partitions of sets, so let us first settle some basic notation and constructions. If ξ is a partition of a set X, we let $\xi(x)$ denote the atom of ξ which contains $x \in X$, that is, $\xi(x)$ is the equivalence class of x with respect to ξ. We will also consider a partial ordering $\leq$ of partitions, defined as follows:

$$\xi \leq \xi' \text{ if } \xi(x) \supset \xi'(x) \text{ for any } x \in X.$$

Given two partitions ξ, η of the same set X, we define their greatest lower bound $\xi\wedge\eta$ to be the largest partition of X which is coarser than both ξ and η. Similarly, we let $\xi\vee\eta$ denote the smallest upper bound of ξ, η, which is the partition $(\xi(x)\cap\eta(x))_{x\in X}$, or the coarsest partition of X which is finer than both ξ and η. Extending these notions, given an ascending chain $\xi_1 \leq \xi_2 \leq \dots$ of partitions of X, we let $\bigvee_{n=1}^{\infty}\xi_n$ denote the smallest upper bound of the sequence $(\xi_n)_{n\geq 1}$, and analogously, $\bigwedge_{n=1}^{\infty}\eta_n$ denotes the greatest lower bound of a descending chain $\eta_1 \geq \eta_2 \geq \dots$. In the first case, we say that $\xi_n \nearrow \xi := \bigvee_{i=1}^{\infty} \xi_i$, and in the second case, we say that $\eta_n \searrow \eta := \bigwedge_{i=1}^{\infty} \eta_i$.

Another important construction regarding partitions arises when we consider a partition ξ of a set X and a subset $B \subset X$. Then ξ_B denotes the partition of B induced by ξ, i.e., $\xi_B(x) = \xi(x) \cap B$ for all $x \in X$. When looking at partitions of a set X, we always have two canonical partitions at each end of the hierarchy induced by our ordering. We have the trivial partition consisting only of the atom X, which we denote by $\Xi_{\min}$, but also the discrete partition of the space into singletons, which we will denote by $\Xi_{\max}$. The mindful reader may at this point guess that the purpose of this section is to show that the partition $\approx$ from the exit boundary is equal to $\Xi_{\min}$ if and only if $h(G, \mu) = 0$. This is, indeed, the case.

To do so we focus our attention on partitions of a complete probability space $(X, \mathcal{F}, \mu)$. More precisely, we consider the so-called *measurable partitions.* A partition ξ of X is said to be *measurable*, if there exists a countable family $(B_n)_{n\geq 1}$ of subsets in ξ, i.e., each B_n is a union of atoms from ξ, such that for every $A, B \in \xi$ there exists $n \geq 1$ satisfying

$$A \in B_n, B \notin B_n \text{ or } A \notin B_n, B \in B.$$

The sets $B_1, B_2, \dots$ are said to be a basis for ξ. Clearly, any countable partition is measurable, but measurable partitions will in general be larger. They are however not arbitrarily large, and will for example appear as limits of finite partitions.

Let ξ be a measurable partition of X, and let $B_1, B_2, \dots$ be a basis for ξ. Consider for each $n \geq 1$ the partition β_n of X consisting of the atoms B_n and B_n^c. We define a new sequence of partitions $(\xi_n)_{n\geq 1}$ by $\xi_n = \bigvee_{i=1}^{n} \beta_i$, i.e., any atom $A \in \xi_n$ is on the form

$$A = C_1 \cap \cdots \cap C_n,$$

where $C_k \in \{B_k, B_k^c\}$. The sequence $(\xi_n)_{n\geq 1}$ is increasing and its limit is ξ.

We will also say that two partitions ξ, η are equal μ-a.s. if $\mu(\xi(x) \Delta \eta(x)) = 0$, for all $x \in X$. By the discussion from Section 3.5, we will not distinguish between two partitions that are almost equal and thus we will simply view them as the same partition. We are now ready for the definition of entropy in the setting of partitions:

Definition 4.4. Let ξ be a measurable partition of a complete probability space $(X, \mathcal{F}, \mu)$, and let $C_1, C_2, \dots$ be the atoms of ξ with strictly positive measure. Note that there are at most countably many such atoms. We define the entropy of ξ, denoted by $H(\xi)$, to be the quantity

$$H(\xi) = \begin{cases} -\sum_{n=1}^{\infty} \mu(C_n) \log(\mu(C_n)), & \mu\big(X \setminus \bigcup_{n\geq 1} C_n\big) = 0 \\ \infty & \text{otherwise} \end{cases}$$

This definition is a generalisation of the one of entropy of a measure on G, since if $\mu \in \mathcal{P}(G)$, then $H(\mu)$ is equal to $H(\Xi_{\max})$. Also, any partition with finite entropy must necessarily be countable modulo μ. For another way of describing this entropy, we introduce the function $m(\cdot, \xi) \colon X \to [0, \infty)$, given by $m(x, \xi) = \mu(\xi(x)), x \in X$. Then

$$H(\xi) = -\int_X \log(m(x, \xi)) \, \mathrm{d}\mu(x).$$

Before going into more details, we will start out by stating and proving a handfuld of basic properties arising from this definition.

Proposition 4.5. *Let ξ, η and $(\xi_n)_{n\geq 1}$ be measurable partitions of a complete probability space $(X, \mathcal{F}, \mu)$. Then*

1. *$H(\xi) \geq 0$, and $H(\xi) = 0$ if and only if $\xi = \Xi_{\min}$.*
2. *If $\xi \leq \eta$, then $H(\xi) \leq H(\eta)$. If $\xi \leq \eta$ and $H(\xi) = H(\eta) < \infty$, then $\xi = \eta$.*
3. *If $\xi_n \nearrow \xi$, then $H(\xi_n) \nearrow H(\xi)$.*
4. *If $\xi_n \searrow \xi$ and $H(\xi_n)$ is finite for some $n \in \mathbb{N}$, then $H(\xi_n) \searrow H(\xi)$.*
5. *If ξ is a finite partition of n atoms, then $H(\xi) \leq \log(n)$. The inequality is strict unless each atom has probability $1/n$.*
6. *$H(\xi \vee \eta) \leq H(\xi) + H(\eta)$. Moreover, if both partitions have finite entropy, then the above is an equality if and only if ξ and η are independent. Here, ξ and η are independent if $\mu(\xi(x) \cap \eta(x)) = \mu(\xi(x))\mu(\eta(x))$ for all $x \in X$.*

Proof. 1. Positivity of the entropy comes from positivity of $-\log$ on $(0, 1]$. For the second claim, we notice that the entropy of a partition is zero if and only if one atom has full measure and the rest are null-sets. Then $\xi = \Xi_{\min}$ μ-almost surely.

2. The relation $\xi \leq \eta$ is defined as $\xi(x) \supset \eta(x)$ and hence $m(x, \xi) \geq m(x, \eta)$, for all $x \in X$. In the case of $H(\xi) = \infty$ it follows that $H(\eta) = \infty$, as well. If $H(\xi)$ is finite, then it becomes clear from our integral representation of entropy that $H(\xi) \leq H(\eta)$. Assume now that $H(\xi) = H(\eta) < \infty$. Then

$$0 = |H(\eta) - H(\xi)| = H(\eta) - H(\xi) = \int_X -(\log(m(x, \xi) - \log(m(x, \eta))) \,\mathrm{d}\mu(x)$$

By the previous discussion, the integrand above is non-negative and hence it must be zero μ-a.s.. In other words, $\xi = \eta$ μ-a.s.

3. If $\xi_n \nearrow \xi$, then $-\log(m(x, \xi_n)) \nearrow -\log(m(x, \xi))$, for all $x \in X$, so by the monotone convergence theorem, $H(\xi_n) \nearrow H(\xi)$.

4. The assumption $\xi_n \searrow \xi$ ensures that $-\log(m(x, \xi_n)) \searrow -\log(m(x, \xi))$, while the existence of $n \in \mathbb{N}$, such that $H(\xi_n) < \infty$ gives us an integrable upper bound for the tail of this sequence. Hence, the Lebesgue dominated convergence theorem yields the desired convergence.

5. We start out by noting that $x \mapsto -x \log(x)$ is a strictly concave function on the closed unit interval. Let now $C_1, \ldots, C_n$ be the n atoms of ξ. Then

$$\begin{aligned} H(\xi) &= -\sum_{i=1}^{n} \mu(C_i) \log(\mu(C_i)) = -n \cdot \left(\frac{1}{n} \sum_{i=1}^{n} \mu(C_i) \log(\mu(C_i)) \right) \\ &\leq -n \left(\frac{1}{n} \sum_{i=1}^{n} \mu(C_i) \right) \cdot \log \left(\frac{1}{n} \sum_{i=1}^{n} \mu(C_i) \right) \\ &= -\sum_{i=1}^{n} \mu(C_i) \log \left(\sum_{i=1}^{n} \mu(C_i) \right) + \log(n) \sum_{i=1}^{n} \mu(C_i) = \log(n), \end{aligned}$$

where the inequality sign therein is due to Jensen's inequality. From there it also follows that we have an equality if and only if $\mu(C_i) = c$, for all $i = 1, \ldots, n$, and some constant c or in other words that each C_i has probability $1/n$.

6. If either ξ or η has infinite entropy, then the inequality is clear from *2*. Suppose now that ξ, η both have finite entropy and let $(A_i)_{i\geq 1}$ and $(B_j)_{j\geq 1}$ denote the atoms of ξ, respectively η, with positive measure. For each $j \in \mathbb{N}$, we let I_j denote the set of natural numbers i such that $\mu(A_i \cap B_j) > 0$. This will, in particularly, give us

$$\mu(B_j) = \sum_{i\geq 1} \mu(A_i \cap B_j) = \sum_{i \in I_j} \mu(A_i \cap B_j),$$

for each $j \in \mathbb{N}$. From here on the proof is a series of manipulations with a single application of Jensen's inequality. Namely,

$$\begin{aligned} H(\xi \vee \eta) - H(\xi) &= -\sum_{i,j\in\mathbb{N}} \mu(A_i \cap B_j) \log(\mu(A_i \cap B_j)) + \sum_{i\in\mathbb{N}} \mu(A_i) \log(\mu(A_i)) \\ &= \sum_{i,j\in\mathbb{N}} \mu(A_i \cap B_j) \big[\mu(A_i) - \log(\mu(A_i \cap B_j))\big] \\ &= \sum_{j\in\mathbb{N}} \sum_{i\in I_j} \mu(A_i \cap B_j) \log\left(\frac{\mu(A_i)}{\mu(A_i \cap B_j)}\right) \\ &= \sum_{j\in\mathbb{N}} \mu(B_j) \sum_{i\in I_j} \frac{\mu(A_i \cap B_j)}{\mu(B_j)} \log\left(\frac{\mu(A_i)}{\mu(A_i \cap B_j)}\right). \end{aligned}$$

At this point we apply Jensen's inequality to the interior sums, to obtain

$$\begin{aligned} H(\xi \vee \eta) - H(\xi) &\leq \sum_{j\in\mathbb{N}} \mu(B_j) \log\left(\sum_{i\in I_j} \frac{\mu(A_i \cap B_j)}{\mu(B_j)} \frac{\mu(A_i)}{\mu(A_i \cap B_j)}\right) \\ &= \sum_{j\in\mathbb{N}} \mu(B_j) \log\left(\frac{1}{\mu(B_j)} \sum_{i\in I_j} \mu(A_i)\right) \\ &= \sum_{j\in\mathbb{N}} \mu(B_j) \log\left(\sum_{i\in I_j} \mu(A_i)\right) - \sum_{j\in\mathbb{N}} \mu(B_j) \log(\mu(B_j) \\ &\leq \sum_{j\in\mathbb{N}} \mu(B_j) \log(\mu(B_j)) = H(\eta). \end{aligned}$$

In order to determine when this inequality is in fact an equality, we have to consider each of the two above inequalities. In the application of Jensen's inequality, we have an equality if and only if $\frac{\mu(A_i)}{\mu(A_i \cap B_j)} = c_j$, for all $i \in I_j$ and $j \geq 1$, where c_j is some constant. The other inequality is an equality if and only if $\sum_{i\in I_j} \mu(A_i) = 1$, for all $j \geq 1$. If both these conditions are satisfied, it is evident that

$$\frac{1}{c_j} = \sum_{i\in I_j} \frac{\mu(A_i)}{c_j} = \sum_{i\in I_j} \mu(A_i \cap B_j) = \mu(B_j),$$

and hence $\mu(A_i \cap B_j) = \mu(A_i)\mu(B_j)$ for all $i, j \geq 1$, i.e., ξ and η are independent. The other implication follows from the same argument. $\square$

4.2 Conditional entropy of partitions

So far we discussed entropy of a single measurable partition, but the tools which will be of most use are the so-called conditional entropies. For this consider two measurable partitions, ξ and η of a complete probability space $(X, \mathcal{F}, \mu)$. If we let π_η denote the projection from X onto X/η, we may define the measure μ_η on X/η as the image measure of μ under π_η. Before moving on to conditional entropy, we will start out with a small proposition providing an alternative approach to integration over a partition.

Proposition 4.6. *Let ξ be a measurable partition of a probability space $(X, \mathcal{F}, \mu)$. Then there exists $\mu_B \in \mathcal{P}(B)$ for μ_ξ-a.e. $B \in \xi$, such that*

$$\int_X f(x)\,\mathrm{d}\mu = \int_{X/\xi}\int_B f|_B(x)\,\mathrm{d}\mu_B(x)\,\mathrm{d}\mu_\xi(B),$$

for any $f \in L^1(X, \mathcal{F}, \mu)$.

Proof. Consider the Hilbert space $L^2(X) = L^2(X, \mathcal{F}, \mu)$ and consider the subspace $L^2(X, \xi)$ consisting of the functions which are constant on each atom of ξ. Then $L^2(X, \xi)$ is a closed subspace of $L^2(X)$ and hence, there exists a projection p_ξ from $L^2(X)$ to $L^2(X, \xi)$. Note also that $L^2(X, \xi)$ is naturally isomorphic to the space $L^2(X/\xi)$.

Let now $B \in \xi$ be given, and consider $f \in C_c(B)$. Then f extends to a map $\tilde{f} \in L^2(X)$ by setting $\tilde{f}$ equal to 0 outside of B. Applying p_ξ to $\tilde{f}$ will then give us a map $p_\xi(\tilde{f})$ which is constant on B and zero outside of B, so by identifying $p_\xi(\tilde{f})$ with the constant $f|_B$, we obtain a Borel probability measure μ_C on C, through the means of the Riesz Representation Theorem.

From here on, the projection p_ξ and the theory of conditional expectations hands us the desired result, that is,

$$\begin{aligned}\int_X f(x)\,\mathrm{d}\mu(x) = \int_X p_\xi(f)\,\mathrm{d}\mu(x) &= \int_{X/\eta} p_\xi(f)(B)\,\mathrm{d}\mu_\eta(B)\\ &= \int_{X/\eta}\int_B f(x)\,\mathrm{d}\mu_B(x)\,\mathrm{d}\mu_\eta(B)\end{aligned}$$

Note also that if $\mu(B) > 0$, then μ_B is simply the conditional measure given knowledge of B, i.e., $\mu_B(A \cap B) = \mu(A \cap B)/\mu(B)$ for all $A \in F$. □

To define the conditional entropy of ξ given η, we will for μ_η-a.e. $B \in \eta$ consider the partition ξ_B of the probability space (B, μ_B). From here, we define the entropy $H(\xi_B)$ as in Definition 4.4, with respect to the probability space (B, μ_B). This gives us a map $B \mapsto H(\xi_B)$, which is well-defined μ_η-a.e. and hence the integral

$$\int_{M/\eta} H(\xi_B)\,\mathrm{d}\mu_\eta(B)$$

makes sense. We will let $H(\xi \mid \eta)$ denote this integral, and this quantity will be the *conditional entropy of ξ given η*. The following proposition gives us a more concrete way of representing this quantity, through the conditional measures from above.

Proposition 4.7. *For $x \in X$, let $m(x, \xi \mid \eta)$ denote the conditional measure of the set $\xi(x)$ restricted to $\eta(x)$, i.e., $m(x, \xi \mid \eta) = \mu_{\eta(x)}(\xi(x) \cap \eta(x))$. Then*

$$H(\xi \mid \eta) = -\int_X \log(m(x, \xi \mid \eta))\, \mathrm{d}\mu(x).$$

Proof. Consider $B \in \eta$ and $y \in B$. Then $m(y, \xi_B) = m(y, \xi \mid \eta)$, and hence

$$\begin{aligned} -\int_X \log(m(x, \xi \mid \eta))\, \mathrm{d}\mu &= -\int_{X/\eta} \int_B m(x, \xi_B)\, \mathrm{d}\mu_B(x)\, \mathrm{d}\mu_\eta(B) \\ &= \int_{X/\eta} H(\xi_B)\, \mathrm{d}\mu_\eta(B), \end{aligned}$$

which is what we wanted to show. □

We have now defined the tools needed to prove the main result of this section, but let us start out with some properties of the conditional entropy, which will be established through the use of Proposition 4.5.

Proposition 4.8. *Let ξ, η, ζ and $(\xi_n)_{n\geq 1}, (\eta_n)_{n\geq 1}$ be measurable partitions of a complete probability space $(X, \mathcal{F}, \mu)$. Then*

1. *$H(\xi \mid \Xi_{\min}) = H(\xi)$.*

2. *If $\zeta \leq \eta$, then*
$$H(\xi \vee \zeta \mid \eta) = H(\xi \mid \eta).$$

3. *$H(\xi \mid \eta) \geq 0$ and $H(\xi \mid \eta) = 0$ if and only if $\xi \leq \eta$.*

4. *If $\xi \leq \zeta$, then $H(\xi \mid \eta) \leq H(\zeta \mid \eta)$. If in addition $H(\xi \mid \eta) = H(\zeta \mid \eta) < \infty$, then $\xi \vee \eta = \zeta \vee \eta$.*

5. *$H(\xi \vee \eta \mid \zeta) \leq H(\xi \mid \zeta) + H(\eta \mid \zeta)$. If in addition $H(\xi \mid \zeta)$ and $H(\eta \mid \zeta)$ are finite, then the inequality in question is an equality if and only if ξ and η are independent with respect to ζ.*

6. *If $\xi_n \nearrow \xi$, then $H(\xi_n \mid \eta) \nearrow H(\xi \mid \eta)$. If $\xi_n \searrow \xi$ and there exists $m \in \mathbb{N}$ such that $H(\xi_m \mid \eta) < \infty$, then $H(\xi_n \mid \eta)$ converges downwards to $H(\xi \mid \eta)$.*

7. *$H(\xi \vee \eta \mid \zeta) = H(\xi \mid \zeta) + H(\eta \mid \xi \vee \zeta)$.*

8. *$H(\xi \mid \eta \vee \zeta) \leq H(\xi \mid \zeta)$. If, in addition, $H(\xi \mid \zeta), H(\eta \mid \zeta) < \infty$, the above is an equality if and only if ξ and η are independent with respect to ζ, that is, ξ_B and η_B are independent with respect to μ_B, for μ_ζ-a.e. $B \in \zeta$.*

9. *If $\eta_n \nearrow \eta$ and that there exists $n \in \mathbb{N}$ with $H(\xi \mid \eta_n) < \infty$. Then $H(\xi \mid \eta_n)$ converges downwards to $H(\xi \mid \eta)$.*

10. *If $\eta_n \searrow \eta$, then $H(\xi \mid \eta_n) \nearrow H(\xi \mid \eta)$.*

Proof. *1.* The first statement follows quite easily from Proposition 4.7 as

$$m(x,\xi \mid \Xi_{\min}) = \mu_{\Xi_{\min}(x)}(\xi(x) \cap \Xi_{\min}(x)) = \mu_X(\xi \cap X) = \mu(\xi) = m(x,\xi),$$

for all $x \in X$, and hence $H(\xi) = H(\xi \mid \Xi_{\min})$.

2. If $\zeta \leq \eta$, then $\zeta(x) \cap \eta(x) = \eta(x)$, for any $x \in X$ and hence $(\xi \vee \zeta)_B = \xi_B$ for any μ_η-a.e. $B \in \eta$. From there on, it is a matter of applying this result together with the definition of conditional entropy, i.e.,

$$H(\xi \vee \zeta \mid \eta) = \int_{X/\eta} H((\xi \vee \zeta)_B)\, \mathrm{d}\mu_\eta(B) = \int_{X/\eta} H(\xi_B)\, \mathrm{d}\mu_\eta(B) = H(\xi \mid \eta).$$

3. As $H(\xi_B)$ is positive for any $B \in \eta$, it is clear that $H(\xi \mid \eta)$ is positive. For the second part of the statement, we note that $H(\xi \mid \eta) = 0$ if and only if $H(\xi_B) = 0$ for μ_η-a.e. $B \in \eta$. This happens if and only if ξ_B is the trivial partition of B, and thus $\xi(x) \cap B = B$ for all $x \in B$, i.e., $\xi \leq \eta$.

4. If $\xi \leq \eta$, then $\xi_B \leq \zeta_B$ for all $B \in \eta$. From here $H(\xi_B) \leq H(\zeta_B)$ for μ_η-a.e. $B \in \eta$ and hence $H(\xi \mid \eta) \leq H(\zeta \mid \eta)$. If furthermore both $H(\xi \mid \eta)$ and $H(\zeta \mid \eta)$ are finite, then $H(\xi \mid \eta) = H(\zeta \mid \eta)$ if and only if $H(\xi_B) = H(\zeta_B)$ for μ_η-a.e. $B \in \eta$, which, as seen previously, happens if and only if ξ_B and ζ_B are equal for μ_η-a.e. $B \in \eta$, or, phrased another way $\xi \vee \eta = \zeta \vee \eta$.

5. This result follows fairly easily from the non-conditional case, once we establish the equality $(\xi \vee \eta)_B = \xi_B \vee \eta_B$ for all $B \in \zeta$. Fortunately, this is not a problem, as for any $x \in X$,

$$\begin{aligned}(\xi \vee \eta)_B(x) &= (\xi \vee \eta)(x) \cap B = (\xi(x) \cap \eta(x)) \cap B = (\xi(x) \cap B) \cap (\eta(x) \cap B)\\ &= \xi_B(x) \cap \eta_B(x) = (\xi_B \vee \eta_B)(x),\end{aligned}$$

6. If $\xi_n \nearrow \xi$, then $(\xi_n)_B \nearrow \xi_B$ for all $B \in \eta$ and hence $H((\xi_n)_B) \nearrow H(\xi_B)$. By the monotone convergence theorem, we conclude that $H(\xi_n \mid \eta) \nearrow H(\xi \mid \eta)$. Assume now that $\xi_n \searrow \xi$ and that there exists $m \in \mathbb{N}$ such that $H(\xi_m \mid \eta) < \infty$. Then also $H((\xi_m)_B) < \infty$ for μ_η-a.e. $B \in \eta$ and hence $H((\xi_n)_B) \searrow H(\xi_B)$, as $(\xi_n)_B \searrow \xi_B$. The desired conclusion will now follow the Lebesgue dominated convergence theorem using the map $B \mapsto H((\xi_m)_B)$ as an integrable upper bound.

7. Let us first consider the case when ζ is the trivial partition $\Xi_{\min}$. The equality in question here is merely

$$H(\xi \vee \eta) = H(\xi) + H(\eta \mid \xi).$$

In the case when $H(\xi) = \infty$, this follows from *4.* If now $H(\xi) < \infty$, but $H(\eta)$ is infinite, we need to show that $H(\eta \mid \xi) = \infty$. If this is not the case, then $H(\eta_B) < \infty$ for μ_ξ-a.e. $B \in \xi$, which gives a countable partition of X, which is larger than both η and ξ and having full measure. As notation, let $(B_i)_{i\geq 1}$ be the sets of positive measure in ξ, and $(A_{i,j})_{j\geq 1}$ be the sets of positive measure in η_{B_i}. Then $(A_j)_{j\geq 1}$, defined as $A_j = \bigcup_i A_{i,j}$ will be the sets of positive measure in η. We deduce

$$\begin{aligned}
H(\eta \mid \xi) &= \sum_i H(\eta_{B_i})\mu(B_i) = -\sum_{i,j} \mu(B_i)\mu_{B_i}(A_{i,j} \cap B_i)\log(\mu_{B_i}(A_{i,j} \cap B_i)) \\
&= -\sum_{i,j} \mu(A_{i,j} \cap B_i)\big(\log(\mu(A_{i,j} \cap B_i)) - \log(\mu(B_i))\big) \\
&= -\sum_{i,j} \mu(A_{i,j} \cap B_i)\log(\mu(A_{i,j} \cap B_i)) + \sum_{i,j} \mu(A_{i,j} \cap B_i)\log(\mu(B_i)) \\
&\geq -\sum_{i,j} \mu(A_{i,j} \cap B_i)\log\left(\sum_k \mu(A_{k,j} \cap B_k)\right) - H(\xi) \\
&= -\sum_{i,j} \mu(A_{i,j} \cap B_i)\log(\mu(A_j)) - H(\xi) = H(\eta) - H(\xi).
\end{aligned}$$

As $H(\xi)$ was assumed to be finite, we must have $H(\eta \mid \xi) = \infty$, whenever $H(\eta)$ is infinite. Let us finally consider the case, where both $H(\xi)$ and $H(\eta)$ are finite, and let $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ denote the respective sequences of sets with positive measure. Then

$$\begin{aligned}
H(\xi \vee \eta) &= -\sum_{i,j} \mu(A_i \cap B_j)\log(\mu(A_i \cap B_j)) \\
H(\xi) &= -\sum_j \mu(B_j)\log(\mu(B_j)) = -\sum_{i,j} \mu(A_i \cap B_j)\log(\mu(B_j) \\
H(\eta \mid \xi) &= -\sum_{i,j} \mu(A_i \cap B_j)\big(\log(\mu(A_i \cap B_j) - \log(B_j)\big).
\end{aligned}$$

Adding these relations up, gives us the desired result. The reason for this intermediate step is to see that we immediately have

$$H(\xi_B \vee \eta_B) = H(\xi_B) + H(\eta_B \mid \xi_B),$$

for all $B \in \zeta$. From here it remains to show that

$$\int_{X/\zeta} H(\eta_B \mid \xi_B)\,\mathrm{d}\mu_\zeta(B) = H(\eta \mid \xi \vee \zeta).$$

As the partition $\xi \vee \zeta$ of X can be viewed as first partitioning X with respect to ζ and afterwards partitioning each atom of ζ with respect to ξ, it is evident that for any $B \in \zeta$, the equality $m(x, \eta_B \mid \xi_B) = m(x, \eta \mid \xi \vee \zeta)$ holds true for any $x \in B$. From there, an application of Proposition 4.7 gives us the desired result.

8. If $H(\xi \mid \zeta) = \infty$, then there is nothing to prove, and the case of $H(\eta \mid \zeta) < \infty$ does not cause much trouble, since by *5.* and *7.*,

$$H(\xi \mid \eta \vee \zeta) + H(\eta \mid \zeta) = H(\xi \vee \eta \mid \zeta) \leq H(\xi \mid \zeta) + H(\eta \mid \zeta).$$

If, in addition, $H(\xi \mid \zeta) < \infty$, then the second part of the statement is a consequence of the second part of *5.* The third and most difficult case is the one where the quantity $H(\xi \mid \zeta)$ is finite, but $H(\eta \mid \zeta) = \infty$. The proof of this case will also prove certain parts of *9.* Assume first that ξ is a finite partition of X, and let $A_1, \ldots, A_m$ denote its atoms. Let now $(\eta_n)_{n\geq 1}$ be an ascending chain of partitions converging to η, and let $f_{i,n}, f_i \colon X \to [0,1]$ for $1 \leq i \leq m, n \geq 1$, be defined as follows for $x \in X$,

$$f_{i,n}(x) = \mu_{\eta_n(x)}(A_i \cap \eta_n(x)), \quad f_i(x) = \mu_\eta(x)(A_i \cap \eta(x)).$$

By the construction, we obtain $f_{i,n} \xrightarrow{L^2} f_i$ as $n \to \infty$ for all $1 \leq i \leq m$, and hence converges also in probability. As ξ was finite, so is each $\xi_{\eta_n(x)}$ and $\xi_{\eta(x)}$, and thus

$$H(\xi_{\eta_n(x)}) = -\sum_{i=1}^{m} f_{i,n}(x)\log(f_{i,n}(x)), \qquad H(\xi_{\eta(x)}) = -\sum_{i=1}^{m} f_i(x)\log(f_i(x)).$$

As convergence in probability is preserved by continuous transformations, we conclude that $H(\xi_{\eta_n(x)}) \to H(\xi_{\eta(x)})$ in probability. Furthermore each $\xi_{\eta_n(x)}$ consists of at most m atoms, and hence $H(\xi_{\eta_n(x)})$ is less than or equal to $\log(m)$ for all $x \in X, n \geq 1$. This gives convergence in L^1, or in other words $H(\xi \mid \eta_n) \to H(\xi \mid \eta)$. Note that this is a actually a particular case of *9.* Assume now that each η_n is finite, which is possible by our discussion on measurable partitions. Then $H(\eta_n \mid \zeta) < \infty$ for each $n \geq 1$, and also

$$H(\xi \mid \eta_n \vee \zeta) \leq H(\xi \mid \zeta).$$

By the convergence established above, we conclude that $H(\xi \mid \eta \vee \zeta) \leq H(\xi \mid \zeta)$. For the general case when ξ is not necessarily finite, we can pick an ascending sequence $(\xi_n)_{n\geq 1}$ such that $\xi_n \nearrow \xi$. Then $H(\xi_n \mid \eta \vee \zeta) \leq H(\xi_n \mid \zeta)$, for all $n \geq 1$, and by *6.* we conclude that $H(\xi \mid \eta \vee \zeta) \leq H(\xi \mid \zeta)$.

9. We have already shown the convergence in the case where ξ is a finite partition, but let us prove it for an arbitrary measurable partition as well. For this let $\delta > 0$ be given and pick a finite partition $\xi' \leq \xi$ such that

$$|H(\xi' \mid \eta) - H(\xi \mid \eta)| < \delta/3, \quad |H(\xi' \mid \eta_1) - H(\xi \mid \eta_1)| < \delta/3.$$

By the above comment, there exists $N \in \mathbb{N}$ such that $|H(\xi' \mid \eta) - H(\xi' \mid \eta_n)| < \delta/3$, whenever $n \geq N$. From there on, we have for $n \geq 1$,

$$\begin{aligned} H(\xi \mid \eta_n) - H(\xi' \mid \eta_n) &= H(\xi \vee \xi_\delta \mid \eta_n) - H(\xi_\delta \mid \eta) \\ &= H(\xi \mid \xi_\delta \vee \eta_n) = H(\xi \mid \xi_\delta \vee \eta_n \vee \eta_1) \\ &\leq H(\xi \mid \xi_\delta \vee \eta_1) \\ &= H(\xi \mid \eta_1) - H(\xi_\delta \mid \eta_1) < \delta/3. \end{aligned}$$

For $n \geq N$, we can now conclude that $|H(\xi \mid \eta_n) - H(\xi \mid \eta)| < \delta$ by applying the triangle inequality twice along with the estimates above. This gives us the desired convergence.

10. If ξ is a finite partition, the proof is analogous to the finite case of *9.*, so we will move on to the general case. First of, we consider the case when $H(\xi \mid \eta) = 0$, for which the result is clear due to *3.* Assume now that $H(\xi \mid \eta) > 0$ and let

$0 < c < H(\xi \mid \eta)$ be given. Note that $H(\xi \mid \eta_n) \leq H(\xi \mid \eta)$ for all $n \geq 1$, so it suffices to show that $H(\xi \mid \eta_n) > c$ from a certain point on. For this we pick a finite partition $\xi' \leq \xi$ such that $c < H(\xi' \mid \eta) \leq H(\xi \mid \eta)$. By the comment above, we may pick $N \in \mathbb{N}$ such that $c < H(\xi' \mid \eta_n) \leq H(\xi' \mid \eta)$, for all $n \geq N$ and thus, also $c < H(\xi' \mid \eta_n) \leq H(\xi \mid \eta_n)$. In conclusion $H(\xi \mid \eta_n) \nearrow H(\xi \mid \eta)$, as $n \to \infty$. □

4.3 The entropy criteria for triviality of the Poisson boundary

We are now ready to relate these concepts of entropy to triviality of the Poisson boundary. We return to our probability space Ω from the construction of the Poisson boundary, equipped with the restriction of $\mathbb{P}_e$ to this space, for now denoted $\mathbb{P}_\Omega$. We associate two sequences of measurable partitions of Ω denoted by $(\alpha_n)_{n\geq 0}$ and $(\eta_n)_{n\geq 0}$. For $n \geq 0$ and $\omega, \omega' \in \Omega$, we define these partitions as the equivalence classes for the following equivalence relations.

$$\omega \overset{\eta_n}{\sim} \omega' \text{ if and only if } \omega_k = \omega'_k \text{ for all } k \geq n$$
$$\omega \overset{\alpha_n}{\sim} \omega' \text{ if and only if } \omega_k = \omega'_k \text{ for all } k \leq n$$

We will in the following not distinguish between equivalence relations and partitions. The sequence $(\alpha_n)_{n\geq 0}$ is an ascending chain of partitions with limit $\Xi_{\max}$. The sequence $(\eta_n)_{n\geq 0}$ is descending with limit η_∞ defined as follows

$$\omega \overset{\eta_\infty}{\sim} \omega' \text{ if and only if there exists } n \geq 0 \text{ such that } \omega_k = \omega'_k \text{ for all } k \geq n.$$

The partition η_∞ looks very much like the partition associated to $\approx$ defined for the exit boundary. Note that this latter partition does not care for the first coordinate, i.e. the coordinate corresponding to the initial distribution of the random walk. In other words each atom in this partition can be written as $G \times A$, with $A \in \eta_\infty$, and hence we only need to study η_∞.

Lemma 4.9. *Assume that μ has finite entropy and let $0 \leq k \leq n < \infty$. Then*

$$H(\alpha_k \mid \eta_n) = (k+1)H(\mu) + H(\mu^{*(n-k)}) - H(\mu^{*n}).$$

Proof. By Proposition 4.7, we may write

$$H(\alpha_k \mid \eta_n) = -\int_\Omega \log(m(\omega, \alpha_k \mid \eta_n))\, \mathrm{d}\mathbb{P}_\Omega(\omega).$$

As $\mathbb{P}_\Omega$-a.e. $x \in \Omega$ is of the form

$$x = (\omega_0, \omega_0\omega_1, \omega_0\omega_1\omega_2, \dots)$$

for $\omega = (\omega_0, \omega_1, \dots) \in \Omega$, the quantity $m(x, \alpha_k \mid \eta_n)$ measures the probability of an element in $\eta_n(x)$ agreeing with x on the first k coordinates, i.e.

$$\mathbb{P}_\Omega(R_m = x_m,\ m \leq k \mid R_n = x_n).$$

These elements may not coincide with x on the coordinates between k and n, but the intermediate steps will create a path from x_k to x_n, or in other words, such an x' will satisfy $\omega'_{k+1}\omega'_{k+2}\cdots\omega'_n = x_k^{-1}x_n$. This translates to the fact that

$$m(x, \alpha_k \mid \eta_n) = \frac{\mu(\omega_0)\cdots\mu(\omega_k)\mu^{*(n-k)}(x_k^{-1}x_n)}{\mu^{*n}(x_n)}$$

where $x = (x_0, x_1, \dots) = (\omega_0, \omega_0\omega_1, \dots)$. Inserting this into our formula for the conditional entropy, we obtain

$$\begin{aligned} H(\alpha_k \mid \eta_n) = -\sum_{i=0}^{k} \int_\Omega \mu(\omega_i)\,\mathrm{d}\mathbb{P}_\Omega(\omega) - \int_\Omega \log(\mu^{*(n-k)}(x_k^{-1}x_n)\,\mathrm{d}\mathbb{P}_\Omega(\omega) \\ + \int_\Omega \log(\mu^{*n}(x_n))\,\mathrm{d}\mathbb{P}_\Omega(\omega). \end{aligned}$$

To see that this expression is in fact the one we are looking for, we apply a suitable version of the abstract change of variable formula. Then for any $n \geq 0$,

$$\begin{aligned} -\int_\Omega \log(\mu^{*n}(x_n))\,\mathrm{d}\mathbb{P}_\Omega(\omega) &= -\int_\Omega \log(\mu^{*n}(R_n(\omega))\,\mathrm{d}\mathbb{P}_\Omega(\omega) \\ &= -\int_G \log(\mu^{*n}(g))\,\mathrm{d}R_n(\mathbb{P}_\Omega)(g) \\ &= -\sum_{g\in G} \log(\mu^{*n}(g))\mu^{*n}(g) = H(\mu^{*n}) = h_n. \end{aligned}$$

Applied to our above result, we obtain the desired conclusion. □

Lemma 4.10. *Assume that μ has finite entropy. Then $(h_{n+1}-h_n)_{n\geq 0}$ is a decreasing sequence with limit $h(G,\mu)$, where $h_n = H(\mu^{*n})$ for $n \geq 1$.*

Proof. Let $n \geq 0$ be given. By Lemma 4.9 we may consider the difference

$$h(\alpha_1 \mid \eta_{n+1}) - h(\alpha_1 \mid \eta_n) = (h_n - h_{n+1}) - (h_{n-1} - h_n),$$

which by the properties of conditional expectations is strictly positive, and hence the sequence in question is decreasing. The sequence $(h_{n+1} - h_n)_{n\geq 0}$ must also be non-negative, since otherwise

$$h_m = \sum_{i=0}^{m-1} h_{i+1} - h_i,$$

would eventually become negative, and which is impossible. Thus the limit of the sequence $(h_{n+1} - h_n)_{n\geq 0}$ exists. This limit will then coincide with the limit of the averages

$$\frac{1}{n}\sum_{i=0}^{n-1}(h_{i+1} - h_i) = \frac{h_n}{n} + \frac{h_0}{n} = \frac{h_n}{n},$$

where we recall that the degenerate measure μ^{*0} has zero entropy. By the very definition of $h(G,\mu)$, this gives us $(h_{n+1} - h_n) \to h(G,\mu)$ as $n \to \infty$. □

Theorem 4.11. *Let μ be a probability measure on G with finite entropy. Then $h(G,\mu) = 0$ if and only if the Poisson boundary is trivial.*

Proof. We only need to prove that the exit boundary $(\mathcal{E}, \alpha_\rho)$ is trivial. We start out by estimating for $0 \leq k \leq n < \infty$,

$$\begin{aligned} H(\alpha_k \mid \eta_n) &= (k+1)h_1 + h_{n-k} - h_n \\ &= (k+1)h_1 - \sum_{i=0}^{k}(h_{n-k+i+1} - h_{n-k+i}), \end{aligned}$$

Thus, $H(\alpha_k \mid \eta_\infty) = \lim_{n\to\infty} H(\alpha_k \mid \eta_n) = kh_1 - (k+1)h(G,\mu)$. Recall that the partition α_k only cares about the first $k+1$ coordinates of the sequences in Ω, but on those first coordinates it is a discrete partition. In other words $\Omega/\alpha_k \cong G^{k+1}$, and hence $H(\alpha_k) = H(\mu \otimes \cdots \otimes \mu) = (k+1)h_1$. By an application of Proposition 4.8 is evident that $h(G,\mu) = 0$ if and only if α_k and η_∞ are independent for all $k \geq 1$. As $(\alpha_k)_{k\geq 1}$ converges to Ξ_{max} this happens if and only if η_∞ is equal to $\Xi_{\min}$. Combined with earlier arguments, we conclude that the partition of $G\times\Omega$ associated to $\approx$ consists of a single atom as well and hence

$$\mathcal{E} = (G \times \Omega)/\!\approx$$

is a singleton, which was the desired result. □

In this entropy criteria, the assumption of finite entropy was crucial. Let us see what might go wrong, if we allow infinite entropy. We consider an abelian group G equipped with a Borel probability measure μ of full support, such that $H(\mu) = \infty$. The existence of such a measure was discusses in Remark 4.2. By an application of Jensen's inequality, we also have $H(\mu^{*n}) = \infty$, for all $n \geq 1$ and thus $h(G,\mu) = \infty$. However, as μ has full support, the Poisson boundary is trivial, which shows that Theorem 4.11 does not hold true for arbitrary measures.

4.4 Growth and entropy

In this section, we introduce the concept of growth of a finitely generated group, and establish how certain types of growth allow us to conclude that well-behaved measures on such groups have zero entropy.

Consider now a finitely generated group G, and a finite symmetric generating set S with identity. We define a function $\gamma_S^G \colon \mathbb{N}_0 \to \mathbb{N}_0$, by $\gamma_S(n) = |S^n|$, i.e., $\gamma_S(n)$ is the number of elements which can be constructed by at most n elements from S. Such a function will be called a *growth function.* Next, we define an ordering $\preceq$ of growth functions as follows:

$$\gamma_S^G \preceq \gamma_{S'}^{G'} \text{ if there exists } k \in \mathbb{N} \text{ such that } \gamma_S^G(n) \leq k\, \gamma_{S'}^{G'}(kn), \text{ for all } n \in \mathbb{N}.$$

One can easily check that this is a partial order and thus inducing a equivalence $\sim$ of growth functions, i.e., $\gamma_S^G \sim \gamma_{S'}^{G'}$ if both $\gamma_S^G \preceq \gamma_{S'}^{G'}$ and $\gamma_{S'}^{G'} \preceq \gamma_S^G$. Now, we would like to talk about *the* growth of a group, so it would be desirable that the growth did not depend on the choice of finite generating set. Fortunately, this problem is solved by the following proposition.

Proposition 4.12. *Let G be a finitely generated group, and let S, S' be finite symmetric generating sets of G. Then $\gamma_S^G \sim \gamma_{S'}^G$.*

Proof. With only finitely many elements in S', there exists some $k \in \mathbb{N}$, such that $S' \subset S^k$. Then $\gamma_{S'}^G(n) \leq \gamma_S^G(kn) \leq k \cdot \gamma_S^G(kn)$, and hence $\gamma_{S'}^G \preceq \gamma_S^G$. The converse inequality can be shown in the exact same manner. □

With this proposition in mind, we let γ^G denote the equivalence class of the growth functions for G, and call this function *the* growth of G. We next spend a bit of time investigating the possible types of growth.

- The first types of growth functions we will consider are the exponential ones, i.e., functions of the form $n \mapsto a^n$ for some $a > 1$. As it turns out, all exponential functions are equivalent in the sense defined above and to prove this, let $a, b > 1$ be given, with $a \leq b$. Clearly $a^n \preceq b^n$ and for the other inequality let k be an integer greater than b/a. Then

  $$b^n \leq (ka)^n \leq k \cdot (ka)^n$$

  and hence $b^n \preceq a^n$. From here on, we let $\exp(n)$ denote the equivalence class of exponential functions. Groups with growth $\exp(n)$ will be said to have *exponential growth.*

  Examples of groups with exponential growth are the free groups. For example, if $G = \mathbb{F}_2 = \langle a, b\rangle$, with a symmetric generating set $S = \{e, a, b, a^{-1}, b^{-1}\}$, it is not difficult to see that $\gamma_S^{\mathbb{F}_2}(n) = 4 \cdot 3^{n-1}$.

- The second interesting type of growth is the polynomial one, i.e., $n \mapsto n^d$ for some $d \in \mathbb{N}$. It is clear that $c \leq d$ implies $n^c \preceq n^d$, but the converse inequality does not hold. Indeed, if $d > c$ and $k \in \mathbb{N}$ is given, such that $n^d \leq k \cdot (kn)^b$ for all $n \in \mathbb{N}$, then $n^{d-b} \leq k^{b+1}$ for all $n \in \mathbb{N}$, which is clearly a contradiction.

  Note that, these types of growth are sufficient to describe any kind of polynomial growth function. Indeed, it is well-known that a polynomial of degree d will behave like n^d, when n gets big enough. In other words, if $\gamma(n)$ is on the form $\sum_{k=0}^d \alpha_k n^k$, with $\alpha_0, \ldots, \alpha_d \in \mathbb{R}$, then $\gamma \sim n^d$.

  We should note that any finitely generated abelian group has polynomial growth. Indeed, given a finitely generated group G, with a finite symmetric generating set $S = \{s_1, \ldots, s_k\}$, the set S^n is contained in the set

  $$\{s_1^{n_1} \cdots s_k^{n_k} \mid n_1, \ldots, n_k \leq n\},$$

  and hence $\gamma_S^G(n) \leq n^k$.

- After introducing the two important types of growth above, we notice that exponential growth is always strictly greater than any polynomial growth. To see that $n^d \preceq \exp(n)$ for any $d \in \mathbb{N}$, notice that $n^d e^{-n} \to 0$, as $n \to \infty$, and hence there exists some integer c, such that $n^d \leq c\exp(n) \leq c\exp(cn)$, for all $n \in \mathbb{N}$, which by definition is $n^d \preceq \exp(n)$. To see that the inequality is strict, assume that there exists some integers $c, d \in \mathbb{N}$, such that $\exp(n) \leq c \cdot (cn)^d$ for all $n \in \mathbb{N}$. Then $e^n n^{-d} \leq c^{d+1}$, for all $n \in \mathbb{N}$, which contradicts the fact that the sequence $(e^n n^{-d})_{n \geq 1}$ is unbounded.

- Given a finite generating set S, it is evident that $S^n = \{s_1 \cdots s_n \mid s_i \in S\}$, which contains at most $|S|^n$ elements, representing the words without any cancellations. In other words, $\gamma^G \preceq \exp(n)$ and hence, any finitely generated group has at most exponential growth. This gives us an upper bound for the growth of a group, so let us investigate a possible lower bound. If G is a finite group, then $\gamma^G(n) \leq |G| \cdot 1$ for all $n \in \mathbb{N}$, and hence $\gamma^G \sim 1$, which we will call constant growth. If G is an infinite group, then
$$S \subsetneq S^2 \subsetneq \cdots$$
and hence $\gamma_S^G(n) + 1 \leq \gamma_S^G(n+1)$ for any $n \in \mathbb{N}$. Then $\gamma^G \succeq n$, i.e. G has at least linear growth. It is now clear that any infinite, finitely generated group has growth somewhere between linear and exponential growth.

- Any group not of exponential growth will be said to have subexponential growth. Alternatively, G has subexponential growth if $\left(\gamma^G(n)\right)^{1/n} \to 1$ as $n \to \infty$. Furthermore, if $\gamma^G \sim n^d$ for some $d \in \mathbb{N}$, we say that G has polynomial growth. The degree of the polynomial is usually not that important, so it makes sense to bundle all these groups together.

  If G is a group, neither of polynomial growth nor of exponential growth, we say that G has intermediate growth. The existence of such groups was a long-time open question, but settled by Rostislav Grigorchuk in 1984 [13].

We are now ready to see, how we can use the growth of the given group to establish if the entropy for certain types of measures is zero.

Proposition 4.13. *Let G be a finitely generated group of subexponential growth, and let μ be a probability measure on G with finite support. Then $h(G, \mu) = 0$.*

Proof. Let K denote the support of μ, and let S be some finite symmetric generating set for G, containing the identity and also K. Then $|K^n| \leq \gamma_S^G(n)$, for all $n \in \mathbb{N}$. As μ^{*n} is the image measure of the n-fold product of elements in K, it is not difficult to see that $\operatorname{supp}(\mu^{*n}) \subset K^n$. Finally, by arguments similar to those of proof of Proposition 4.5, part *5.*, we have $H(\mu^{*n}) \leq \log(|K^n|)$, and hence
$$\frac{H(\mu^{*n})}{n} \leq \frac{\log(|K^n|)}{n} = \log\left(|K^n|^{1/n}\right) \leq \log\left(\gamma_S^G(n)^{1/n}\right).$$
By assumption of subexponential growth, we may conclude that the entropy $h(G, \mu)$, is equal to the logarithm of 1, which is zero. $\square$

Note that combining this result with the previously established entropy criteria and the example in Section 3.2, actually provides an alternative proof for the fact that

$\mathbb{F}_2$ has exponential growth. However, this argument is infinitely more complicated than the direct one.

The next goal is to show that we can replace the requirement of μ having finite support with the weaker condition of finite first moment. In order to talk about a measure μ on the group G having a first moment, we will have to find a way of assigning a numerical value to each group element. The idea is to assign each group element with a length, so let S be a finite, symmetric generating set for G containing the identity. We then define a map $\ell_S^G : G \to \mathbb{R}$, by

$$\ell_S^G(g) = \inf\{n \in \mathbb{N}_0 \mid \exists s_1, \ldots, s_n \in S : g = s_1 \cdots s_n\}, \quad g \in G,$$

that is, $\ell_S^G(g)$ is the minimal number of elements in S required produce g. In this way, we could alternatively have defined $\gamma_S^G(n)$ to be the number of elements with length less than or equal to n, with respect to S. As with the growth functions, it is desirable that we could ignore the generating set S, so let us see how the length functions for two generating sets relate.

Proposition 4.14. *Let S_1, S_2 be two finite, symmetric generators for G, both containing the identity. Then there exists $C, C' > 0$ such that*

$$\frac{1}{C'}\ell_{S_1}^G(g) \le \ell_{S_2}^G(g) \le C\ell_{S_1}^G(g), \quad g \in G.$$

Proof. As both S_1 and S_2 are finite generating sets, let C denote the number

$$\max\{\ell_{S_2}^G(s) \mid s \in S_1\},$$

which is finite. Then $\ell_{S_2}^G(g) \le C\ell_{S1}^G(g)$ for all g, as any generator in S_1 can be build up from at most C generators from S_2. Similarly we define $C' = \max\{\ell_{S_1}^G(s) \mid s \in S_2\}$, to get the other inclusion. □

This result tells us that the length functions are all equal up to multiplication by some constant. This constant will not have any effect on the results to come, so for the remainder of this section, we consider some fixed finite, symmetric generating set S and denote the length function by ℓ^G. We are now ready to define the first moment of a probability measure on G.

Definition 4.15. Let μ be a probability measure on a finitely generated group G. We define the first moment of μ, denoted by $L(\mu)$, to be the quantity

$$\sum_{g\in G} \ell^G(g)\mu(g).$$

If this is finite, we say that μ has finite first moment.

Since having finite entropy was crucial for the entropy criteria, we start out by showing that finite first moment automatically gives us finite entropy. To do this, we first rewrite the the first moment, as

$$L(\mu) = \sum_{k\ge 0}\sum_{g\in S_k} k\mu(g),$$

where $S_k = \{g \in G \mid \ell^G(g) = k\}$. As $(S_k)_{k\geq 0}$ is a partition of G, we may write

$$H(\mu) = -\sum_{k\geq 0}\sum_{g\in S_k} \log(\mu(g))\mu(g).$$

For any $k \geq 0$ with $\mu(S_k) > 0$ we rewrite the inner sums as follows

$$\begin{aligned}
-\sum_{g\in S_k} \mu(g)\log(\mu(g)) &= \mu(S_k)\Big(-\sum_{g\in S_k} \frac{\mu(g)}{\mu(S_k)}\log(\mu(g))\Big) \\
&= \mu(S_k)\Big(-\sum_{g\in S_k}\Big[\frac{\mu(g)}{\mu(S_k)}\log(\mu(g))\Big] + \log(\mu(S_k))\Big) \\
&\quad - \log(\mu(S_k))\mu(S_k) \\
&= \mu(S_k)\Big(-\sum_{g\in S_k} \frac{\mu(g)}{\mu(S_k)}\log\Big(\frac{\mu(g)}{\mu(S_k)}\Big)\Big) - \log(\mu(S_k))\mu(S_k).
\end{aligned}$$

The sum above is now representing the entropy of the normalisation of μ to the finite set S_k and is thus less than or equal to $\log(|S_k|)$, as seen in Proposition 4.5. Moving on from here, we see that

$$\begin{aligned}
H(\mu) &\leq \sum_{k\geq 0}\big(\mu(S_k)\log(|S_k|) - \mu(S_k)\log(\mu(S_k))\big) \\
&\leq \sum_{k\geq 0}\big(k\mu(S_k)\log(|S|) - \mu(S_k)\log(\mu(S_k))\big) \\
&= L(\mu)\log(|S|) - \sum_{k\geq 0}\mu(S_k)\log(\mu(S_k)).
\end{aligned}$$

In the second equality, we have applied the fact that $S_k \subset S^k$, to obtain the inequality $|S_k| \leq |S^k| \leq |S|^k$. Let now $N = \{k \geq 0 \mid \mu(S_k) < e^{-k}\}$ and recall that $t \mapsto -t\log(t)$ is an increasing function on $[0, e^{-1}]$. Then

$$\begin{aligned}
\sum_{k\geq 0} -\mu(S_k)\log(\mu(S_k)) &= \sum_{k\in N} -\mu(S_k)\log(\mu(S_k)) + \sum_{k\in N^c} -\mu(S_k)\log(\mu(S_k)) \\
&\leq \sum_{k\in N} ke^{-k} + \sum_{k\in N^c} k\mu(S_k) \leq \sum_{k\geq 0} ke^{-k} + \sum_{k\geq 0} k\mu(S_k),
\end{aligned}$$

where the first sum is a convergent infinite series and the second sum is simply $L(\mu)$. All in all, we have shown that $H(\mu)$ is finite, whenever $L(\mu)$ is finite. This however, is not the result we were looking for, but merely a short step along the way. We now consider a small lemma.

Lemma 4.16. *Let $\varepsilon > 0$ be given, and let $(B_n)_{n\geq 1}$ be a sequence of finite subsets of G such that $|B_n| < e^{(h(G,\mu)+\varepsilon)n}$, eventually. Then $R_n \notin B_n$ eventually.*

Proof. Assume in order to reach a contradiction that $R_n \in B_n$ infinitely often. Then μ^{*n} would be a probability measure on B_n for any such $n \in \mathbb{N}$, and hence

$H(\mu^{*n}) \leq \log(|B_n|)$. However, as the numbers $H(\mu^{*n})/n$ eventually drop below $h(G,\mu)+\varepsilon$, we must also have

$$\log(|B_n|) \geq H(\mu^{*n})n > (h(G,\mu)+\varepsilon)n,$$

from some n onwards. With two contradicting inequalities happening infinitely often and eventually, we are left with a contradiction, which proves the claim. $\square$

Theorem 4.17. *Let G be a group with subexponential growth and let μ be a probability measure on G with finite first moment. Then $h(G,\mu) = 0$.*

Proof. Let us first consider the number $c = \limsup_{n\to\infty} \log(\gamma^G(n))/n$, which as noted earlier will be equal to zero, whenever the group has sub-exponential growth. The goal is now to prove that $h(G,\mu)$ is bounded by c multiplied by some finite constant, whenever μ has finite entropy. To find this constant, we recall that

$$R_{n+m}(\omega) = R_n(\omega)\cdots R_m(T^n(\omega)),$$

for all $n,m \geq 0$ and $\omega \in \Omega$, where $T\colon \Omega \to \Omega$ again denotes the shift map. Looking at the definition of ℓ^G, it is not difficult to see that

$$\ell^G(R_{n+m}(\omega)) \leq \ell^G(R_n(\omega)) + \ell^G(R_m(T^n(\omega))),$$

and hence $\ell^G(R_{n+m}) \leq \ell^G(R_n) + \ell^G(R_m)\circ T^n$. In order to apply Kingman's Subadditive Ergodic Theorem (see Theorem B.1), we still need to show that $E\big(\ell^G(R_n)\big)$ is finite for all $n \geq 1$, where the expectation is respect to the measure $\mathbb{P}$ introduced in Section 3.2. To do this, we start out by noticing that

$$E\big(\ell^G(R_1)\big) = \int_\Omega \ell^G(R_1)\,\mathrm{d}\mathbb{P} = \int_G \ell^G(g)\,\mathrm{d}R_1(g) = \sum_{g\in G} \ell^G(g)\mu(g) = L(\mu).$$

Furthermore, we recall that T is $\mathbb{P}$-measure preserving, and hence

$$E\big(\ell^G(R_{n+1})\big) \leq E\big(\ell^G(R_n)\big) + E\big(\ell^G(R_1)\circ T\big) = E\big(\ell^G(R_n)\big) + L(\mu),$$

for any $n \geq 0$. From here on it follows by induction that $E\big(\ell^G(R_n)\big) < \infty$, for all $n \geq 0$. All in all, this lets us define the number $\ell(G,\mu)$ as

$$\ell(G,\mu) = \lim_{n\to\infty} E\left(\frac{\ell^G(R_n)}{n}\right)$$

The goal is now to prove that $h(G,\mu) \leq c\cdot\ell(G,\mu)$. To do this, let $\varepsilon > 0$ be given, and consider the subsets $(A_n)_{n\geq 1}$ of G, given by $A_n = S^{(\ell(G,\mu)+\varepsilon)n}$. From the definition of c, we must have $\log(\gamma^G(n)) \leq (c+\varepsilon)n$ eventually, and thus

$$\log|A_n| \leq (\ell(G,\mu)+\varepsilon)\log(\gamma^G(n)) \leq n(\ell(G,\mu)+\varepsilon)(c+\varepsilon),$$

eventually. Applying the exponential function on both sides, will eventually give us the inequality

$$|A_n| \leq \exp(n(\ell(G,\mu)+\varepsilon)(c+\varepsilon)).$$

On the other hand, the choice of A_n gives us $R_n \in A_n$ eventually, so by the above lemma, we may conclude that $|A_n| \geq e^{(h(G,\mu)+\varepsilon)n}$, infinitely often. In particular we can conclude that $(h(G,\mu)+\varepsilon) \leq (c+\varepsilon)(\ell(G,\mu)+\varepsilon)$, which after letting ε tend to zero will result in $h(G,\mu) \leq c \cdot \ell(G,\mu)$. As $c = 0$, whenever G has subexpontential growth the proof is complete. □

To finish off this section, we should note that the above results do not provide a complete description of boundary triviality. The following theorem of interest is mentioned in [7]:

Theorem 4.18.

1. *There exist groups of subexponential growth, which admit Borel probability measures with non-trivial Poisson boundary.*
2. *There exist groups G of subexponential growth, which admit Borel probability measures μ with finite entropy, but with $h(G,\mu) > 0$.*

5 Amenability

This section is devoted to investigating two connections between μ-boundaries and amenability. The first one is a alternative way of defining amenability of a second countable, locally compact groups, solely by the appearance of the μ-boundaries. Harry Furstenberg proved in 1973 [11] that any non-amenable group admits a probability measure with full support and non-trivial boundary, whereafter he conjectured that the converse implication was true as well. The second half of the statement was later proved to be true as well. this was later proved by Joseph Rosenblatt in 1981 in [23], and independently by Kaimanovich-Vershik in [16]. We will not go into technical details in the part, but rather give a brief survey of the important results. A thorough discussion of the proofs can be found in [21].

Another approach to amenability is that of amenable actions. The idea behind this concept is to extend the notion of an amenable groups to a more general case of a G-space. This can be done in two different ways, where we follow the definition of Robert Zimmer, who also proved that any group G acts amenably on its Poisson boundary, whenever the measure μ is absolutely continuous with respect to the Haar measure λ. We will, however, provide a different proof for this result, a proof which can be found in [2].

Before starting all this, we give a brief introduction to amenable locally compact groups, as well as a list of equivalent ways to characterise amenability of a group. Again, this will be done without many details, but the curious reader can look at [21], [12] for more explanations.

Definition 5.1. A locally compact group G is said to be amenable if there exists a right invariant mean m on $L^\infty(G, \lambda)$, i.e. m is a positive linear functional on $L^\infty(G, \lambda)$, such that $m(1) = 1$ and $m(g.f) = f$ for any $g \in G$ and $f \in L^\infty(G, \lambda)$.

As mentioned, this is not the only way of defining amenability for locally compact groups, as we have a wide range of different ways to do this. We see a number of these in the proposition below, which bundles together a series of results, whose proofs can be found in [21]. We also introduce the following notation,

$$\mathrm{Prob}(G) = \{f \in L^1(G, \lambda) \mid f \geq 0, \|f\|_1\}$$

Proposition 5.2. *Let G be a locally compact group. Then the following are equivalent.*

1. *G is amenable.*

2. ***Følner's condition:** For every compact set $F \subset G$ and $\varepsilon > 0$, there exists compact set $K \subset G$ such that $\lambda(K\Delta(xK)) < \varepsilon\lambda(K)$ for any $x \in F$.*

3. ***Reiter's condition:** For every compact set $F \subset G$ and $\varepsilon > 0$, there exists $\varphi \in \mathrm{Prob}(G)$, such that $\|\varphi_x - \varphi\| < \varepsilon$ for all $x \in F$.*

4. ***Fixed point property:** For any affine action $G \curvearrowright X$, where X is a convex compact subset of a locally convex space E, there exists $x \in X$ such that $g.x = x$, for all $g \in G$.*

5. *If G acts continuously on a compact set X, then there exists a G-invariant probability measure ν on X.*

6. *There exists a net $(\varphi_j)_{j\in I}$ in $\mathrm{Prob}(G)$, such that $((\varphi_j)_x - \varphi_j) \to 0$ weakly, for all $x \in G$.*

7. *There exists a net $(\varphi_j)_{j\in I}$ in $\mathrm{Prob}(G)$, such that $\|(\varphi_j)_x - \varphi_j\|_1 \to 0$, for all $x \in G$.*

Note that the proof provided in [21] for the equivalence of *5.* to the others (cf. Proposition 4.2 therein) is formulated only for σ-compact groups. However, the additional assumption of σ-compactness is not used anywhere in the proof and thus, the statement holds true in general.

5.1 Furstenberg's conjecture

For the paper in question, Furstenberg had his attention on σ-compact locally compact groups and it was in this setting, that he proposed the following conjecture:

> *G possesses a measure μ whose support is all of G and for which no nontrivial μ-boundaries exists iff G is amenable.*

To fully understand this statement, we notice the following. The measure μ is of course a probability measure on G, and triviality of a μ-boundary (B, ν) is to be understood as $\nu = \delta_b$, for some $b \in B$. It is also worth noticing that Furstenberg focused on the topological μ-boundaries.

In his paper from 1973 [11], Furstenberg proved that for a non-amenable group, any $\mu \in \mathcal{P}(G)$ with full support admit a non-trivial μ-boundary. His proof uses a compact G-space with no G-invariant measure to describe the limit behaviour of the random walk associated to μ. In this thesis, we will present a different proof from [2], which relies on the Poisson boundary and the μ-harmonic functions.

Proposition 5.3. *Let G be an non-amenable locally compact group and let $\mu \in \mathcal{P}(G)$ be given, such that the semigroup generated by* $\mathrm{supp}(\mu)$ *is all of G. Then $(\Pi_\mu, \overline{\nu})$ is non-trivial.*

Proof. Let K be a compact space, with no G-invariant probability measure and let ν be a stationary measure on K. Let $g \in G$ and $A \subset K$ be given such that $\nu(A) \neq \nu(g.A)$. Define $f \in L^\infty(K, \nu)$, by $f = 1_A$. Then $P_\nu(f)(g)$ is not equal to $P_\nu(f)(e)$, and by density of $C(K)$ in $L^1(K, \nu)$ we may pick $\varphi \in C(K)$, such that $P_\nu(\varphi)(g) \neq P_\nu(\varphi)(e)$. As φ is continuous $P_\nu(\varphi)$ is a non constant function in $H^\infty_{\mathrm{luc}}(G, \mu)$, so $(\Pi_\mu, \overline{\nu})$ is not a singleton. By assumption on μ and Proposition 3.4, the measure $\overline{\nu}$ is not a one-point measure, thus concluding the proof. □

The converse implication was left as an open question, hence only presented as a conjecture. The conjecture has since been confirmed, and we will now present the main ideas of the proof provided by Joseph Rosenblatt in [23]. Rosenblatt's argument can be found in more details in [21]. He proved the result in the setting, where λ is a left Haar measure, so for the remaining part of this section, we adopt this notation. His idea includes the notion of a measure being ergodic by convolutions, so let us start out with a definition.

Definition 5.4. Let G be a locally compact group and let $\mu \in \mathcal{P}(G)$ be given. We say that

- μ is ergodic by convolutions if $\|f * \frac{1}{N}\sum_{n=1}^{N} \mu^{*n}\| \to 0$ as $n \to \infty$, for every f in $L^1(G,\lambda)$, with $\int f \,\mathrm{d}\lambda = 0$.
- μ is mixing by convolutions if $\|f * \mu^{*n}\| \to 0$ as $n \to \infty$, for every $f \in L^1(G,\lambda)$, with $\int f \,\mathrm{d}\lambda = 0$.

These concepts turns out to be equivalent, but we only need the easy implication of mixing measures being ergodic as well. The next step in the proof is to show that if $\mu \in \mathcal{P}(G)$ is a measure which is absolutely continuous with respect to λ, has symmetric density and is ergodic by convolutions, then every continuous μ-harmonic function on G is constant. This is Proposition 1.2 in [23], in conjunction with the appearance of μ-harmonic functions, when μ has a symmetric density with respect to λ. From there, we move on to the main theorem, which yields the existence of such a measure.

Theorem 5.5 (Theorem 1.10, [23])**.** *Let G be a σ-compact amenable locally compact group. Then there exists $\mu \in \mathcal{P}(G)$, such that μ is mixing by convolutions, absolutely continuous with respect to λ and has a symmetric density.*

The proof os this theorem is rather technical and involved, so we will only briefly sketch the main ideas: We apply Følner's criteria for amenability to pick a sequence $(F_n)_{n\geq 1}$ of symmetric compact subsets of G, and define $f \in L^1(G)$ by

$$f = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{\lambda(F_n)} 1_{F_n},$$

where $(\varepsilon)_{n\geq 1}$ is a sequence of positive numbers with a sum of one. From here we define a measure $\mu = f \cdot \lambda$, and show that the choice of $(F_n)_{n\geq 1}$ gives us

$$\|\delta_g * \mu^{*n} - \mu^{*n}\| \to 0, \quad n \to \infty,$$

for all $g \in G$. This argument is fairly combinatorial and relies heavily on the clever way of picking the sequence $(F_n)_{n\geq 1}$. Finally, we apply Lemma 1.11 in [23], which states that the above convergence is equivalent to μ being mixing by convolutions.

The construction above does not ensure that μ has full support, but it is shown in [15] that any measure which is ergodic by convolutions has the property that its support generates a dense subgroup in G. From here on, it is a matter of showing that the associated measure $\tilde{\mu}$ as defined in Section 2.4 has full support and inherits all the desired properties from μ. These arguments can be found in [21].

5.2 Amenable actions

As mentioned earlier, we should be cautioned when dealing with amenable actions, since there are two very different ways of defining an amenable action. In one case, we say that a group G acts amenably on a space X, if there exists a G-invariant probability measure on X. This is the definition, which relates to the Banach-Tarski paradox, where the group of rotations in $\mathbb{R}^3$ acts non-amenably on the unit sphere. It is, however, the second definition that we will consider in the thesis. This notion was originally introduced by Robert Zimmer in [28]. We start out by proving that in this context an amenable group will always act amenably on any space, and that only amenable groups acts amenably on a singleton. This should provide a relation between amenability and amenable actions. From here, we prove that any group G acts amenably on the Poisson boundary associated to any measure, which is absolutely continuous with respect to the Haar measure. This latter result was originally proved by Zimmer in the discrete case, and later in the locally compact case by Scot Adams, George Elliot and Thierry Giordano in [1]. We will follow the definition and prove found in [2].

Definition 5.6. Let G be a locally compact group, and let (X, μ) be a probability space on which G acts in a measurable manner. The action of G on X is said to be measurably amenable if there exists a G-equivariant projection from $L^\infty(G\times X, \lambda\otimes\mu)$ onto $L^\infty(X, \mu)$, with norm one.

In order for such a projection to even make sense, we need to view $L^\infty(X, \mu)$ as a subspace of $L^\infty(G \times X, \lambda \otimes \mu)$. To do so, we will simply identify $f \in L^\infty(X, \mu)$, with the map $\tilde{f} \colon G \times X \to \mathbb{C}$ given by $\tilde{f}(g, x) = f(x)$, for all $x \in X$ and $g \in G$. In other words we can identify $L^\infty(X, \mu)$ with the space of functions from $G \times X \to \mathbb{C}$ that are measurable, once we equip G with the trivial σ-algebra $\{\emptyset, G\}$. In this way, the projection in question is a conditional expectation. Furthermore, the action of G on $L^\infty(G \times X, \lambda \otimes \mu)$ and $L^\infty(X, \nu)$, is the action inherited from $G \times X$ and X respectively, where G acts on $G \times X$ by $g.(h, x) = (gh, g.x)$ for all $g, h \in G$ and $x \in X$.

If X is a singleton, then $L^\infty(G \times X, \lambda \otimes \mu) \cong L^\infty(G, \lambda)$ and $L^\infty(X, \mu) \cong \mathbb{C}$, and hence a projection as in the definition above is merely a left-invariant mean on $L^\infty(G, \lambda)$. In other words, amenability of the action of G on a singleton is simply amenability of G. Another connection between amenable groups and amenable actions can be found in the proposition below.

Proposition 5.7. *Let G be an locally compact amenable group. Then G acts amenably on any G-space X.*

Proof. Let m be a left invariant mean on $L^\infty(G, \lambda)$ and let (X, ν) be a G-space. For $f \in L^\infty(G \times X, \lambda \otimes \nu)$ and $x \in X$, let $f_x \colon G \to \mathbb{C}$ denote the map $f_x(g) = f(g, x)$. We will now set out to prove that the map

$$P \colon L^\infty(G \times X, \lambda \otimes \nu) \to L^\infty(X, \nu),$$

defined as $P(f)(x) = m(f_x)$, is a projection with norm one. The map P is clearly linear and to see that it is a projection pick $f \in L^\infty(X, \nu)$. Then f_x is constantly equal to $f(x)$, and thus $P(f)(x) = m(f_x) = f(x)$.

To see that the P has norm equal to one, we should first notice that $P(1) = 1$ and hence $\|P\| \geq 1$. On the other hand, if $f \in L^\infty(G \times X, \lambda \otimes \nu)$ is given, then

$$\|P(f)\|_\infty = \sup_{x\in X} |m(f_x)| \leq \sup_{x\in X} \|f_x\|_\infty = \sup_{x\in X} \sup_{g\in G} |f_x(g)| = \|f\|_\infty$$

and hence $\|P\| \leq 1$. For G-equivariance of P, let $f \in L^\infty(G \times X, \lambda \otimes \nu)$ and $g \in G$ be given. For $g, h \in G$ and $x \in X$, we see that $g.(f_{g^{-1}.x})(h) = (g.f)_x(h)$ and thus

$$g.P(f)(x) = P(f)(g^{-1}.x) = m(f_{g^{-1}.x}) = m(g.(f_{g^{-1}.x})) = m((g.f)_x) = P(g.f)(x),$$

by left invariance of m. In conclusion, P is the desired G-equivariant projection with norm one. □

The main result of this section is to prove that **any** group acts amenably on the Poisson boundary, under certain assumptions on the corresponding probability measure $\mu \in \mathcal{P}(G)$. Note that the previous comment, along with Furstenberg's conjecture, require a certain caution when dealing with non-amenable groups. Before proving this result, we start out with a couple of lemmas.

Lemma 5.8. *Let X, Y be Banach spaces, and let $B(X, Y^*)$ denote the space of linear maps from X to Y^*. Then, the unit ball of $B(X, Y^*)$ is compact in the point-weak* topology. We recall $T_\alpha \to T$ in the point-weak* topology, if $T_\alpha x$ converges to Tx in the weak*-topology on Y^*, for all $x \in X$.*

Proof. For $r \geq 0$, we let $(Y^*)_r$ denote the set $\{\varphi \in Y^* \mid \|\varphi\| \leq r\}$, which is compact by the Banach-Alaoglu theorem. By Tychonoff's theorem, the product $\prod_{x\in X}(Y^*)_{\|x\|}$ is compact in the product topology corresponding to the weak$*$-topology on each copy of Y^*. As convergence in the product topology is equivalent to convergence in each coordinate, we consider the embedding Φ from $(B(X, Y^*))_1$ to $\prod_{x\in X}(Y^*)_{\|x\|}$, defined as

$$\Phi(T) = (T(x))_{x\in X}.$$

We will check that this in fact an embedding, i.e., Φ is injective, continuous and open. The injectivity follows from the fact that the $(B(X, Y^*))_1$ separates points in X, and for the rest pick a net $(T_\alpha)_{\alpha\in A}$ and an element T in $B(X, Y^*)$. Then T_α converges to T in the point-weak* topology if and only if $T_\alpha(x) \xrightarrow{w^*} T(x)$ for all $x \in X$, which again happens if and only if $(T_\alpha(x))_{x\in X} \to (T(x))_{x\in X}$, or in other words, $\Phi(T_\alpha) \to \Phi(T)$. All in all, this implies that $(B(X, Y^*))_1$ is homeomorphic to a closed subset of $\prod_{x\in X}(Y^*)_{\|x\|}$ and thus also compact. □

Lemma 5.9. *Let $(f_\alpha)_{\alpha\in A}$ and f be in $L^\infty(G, \lambda)$, and assume that $f_\alpha \xrightarrow{w^*} f$ when viewed as elements of $(L^1(G, \lambda))^*$. Then $f_\alpha * \varphi$ converges pointwise to $f * \varphi$ for any $\varphi \in L^1(G, \lambda)$.*

Proof. This proof relies heavily on computations with the Haar measure, so let φ be an element in $L^1(G, \lambda)$. Then for any $g \in G$, the map $t \mapsto \varphi(t^{-1}g)\Delta(t^{-1})$, $t \in G$ will also belong to $L^1(G,\lambda)$. Indeed, for any $g \in G$,

$$\begin{aligned}\int_G |\varphi(t^{-1}g)|\Delta(t^{-1})\,\mathrm{d}\lambda(t) &= \Delta(g)\int_G |\varphi(t^{-1})|\Delta(t^{-1})\,\mathrm{d}\lambda(t)\\ &= \Delta(g)\int_G |\varphi(t)|\,\mathrm{d}\lambda(t) = \Delta(g)\|\varphi\|_1 < \infty.\end{aligned}$$

From there on, we notice that for any $\psi \in L^\infty(G,\lambda), \varphi \in L^1(G,\lambda)$ and $g \in G$,

$$\begin{aligned}(\psi * \varphi)(g) &= \int_G \psi(gt^{-1})\varphi(t)\,\mathrm{d}\lambda(t) = \int_G \psi(t^{-1})\varphi(tg)\,\mathrm{d}\lambda(t)\\ &= \int_G \psi(t)\varphi(t^{-1}g)\Delta(t^{-1})\,\mathrm{d}\lambda(t),\end{aligned}$$

so by applying the weak* convergence of $(f_\alpha)_{\alpha\in A}$ to f, the proof is complete. □

Theorem 5.10. *Let μ be a Borel probability measure on G, which is absolutely continuous with respect to λ, and let (X,ν) be some version of the corresponding measurable Poisson boundary. Then G acts amenably on (X,ν).*

Proof. For any $n \geq 1$, we define $\mu_n := n^{-1}\sum_{k=1}^n \mu^{*k}$. Then $\|(\mu-\delta_e) * \mu_n\|_1 \to 0$, when viewing μ and its convolution powers as L^1 functions. Let furthermore α be a Borel probability measure on $G \times X$, which is absolutely continuous with respect to $\lambda \otimes \nu$. For $n \geq 1$, we define $H_n\colon L^\infty(G\times X) \to L^\infty(G)$ by

$$H_n(\psi)(g) = (\delta_g * \mu_n * \rho)(\psi), \quad \psi \in L^\infty(G\times X), g \in G.$$

Then H_n is a positive, contractive, linear operator that satisfies $H(1) = 1$ and thus also an element in $B(L^\infty(G\times X), L^\infty(G))$. Furthermore, for $g, t \in G$ and $\psi \in L^\infty(G\times X)$,

$$\begin{aligned}t.H_n(\psi)(g) = H_n(\psi)(t^{-1}g) &= (\delta_{t^{-1}g} * \mu_n * \rho)(\psi) = (\delta_{t^{-1}} * \delta_g * \mu_n * \rho)(\psi)\\ &= (\delta_g * \mu_n * \rho)(t.\psi) = H_n(t.\psi)(g),\end{aligned}$$

and thus H_n is G-equivariant. Furthermore, for all $g \in G$ and $\psi \in L^\infty(G\times X)$, we see that

$$\begin{aligned}H_n(\psi) \star (\mu-\delta_e)(g) &= \int_G H_n(\psi)(gt)\,\mathrm{d}(\mu-\delta_e)(t)\\ &= \int_G (\delta_{gt} * \mu_n * \rho)(\psi)\,\mathrm{d}(\mu-\delta_e)(t)\\ &= \int_G (\delta_g * \delta_t * \mu_n * \rho)(\psi)\,\mathrm{d}(\mu-\delta_e)(t)\\ &= (\delta_g * (\mu-\delta_e) * \mu_n * \rho)(\psi).\end{aligned}$$

For here on it follows that $|H_n(\psi) \star (\mu-\delta_e)(g)| \leq \|(\mu-\delta_e) * \mu_n)\|_1\|\psi\|_\infty$, for all $g \in G$, and hence $|H_n(\psi) \star (\mu-\delta_e)(g)| \to 0$ as $n \to \infty$ for all $g \in G$. For the next part of the proof, we identify $L^\infty(G)$ with the dual of $L^1(G)$ and apply

Lemma 5.8 to pick a subnet $(H_{n_i})_{i\in I}$ of $(H_n)_{n\geq 1}$, which converges towards some $H \in B(L^\infty(G\times X), L^\infty(G))$ in the point-weak* topology.

The claim will now be that $H\psi$ is μ-harmonic for all $\psi \in L^\infty(G\times X)$, so let $\psi \in L^\infty(G\times X)$ be given. By the remark in connection to Definition 2.2, for any $\varphi \in L^1(G)$ and $i \in I$, we see that

$$H_{n_i}(\psi) \star (\mu - \delta_e) * \varphi = H_i(\psi) * (\tilde{\mu} - \delta_e) * \varphi.$$

Since $(\tilde{\mu}-\delta_e)*\varphi \in L^1(G)$, we conclude that $H_{n_i}(\psi)\star(\mu-\delta_e)*\varphi$ converges pointwise to $H(\psi)\star(\tilde{\mu}-\delta_e)*\varphi$ on G. On the other hand, the convergence of $\|H_{n_i}(\psi)\star(\mu-\delta_e)\|_\infty$ to 0 will also give us that $H_{n_i}(\psi)\star(\mu-\delta_e)*\varphi \to 0$ pointwise on G, so by uniqueness of limits, we must have $H(\psi)\star(\mu-\delta_e)*\varphi(g) = 0$ for all $g \in G$. We would now like to show that $H(\psi)\star(\mu-\delta_e) = 0$ almost surely, and for this it suffices to show that the equality holds true almost surely on all compact subsets of G. Let $K \subset G$ be compact and define $\varphi \in L^1(G)$ by $\varphi(t) = \overline{H(\psi)\star(\mu-\delta_e)}(t^{-1})\cdot 1_K$. The result above can then be written as

$$\begin{aligned} 0 = H(\psi)\star(\mu-\delta)*\varphi(e) &= \int_K |H(\psi)\star(\mu-\delta_e)(t^{-1})|^2 \,\mathrm{d}\lambda(t) \\ &= \int_K |H(\psi)\star(\mu-\delta_e)(t)|^2 \Delta(t^{-1}) \,\mathrm{d}\lambda(t), \end{aligned}$$

which happens if and only if $H(\psi)\star(\mu-\delta_e) = 0$ almost surely on K. From there on we conclude that $H(\psi)\star(\mu-\delta_e) = 0$ almost surely, or in other words that $H(\psi) \in H^\infty_{\text{a.s.}}(G,\mu)$, for any $\psi \in L^\infty(G\times X)$.

As (X,ν) is the measurable Poisson boundary, we can for any $\psi \in L^\infty(G\times X)$ find a unique $\tilde{\psi} \in L^\infty(X)$ such that $H(\psi) = P_\nu(\tilde{\psi})$. If we define a map M from $L^\infty(G\times X)$ to $L^\infty(X)$, by $M(\psi) = \tilde{\psi}$ for any $\psi \in L^\infty(G\times X$, then M is a positive contraction with $M(1) = 1$. To show that M is a projection, we will now assume that $\alpha = \beta\otimes\nu$, for some Borel probability measure β on G, which is absolutely continuous with respect to λ. For $\varphi \in L^\infty(X)$, where we view $L^\infty(X) \subset L^\infty(G\times X)$ as above, we use the μ-stationarity of ν to see that $H_n(\varphi)(g) = g.\nu(\psi) = P_\nu(\varphi)(g)$, for all $g \in G$ and $n \geq 1$. Then also $H(\varphi)(g) = P_\nu(\varphi)(g)$, which implies $M(\varphi) = \varphi$, as we set out prove. □

Note that the above theorem along with the discussion on groups acting amenably on a singleton actually provides an alternative proof of Proposition 5.3. Indeed, if the Poisson boundary is trivial and G is acting amenably on this singleton, G must be amenable itself or in other words, any non-amenable group gives rise to a non-trivial Poisson boundary.

6 G-boundaries and recent applications

This final section will serve as an appetiser for a different kind of group theoretic boundary, and some very recent applications in the field of C^*-algebras. We will not provide any proofs and 'a deeper understanding and working knowledge of C*-algebras will be assumed throughout this section. For details, see [19], [18]. The type of boundaries we will discuss are the so-called G-boundaries and we will work in the setting of a discrete, countable group G.

Definition 6.1. A compact, topological G-space B is said to be a G-boundary if the following is satisfied:

- B is minimal, i.e., there are no proper G-invariant subset of B.
- For every $\nu \in \mathcal{P}(B)$, the w^*-closure of $\{g.\nu \mid g \in G\}$ contains δ_B.

We notice that there are some similarities between the definition of a G-boundary and a μ-boundary, but there are no natural way of comparing the two. However, they do share some of the same properties. For instance, every group G gives rise to a universal G-boundary.

Proposition 6.2. *There exists a G-boundary B, such that every other G-boundary is an equivariant image of B. This G-boundary is called the Furstenberg boundary, and is denoted by $\partial_F G$. Furthermore, the Furstenberg boundary is unique up to homeomorphism.*

If we consider the case of $G = \mathbb{F}_2$, the Furstenberg boundary $\partial_F \mathbb{F}_2$ actually coincides with the Poisson boundary constructed in Section 3.2, i.e. $\partial_F \mathbb{F}_2$ is the space of infinite, reduced words in the generators of $\mathbb{F}_2$. Furthermore, there is a result for G-boundaries, which somewhat resembles Furstenberg's conjecture.

Proposition 6.3 (Kennedy and Kalantar, [19])**.** *The Furstenberg boundary $\partial_F G$ is a singleton if and only if G is amenable. In fact, $\partial_F G$ is non-metrisable if G is not amenable.*

As mentioned earlier, the Furstenberg boundary has recently been used to answer a series of questions in the theory of C^*-algebras. More specifically, it has been used to characterise groups for which the reduced group C^*-algebra is simple, the so-called C^*-simple groups or has a unique tracial state, in which case we say that the group has the unique trace property. Furthermore, it has been an open question whether C^*-simplicity and the unique trace property are equivalent, and here the Furstenberg boundary was used to prove that C^*-simple groups have the unique trace property. The converse implication is officially still an open question, but rumor has it that a counterexample has been discovered, thus leaving it a one-way implication. This proof of C^*-simplicity implying the unique trace property involves an object called the injective envelope, so for this we introduce a handful of definitions.

Definition 6.4. Let G be a countable, discrete group.

1. A self-adjoint, unital, linear subspace of a unital C^*-algebra is called an operator system.

2. An operator system $\mathcal{S}$ is said to be a G-operator system, if G acts on $\mathcal{S}$ by completely positive isomorphisms.

3. An operator system $\mathcal{S}$ is said to be G-injective if for every unital, completely isometric, G-equivariant map $\iota\colon \mathcal{E} \to \mathcal{F}$ and every unital, completely positive, G-equivariant map $\varphi\colon \mathcal{E} \to \mathcal{S}$, there exists a unital, completely positive, G-equivariant map $\psi\colon \mathcal{F} \to \mathcal{S}$ such that $\psi \circ \iota = \varphi$.

4. A G-extension of a G-operator system $\mathcal{S}$ is a pair $(\mathcal{T}, \iota)$, where $\mathcal{T}$ is a G-operator system and $\iota\colon \mathcal{S} \to \mathcal{T}$ is a completely isometric, G-equivariant map. The extension is said to be G-injective if $\mathcal{T}$ is G-injective. The extension is said to be G-essential, if for every unital, completely positive, G-equivariant map $\varphi\colon \mathcal{T} \to \mathcal{E}$, such that $\varphi \circ \iota$ is completely isometric, then φ is completely isometric.

In 1985, Masamichi Hamana provided a proof (see [14]) for existence and uniqueness of a G-injective and G-essential extension of any operator system $\mathcal{S}$. This extension is called the G-injective envelope of $\mathcal{S}$, and is denoted by $I_G(\mathcal{S})$. Later, it was proved that $I_G(\mathbb{C}) \cong C(\partial_F G)$ for any discrete group G, and it is this identification, which is used in the recent results mentioned earlier. However, before looking at those, we start out with yet another definition

Definition 6.5. Let G be a discrete group, and let X be a G-space.

- We say that G acts topologically free on X, if the set $\{x \in X \mid s.x \neq x\}$ is dense in X for every $s \in G \backslash \{e\}$.

- We say that G acts faithfully on X, if for any $g \neq h$ in G there exists $x \in X$ such that $g.x \neq h.x$.

We are now ready to state the three recent results, for which the Furstenberg boundary has found applications in the theory of C^*-algebras.

Theorem 6.6 (Kennedy and Kalantar, [19])**.** *Let G be a discrete group, and let $\partial_F G$ denote its Furstenberg boundary. Then G is C^*-simple if and only if G acts topologically free on $\partial_F G$.*

Theorem 6.7 (Kennedy and Kalatar, [19])**.** *A group G acts amenably on $\partial_F G$ if and only if G is exact, i.e. the reduced group C^*-algebra $C_r^*(G)$ is exact.*

Theorem 6.8 (Kennedy, Kalantar, Breuillard and Ozawa, [18])**.** *Let G be a discrete group, and let $\partial_F G$ denote its Furstenberg boundary. Then G has the unique trace property if and only if G acts faithfully on $\partial_F G$.*

A The Haar measure and convolutions

This appendix will serve as a short survey of the existence and the basic properties of the right Haar measure on a locally compact group, and how we use the Haar measure to define a series of convolutions. The goal is to provide the tools needed for reading through the thesis at hand. For proofs, details regarding the Haar measure can be found in Chapter 11 in [8], while details on convolutions can be found in Chapter 8 in [8] and in [12]. Let us first recall the following definition.

Definition A.1. Let X be a locally compact Hausdorff space. A Radon measure μ on X is a Borel measure with the following properties

- $\mu(K) < \infty$ for all compact subsets $K \subset X$.
- $\mu(E) = \inf\{\mu(U) \mid U \subset X \text{ open }, E \subset U\}$, for any Borel set $E \subset X$.
- $\mu(U) = \sup\{\mu(K) \mid K \subset X \text{ compact }, K \subset U\}$, for any open set $U \subset X$

We are now ready to define the right Haar measure, using a theorem which is proved by the means of a number of Functional Analysis result, including Urysohn's Lemma, the Riesz Representation Theorem and Caratheodory's Theorem.

Theorem A.2. *Let G be a locally compact group. Then there exists a right invariant Radon measure μ on G, where right invariance means $\mu(A) = \mu(Ag)$ for all $g \in G$ and $A \subset G$ Borel. Furthermore, this measure λ is unique up to multiplication with a strictly positive constant.*

For a locally compact group G, we will let λ_G denote one of these measures and we call it the right Haar measure. If there is no ambiguity, we will omit the subscript and simply write λ. We will in general not worry about which right Haar measure we choose, but there are certain exceptions.

- If G is a compact group, λ is a finite measure and thus, we choose the version, which is also a probability measure.
- If G is the real line with addition, the Lebesgue measure is a right Haar measure and it will be our canonical choice.
- If G is an infinite, discrete group, then λ is the counting measure on G and thus, we choose the version which gives each singleton measure one.

Even though we do not worry about the choice of right Haar measure, there is a certain relation between these measures. The idea here is that every left translate of a right Haar measure, is a right Haar measure and thus they are all equal up to multiplication by some positive constant.

Proposition A.3. *Let G be a locally compact group and let λ be the right Haar measure. Then there exists a continuous function $\Delta\colon G \to (0, \infty)$ satisfying*

1. *$\lambda(xA) = \Delta(x)\lambda(A)$, for all $x \in G$, and $A \subset G$ Borel,*

2. $\Delta(xy) = \Delta(x)\Delta(y)$,

3. Δ *is independent of the choice of* λ.

The point of the modular function is in some sense to measure, how close our right Haar measure is to being left-invariant as well. There are certain cases where Δ is constantly equal to 1 and such groups will be called unimodular. It is clear that abelian groups are unimodular, but it is true for types of groups as well. If G is a compact group, then $\Delta(G)$ is a compact subgroup of $(\mathbb{R}, \cdot)$, which is only possible when $\Delta(G) = \{1\}$. The modular function Δ will also give us some tools for integrating against λ. Here we have the following identities for any function $f \colon G \to \mathbb{R}$, for which the integrals make sense

- $\int f(gh)\,\mathrm{d}\lambda(g) = \int f(g)\,\mathrm{d}\lambda(g), h \in G$.
- $\int f(hg)\,\mathrm{d}\lambda(g) = \int f(g)\Delta(h^{-1})\,\mathrm{d}\lambda(g), h \in G$.
- $\int f(g^{-1})\,\mathrm{d}\lambda(g) = \int f(g)\Delta(g^{-1})\,\mathrm{d}\lambda(g)$.

Convolutions

The convolution of two Borel probability measures μ, ν on a group G is the image of the product measure $\mu \otimes \nu$ with respect to the product on G. In other words, the convolution of μ and ν, denoted by $\mu * \nu$ is defined as

$$\mu * \nu(\varphi) = \int_G \int_G \varphi(gh)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(h), \quad \varphi \in C_c(G).$$

We will also let μ^{*n} denote the convolution of n copies of a measure μ. By having a canonical measure on G, the Haar measure lets us extend the definition of convolutions as follows. For a probability measure μ on G and a function $f \in L^1(G, \lambda)$, we want to define integrable functions $f*\mu, \mu*f \colon G \to \mathbb{R}$, such that $(f\cdot\lambda)*\mu = (f*\mu)\cdot\lambda$ and $\mu * (f \cdot \lambda) = (\mu * f) \cdot \lambda$. These functions must necessarily be defined as follows

$$(\mu * f)(g) = \int_G f(t^{-1}g)\Delta(t)\,\mathrm{d}\mu(t)$$
$$(f * \mu)(g) = \int_G f(gt^{-1})\,\mathrm{d}\mu(t)$$

This construction also works for functions in $L^p(G, \lambda)$ for any $1 \le p \le \infty$, and the inequalities $\|\mu * f\| \le \|\mu\| \cdot \|f\|_p, \|f * \mu\| \le \|f\|_p\|\mu\|$ holds true for any $f \in L^p(G, \lambda)$ and Borel propability measure μ on G, where $\|\mu\|$ denotes the total variation of μ. We can extend this even further to the case, where $\mu = \varphi \cdot \lambda$ for some $\varphi \in L^1(G, \lambda)$. Then $f * \varphi \colon G \to \mathbb{R}$ is defined as

$$(f * \varphi)(g) = \int_G f(gt^{-1})\varphi(t)\,\mathrm{d}\lambda(t), \quad g \in G.$$

If the reader is familiar with traditional convolutions of functions $f, g \colon \mathbb{R} \to \mathbb{R}$ or $\varphi, \psi \in \ell^1(G)$ for some countable discrete group G, then it is not difficult to check that these definitions are special cases of with the general one above. Lastly, for $f \in L^\infty(G)$ and $\varphi \in L^1(G)$ the convolution $f * \varphi$ as above will give us a function in $L^1(G)$, and this function will in fact be continuous. As a final remark to this section, we should note that the above discussion and concrete formulas only concerned the right Haar measure. Any locally compact group will also admit a left invariant Radon measure, the left Haar measure, for which all the tools above can be constructed in an analogous way, but the concrete formulas for integration and convolutions will have to be changed appropriately.

B Miscellaneous results

This appendix will serve as a catalogue of different results that will be used but not proven in the thesis. The first theorem is a result from ergodic theory, which come in handy, when dealing with random walks and the like.

Theorem B.1 (Kingman's Subadditive Ergodic Theorem, [20])**.** *Let $(\Omega, \mathbb{P})$ be a probability space, and let $\theta : \Omega \to \Omega$ be a measure preserving transformation. Let $(X_n)_{n\geq 0}$ be a sequence of real-valued stochastic variables on Ω, satisfying*

- $EX_n < \infty$*, for all* $n \geq 0$.
- $EX_n \geq -An$*, for all* $n \geq 1$ *and some constant* A.
- $X_{n+m}(\omega) \leq X_n(\omega) + X_m(\theta^n(\omega))$*, for all* $n, m \geq 0$ *and* $\omega \in \Omega$.

Then X_n/n converges almost surely and in L^1 to some random variable X.

The next theorem is a result describing the possible behaviour of convolution powers of a probability measure.

Theorem B.2 (The zero-two law, [6])**.** *Let μ be a probability measure on a locally compact group G. Then $\lim_{n\to\infty} \|\mu^{*(n+1)} - \mu^{*n}\|$ exists, and is either zero or two. In particular, if $\|\mu^{*2} - \mu\| < 2$, then $\lim_{n\to\infty} \|\mu^{*(n+1)} - \mu^{*n}\| = 0$.*

Note that the last claim follows directly from the first, when remembering that $\mathcal{P}(G)$ is a Banach algebra with respect to the convolution. The zero-two law can be formulated in the much more general setting of Markov processes, but the above version is the one we need for this thesis. Finally, we have a result, which is of great use once we consider random walks on an abelian group.

Theorem B.3 (Hewitt-Savage 0-1 law, [17])**.** *Let $(X_n)_{n\geq 1}$ be a sequence of independent, identically distributed random variables with values in some measurable space X, and let $\mathbb{P}$ denote the distribution of $(X_n)_{n\geq 1}$ on the sequence space $X^{\mathbb{N}}$. If $\varphi\colon X^{\mathbb{N}} \to X$ is a function of $(X_n)_{n\geq 1}$, which is invariant under any finite permutation of $(X_n)_{n\geq 1}$ $\mathbb{P}$-a.s., then φ is constant $\mathbb{P}$-a.s.*

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