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Bachelor Thesis in Mathematics

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The Asymptotic Quantum Birkhoff Conjecture

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Abstract

The Asymptotic Quantum Birkhoff Conjecture, formulated in 2005 by J.A. Smolin, F. Verstraete and A. Winter, was an open problem in Quantum Information Theory until it was solved in the negative by U. Haagerup and M. Musat in 2011. The key tool for disproving it is the analysis of the class of factorizable completely positive maps, a notion introduced in 2006 by C. Anantharaman-Delaroche in the setting of von Neumann algebras. We present a historical context for the conjecture as well as the proof that it fails. Along the way we develop the necessary background on factorizable completely positive maps on matrix algebras. An important tool in the analysis is the theory of ultrafilters and ultraproducts, which we construct. We also prove that the tracial ultraproduct of tracial von Neumann algebras is a von Neumann algebra.

Resumé

The Asymptotic Quantum Birkhoff Conjecture blev formuleret i 2005 af J.A. Smolin, F. Verstraete and A. Winter. Formodningen var et åbent problem i Quantum Information Theory, indtil den i 2011 blev modbevist af U. Haagerup and M. Musat. Et centralt redskab i deres argument er en analyse klassen af faktoriserbare fuldstændigt positive afbildninger; en klasse af afbildninger som introduceredes i 2006 af C. Anantharaman-Delaroche i konteksten af von Neumann-algebraer. Vi præsenterer den historiske kontekst for formodningen samt beviset for, at den ikke holder. Undervejs udvikler vi den nødvendige baggrund omkring faktoriserbare fuldstændigt positive afbildninger på matrix-algebraer. Et vigtigt redskab i analysen er teorien omkring ultrafiltre og ultraprodukter, som vi konstruerer. Vi viser også, at vi kan konstruere et ultraprodukt af von Neumann-algebraer med et trace, således at ultraproduktet igen bliver en von Neumann-algebra.

Contents

Introduction

The Asymptotic Quantum Birkhoff Conjecture, formulated in 2005 by John A. Smolin, Frank Verstraete and Andreas Winter [1], was an open problem in Quantum Information Theory listed on Reinhard Werners web page [2] until it was solved in the negative by Uffe Haagerup and Magdalena Musat in 2011 [3]. The classical Birkhoff theorem is a well-known result in the literature due to Garrett Birkhoff [4]. It states that any doubly stochastic matrix, which is a positive map preserving probability distributions on a finite dimensional discrete alphabet, can be written as a convex combination of permutation matrices. The Quantum Birkhoff Conjecture is a natural generalization of the classical theorem to the non-commutative case; it makes the analogous statement about completely positive maps preserving quantum states. Though it holds in dimension 2, it was disproved by Burkhard Kümmerer and Hans Maassen [5] in all dimensions greater than or equal to 3. The Asymptotic Quantum Birkhoff Conjecture can in this historical context be viewed as an attempt to restore the quantum generalization of the classical Birkhoff theorem in the asymptotic limit. In this thesis we present both the historical context of the conjecture, as well the proof that it fails. The thesis is organized as follows:

In Chapter 1 we establish the terminology used when discussing matrix algebras, and develop the necessary background on completely positive and completely bounded maps between matrix algebras. Of particular importance is the analysis of completely positive maps, which is due to Man-Duen Choi [6].

In Chapter 2 we will present the historical context for the Asymptotic Quantum Birkhoff Conjecture. We will on the way prove the classical result of Birkhoff, and we will prove that the Quantum Birkhoff Conjecture holds in dimension 2.

The key tool for disproving the Asymptotic Quantum Birkhoff Conjecture is the existence of *non-factorizable* completely positive maps on $M_n(\mathbb{C})$. The notion of *factorizable maps* completely positive maps was introduced by Claire Anantharaman-Delaroche in [7] in the setting of von Neumann algebras. We present in Chapter 4 a thorough analysis of factorizable maps on matrix algebras, based on the work of Uffe Haagerup and Magdalena Musat in [3] and [8]. In particular we prove that the set $\mathfrak{F}_n$ of factorizable completely positive maps on $M_n(\mathbb{C})$ is a convex and closed set containing the convex hull of automorphisms of $M_n(\mathbb{C})$. We also show that the completely bounded distance to $\mathfrak{F}_n$ does not decrease when tensoring a map with itself. Finally, we provide counterexamples to the Asymptotic Quantum Birkhoff Conjecture in dimensions 3 and 4.

In Chapter 3 we present the results from the theory on von Neumann algebras that we shall need in our work with factorizable maps. In particular, we need to understand what a finite von Neumann algebra equipped with a faithful normal trace is. We show that $L_\infty([0,1])$ is an example of such. The proof that the set of factorizable maps is closed, which is a key tool in disproving the Asymptotic Quantum Birkhoff Conjecture, turns out to be quite involved. Chapter 3 therefore contains some rather deep results on tracial von Neumann algebras, as well as on ultraproducts of von Neumann algebras.

The theory on ultrafilters and ultraproducts is not core curriculum in the mathematics program at the University of Copenhagen. We therefore offer a quite detailed introduction to this theory in Appendix B, which is intended to be self-contained.

The reader is assumed to be familiar with linear algebra, as well as functional analysis and some theory of operator algebras. In particular, the reader is assumed to be familiar with the basic concepts of Banach algebras and C^*-algebras, and we shall only recap the facts that are of key relevance to this thesis.

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Chapter 1

Prerequisites

1.1 Matrix algebras

Let V be a linear space over $\mathbb{C}$. For $n, m \in \mathbb{N}$, we denote by $M_{m,n}(V)$ the set of $m \times n$-matrices with entries in V. This is again a linear space over $\mathbb{C}$ with pointwise operations. When $n = m$, this set is simply denoted by $M_n(V)$. Elements of $M_{m,n}(V)$ are usually denoted by $[v_{ij}]$; here $v_{ij} \in V$ is the element in the i'th row and j'th column. We write $[v_{ij}]_{i,j}$ when there would otherwise be a risk of ambiguity. We shall mostly be interested in the case where $V = \mathbb{C}$ or $V = M_k(\mathbb{C})$, for some $k \geq 1$, and later where V is a C^*-algebra.

The *matrix units* of $M_{m,n}(\mathbb{C})$ are denoted by e_{ij}, for $1 \leq i \leq m$ and $1 \leq j \leq n$, or by $e_{ij}^{(m,n)}$, when specifying the dimensions is necessary. These are the $m \times n$-matrices with a 1 at the (i, j)'th entry and zeros elsewhere. The set of matrix units forms a vector basis for $M_{m,n}(\mathbb{C})$. When $n = m$, one can easily check that $M_n(\mathbb{C})$ is an algebra over $\mathbb{C}$, where the multiplication is the usual matrix multiplication. The multiplicative and additive identities in $M_n(\mathbb{C})$ are denoted by $\mathbf{1}_n$ and $\mathbf{0}_n$, respectively. The matrix units in $M_n(\mathbb{C})$ have the following properties:

(a) $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}, \quad 1 \leq i, j, k, \ell \leq n,$

(b) $\sum_{i=1}^{n} e_{ii} = \mathbf{1}_n$.

The space $M_{m,n}(V)$ can be identified with the tensor product space $M_{m,n}(\mathbb{C}) \otimes V$ (and likewise with $V \otimes M_{m,n}(\mathbb{C})$). Indeed, consider the map $M_{m,n}(V) \to M_{m,n}(\mathbb{C}) \otimes V$ given by

$$v = [v_{ij}] \mapsto \sum e_{ij} \otimes v_{ij} \tag{1.1}$$

It is easy to see that this defines an injective linear map. Further, given an elementary tensor, i.e., an element of the form $\gamma \otimes v$, where $\gamma = [\gamma_{ij}] \in M_n(\mathbb{C})$ and $v \in V$, we can construct the matrix $[\gamma_{ij}v]$ in $M_{m,n}(V)$. Then

$$[\gamma_{ij}v] \mapsto \sum e_{ij} \otimes (\gamma_{ij}v) = \sum \gamma_{ij}e_{ij} \otimes v = \gamma \otimes v.$$

This shows by linearity that the map given in equation (1.1) is surjective. The identifications

$$M_{m,n}(V) \cong M_{m,n}(\mathbb{C}) \otimes V \cong V \otimes M_{m,n}(\mathbb{C}) \tag{1.2}$$

are referred to as the *canonical shuffle*, and will often be used without mentioning.

We define the *direct sum* of two elements $x \in M_{m,n}(V)$ and $w \in M_{p,q}(V)$ by

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{m+p,n+q}(V).$$

We have already mentioned that $M_n(\mathbb{C})$ is an algebra over $\mathbb{C}$. Furthermore, $M_n(\mathbb{C})$ is a *-algebra with the conjugate transpose map as the involution, and it is a C^*-algebra with the operator norm given by

$$\|x\| := \sup\{\|xv\|_2 : v \in \mathbb{C}^n, \|v\|_2 \leq 1\}, \qquad \text{for all } x \in M_n(\mathbb{C}). \tag{1.3}$$

Throughout this thesis we shall use the terms *self-adjoint*, *normal*, *unitary*, *positive* and *projection* about the elements of $M_n(\mathbb{C})$ in the way that they are defined, more generally, for C^*-algebras.

Definition 1.1. Let $\mathcal{A}$ be a unital C^*-algebra and let $x \in \mathcal{A}$. We say that x is

(a) *self-adjoint*, if $x^* = x$,

(b) *normal*, if $x^*x = xx^*$,

(c) *unitary*, if $x^*x = xx^* = \mathbf{1}_{\mathcal{A}}$,

(d) *positive*, if $x = y^*y$, for some $y \in \mathcal{A}$,

(e) a *projection*, if $x = x^* = x^2$.

The set of self-adjoint elements of $\mathcal{A}$ is denoted by $\mathcal{A}_{sa}$, and the set of positive elements is denoted by $\mathcal{A}_+$. If $x \in \mathcal{A}_+$, we shall sometimes write $x \geq 0$.

Notice that this practice differs from the usual terminology in linear algebra, where self-adjoint elements are referred to as Hermitian, and where *positive* and *positive semidefinite* matrices are Hermitian matrices with positive, respectively, non-negative eigenvalues. The following lemma shows that positive as defined in the setting of C^*-algebras is equivalent to positive semidefinite in the usual terminology of linear algebra.

Lemma 1.1. *A matrix $x \in M_n(\mathbb{C})$ is positive semidefinite if and only if there exists $y \in M_n(\mathbb{C})$ such that $x = y^*y$.*

Proof. Suppose that $x = y^*y$. Then $v^*xv = v^*y^*yv = \|xv\|^2 \geq 0$, for all $v \in \mathbb{C}^n$, which is equivalent to all eigenvalues of x being non-negative. Hence x is a positive semidefinite matrix. On the other hand, if x is positive semidefinite, then $x = \sum_\lambda \lambda p_\lambda$, where p_λ is the projection onto the eigenspace corresponding to the eigenvalue λ, and $\lambda \geq 0$. Define $y = \sum_\lambda \sqrt{\lambda} p_\lambda$. Then $x = y^*y$. □

Proposition 1.2. *The norm defined in equation (1.3) has the following properties:*

(a) $\|x \oplus y\| = \max\{\|x\|, \|y\|\}$, *for all* $x \in M_n(\mathbb{C})$ *and* $y \in M_m(\mathbb{C})$.

(b) $\|x \otimes \mathbf{1}_m\| = \|x\|$, *for all* $x \in M_n(\mathbb{C})$ *and* $m \in \mathbb{N}$.

(c) $\|x \otimes y\| = \|x\| \|y\|$, *for all* $x \in M_n(\mathbb{C})$ *and* $y \in M_m(\mathbb{C})$.

(d) $\|xu\| = \|ux\| = \|x\|$, *for all* $x \in M_n(\mathbb{C})$ *and all unitary* $u \in M_n(\mathbb{C})$.

(e) $\|x\| = \max\{\varsigma : \varsigma$ *is a singular value of* $x\}$, *for all* $x \in M_n(\mathbb{C})$.

Proof. (a): Let $v \in \mathbb{C}^{n+m}$ with $\|v\|_2 \leq 1$. Let $v_1 \in \mathbb{C}^n$ be the vector consisting of the first n entries of v and $v_2 \in \mathbb{C}^m$ be the vector consisting of the last m entries of v. We have the equality $\|v\|_2^2 = \|v_1\|_2^2 + \|v_2\|_2^2$, and it is then straightforward to derive that

$$\|(x \oplus y)(v)\|_2^2 = \|xv_1\|_2^2 + \|yv_2\|_2^2 \leq \max\{\|x\|^2, \|y\|^2\}.$$

On the other hand, we can extend each $v \in \mathbb{C}^n$ to a vector $\widetilde{v} \in \mathbb{C}^{n+m}$ by adding zeros. Then

$$\begin{aligned}\|x \oplus y\| &= \sup\{\|(x \oplus y)(v)\|_2 : v \in \mathbb{C}^{n+m}, \|v\|_2 \leq 1\} \\ &\geq \sup\{\|(x \oplus y)(\widetilde{v})\|_2 : v \in \mathbb{C}^{n}, \|v\|_2 \leq 1\} = \|x\|,\end{aligned}$$

and we can in the same way derive that $\|x \oplus y\| \geq \|y\|$.

(b): Observe that $x \otimes \mathbf{1}_m$ is nothing but $x \oplus x \oplus \cdots \oplus x$, where there are m terms in this direct sum. It then follows directly from (a) that $\|x \otimes \mathbf{1}_m\| = \|x\|$.

(c): Observe that $x \otimes y = (x \otimes \mathbf{1}_m)(\mathbf{1}_n \otimes y)$. The inequality "$\leq$" follows from the fact that $\|ab\| \leq \|a\| \|b\|$, for all $a, b \in M_k(\mathbb{C})$, together with property (b). On the other hand, $\|x \otimes y\| \geq \sup\{\|(x \otimes y)(v \otimes w)\| : v \in \mathbb{C}^n, w \in \mathbb{C}^m, \|v\|_2, \|w\|_2 \leq 1\} = \|x\| \|y\|$.

(d): It is an easy consequence of the C^*-identity that $\|u\|^2 = \|u^*u\| = 1$, for any unitary matrix $u \in M_n(\mathbb{C})$. Then $\|ux\| \leq \|u\| \|x\| = \|x\| = \|u^*ux\| \leq \|u^*\| \|ux\| = \|ux\|$. The proof of $\|xu\| = \|x\|$ is similar.

(e): Let $x = usv$ be the singular value decomposition of x. It follows directly from property (d) that $\|x\| = \|s\| = \max\{\varsigma : \varsigma \text{ is a singular value of } x\}$. ☐

Remark 1.3. The properties (a-d) in the above proposition can be proved in the more general setting of $M_n(\mathcal{A})$, where $\mathcal{A}$ is a C^*-algebra.

Definition 1.2. A matrix $x \in M_n(\mathbb{C})$ is said to be a *contraction* if $\|x\| \leq 1$.

Lemma 1.4. *Let $x \in M_n(\mathbb{C})$. Then x is a contraction if and only if the block matrix*

$$\begin{pmatrix} \mathbf{1}_n & x \\ x^* & \mathbf{1}_n \end{pmatrix}$$

is positive in $M_{2n}(\mathbb{C})$.

Proof. If $n = 1$, the matrix $\begin{pmatrix} 1 & x \\ \overline{x} & 1 \end{pmatrix}$ has eigenvalues $1 \pm |x|$, which are non-negative if and only if $|x| \leq 1$. The claim follows by Lemma 1.1. For $n \geq 2$, let $x = usv$ be the singular value decomposition of x. Then

$$\tilde{s} := \begin{pmatrix} \mathbf{1}_n & s \\ s & \mathbf{1}_n \end{pmatrix} = \begin{pmatrix} u^* & \mathbf{0}_n \\ \mathbf{0}_n & v \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & usv \\ v^*su^* & \mathbf{1}_n \end{pmatrix} \begin{pmatrix} u & \mathbf{0}_n \\ \mathbf{0}_n & v^* \end{pmatrix} = \begin{pmatrix} u^* & \mathbf{0}_n \\ \mathbf{0}_n & v \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & x \\ x^* & \mathbf{1}_n \end{pmatrix} \begin{pmatrix} u & \mathbf{0}_n \\ \mathbf{0}_n & v^* \end{pmatrix}.$$

Hence, our matrix of interest is unitarily equivalent to $\tilde{s}$, which in turn is unitarily equivalent (via the canonical shuffle) to the direct sum

$$\begin{pmatrix} 1 & \varsigma_1 \\ \varsigma_1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & \varsigma_2 \\ \varsigma_2 & 1 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 1 & \varsigma_n \\ \varsigma_n & 1 \end{pmatrix},$$

where $\varsigma_1, \varsigma_2, \ldots, \varsigma_n$ are the singular values of x. This direct sum is positive in $M_{2n}(\mathbb{C})$ if and only if $\varsigma_i \leq 1$, for all $1 \leq i \leq n$, by the first part of the proof. By Proposition 1.2(e), and since positivity is preserved under unitary equivalence, the assertion follows. ☐

Lemma 1.5. *If $x \in M_n(\mathbb{C})$ is a contraction, then there exist unitary matrices $u, v \in M_n(\mathbb{C})$ such that $x = (u + v)/2$.*

Proof. Let $x = \tilde{u}s\tilde{v}$ be the singular value decomposition of x, and let $\varsigma_1, \varsigma_2, \ldots, \varsigma_n$ denote the singular values of x. Since the norm of x is less than or equal to 1 by assumption, we infer that $\varsigma_1, \varsigma_2, \ldots, \varsigma_n \in [0, 1]$. For each $1 \leq j \leq n$, we may therefore write $\varsigma_j = (e^{i\theta_j} + e^{-i\theta_j})/2$, for some $\theta_j \in [0, \pi/2]$. Define $u := \tilde{u}\,\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})\tilde{v}$ and $v = \tilde{u}\,\mathrm{diag}(e^{-i\theta_1}, e^{-i\theta_2}, \ldots, e^{-i\theta_n})\tilde{v}$. It is straightforward to check that u and v are unitary matrices and that $x = (u + v)/2$. ☐

We end this section by discussing traces on $M_n(\mathbb{C})$.

Definition 1.3. Let φ be a linear functional on $M_n(\mathbb{C})$. We say that φ is *positive*, if $\varphi(x) \geq 0$, for all $x \geq 0$, and that φ is a *state*, if φ is positive and $\varphi(\mathbf{1}_n) = 1$. Moreover, we say that φ is a *trace*, or that φ satisfies the *trace property*, if $\varphi(xy) = \varphi(yx)$, for all $x, y \in M_n(\mathbb{C})$.

Definition 1.4. The *standard trace* on $M_n(\mathbb{C})$ is the map $\mathrm{Tr} : M_n(\mathbb{C}) \to \mathbb{C}$ given by $\mathrm{Tr}(x) = \sum_{i=1}^n x_{ii}$, and the *normalized trace* on $M_n(\mathbb{C})$ is the map $\tau_n = \frac{1}{n}\,\mathrm{Tr}$.

Remark 1.6. It is straightforward to check that the standard trace is, in fact, a trace and that the normalized trace is a tracial state. The standard trace has the property $\mathrm{Tr}(e_{ij}) = \delta_{ij}$, for all $1 \leq i, j \leq n$, and $\mathrm{Tr}(\mathbf{1}_n) = n$.

Theorem 1.7. *The standard trace is unique up to a scalar factor among the linear functionals on $M_n(\mathbb{C})$ that satisfy the trace property.*

Proof. Let $\tau : M_n(\mathbb{C}) \to \mathbb{C}$ be a linear functional satisfying the trace property. We have to show that $\tau = \lambda\,\mathrm{Tr}$, for some $\lambda \in \mathbb{C}$. If $i \neq j$, then $\tau(e_{ij}) = \tau(e_{ii}e_{ij}e_{jj}) = \tau(e_{jj}e_{ii}e_{ij}) = 0$, and for all $1 \leq i, j \leq n$, we have $\tau(e_{ii}) = \tau(e_{ij}e_{ji}) = \tau(e_{ji}e_{ij}) = \tau(e_{jj})$. Let $\lambda = \tau(e_{11})$, then by the above we have $\tau(e_{ij}) = \lambda\delta_{ij}$. As $\{e_{ij} : 1 \leq i, j \leq n\}$ is a basis for $M_n(\mathbb{C})$, the claim that $\tau(x) = \lambda\,\mathrm{Tr}(x)$, for all $x \in M_n(\mathbb{C})$, follows by linearity. □

1.2 Completely positive maps

We denote by ι_n the identity map on $M_n(\mathbb{C})$.

Definition 1.5. Let $T : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a linear map. We say that

(a) T is *positive*, if $T(x)$ is positive, whenever x is,

(b) T is *k-positive*, if $\iota_k \otimes T$ is positive on $M_k(\mathbb{C}) \otimes M_n(\mathbb{C})$,

(c) T is *completely positive*, if T is k-positive, for all $k \geq 1$.

Definition 1.6. A linear map $T : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is said to be *unital* if $T(\mathbf{1}_n) = \mathbf{1}_m$.

Definition 1.7. Let τ be a trace on $M_n(\mathbb{C})$ and let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. Then T is said to be *τ-preserving* if $\tau \circ T(x) = \tau(x)$, for all $x \in M_n(\mathbb{C})$.

We let $\mathrm{CP}(M_n(\mathbb{C}), M_m(\mathbb{C}))$ denote the set of completely positive maps from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$. The set of completely positive and unital maps from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$ is denoted by $\mathrm{UCP}_{n,m}$, or simply by UCP_n, when $n = m$. The set of completely positive, unital and τ_n-preserving maps on $M_n(\mathbb{C})$ is denoted by UCPT_n.

Theorem 1.8 (Choi)**.** *Let $T : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a linear map. Then T is completely positive if and only if there exist $a_1, a_2, \ldots, a_d \in M_{n,m}(\mathbb{C})$ such that*

$$Tx = \sum_{i=1}^{d} a_i^* x a_i, \qquad \textit{for all } x \in M_n(\mathbb{C}). \tag{1.4}$$

Proof. A row matrix $v \in M_{1,nm}(\mathbb{C})$ can be viewed as a block matrix $v = (v_1, v_2, \ldots, v_n) \in M_{1,n}(M_{1,m}(\mathbb{C}))$. We associate to v the $n \times m$ matrix a which has v_j in the j'th row. One can easily verify that

$$v^*v = \left[v_j^* v_k\right]_{j,k=1}^n = \left[a^* e_{jk} a\right]_{j,k=1}^n. \tag{1.5}$$

The matrix $\left[e_{jk}\right]_{j,k=1}^n$ is self-adjoint and has eigenvalues 0 and n. Suppose that $T : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is completely positive. Then $\iota_n \otimes T\left(\left[e_{jk}\right]_{j,k=1}^n\right) = \left[T\left(e_{jk}\right)\right]_{j,k=1}^n$ is positive in $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$. By the (finite dimensional) spectral theorem we can find vectors $v_1, v_2, \ldots, v_d$ in $M_{1,nm}(\mathbb{C})$ such that

$$\left[T\left(e_{jk}\right)\right]_{j,k=1}^n = \sum_{i=1}^d v_i^* v_i.$$

For each $1 \leq i \leq d$, we can by equation (1.5) write $v_i^* v_i = \left[a_i^* e_{jk} a_i\right]_{j,k=1}^n$, where a_i is the $n \times m$ matrix associated with v_i as explained previously. Hence

$$\left[T\left(e_{jk}\right)\right]_{j,k=1}^n = \sum_{i=1}^d \left[a_i^* e_{jk} a_i\right]_{j,k=1}^n.$$

Since $M_n(\mathbb{C}) = \operatorname{span}\{e_{ij} : 1 \leq i, j \leq n\}$, it follows by the linearity of T that $Tx = \sum_{i=1}^d a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$. Conversely, suppose that $T : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is given by $Tx = a^* x a$, for all $x \in M_n(\mathbb{C})$. Let $k \in \mathbb{N}$. If $x \in M_k(\mathbb{C}) \otimes M_n(\mathbb{C})$ is positive, we can find $y \in M_k(\mathbb{C}) \otimes M_n(\mathbb{C})$ such that $x = y^* y$. Then $(\iota_k \otimes T)(x) = (\mathbf{1}_k \otimes a)^* x (\mathbf{1}_k \otimes a) = (y(\mathbf{1}_k \otimes a))^* y(\mathbf{1}_k \otimes a)$, which is positive by definition. Hence T is completely positive. Since sums of completely positive maps are completely positive, the assertion follows. □

Definition 1.8. Let $T : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a linear map. The *Choi matrix* associated to T is the matrix $C_T \in M_{nm}(\mathbb{C})$ defined by

$$C_T := (\iota_n \otimes T)\left(\left[e_{ij}^{(n)}\right]_{i,j=1}^n\right) = \left[T\left(e_{ij}^{(n)}\right)\right]_{i,j=1}^n,$$

or, equivalently,

$$C_T = (\iota_n \otimes T)\left(\sum_{i,j} e_{ij} \otimes e_{ij}\right) = \sum_{i,j} e_{ij} \otimes T(e_{ij})$$

Proposition 1.9. *A linear map $T : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ is completely positive if and only if its associated Choi matrix C_T is positive in $M_{nm}(\mathbb{C})$. Moreover, the map given by $T \mapsto C_T$ is a bijection between* $\mathrm{CP}(M_n(\mathbb{C}), M_m(\mathbb{C}))$ *and* $M_{nm}(\mathbb{C})_+$.

Proof. The implication "⇒" is obvious from the definition. Conversely, if C_T is positive we can find $1 \times nm$ matrices $v_1, v_2, \ldots, v_d$ such that $C_T = \sum_{i=1}^d v_i^* v_i$. Following the proof of Theorem 1.8, it is clear that T has the form $Tx = \sum_{i=1}^d a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$, where a_i is the $n \times m$ matrix associated to v_i. Hence T is completely positive. Consider finally the map $\mathrm{CP}(M_n(\mathbb{C}), M_m(\mathbb{C})) \to M_{nm}(\mathbb{C})_+$ given by $T \mapsto C_T$. This is a well-defined map by the above argument, and both injectivity and surjectivity are clear. □

Remark 1.10 (On the uniqueness of the representation of T). Given the Choi matrix C_T associated to a completely positive map T, we constructed in the proof of Theorem 1.8 matrices $a_1, a_2, \ldots, a_d$ by first decomposing C_T into a sum of projections onto its eigenspaces. However, the choice of vectors spanning the respective eigenspaces is not unique. Hence, the representation of T in terms of $\{a_i\}$ is not unique, either. We may require that the eigenvectors of C_T are chosen to be linearly independent. In this case, the set $\{a_i\}$ is linearly independent, as well, and $Tx = \sum_{i=1}^d a_i^* x a_i$, $x \in M_n(\mathbb{C})$, is called a *canonical form* for T.

Lemma 1.11. *A subset $\{v_i\}$ of $M_{1,n}(\mathbb{C})$ is linearly independent if and only if $\{v_i^* v_j\}$ is linearly independent in $M_n(\mathbb{C})$.*

Proof. Assume that $\{v_i\}$ is a linearly independent set, and suppose that $\sum_{i,j} \lambda_{ij} v_i^* v_j = \mathbf{0}_n$. We identify $M_{1,n}(\mathbb{C})$ with the dual of $\mathbb{C}^n$ by $v\alpha = \langle \alpha , v^* \rangle$, for $v \in M_{1,n}(\mathbb{C})$ and $\alpha \in \mathbb{C}^n$. Then for all $\alpha \in \mathbb{C}^n$,

$$\left(\sum_{i,j} \lambda_{ij} v_i^* v_j \right) \alpha = \sum_{i,j} \lambda_{ij} v_i^* (v_j \alpha) = \sum_i \left(\sum_j \lambda_{ij} \langle \alpha , v_j^* \rangle \right) v_i^* = 0.$$

As $\{v_i\}$ is a linearly independent set so is $\{v_i^*\}$, and it follows that $\sum_j \lambda_{ij} \langle \alpha , v_j^* \rangle = 0$, for all i and for all $\alpha \in \mathbb{C}^n$. We conclude that $\sum_j \lambda_{ij} v_j^* = 0$, and therefore that $\lambda_{ij} = 0$, for all i, j.

Assume on the other hand that $\{v_i^* v_j\}$ is a linearly independent set, and suppose that $\sum_i \lambda_i v_i = 0$. Then $\sum_{i,j} \overline{\lambda}_i \lambda_j v_i^* v_j = 0$, and thus $\overline{\lambda}_i \lambda_j = 0$, for all i, j. In particular $|\lambda_i|^2 = 0$ and hence $\lambda_i = 0$, for all i. □

Proposition 1.12. *Let $T \in \mathrm{CP}(M_n(\mathbb{C}), M_m(\mathbb{C}))$ and suppose that $Tx = \sum_{i=1}^d a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$, is a canonical form for T. Let $\{b_r : 1 \leq r \leq d'\}$ be a set of $n \times m$ matrices. Then $Tx = \sum_{r=1}^{d'} b_r^* x b_r$, $x \in M_n(\mathbb{C})$, if and only if there exists a $d' \times d$ matrix $[\mu_{ri}]_{r,i}$, that is an isometry, such that $b_r = \sum_{i=1}^d \mu_{ri} a_i$, for all $1 \leq r \leq d'$. Moreover, if $\{b_r : 1 \leq r \leq d'\}$ is also a linearly independent set, then $d' = d$ and $[\mu_{ri}]_{r,i}$ is unitary.*

Proof. Suppose that $Tx = \sum_{r=1}^{d'} b_r^* x b_r$, for all $x \in M_n(\mathbb{C})$. For each r, the $n \times m$ matrix b_r corresponds to a $1 \times nm$ matrix w_r constructed by placing the rows of b_r besides each other in order. In the same way, we construct for each i a $1 \times nm$ matrix v_i corresponding to the $n \times m$ matrix a_i. Following the proof of Theorem 1.8, we get that

$$\sum_{r=1}^{d'} w_r^* w_r = \left[T(e_{jk}^{(n)}) \right]_{j,k} = \sum_{i=1}^{d} v_i^* v_i.$$

If $\alpha \in \mathbb{C}^{nm}$ is in the kernel of $\sum_i v_i^* v_i$, then $\alpha^* (\sum_r w_r^* w_r) \alpha = \sum_r | \langle \alpha , w_r^* \rangle |^2 = 0$. In particular, it follows that $w_r^* \in \ker(\sum_i v_i^* v_i)^\perp$, or, equivalently, that $w_r^* \in \mathrm{span}\{v_i^*\}$, for all $1 \leq r \leq d'$. Hence there exist scalars μ_{ri} such that $w_r^* = \sum_i \overline{\mu}_{ri} v_i^*$, for all $1 \leq r \leq d'$. We can arrange these scalars into a $d' \times d$-matrix $[\mu_{ri}]$. We have now that

$$\sum_i v_i^* v_i = \sum_r w_r^* w_r = \sum_{i,j} \left(\sum_r \overline{\mu}_{ri} \mu_{rj} \right) v_i^* v_j.$$

The set $\{v_i\}$ is linearly independent by assumption, and so $\{v_i^* v_j\}$ is linearly independent by Lemma 1.11. Hence $\sum_r \overline{\mu}_{ri} \mu_{rj} = \delta_{ij}$, or equivalently $[\mu_{ri}]^* [\mu_{ri}] = \mathbf{1}_d$. This shows that $[\mu_{ri}]$ is an isometry. Suppose next that $\{b_r\}$ and thereby that $\{w_r^*\}$ is a linearly independent set. We need to show that $[\mu_{ri}]$ is unitary. We have already shown that $\mathrm{span}\{w_r\} \subseteq \mathrm{span}\{v_i\}$, and by a similar argument we can derive the opposite inclusion, and hence that $\mathrm{span}\{w_r\} = \mathrm{span}\{v_i\}$. If both sets are linearly independent, it follows immediately that $d' = d$. Now $[\mu_{ri}]$ is a linear map from $\mathrm{span}\{v_i\}$ to $\mathrm{span}\{w_r\}$. Being an isometry, the kernel of this map is trivial, and, as the two spaces have equal and finite dimension, it follows that $[\mu_{ri}]$ is a bijection, and hence it has a unique inverse. As we already know that $[\mu_{ri}]^*$ is a left inverse, it follows that $[\mu_{ri}]$ is unitary.

For the converse, let $[\mu_{ri}]$ is an isometry in $M_{d',d}(\mathbb{C})$. Then $\sum_{r=1}^{d'} \overline{\mu}_{ri} \mu_{rj} = \delta_{ij}$, for all $1 \leq i, j \leq d$. Define for each r the $n \times m$-matrix $b_r := \sum_i \mu_{ri} a_i$. For any $x \in M_n(\mathbb{C})$, we have

$$\sum_{r=1}^{d'} b_r^* x b_r = \sum_{i,j=1}^{d} \left(\sum_{r=1}^{d'} \overline{\mu}_{ri} \mu_{ri} \right) a_i^* x a_j = \sum_{i=1}^{d} a_i^* x a_i.$$

Hence $Tx = \sum_{r=1}^{d'} b_r^* x b_r$, for all $x \in M_n(\mathbb{C})$. □

Theorem 1.13 (Choi)**.** *Let $T : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a linear map. Then T is extreme in* $\mathrm{UCP}_{n,m}$ *if and only if T can be written as*

$$Tx = \sum_{j=1}^{d} a_j^* x a_j, \qquad \text{for all } x \in M_n(\mathbb{C}),$$

where the $a_1, a_2, \ldots, a_d$ are complex $n \times m$-matrices, $\sum_{j=1}^d a_j^* a_j = \mathbf{1}_m$ and $\{a_j^* a_k : 1 \le j, k \le d\}$ is a linearly independent set.

Proof. Assume first that T is extreme in $\mathrm{CP}(M_n(\mathbb{C}), M_m(\mathbb{C}), \mathbf{1}_m)$, and express T in a canonical form, i.e., $Tx = \sum_{j=1}^d a_j^* x a_j$, for all $x \in M_n(\mathbb{C})$, where $\{a_j : 1 \le j \le d\}$ is a linearly independent set. Then $\sum_{j=1}^d a_j^* a_j = T(\mathbf{1}_n) = \mathbf{1}_m$. What we need to show is the linear independence of $\{a_j^* a_k : 1 \le j, k \le d\}$. Suppose that $\sum_{j,k=1}^d \lambda_{jk} a_j^* a_k = 0$, for some matrix $[\lambda_{jk}]_{j,k}$. Note that this is equivalent to having $\sum_{j,k=1}^d (\lambda_{jk} + \overline{\lambda}_{kj}) a_j^* a_k = 0$ and $\sum_{j,k=1}^d i(\lambda_{jk} - \overline{\lambda}_{kj}) a_j^* a_k = 0$. We may therefore without loss of generality assume that $[\lambda_{jk}]_{j,k}$ is self-adjoint. By rescaling, we may furthermore assume that $-\mathbf{1}_n \le [\lambda_{jk}]_{j,k} \le \mathbf{1}_n$.

Define $S_\pm : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ by $S_\pm(x) = \sum_{j=1}^d a_j^* x a_j \pm \sum_{j,k=1}^d \lambda_{jk} a_j^* x a_k$, for all $x \in M_n(\mathbb{C})$. Clearly, $S_\pm(\mathbf{1}_n) = \mathbf{1}_m$. We show next that these maps are also completely positive. The matrix $\mathbf{1}_d + [\lambda_{jk}]_{j,k}$ is self-adjoint as it is the sum of two self-adjoint matrices. Moreover, it has non-negative eigenvalues, since we assumed that $-\mathbf{1}_d \le [\lambda_{jk}]_{j,k}$. We can then find a $d \times d$ matrix $[\alpha_{jk}]_{j,k}$ such that $\mathbf{1}_d + [\lambda_{jk}]_{j,k} = [\alpha_{jk}]_{j,k}^* [\alpha_{jk}]_{j,k}$. Defining $b_j = \sum_{k=1}^d \alpha_{jk} a_k$,

$$\begin{aligned}\sum_{j=1}^d b_j^* x b_j &= \sum_{j=1}^d \left(\sum_{k,\ell=1}^d \overline{\alpha}_{jk} \alpha_{j\ell} a_j^* x a_\ell \right) = \sum_{k,l=1}^d \left(\sum_{j=1}^d \overline{\alpha}_{jk} \alpha_{j\ell} \right) a_k^* x a_\ell \\ &= \sum_{k,\ell=1}^d (\delta_{k\ell} + \lambda_{k\ell}) a_k^* x a_\ell = S_+(x),\end{aligned}$$

for all $x \in M_n(\mathbb{C})$. It follows by Theorem 1.8 that S_+ is completely positive. Showing that S_- is completely positive is analogous, only here we instead use the assumption $[\lambda_{jk}]_{j,k} \le \mathbf{1}_d$. Clearly $T = \frac{1}{2} S_+ + \frac{1}{2} S_-$, and since T is extreme, we obtain $T = S_+ = S_-$. It follows by Proposition 1.12 that $[\alpha_{jk}]_{j,k}$ is an isometry, and hence that $\mathbf{1}_d + [\lambda_{jk}]_{j,k} = [\alpha_{jk}]_{j,k}^* [\alpha_{jk}]_{j,k} = \mathbf{1}_d$. We conclude that $[\lambda_{jk}]_{j,k} = \mathbf{0}_d$, as required.

Conversely, assume that $Tx = \sum_{j=1}^d a_j^* x a_j$ where $\{a_j^* a_k : 1 \le j, k \le d\}$ is a linearly independent set and $\sum_{j=1}^d a_j^* a_j = \mathbf{1}_m$. If $T = pS + (1-p)R$, for some $p \in (\frac{1}{2}, 1)$, then $T = \frac{1}{2} S + \frac{1}{2}(2(p - \frac{1}{2}) S + 2(1-p) R)$. It is therefore enough to check that $T = \frac{1}{2} S + \frac{1}{2} R$ implies $T = S$. Suppose that $T = \frac{1}{2} S + \frac{1}{2} R$, for maps $S, R \in \mathrm{UCP}_{n,m}$. Write $Sx = \sum_{s=1}^{d'} b_s^* x b_s$ and $Rx = \sum_{r=1}^{d''} c_r^* x c_r$, for all $x \in M_n(\mathbb{C})$. Since $Tx = \frac{1}{2} \sum_{s=1}^{d'} b_s^* x b_s + \frac{1}{2} \sum_{r=1}^{d''} c_r^* x c_r$, for all $x \in M_n(\mathbb{C})$, we can by Proposition 1.12 express b_s and c_r in terms of $a_1, a_2, \ldots, a_d$, for each $1 \le s \le d'$ and each $1 \le r \le d''$. Let $b_s = \sum_{j=1}^d \mu_{sj} a_j$, for each s. Then

$$\sum_{j=1}^d a_j^* a_j = T(\mathbf{1}_n) = S(\mathbf{1}_n) = \sum_{s=1}^{d'} b_s^* b_s = \sum_{s,j,k} \overline{\mu}_{sj} \mu_{sk} a_j^* a_k.$$

By linear independence of the set $\{a_j^* a_k : 1 \le j, k \le d\}$, we get that $\sum_{s=1}^{d'} \overline{\mu}_{sj} \mu_{sk} = \delta_{jk}$. For any $x \in M_n(\mathbb{C})$, we have

$$Sx = \sum_{s=1}^{d'} b_s^* x b_s = \sum_{s=1}^{d'} \left(\sum_{j,k=1}^d \overline{\mu}_{sj} \mu_{sk} a_j^* x a_k \right) = \sum_{j,k=1}^d \left(\sum_{s=1}^{d'} \overline{\mu}_{sj} \mu_{sk} \right) a_j^* x a_k = \sum_{j=1}^d a_j^* x a_j = Tx.$$

It follows that $T = S$, and hence that T is extreme. $\square$

Theorem 1.14. *Let $T \in \mathrm{UCPT}_n$ be given by $Tx = \sum_{i=1}^d a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$. Then T is extreme in UCPT_n if and only if the condition $\sum_{i,j=1}^d \lambda_{ij} a_i^* a_j = \sum_{i,j=1}^d \lambda_{ij} a_j a_i^* = 0$ implies $\lambda_{ij} = 0$, for $1 \leq i, j \leq d$.*

Proof. Suppose that $T \in \mathrm{UCPT}_n$ is extreme and that $\sum_{i,j=1}^d \lambda_{ij} a_i^* a_j = \sum_{i,j=1}^d \lambda_{ij} a_j a_i^* = 0$. We want to show that $[\lambda_{ij}] = \mathbf{0}_d$. Just as in the proof of Theorem 1.13, we may assume without loss of generality that $[\lambda_{ij}]$ is self-adjoint and that $-\mathbf{1}_d \leq [\lambda_{ij}] \leq \mathbf{1}_d$. Define $S_\pm : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ by $S_\pm(x) = \sum_{i=1}^d a_i^* x a_i \pm \sum_{i,j=1}^d \lambda_{ij} a_i^* x a_j$, $x \in M_n(\mathbb{C})$. Clearly, $S_\pm(\mathbf{1}_n) = \mathbf{1}_n$ and

$$\begin{aligned}\tau_n \circ S_\pm(x) &= \tau_n(T(x)) \pm \sum_{i,j=1}^d \lambda_{ij} \tau_n(a_i^* x a_j) = \tau_n(x) \pm \sum_{i,j=1}^d \lambda_{ij} \tau_n(a_j a_i^* x) \\ &= \tau_n(x) \pm \tau_n\left(\left(\sum_{i,j=1}^d \lambda_{ij} a_j a_i^*\right) x\right) = \tau_n(x).\end{aligned}$$

From here the argument that $[\lambda_{ij}] = \mathbf{0}_d$ is exactly the same as in the proof of Theorem 1.13.

Conversely, suppose that $\sum_{i,j=1}^d \lambda_{ij} a_i^* a_j = \sum_{i,j=1}^d \lambda_{ij} a_j a_i^* = 0$ implies $\lambda_{ij} = 0$, for $1 \leq i, j \leq d$, and suppose that $T = \frac{1}{2}S + \frac{1}{2}R$, for some $S, R \in \mathrm{UCPT}_n$. We write $Sx = \sum_{s=1}^{d'} b_s^* x b_s$ and $Rx = \sum_{r=1}^{d''} c_r^* x c_r$, for all $x \in M_n(\mathbb{C})$, and use Proposition 1.12 to express b_s in terms of $a_1, a_2, \ldots, a_d$ as $b_s = \sum_{i=1}^d \mu_{si} a_i$, for each $1 \leq s \leq d'$. Then

$$\begin{aligned}\sum_{i=1}^d a_i^* a_i &= T(\mathbf{1}_n) = S(\mathbf{1}_n) = \sum_{s=1}^{d'} b_s^* b_s = \sum_{s,i,j} \overline{\mu}_{si} \mu_{sj} a_i^* a_j, \\ \sum_{i=1}^d a_i a_i^* &= \mathbf{1}_n = \sum_{s=1}^{d'} b_s b_s^* = \sum_{s,i,j} \overline{\mu}_{si} \mu_{sj} a_j a_i^*.\end{aligned}$$

The second equality comes from observing that the map $x \mapsto \sum_{i=1}^d a_i^* x a_i$ on $M_n(\mathbb{C})$ is τ_n-preserving if and only if $\sum_{i=1}^d a_i a_i^* = \mathbf{1}_n$. Define $\lambda_{ij} := \delta_{ij} - \sum_{s=1}^{d'} \overline{\mu}_{si} \mu_{sj}$, for all $1 \leq i, j \leq d$. Then the above equalities become

$$\sum_{i,j=1}^d \lambda_{ij} a_i^* a_j = \sum_{i=1}^d \lambda_{ij} a_j a_i^* = \mathbf{0}_n.$$

By assumption, this implies that $\lambda_{ij} = 0$, or, equivalently, $\sum_{s=1}^{d'} \overline{\mu}_{si} \mu_{sj} = \delta_{ij}$, for $1 \leq i, j \leq d$. As in the proof of Theorem 1.13, this implies that $T = S$, and hence that T is extreme. □

1.3 Norms and completely bounded maps

Let $\mathcal{L}(M_n(\mathbb{C}), M_m(\mathbb{C}))$ be the space of linear maps from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$. If $n = m$, we denote this space by $\mathcal{L}(M_n(\mathbb{C}))$. As matrix algebras are normed spaces with a norm given by equation (1.3), we can define a norm on $\mathcal{L}(M_n(\mathbb{C}), M_m(\mathbb{C}))$ as follows:

$$\|T\| := \sup\{\|Tx\| : x \in M_n(\mathbb{C}), \|x\| \leq 1\}, \qquad \text{for all } x \in M_n(\mathbb{C}).$$

Remark 1.15. Let V and W be normed vector spaces and let $T : V \to W$ be a linear map. If V is finite dimensional, then T is automatically bounded. Indeed, suppose that $n \geq 1$ is the dimension of V and let $\{v_i : 1 \leq i \leq n\}$ be a basis for V. Since all norms are equivalent on finite dimensional vector spaces, we may without loss of generality choose the norm on

V to be the maximum norm, i.e., $\|v\| = \|\sum_{i=1}^n \alpha_i v_i\| = \max\{|\alpha_i| : 1 \le i \le n\}$, for $v \in V$. Let $C = \sum_{i=1}^n \|Tv_i\|$ and let $v = \sum_{i=1}^n \alpha_i v_i$ be any element of V. Then

$$\|Tv\| \le \sum_{i=1}^n |\alpha_i| \, \|Tv_i\| \le \max\{|\alpha_i| : 1 \le i \le n\} \sum_{i=1}^n \|Tv_i\| = C \, \|v\| .$$

This shows that T is bounded.

Proposition 1.16. *The set of completely positive, unital and τ_n-preserving maps on $M_n(\mathbb{C})$ is a closed subset of $\mathcal{L}(M_n(\mathbb{C}))$ with respect to the norm topology.*

Proof. Let $(T_k)_{k\ge1}$ be a sequence in UCPT_n and suppose that $(T_k)_{k\ge1}$ converges in norm to a map T on $M_n(\mathbb{C})$. It is straightforward to show that T is linear unital and τ_n-preserving. We proceed to show that T is completely positive. The map T is completely positive by Proposition 1.9 if and only if its associated Choi matrix, C_T, is positive, which is equivalent to the condition that $\langle C_T v \,, v\rangle \ge 0$ for all $v \in \mathbb{C}^{n^2}$. We can derive the following estimate for the distance between C_T and C_{T_k} using Proposition 1.2(b):

$$\begin{aligned}
\|C_{T_k} - C_T\| &= \Big\| \sum_{i,j=1}^n e_{ij} \otimes (T_k - T)(e_{ij}) \Big\| \le \sum_{i,j=1}^n \|e_{ij} \otimes (T_k - T)(e_{ij})\| \\
&= \sum_{i,j=1}^n \|(T_k - T)(e_{ij})\| \le n^2 \, \|T_k - T\| .
\end{aligned}$$

Hence, the sequence $(C_{T_k})_{k\ge1}$ in $M_{n^2}(\mathbb{C})$ converges in norm to C_T. Let v be any vector in $\mathbb{C}^{n^2}$. By the Cauchy-Schwarz inequality, we have that

$$|\langle (C_{T_k} - C_T)v \,, v\rangle| \le \|(C_{T_k} - C_T)v\| \, \|v\| \le \|C_{T_k} - C_T\| \, \|v\|^2 .$$

This implies that $\langle C_T v \,, v\rangle$ is the limit of the sequence $(\langle C_{T_k} v \,, v\rangle)_{k\ge1}$. The assertion follows since $(\langle C_{T_k} v \,, v\rangle)_{k\ge1}$ is a sequence in $\mathbb{R}$ bounded below by 0. ☐

Theorem 1.17 (Russo-Dye)**.** *Let $T : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a positive, linear and unital map. Then $\|T\| = 1$.*

Proof. Let $u \in M_n(\mathbb{C})$ be a unitary matrix. By the finite dimensional spectral theorem for normal matrices we can write u as the sum $u = \sum_{i=1}^k \lambda_i p_i$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all the distinct eigenvalues of u, and p_i is the projection onto the eigenspace corresponding to the eigenvalue λ_i. Since u is unitary, the eigenvalues are complex numbers of modulus 1. Moreover, the rank of u is n, and so $\mathbf{1}_n = \sum_{i=1}^k p_i$. Then

$$(\iota_2 \otimes T) \begin{pmatrix} \mathbf{1}_n & u \\ u^* & \mathbf{1}_n \end{pmatrix} = \begin{pmatrix} T(\mathbf{1}_n) & T(u) \\ T(u)^* & T(\mathbf{1}_n) \end{pmatrix} = \sum_{i=1}^k \begin{pmatrix} T(p_i) & \lambda_i T(p_i) \\ \overline{\lambda}_i T(p_i) & T(p_i) \end{pmatrix} = \sum_{i=1}^k \begin{pmatrix} 1 & \lambda_i \\ \overline{\lambda}_i & 1 \end{pmatrix} \otimes T(p_i),$$

which is positive by Lemma 1.4, the positivity of T and since the set of positive elements is a cone. Since T is unital, we have that

$$(\iota_2 \otimes T) \begin{pmatrix} \mathbf{1}_n & u \\ u^* & \mathbf{1}_n \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n & T(u) \\ T(u)^* & \mathbf{1}_n \end{pmatrix} .$$

Hence $\|Tu\| \le 1$, by Lemma 1.4. Suppose next that $x \in M_n(\mathbb{C})$ is any contraction. We may then by Lemma 1.5 write $a = (u + v)/2$, where u and v are unitary elements of $M_n(\mathbb{C})$. It follows by linearity of T and by the triangle inequality that $\|T\| \le 1$. On the other hand, $\|T\| \ge \|T(\mathbf{1}_n)\| = \|\mathbf{1}_n\| = 1$ and the assertion follows. ☐

Corollary 1.18. *The set* UCPT_n *is compact.*

Proof. The set $\mathcal{L}(M_n(\mathbb{C}))$ of linear maps on the matrix algebra $M_n(\mathbb{C})$ is a finite dimensional vector space. Hence, the compact sets are precisely the sets that are closed and bounded. The assertion follows directly from Proposition 1.16 and Theorem 1.17. □

We proceed from here to discuss a whole family of norms on $\mathcal{L}(M_n(\mathbb{C}), M_m(\mathbb{C}))$; in particular we will introduce the completely bounded norm. Note that the theory of completely bounded maps can be discussed in the much more general setting of linear maps between operator spaces. For the purpose of this thesis we shall restrict our attention to only discussing $\mathcal{L}(M_n(\mathbb{C}), M_m(\mathbb{C}))$. Let $T : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a linear map. For each $k \in \mathbb{N}$, we define a linear map $T_k : M_k(M_n(\mathbb{C})) \to M_k(M_m(\mathbb{C}))$ by

$$T_k([x_{ij}]) = [T(x_{ij})], \qquad \text{for all}[x_{ij}] \in M_n(\mathbb{C}). \tag{1.6}$$

With the identification in equation (1.2), we can equivalently define this map in terms of elementary tensors as follows:

$$T_k(\gamma \otimes x) = \gamma \otimes T(x), \qquad \gamma \in M_k(\mathbb{C}), x \in M_n(\mathbb{C}),$$

and we see that T_k is simply the tensor product map $T_k = \iota_k \otimes T$.

Definition 1.9. Let $T : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a linear map. The *completely bounded norm* of T is defined by

$$\|T\|_{cb} = \sup_{k\geq 1} \|T_k\| ,$$

and T is said to be *completely bounded* if $\|T\|_{cb} < \infty$.

It is straightforward to check that $\|\cdot\|_{cb}$ indeed defines a norm on the set of completely bounded maps. Moreover, the family of norms on T we have just defined satisfies

$$\|T\| \leq \|T_2\| \leq \ldots \leq \|T_k\| \leq \ldots \leq \|T\|_{cb} .$$

To see this, associate to each block matrix $x = [x_{ij}]_{i,j=1}^{k} \in M_k(M_n(\mathbb{C}))$ the block matrix $\widetilde{x} = [\widetilde{x}_{ij}]_{i,j=1}^{k+1} \in M_{k+1}(M_n(\mathbb{C}))$, where $\widetilde{x}_{ij} = x_{ij}$, for $1 \leq i, j \leq k$, and $\mathbf{0}_n$ otherwise. It is clear from the definition of the operator norm on the matrix algebra that $\|\widetilde{x}\| \leq \|x\|$, and it is a consequence of property (a) of Proposition 1.2 that $\|T_{k+1}\widetilde{x}\| = \|T_k x\|$. We can now derive that $\|T_{k+1}\| \geq \sup\{\|(\iota_{k+1} \otimes T)\widetilde{x}\| : x \in M_k(M_n(\mathbb{C})), \|x\| \leq 1\} = \|T_k\|$, for all $k \geq 1$.

Definition 1.10. Let $T : M_n(\mathbb{C}) \to M_m(\mathbb{C})$ be a linear map and let $\mathcal{S}$ be a subset of $\mathcal{L}(M_n(\mathbb{C}), M_m(\mathbb{C}))$. We define the *completely bounded distance* between T and $\mathcal{S}$ as follows:

$$d_{cb}(T, \mathcal{S}) = \inf\{\|T - R\|_{cb} : R \in \mathcal{S}\}.$$

1.4 Literature

Section 1.1 as well as the discussion regarding completely bounded maps in Section 1.3 is inspired by [9, Chapter 1 and 2]. The proofs of Lemma 1.4 and Theorem 1.17 are taken from [10]. All results on completely positive maps are due to the work of Man-Duen Choi, and Section 1.2 is based on his original article [6].

Chapter 2

Birkhoff

2.1 A classical theorem of Birkhoff

Definition 2.1. A *doubly stochastic* matrix is a square matrix such that all entries are real and non-negative, and that each row and each column sum up to 1.

Remark 2.1. Observe that the constraints given in Definition 2.1 define a closed convex polytope (see Definition A.4) in the relevant matrix algebra. Moreover, the constraints imply that all entries of lie in $[0, 1]$. Hence, the set of doubly stochastic matrices is bounded.

If K is a *convex* subset of a vector space (see Definition A.1), we denote by $\operatorname{Ext}(K)$ the set of *extreme points* of K (see Definition A.2).

Theorem 2.2 (Birkhoff's theorem)**.** *Every doubly stochastic matrix is a convex combination of permutation matrices.*

Proof. Fix $n \in \mathbb{N}$ and let B_n be the set of doubly stochastic matrices in $M_n(\mathbb{R})$. Suppose that $X = [x_{ij}] \in B_n$ is not a permutation matrix. Then there must be an entry $0 < x_{i_1 j_1} < 1$. Because of the row constraint, $\sum_{j=1}^{n} x_{i_1 j} = 1$, there must be a column $j_2 \neq j_1$ such that $0 < x_{i_1 j_2} < 1$. Because of the column constraint, $\sum_{i=1}^{n} x_{i j_2} = 1$, we can find a row $i_2 \neq i_1$ such that $0 < x_{i_2 j_2} < 1$. This process of alternating between moving horizontally and vertically from entry to entry in the matrix can be iterated. Viewing the entries of the matrix as points on a grid, we are in this way drawing a path between the points of the grid consisting of only horizontal and vertical lines. The vertices of this path all correspond to entries of the matrix strictly between zero and one. After a finite number of iterations, we will choose a vertex that lies on the already drawn path. At this point the iteration terminates, and the resulting path will contain a closed path with an even number of vertices. List and relabel the entries at the vertices of this closed path, taking them in order: $\{x_{i_1 j_1}, x_{i_1 j_2}, x_{i_2, j_2}, x_{i_2 j_3}, \dots, x_{i_k j_{k+1}}\}$. Let $\varepsilon_0 = \min_{1 \leq \ell \leq k} \{x_{i_\ell j_\ell}, x_{i_\ell j_{\ell+1}}, 1 - x_{i_\ell j_\ell}, 1 - x_{i_\ell j_{\ell+1}}\} > 0$, and let C be the matrix with entries

$$c_{ij} := \begin{cases} 1, & \text{if } (i,j) = (i_\ell, j_\ell) \text{ for some } 1 \leq \ell \leq k \\ -1, & \text{if } (i,j) = (i_\ell, j_{\ell+1}) \text{ for some } 1 \leq \ell \leq k \\ 0, & \text{otherwise} \end{cases}.$$

Let $0 < \varepsilon < \varepsilon_0$ and define $X^+(\varepsilon) := X + \varepsilon C$ and $X^-(\varepsilon) := X - \varepsilon C$. By construction, $X^+(\varepsilon)$ and $X^-(\varepsilon)$ are doubly stochastic, and clearly $X = \frac{1}{2}X^+(\varepsilon) + \frac{1}{2}X^-(\varepsilon)$. Hence X is not extreme in B_n. By contraposition, this shows that all extreme points of B_n are permutation matrices. Conversely, any permutation matrix is clearly an extreme point of B_n.

The set B_n of doubly stochastic matrices in $M_n(\mathbb{R})$ is a closed and bounded convex polytope by Remark 2.1. Since the matrix algebra $M_n(\mathbb{R})$ is a finite dimensional vector space over $\mathbb{R}$, it follows by the Heine-Borel theorem that B_n is compact. Hence B_n equals the closed convex hull of its extreme points by the Krein-Milman theorem (see Theorem A.3). The set of permutation matrices in $M_n(\mathbb{C})$ is finite, hence trivially compact. This implies that the convex hull of this set is compact by Proposition A.2 and then, in particular, closed. We conclude that $B_n = \text{conv}(\text{Ext}(B_n))$, and the assertion follows. □

Consider the vector space $D := \ell_\infty(\{1, 2, \ldots, n\})$. This is an algebra over $\mathbb{C}$ with a multiplicative identity given by $\mathbf{1} := \mathbf{1}_{\{1,2,\ldots,n\}}$. Let τ be the *normalized trace* on D, i.e., $\tau : D \to \mathbb{C}$ is the linear functional given by $\tau(\mathbf{1}_{\{i\}}) = 1/n$, for $1 \le i \le n$. An element in D is said to be positive if all its entries are non-negative, and a linear map $T : D \to \mathbb{C}$ is positive if it maps positive elements to positive elements. Moreover, T is said to be unital if $T(\mathbf{1}) = \mathbf{1}$ and τ-preserving if $\tau(Tx) = \tau(x)$, for all $x \in D$. If we identify a linear map with its associated matrix, observe that the positive, unital and τ-preserving linear maps on D are precisely the doubly stochastic $n \times n$-matrices.

Theorem 2.3. *The automorphisms on D are precisely the permutations of* $\{1, 2, \ldots, n\}$.

Proof. The permutations of $\{1, 2, \ldots, n\}$ correspond exactly to the permutation matrices in $M_n(\mathbb{C})$. It is clear that a map on D given by $x \mapsto Ax$, for all $x \in D$, where A is a permutation matrix in $M_n(\mathbb{C})$, is linear, multiplicative and bijective, and hence an automorphism. We show the other direction. Suppose that $A \in M_n(\mathbb{C})$ is not a permutation matrix. If A is not of full rank, then A is not surjective, and hence not an automorphism. So assume that A has full rank. Then A is a linear bijection. We aim to show that A is not multiplicative. Suppose that A has an entry which is not zero and not one, i.e., $a_{ij} \notin \{0, 1\}$, for some $1 \le i, j \le n$. Let $x \in D$ be the vector with 1 in the j'th entry and zeros elsewhere. Then

$$(Ax^2)_i = (Ax)_i = a_{ij} \neq a_{ij}^2 = (Ax)_i(Ax)_i,$$

and hence A is not an automorphism. Next, suppose that all entries of A are either zero or one, but that A has a row with more that one non-zero entry. Let this be the i'th row, let m be the number of non-zero entries, and let j_0 be an index such that $a_{ij_0} = 1$. Let $x = \mathbf{1}_{\{1,2,\ldots,n\}}$, and let $y \in D$ be the vector with 1 in the j_0'th entry and zeros elsewhere. Then $(Ax)_i(Ay)_i = m \cdot 1 = m \neq 1 = (A(xy))_i$, and hence A is not an automorphism. This exhausts all possibilities for what the matrix A can be if A is not a permutation matrix. Thus the proof ends. □

Based on the above discussion, we can rephrase Birkhoff's classical theorem as follows:

Theorem 2.4 (Birkhoff's theorem reformulated)**.** *Every positive, unital and τ-preserving linear map on $D = \ell_\infty(\{1, 2, \ldots, n\})$ is a convex combination of automorphisms on D.*

2.2 The Quantum Birkhoff Conjecture

The classical theorem of Birkhoff can be thought of as a statement about positive unital linear maps which preserve probability distributions on a finite dimensional alphabet. In quantum physics the analogue of finite probability distributions are quantum states, which can be represented by self-adjoint positive matrices of standard trace 1. The linear maps which preserve the mentioned properties of quantum states (even when only applied to part of the physical system) are the completely positive unital τ_n-preserving maps, commonly referred to as *quantum channels* in the language of Quantum Information Theory. The Quantum Birkhoff Conjecture asks if the classical theorem of Birkhoff can be generalized to the non-commutative quantum setting.

2.2.1 The *-automorphisms of the matrix algebra $M_n(\mathbb{C})$

Definition 2.2. The set of *-automorphisms of $M_n(\mathbb{C})$ is denoted by $\mathrm{Aut}(M_n(\mathbb{C}))$, and it is the set of bijective, linear, multiplicative and *-preserving maps on $M_n(\mathbb{C})$.

Definition 2.3. Let $\mathcal{A}$ be a unital C^*-algebra. We say that two projections $p, q \in \mathcal{A}$ are equivalent, and write $p \sim q$, if there exists an element $v \in \mathcal{A}$ such that $p = v^*v$ and $q = vv^*$.

Lemma 2.5. *Let $p, q \in M_n(\mathbb{C})$ be projections. If $\mathrm{Tr}(p) = \mathrm{Tr}(q)$, then $p \sim q$.*

Proof. Projections are in particular self-adjoint. We can by the (finite dimensional) spectral theorem find unitary matrices u and v such that upu^* and vqv^* are diagonal with the eigenvalues of p and q, respectively, listed in decreasing order. Since p and q are projections, the eigenvalues are either zero or one. Now $\mathrm{Tr}(p) = \mathrm{Tr}(u^*up) = \mathrm{Tr}(upu^*) = m_p$, where m_p is the multiplicity of the eigenvalue one for p. Likewise, $\mathrm{Tr}(q) = m_q$, where m_q is the multiplicity of the eigenvalue one for q. It then follows from the assumption $\mathrm{Tr}(p) = \mathrm{Tr}(q)$ that the eigenvalue one has the same multiplicity for p and q. Therefore $upu^* = vqv^*$. Setting $w = v^*up$ it is straightforward to check that $p = w^*w$ and $q = ww^*$. $\square$

Theorem 2.6. *Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. Then $T \in \mathrm{Aut}(M_n(\mathbb{C}))$ if and only if there exists a unitary matrix $u \in M_n(\mathbb{C})$ such that*

$$Tx = u^*xu, \qquad \text{for all } x \in M_n(\mathbb{C}).$$

Proof. Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be given by $Tx = u^*xu$, for all $x \in M_n(\mathbb{C})$, and for some unitary matrix $u \in M_n(\mathbb{C})$. It is clear that this is a linear, multiplicative and *-preserving map. Surjectivity follows immediately since the vector space is finite dimensional. For injectivity, observe that $x \mapsto uxu^*$ is an inverse of T.

Conversely, let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a *-automorphism of $M_n(\mathbb{C})$. The map $\mathrm{Tr} \circ T : M_n(\mathbb{C}) \to \mathbb{C}$ is a trace on $M_n(\mathbb{C})$ since T is multiplicative and Tr is a trace. Indeed,

$$\mathrm{Tr} \circ T(xy) = \mathrm{Tr}(T(x)T(y)) = \mathrm{Tr}(T(y)T(x)) = \mathrm{Tr} \circ T(yx), \qquad \text{for all } x, y \in M_n(\mathbb{C}).$$

The map T is unital since T is a homomorphism, and therefore $\mathrm{Tr} \circ T(\mathbf{1}_n) = \mathrm{Tr}(\mathbf{1}_n) = n$. We conclude by Theorem 1.7 that $\mathrm{Tr} \circ T = \mathrm{Tr}$. In particular, it follows that $\mathrm{Tr}(T(e_{11})) = \mathrm{Tr}(e_{11})$. Note that e_{11} is a projection, and that *-preserving homomorphisms map projections to projections. By Lemma 2.5 we conclude that there exists $v \in M_n(\mathbb{C})$ such that $e_{11} = v^*v$ and $T(e_{11}) = vv^*$. Set $u = \sum_{k=1}^n e_{k1}v^*T(e_{1k})$. Then

$$\begin{aligned}
uu^* &= \left(\sum_{k=1}^n e_{k1}v^*T(e_{1k})\right)\left(\sum_{l=1}^n T(e_{l1})ve_{1l}\right) = \sum_{k,l=1}^n e_{k1}v^*T(e_{1k})T(e_{l1})ve_{1l} \\
&= \sum_{k,l=1}^n e_{k1}v^*T(e_{11})\delta_{kl}ve_{1l} = \sum_{k=1}^n e_{k1}(v^*v)(v^*v)e_{1l} = \sum_{k=1}^n e_{kk} = \mathbf{1}_n, \\
u^*u &= \left(\sum_{k=1}^n T(e_{k1})ve_{1k}\right)\left(\sum_{l=1}^n e_{l1}v^*T(e_{1l})\right) = \sum_{k,l=1}^n T(e_{k1})ve_{1k}e_{l1}v^*T(e_{1l}) \\
&= \sum_{k=1}^n T(e_{k1})ve_{11}v^*T(e_{1k}) = \sum_{k=1}^n T(e_{kk}) = T(\mathbf{1}_n) = \mathbf{1}_n.
\end{aligned}$$

Hence u is unitary, and for all $1 \le i, j \le n$,

$$\begin{aligned}
u^*e_{ij}u &= \left(\sum_{k=1}^n T(e_{k1})ve_{1k}\right)e_{ij}\left(\sum_{k=1}^n e_{k1}v^*T(e_{1k})\right) = (T(e_{ij})ve_{1j})\left(\sum_{k=1}^n e_{k1}v^*T(e_{1k})\right) \\
&= T(e_{i1})ve_{11}v^*T(e_{1j}) = T(e_{ij}).
\end{aligned}$$

Since T is linear and $\{e_{ij}\}_{i,j}$ spans $M_n(\mathbb{C})$, it follows that $T(x) = u^*xu$, for all $x \in M_n(\mathbb{C})$. $\square$

Proposition 2.7. *The set* $\mathrm{Aut}(M_n(\mathbb{C}))$ *is compact in the norm topology.*

Proof. It follows from Theorem 2.6 that if $T \in \mathrm{Aut}(M_n(\mathbb{C}))$, then $\|Tx\| \leq \|x\|$, for all $x \in M_n(\mathbb{C})$. Hence $\|T\| = 1$, for all $T \in \mathrm{Aut}(M_n(\mathbb{C}))$. This shows that $\mathrm{Aut}(M_n(\mathbb{C}))$ is bounded. We proceed to show that $\mathrm{Aut}(M_n(\mathbb{C}))$ is a closed set with respect to the operator norm. Let $(T_n)_{n\geq 1}$ be a sequence in $\mathrm{Aut}(M_n(\mathbb{C}))$ and suppose that T_n converges to T in norm. The map T is *-preserving since the involution is a linear isometry. Since addition and multiplication are both jointly continuous in the norm-topology on $M_n(\mathbb{C})$, it follows that T is linear and multiplicative. Let $x \in M_n(\mathbb{C})$, $x \neq 0$, and let $c = \|x\|$. Observe that if u is any unitary matrix, then $\|u^*xu\| = c$. Then $\|Tx\| = \lim_{n\to\infty} \|T_n x\| = c > 0$, showing that $x \notin \ker T$. Since $M_n(\mathbb{C})$ is a finite dimensional vector space, surjectivity is automatic. We conclude that $T \in \mathrm{Aut}(M_n(\mathbb{C}))$. Hence $\mathrm{Aut}(M_n(\mathbb{C}))$ is closed. It follows that $\mathrm{Aut}(M_n(\mathbb{C}))$ is compact in the norm topology because the bounded linear operators on $M_n(\mathbb{C})$ form a finite dimensional normed vector space. □

Remark 2.8. The convex hull of *-automorphisms on $M_n(\mathbb{C})$ is a closed subset of $\mathcal{L}(M_n(\mathbb{C}))$. Indeed, $\mathrm{Aut}(M_n(\mathbb{C}))$ is compact by Proposition 2.7, and as $\mathcal{L}(M_n(\mathbb{C}))$ is a finite dimensional vector space, it follows by Proposition A.2 that $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$ is compact, and then in particular closed.

2.2.2 The Quantum Birkhoff Conjecture

Conjecture 1 (The Quantum Birkhoff Conjecture)**.** *Every completely positive, unital and* τ_n*-preserving linear map on* $M_n(\mathbb{C})$ *is a convex combination of* **-automorphisms on* $M_n(\mathbb{C})$*.*

Remark 2.9. We know that the set of completely positive, unital and τ_n-preserving linear maps on $M_n(\mathbb{C})$ is compact by Corollary 1.18, and hence equals the closed convex hull of its extreme points by the Krein-Milman theorem (see Theorem A.3). We also know that the convex hull of the set of *-automorphisms on $M_n(\mathbb{C})$ is closed. Moreover, it follows by Theorem 1.14 that the *-automorphisms are extreme points of UCPT_n. If there were an extreme point of UCPT_n that was not a *-automorphism, it would by definition not be a convex combination of *-automorphisms. The Quantum Birkhoff conjecture is, in other words, stating that the extreme points of UCPT_n are exactly the *-automorphisms.

Theorem 2.10 (The Quantum Birkhoff Conjecture in dimension 2)**.** *The Quantum Birkhoff Conjecture is true for* $n = 2$*.*

Proof. Let $T \in \mathrm{UCPT}_2$ be given by $Tx = \sum_{i=1}^d a_i^* x a_i$, for all $x \in M_2(\mathbb{C})$. By Remark 1.10, we may choose the matrices $a_1, a_2, \ldots, a_d \in M_2(\mathbb{C})$ to be linearly independent. We may therefore assume that $1 \leq d \leq 4$.

Case 1: $d = 3$ or 4. Suppose $d = 3$ and consider for an element $[\lambda_{ij}]_{i,j} \in M_3(\mathbb{C})$ the condition $\sum_{i,j=1}^3 \lambda_{ij} a_i^* a_j = 0$. This condition is satisfied if and only if $[\lambda_{ij}]_{i,j}$ is orthogonal to the subspace spanned by $\{[a_i^* a_j(k,\ell)]_{i,j} : 1 \leq k, \ell \leq 2\}$. Hence the condition determines a subspace of dimension greater than or equal to 5. Similarly, the condition $\sum_{i,j=1}^3 \lambda_{ij} a_j a_i^* = 0$ determines a subspace of $M_3(\mathbb{C})$ of dimension greater than or equal to 5. Now $M_3(\mathbb{C})$ is a vector space of dimension 9, so any two 5-dimensional subspaces hereof have non-trivial intersection. By Theorem 1.14 this implies that T is not extreme. The same argument holds for $d = 4$, only here $[\lambda_{ij}]_{i,j}$ is an element of a 16-dimensional vector space, and the two conditions $\sum_{i,j=1}^4 \lambda_{ij} a_i^* a_j = \sum_{i,j=1}^4 \lambda_{ij} a_j a_i^* = 0$ are true if and only if $[\lambda_{ij}]_{i,j}$ is in the intersection between two subspaces of dimension 12 (at least). Again, it is perfectly possible to find a non-trivial element of $M_4(\mathbb{C})$ satisfying this, and we therefore conclude by Theorem 1.14 that T is not extreme.

Case 2: $d = 2$. We have $Tx = a_1^* x a_1 + a_2^* x a_2$, for all $x \in M_2(\mathbb{C})$, where $a_1^* a_1 + a_2^* a_2 = a_1 a_1^* + a_2 a_2^* = \mathbf{1}_2$ as T is unital and τ_2-preserving. We aim to show that the set $\{a_i^* a_j : 1 \leq i, j \leq 2\}$ is not linearly independent. Let $a_1 = u_1|a_1|$ and $a_2 = u_2|a_2|$ be the polar decompositions of a_1 and a_2, respectively, and define $v := u_1^* u_2$. Then v is unitary and $a_2 = u_1 v|a_2|$. From the condition $a_1^* a_1 + a_2^* a_2 = \mathbf{1}_2$ we infer that $|a_1|^2 = \mathbf{1}_2 - a_2^* a_2$. Then

$$\mathbf{1}_2 = a_1 a_1^* + a_2 a_2^* = u_1 |a_1|^2 u_1^* + a_2 a_2^* = u_1(\mathbf{1}_2 - a_2^* a_2) u_1^* + a_2 a_2^* = \mathbf{1}_2 - u_1 a_2^* a_2 u_1^* + a_2 a_2^*,$$

and hence $u_1 a_2^* a_2 u_1^* = a_2 a_2^*$. Plugging in the polar decomposition of a_2, we obtain $u_1|a_2|^2 u_1^* = u_1 v|a_2|^2 v^* u_1^*$, and hence that $|a_2|^2 v = v|a_2|^2$. From the condition $\mathbf{1}_2 = |a_1|^2 + |a_2|^2$ we obtain that v commutes with $|a_1|^2$, as well. The same is true for v^*. Let $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be all the distinct eigenvalues of $|a_1|$. Since $|a_1|$ is positive, these are all real and non-negative. By the (finite dimensional) spectral theorem we may write $|a_1| = \sum_i \lambda_i p_i$, where p_i is the projection onto the eigenspace corresponding to the eigenvalue λ_i. Then $|a_1|^2 = \sum_i \lambda_i^2 p_i$, and since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all distinct and non-negative, so are $\lambda_1^2, \lambda_2^2, \ldots, \lambda_k^2$. Suppose x_j is an eigenvector of $|a_1|^2$ with eigenvalue λ_j^2. Then $p_i x_j = \delta_{ij} x_j$. Moreover, since v commutes with $|a_1|^2$, we have that $|a_1|^2 v x_j = v|a_1|^2 x_j = \lambda_j^2 v x_j$, and hence that $v x_j$ is an eigenvector of $|a_1|^2$ with eigenvalue λ_j^2, as well. From the eigenvalue decomposition of $|a_1|$, we conclude that x_j and $v x_j$ are both eigenvectors of $|a_1|$ with eigenvalue λ_j, and it follows that $|a_1| v x_j = \lambda_j v x_j = v|a_1| x_j$. Since $|a_1|$ is positive, and, in particular, then self-adjoint, the Hilbert space $\mathbb{C}^2$ is spanned by the eigenvectors of $|a_1|$. Hence, by linearity, $|a_1| v = v|a_1|$. The same derivation can be made for $|a_2|$. Furthermore, $|a_1|^2$ and $|a_2|^2$ commute. Indeed,

$$|a_1|^2 |a_2|^2 = (\mathbf{1}_2 - |a_2|^2)(\mathbf{1}_2 - |a_1|^2) = \mathbf{1}_2 + |a_2|^2 |a_1|^2 - |a_1|^2 - |a_2|^2 = |a_2|^2 |a_1|^2.$$

As $\sqrt{|a_1|^2}\sqrt{|a_2|^2} = \sqrt{|a_1|^2 |a_2|^2}$, it follows that $|a_1|$ and $|a_2|$ commute, as well. It is now straightforward to check that $a_1^* a_1, a_2^* a_2, a_1^* a_2$ and $a_2^* a_1$ all commute. As $M_2(\mathbb{C})$ is a four dimensional vector space, and it is possible to find 2×2-matrices that do not commute, it follows that $\{a_i^* a_j : 1 \leq i, j \leq 2\}$ is not a linearly independent set. Hence we can find a non-trivial element $[\lambda_{ij}] \in M_2(\mathbb{C})$ such that $\sum_{i,j=1}^{2} \lambda_{ij} a_i^* a_j = 0$. We conclude by Theorem 1.14 that T is not extreme.

Case 3: $d = 1$. Finally in the case where $Tx = a_1^* x a_1$, for all $x \in M_2(\mathbb{C})$, the conditions that T is unital and τ_2-preserving imply that a_1 is a unitary matrix. Hence T is a *-automorphism, by Theorem 2.6. Moreover, T is extreme in UCPT_2, by Theorem 1.14.

We conclude by the cases 1 and 2 that if T is not a *-automorphism, then T is not extreme, and by case 3 that if T is a *-automorphism, then T is extreme. Hence

$$\mathrm{Ext}(\mathrm{UCPT}_2) = \mathrm{Aut}(M_2(\mathbb{C})).$$

It follows by Remark 2.9 that the Quantum Birkhoff Conjecture is true for $n = 2$. □

A counterexample to the Quantum Birkhoff Conjecture for $n = 3$ was first provided by Burkhard Kümmerer in 1986, and counterexamples for all $n \geq 4$ were then provided by Hans Maassen and Burkhard Kümmerer in 1987. We shall not go into detail with these counterexamples here, but note that Example 4.2 also provide a counterexample for $n = 6$. Instead we proceed to the main topic of this thesis, namely the asymptotic version of the Quantum Birkhoff Conjecture.

2.3 The Asymptotic Quantum Birkhoff Conjecture

A map $T \in \mathrm{UCPT}_n$ is said to satisfy the *asymptotic quantum Birkhoff property* if

$$\lim_{k\to\infty} d_{\mathrm{cb}}\left(\bigotimes_{i=1}^{k} T,\ \mathrm{conv}\left(\mathrm{Aut}\left(\bigotimes_{i=1}^{k} M_n(\mathbb{C})\right)\right)\right) = 0. \tag{2.1}$$

Work of John A. Smolin, Frank Verstraete and Andreas Winter in [1, Conjecture 13] led them to formulate in 2005 the following restoration of the Quantum Birkhoff Conjecture in the asymptotic limit.

Conjecture 2. *Every completely positive, unital and τ_n-preserving linear map on $M_n(\mathbb{C})$ for $n \geq 3$ satisfies the asymptotic quantum Birkhoff property.*

This conjecture was listed on Reinhard Wernes web page of open problems in Quantum Information Theory [2].

In 2009, Christian Mendl and Michael Wolf exhibited in [11] a family of maps in UCPT_3 which are not in the convex hull of automorphisms on $M_3(\mathbb{C})$, but which exhibits the property that when tensored with themselves they are. This was thought of as evidence in support of the validity of the Asymptotic Quantum Birkhoff Conjecture.

A couple of years later, the conjecture was shown to fail in all dimensions greater than or equal to 3 by Uffe Haagerup and Magdalena Musat. We will discuss this in the remaining of the thesis.

2.4 Literature

The classical Birkhoff theorem is a theorem by Garrett Birkhoff [4]. The proof presented in this thesis is based on [12]. The proof that the Quantum Birkhoff holds in dimension 2 is based on [13, Theorem 2.3]. For counterexamples in all dimensions greater than or equal to 3 we refer to [5]. The original statement of the Asymptotic Quantum Birkhoff Conjecture can be found in [1, Conjecture 13].

Chapter 3

Von Neumann algebras

3.1 Preliminaries

The aim of this chapter is to give an overview of the prerequisites in the theory of von Neumann algebras necessary for this thesis. For a more thorough introduction to the subject, we refer to [14], and for deeper results to [15] and [16]. We shall follow the customary notation and use $B(H)$ to refer to the set of bounded linear maps on a Hilbert space H.

Definition 3.1. The *strong operator topology*, abbreviated as SOT, is the topology on $B(H)$ generated by the family of seminorms $\|\cdot\|_\xi$ for $\xi \in H$, where

$$\|x\|_\xi = \|x\xi\|, \qquad \text{for all } x \in B(H).$$

The *weak operator topology*, abbreviated as WOT, is the topology on $B(H)$ generated by the family of seminorms $\|\cdot\|_{\xi,\eta}$ for $\xi, \eta \in H$, where

$$\|x\|_{\xi,\eta} = |\langle x\xi\,, \eta\rangle|, \qquad \text{for all } x \in B(H).$$

Remark 3.1. Any topology generated by a family of seminorms is locally convex.

Remark 3.2. Let (x_α) be a net in $B(H)$. Then x_α converges in SOT to x if and only if $x_\alpha\xi$ converges to $x\xi$ in norm, for all $\xi \in H$. Respectively, x_α converges in WOT to x if and only if $x_\alpha\xi$ converges weakly to $x\xi$, for all $\xi \in H$, i.e., $\langle x_\alpha\xi\,, \eta\rangle$ converges to $\langle x\xi\,, \eta\rangle$, for all $\eta \in H$.

Definition 3.2. A *von Neumann algebra* is a C^*-subalgebra of $B(H)$ which contains the unit and which is closed in the strong operator topology.

Remark 3.3. One can show that if C is a convex subset of $B(H)$, then the WOT-closure of C coincides with the SOT-closure of C (see [14, Theorem 16.2]).

If H is any Hilbert space, then all of $B(H)$ is a von Neumann algebra. In particular, since $M_n(\mathbb{C})$ is isometrically isomorphic $B(\mathbb{C}^n)$, we see that the matrix algebra is a von Neumann algebra. Furthermore, if $\mathcal{A}$ is any unital C^*-subalgebra of $B(H)$, then the strong operator closure of $\mathcal{A}$ is a von Neumann algebra. It can be shown that any C^*-algebra can be embedded into $B(H)$, for some Hilbert space H. This can be achieved by using the GNS-construction (Theorem 3.18).

For two von Neumann algebras N and M acting on the Hilbert spaces H_N and H_M, respectively, let $N \odot M$ denote the algebraic tensor product of N and M. Then $N \odot M$ is a self-adjoint and unital subalgebra of $B(H_N \oplus H_M)$.

Definition 3.3. Let N and M be von Neumann algebras. The von Neumann algebra tensor product of N and M, denoted by $N \otimes M$, is defined to be the strong operator closure of the algebraic tensor product, i.e., $N \otimes M := \overline{N \odot M}^{SOT}$.

Definition 3.2 of a von Neumann algebra is topological. It turns out that one can equally well define von Neumann algebras algebraically. For a C^*-subalgebra $\mathcal{A}$ of $B(H)$, we denote by $\mathcal{A}'$ the commutant of $\mathcal{A}$, i.e., $\mathcal{A}' = \{x \in B(H) : xy = yx \ \forall y \in \mathcal{A}\}$.

Theorem 3.4 (Von Neumann's double commutant theorem)**.** *A C^*-subalgebra $\mathcal{A}$ of $B(H)$ is a von Neumann algebra if and only if $\mathcal{A} = \mathcal{A}''$.*

Proof. See [14, Theorem 18.6]. □

For two self-adjoint elements x, y in a von Neumann algebra N, we write $x \geq y$ if $x - y \in N_+$. This defines a partial order on N_{sa}, as one can easily check. Consider now the set of projections.

Definition 3.4. Let p and q be projections in a von Neumann algebra N. If there exists a projection $q_0 \in N$ such that $p \sim q_0 \leq q$, we write $p \preceq q$. If $p \preceq q$ and $p \not\sim q$, we write $p \prec q$.

One can show that this is a partial order on the set of equivalence classes of projections induced by the relation from Definition 2.3 (see [14, Theorem 25.3]). All von Neumann algebras occurring in this thesis will be *finite* in the sense defined below.

Definition 3.5. Let N be a von Neumann algebra. A projection $p \in N$ is said to be *finite* if $p \sim q \leq p$ implies that $p = q$. Otherwise p is said to be *infinite*. If the identity $\mathbf{1}_N$ of N is finite, we say that N is a *finite von Neumann algebra*.

Proposition 3.5. *Suppose that N is a finite von Neumann algebra. If $u^*u = \mathbf{1}_N$, for some element $u \in \mathbf{1}_N$, then u is a unitary element of N.*

Proof. We aim to show that $uu^* = \mathbf{1}_N$. Clearly uu^* is a self-adjoint element and $(uu^*)^2 = u(u^*u)u^* = uu^*$ by assumption. Hence uu^* is a projection in N and we therefore get that $uu^* \leq \mathbf{1}_N$. By definition of equivalence of projections. $uu^* \sim u^*u$. Now $\mathbf{1}_N \sim uu^* \leq \mathbf{1}_N$, and we conclude by finiteness of N that $uu^* = \mathbf{1}_N$. □

We now turn to consider properties of linear functionals on von Neumann algebras. Most of these can be defined in the more general setting of C^*-algebras.

Definition 3.6. Let $\mathcal{A}$ be a unital C^*-algebra. A linear functional φ on $\mathcal{A}$ is said to be

(a) *hermitian*, if $\varphi(x^*) = \overline{\varphi(x)}$, for all $x \in \mathcal{A}$,

(b) *positive*, if $\varphi(x) \geq 0$, for all $x \geq 0$,

(c) a *state*, if φ is positive and $\varphi(\mathbf{1}_\mathcal{A}) = 1$,

(d) a *trace*, if $\varphi(xy) = \varphi(yx)$, for all $x, y \in \mathcal{A}$,

(e) *faithful*, if $\varphi(x^*x) = 0$ if and only if $x = 0$.

Remark 3.6. One can show that positive linear functionals are automatically hermitian.

Proposition 3.7. *Let φ be a linear functional on the unital C^*-algebra $\mathcal{A}$. If φ is positive, then φ is bounded with $\|\varphi\| = \varphi(\mathbf{1}_\mathcal{A})$. In particular, if φ is a state, then $\|\varphi\| = 1$.*

Proof. Suppose that φ is a positive linear functional on the unital C^*-algebra $\mathcal{A}$. For an element $x \in \mathcal{A}_{sa}$ we have the inequalities $-\|x\| \mathbf{1}_\mathcal{A} \leq x \leq \|x\| \mathbf{1}_\mathcal{A}$, and since inequalities are preserved by positive linear functionals, it follows that

$$-\|x\| \varphi(\mathbf{1}_\mathcal{A}) \leq \varphi(x) \leq \|x\| \varphi(\mathbf{1}_\mathcal{A}). \tag{3.1}$$

Let now $x \in \mathcal{A}$ be any element. Define $x_1 = (x+x^*)/2$ and $x_2 = (x-x^*)/(2i)$. Clearly x_1 and x_2 are self-adjoint and $\|x_1\|, \|x_2\| \leq \|x\|$. We can easily derive using equation (3.1) together with the triangle inequality that $|\varphi(x)| \leq 2\|x\| \varphi(\mathbf{1}_{\mathcal{A}})$. This shows that φ is bounded. Given $\varepsilon > 0$, choose $x \in \mathcal{A}$ with $\|x\| = 1$ such that $\|\varphi\| - \varepsilon < |\varphi(x)|$. We may assume without loss of generality that $\varphi(x) \geq 0$ since, otherwise, we could just multiply x with a unimodulus scalar. Define x_1 and x_2 as before. Since φ is hermitian (by Remark 3.6) we have the identity $\varphi(x) = \overline{\varphi(x)} = \varphi(x^*)$, and it follows by linearity of φ that $\varphi(x_2) = 0$. We obtain that

$$\|\varphi\| - \varepsilon < \varphi(x) = \varphi(x_1) \leq \|x_1\| \varphi(\mathbf{1}_{\mathcal{A}}) \leq \varphi(\mathbf{1}_{\mathcal{A}}).$$

As this holds for any $\varepsilon > 0$, we conclude that $\|\varphi\| = \varphi(\mathbf{1}_{\mathcal{A}})$. If φ is a state, it follows directly that φ is bounded with $\|\varphi\| = 1$. □

Remark 3.8. The opposite implication also holds, i.e., if φ is bounded with $\|\varphi\| = \varphi(\mathbf{1}_{\mathcal{A}})$ then φ is positive, and if, further, $\|\varphi\| = 1$ then φ is a state (see [14, Theorem 13.5]).

Remark 3.9. One can prove the more general result that any positive linear map $\varphi : \mathcal{A} \to \mathcal{B}$ between unital C^*-algebras is bounded with $\|\varphi\| = \|\varphi(\mathbf{1}_{\mathcal{A}})\|$. The proof of this is more involved and we will omit it. We refer the interested reader to [17, Corollary 2.9]. Notice that this result also generalizes Theorem 1.17.

In this thesis we are only interested in von Neumann algebras which possess a normal faithful tracial state. The normality condition of a linear functional is specific to von Neumann algebras, and before defining it, we shall need a few prerequisites. Let N be a von Neumann algebra. We denote by $(x_i)_{i\in\Lambda}$, or simply by (x_i), a net in N. The net (x_i) is said to be bounded above (below) if there exists $x \in N$ such that $x_i \leq x$ ($x_i \geq x$), for all $i \in \Lambda$, and (x_i) is said to be increasing (decreasing) if $x_i \leq x_j$ ($x_i \geq x_j$) whenever $i \leq j$.

Theorem 3.10. *Let (x_i) be an increasing net of self-adjoint operators in N which is bounded above. Then (x_i) is strong operator convergent to a self-adjoint operator in N, and the strong limit is the least upper bound of (x_i).*

Proof. See [14, Theorem 17.1]. □

We denote by $\sup_i x_i$ the strong limit of an increasing net (x_i) bounded from above in N. Note that this limit is unique, as the strong operator topology is Hausdorff.

Definition 3.7. Let N be a von Neumann algebra and let φ be a bounded linear functional on N. Then φ is said to be *normal* if for any increasing net (x_i) of self-adjoint operators in N which is bounded from above, the net $(\varphi(x_i))$ in $\mathbb{C}$ converges to $\varphi(\sup_i x_i)$.

Definition 3.8. Let φ be a state on a unital C^*-algebra $\mathcal{A}$ acting on a Hilbert space H. We say that φ is a *vector state*, or that φ is *implemented by a vector* $\xi \in H$, if $\varphi(x) = \langle x\xi, \xi\rangle$ for all $x \in \mathcal{A}$. We shall sometimes use the notation φ_ξ to denote this state.

Lemma 3.11. *Let N be a von Neumann algebra acting on a Hilbert space H and let φ be a bounded linear functional on N. If φ is a vector state, then φ is normal.*

Proof. We show that vector states are strongly continuous, and it will follow from Theorem 3.10 that they are normal. Suppose that φ is a vector state on N, and take $\xi \in H$ such that $\varphi(x) = \langle x\xi, \xi\rangle$, for all $x \in N$. Let (x_i) be a net in N converging strongly to $x \in N$. We have by the Cauchy-Schwarz inequality that

$$|\varphi(x_i) - \varphi(x)| = |\langle (x_i - x)\xi, \xi\rangle| \leq \|(x_i - x)\xi\| \|\xi\|.$$

This shows strong continuity of φ. □

3.2 A characterization of tracial von Neumann algebras

Let $\mathcal{A} \subseteq B(H)$ be a unital C^*-algebra. We already have two ways of checking whether $\mathcal{A}$ is also a von Neumann algebra, namely through Definition 3.2 and Theorem 3.4. If $\mathcal{A}$ possesses a faithful tracial state, we shall establish a third useful criteria. But first we shall develop some prerequisites in this context. In particular, we shall introduce the *trace norm.*

Suppose that τ is a faithful tracial state on $\mathcal{A}$. Then τ induces an inner product and a norm on $\mathcal{A}$, referred to as the *trace norm*, defined as follows:

$$\langle x\,, y\rangle_\tau = \tau(y^*x), \qquad \|x\|_2 = \sqrt{\tau(x^*x)}. \tag{3.2}$$

It is straightforward to check that this is indeed an inner product, and hence that the induced norm is, indeed, a norm. Notice that the definiteness condition on the inner product is ensured by the requirement that τ is faithful. Moreover, it is clear that $\|x^*\|_2 = \|x\|_2$, for all $x \in \mathcal{A}$ since τ satisfies the trace property.

Lemma 3.12. *Let N be a von Neumann algebra with norm $\|\cdot\|$ and trace τ. The norm defined in equation (3.2) satisfies $\|xy\|_2 \le \|x\|\,\|y\|_2$, for all $x, y \in N$. In particular, if τ is a tracial state, then $\|x\|_2 \le \|x\|$, for all $x \in N$.*

Proof. Recall first that for any $x \in N_{sa}$ we have the inequality $x \le \|x\|\,\mathbf{1}_N$. Further for any $x, y \in N_{sa}$ and for any $z \in N$ we have that if $x \le y$ then $z^*xz \le z^*yz$. Let $x, y \in N$ be any elements. By the mentioned facts $x^*x \le \|x^*x\|\,\mathbf{1}_N = \|x\|^2\,\mathbf{1}_N$ and hence $y^*x^*xy \le \|x\|^2\,y^*y$. Since positive linear functionals preserve inequalities, and since a trace is a positive linear functional by definition, we obtain

$$\|xy\|_2^2 = \tau(y^*x^*xy) \le \|x\|^2\,\tau(y^*y) = \|x\|^2\,\|y\|_2^2\,.$$

The asserted inequality follows by taking the square root on both sides. If the trace is normalized, i.e., $\tau(\mathbf{1}_N) = 1$, we get that $\|x\|_2 = \|x\mathbf{1}_N\|_2 \le \|x\|\,\|\mathbf{1}_N\|_2 = \|x\|$ as claimed. □

Definition 3.9. Let $\mathcal{A}$ be a unital C^*-algebra. A C^*-homomorphism, φ, from $\mathcal{A}$ to $B(H)$ for some Hilbert space H is said to be *non-degenerate* if $\operatorname{span}\{\varphi(\mathcal{A})H\}$ is dense in H.

Remark 3.13. If $\mathcal{A} \subseteq B(H)$ we say that $\mathcal{A}$ acts non-degenerately on H if $\operatorname{span}\{\mathcal{A}H\}$ is dense in H. Observe that this is equivalent with $\mathbf{1}_\mathcal{A}$ being the identity operator on H. Indeed, if $\mathbf{1}_\mathcal{A}$ is the identity operator on H, then $H = \mathbf{1}_\mathcal{A}H \subseteq \operatorname{span}\{\mathcal{A}H\} \subseteq H$. Conversely if $\mathbf{1}_\mathcal{A}$ is not the identity operator on H, then it is a projection onto a subspace which is not all of H. Since all other projections are less than $\mathbf{1}_\mathcal{A}$, $\operatorname{span}\{\mathcal{A}H\}$ cannot be dense in H.

Definition 3.10. Let $\mathcal{A}$ be a unital C^*-algebra acting on a Hilbert space H, and let $\xi \in H$.

(a) We say that ξ is *separating* for $\mathcal{A}$ if the map $x \mapsto x\xi$ from $\mathcal{A}$ into H is injective.

(b) We say that ξ is *cyclic* for $\mathcal{A}$ if $\mathcal{A}\xi$ is dense in H.

Remark 3.14. It is an easy observation, that $\mathcal{A}$ necessarily acts non-degenerately on H in the presence of a cyclic vector.

Remark 3.15. If a faithful state φ is implemented by a vector $\xi \in H$, then ξ is necessarily separating. To see this, suppose that ξ is not separating. Let $\Phi : \mathcal{A} \to H$ be the map given by $\Phi(x) = x\xi$, for all $x \in \mathcal{A}$, and take $x_0 \in \ker(\Phi)$ such that $x_0 \neq 0$. Then $\varphi(x_0^*x_0) = \langle x_0\xi\,, x_0\xi\rangle = 0$. Hence ξ cannot implement a faithful state.

Proposition 3.16. *Let $\mathcal{A}$ be a C^*-algebra acting on a Hilbert space H and let ξ be a vector in H. If ξ is cyclic for $\mathcal{A}$ then ξ is separating for $\mathcal{A}'$. In particular, if $\mathcal{A}$ is commutative, then cyclic implies separating.*

Proof. Assume that ξ is cyclic and let $y \in \mathcal{A}'$. If $y\xi = 0$ then $yx\xi = xy\xi = 0$, for all $x \in \mathcal{A}$. Since $\mathcal{A}\xi$ is dense in H it follows that $y = 0$. Hence ξ is separating for $\mathcal{A}'$. If $\mathcal{A}$ is commutative we have the inclusion $\mathcal{A} \subseteq \mathcal{A}'$, and it follows immediately that ξ is also separating for $\mathcal{A}$. □

Suppose that $\mathcal{A}$ acts non-degenerately on a Hilbert space H and suppose that the faithful state τ is implemented by a unit vector $\xi_\tau \in H$. Then

$$\langle x\,, y\rangle_\tau = \langle x\xi_\tau\,, y\xi_\tau\rangle \text{ and } \|x\|_2 = \|x\xi_\tau\|\,, \text{ for all } x, y \in \mathcal{A}. \tag{3.3}$$

$$\|xa\xi_\tau\| \leq \|a\|\,\|x\xi_\tau\| = \|a\|\,\|x\|_2\,, \text{ for all } a, x \in \mathcal{A}. \tag{3.4}$$

We denote by $(\mathcal{A})_1$ the norm-closed unit ball of $\mathcal{A}$.

Theorem 3.17. *Let $\mathcal{A}$ be a unital C^*-algebra acting on a Hilbert space H. If τ is a faithful tracial state on $\mathcal{A}$ implemented by a cyclic vector $\xi_\tau \in H$, then the following are equivalent:*

(i) $(\mathcal{A})_1$ is complete with respect to $\|\cdot\|_2$,

(ii) $(\mathcal{A})_1$ is SOT-closed.

(iii) $\mathcal{A}$ is a von Neumann algebra,

Proof. For the proof of (ii)⇔(iii) we refer to [14, corollary 19.6].

(i)⇒(ii). Let x be in the SOT-closure of $(\mathcal{A})_1$ and let $(x_i)_{i\in\Lambda}$ be a net in $(\mathcal{A})_1$ converging to x in the strong operator topology. In particular $x_i\xi_\tau \to x\xi_\tau$ in norm. It follows by equation (3.3) that $(x_i)_{i\in\Lambda}$ is Cauchy with respect to $\|\cdot\|_2$, and by assumption (i) this implies that there exists a $y \in (\mathcal{A})_1$ such that $\|x_i - y\|_2 \to 0$. For each $a \in \mathcal{A}$, $\lim_i x_i a\xi_\tau = xa\xi_\tau$ by strong operator convergence of $(x_i)_{i\in\Lambda}$. Further, we infer by equation (3.4) that $\|x_i a\xi_\tau - ya\xi_\tau\| \leq \|a\|\,\|x_i - y\|_2$, and hence that $xa\xi_\tau = ya\xi_\tau$ for all $a \in \mathcal{A}$. We conclude that x and y are equal on the dense subset $\mathcal{A}\xi_\tau$ of H. They must therefore be equal on all of H by continuity. This implies that $x \in (\mathcal{A})_1$ and hence that $(\mathcal{A})_1$ is SOT-closed.

(ii)⇒(i). Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in $(\mathcal{A})_1$ with respect to $\|\cdot\|_2$. Then for any $a \in \mathcal{A}$ we have by equation (3.4) that $(x_n a\xi_\tau)$ is Cauchy in H, and hence convergent. Let η_a be the limit. Define a map $y_0 : \mathcal{A}\xi_\tau \to H$ by $y_0 a\xi_\tau = \eta_a$. This map is well-defined because ξ_τ is separating; we can very well have that $\eta_a \neq \eta_b$ for distinct elements $a, b \in \mathcal{A}$, so we need the separating property to secure that $a\xi_\tau \neq b\xi_\tau$ as well. By linearity of the limit $\eta_{a+b} = \eta_a + \eta_b$, hence y_0 is a linear map. Moreover, $\|\eta_a\| \leq \|\eta_a - x_n a\xi_\tau\| + \|x_n a\xi_\tau\|$ and we derive by a standard argument that $\|\eta_a\| \leq \|x_n a\xi_\tau\| \leq \|a\xi_\tau\|$. The latter inequality is by the definition of the operator norm. This holds for all $a \in \mathcal{A}$ and hence $\|y_0\| \leq 1$. Since ξ_τ is cyclic we can extend y_0 to a linear contraction $y \in B(H)$ by continuity.

We will show now that $(x_n)_{n\geq 1}$ converges strongly to y, and hence that $y \in \mathcal{A}$ by assumption (ii). Let $\eta \in H$ and let $\varepsilon > 0$. Because ξ_τ is cyclic we can find an element $a \in \mathcal{A}$ such that $\|\eta - a\xi_\tau\| < \varepsilon/3$. We know that $x_n a\xi_\tau \to \eta_a = y_0 a\xi_\tau = ya\xi_\tau$, so we can take $n_0 \geq 1$ such that $\|x_n a\xi_\tau - ya\xi_\tau\| < \varepsilon/3$, for all $n \geq n_0$. Then

$$\begin{aligned}\|y\eta - x_n\eta\| &\leq \|y\eta - ya\xi_\tau\| + \|ya\xi_\tau - x_n a\xi_\tau\| + \|x_n a\xi_\tau - x_n\eta\| \\ &\leq \|\eta - a\xi_\tau\| + \|ya\xi_\tau - x_n a\xi_\tau\| + \|a\xi_\tau - \eta\| < \varepsilon,\end{aligned}$$

where we have used that $\|y\| \leq 1$ and $\|x_n\| \leq 1$. This shows that $y \in \mathcal{A}$. Finally, as $x_n\xi_\tau \to \eta_{\mathbf{1}_\mathcal{A}} = y\xi_\tau$ it follows from equation (3.3) that $\|y - x_n\|_2 \to 0$. □

Theorem 3.18 (The GNS construction)**.** *Let $\mathcal{A}$ be a unital C^*-algebra and let φ be a state on $\mathcal{A}$. There exist a Hilbert space H_φ, a C^*-homomorphism π_φ from $\mathcal{A}$ onto a C^*-subalgebra of $B(H_\varphi)$ and a cyclic unit vector $\xi_\varphi \in H_\varphi$ such that $\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi\,, \xi_\varphi\rangle$, for all $a \in \mathcal{A}$. The triple $(H_\varphi, \pi_\varphi, \xi_\varphi)$ is called the GNS-representation of $\mathcal{A}$ with respect to φ.*

Proof. See [14, Theorem 14.4] or [15, Theorem 4.5.2]. □

Remark 3.19. Notice that it is a part of the above theorem that ξ_φ is cyclic for $\pi_\varphi(\mathcal{A})$.

Remark 3.20. If φ is faithful then π_φ is, moreover, a C^*-isomorphism. To see this, suppose $\pi_\varphi(a) = 0$. Then $\pi_\varphi(a^*a) = \pi_\varphi(a)^*\pi_\varphi(a) = 0$, from which it follows that $\varphi(a^*a) = 0$. If φ is faithful we infer that $a = 0$, showing that π_φ is injective.

Corollary 3.21 (to Theorem 3.17)**.** *Let $\mathcal{A}$ be a unital C^*-algebra with a faithful tracial state τ, and let $(H_\tau, \pi_\tau, \xi_\tau)$ be the GNS-representation of $\mathcal{A}$ with respect to τ.*

(i) If $(\mathcal{A})_1$ is complete with respect to $\|\cdot\|_2$ then $\pi_\tau(\mathcal{A})$ is a von Neumann algebra and τ is a normal state on $\mathcal{A}$.

(ii) If $\mathcal{A}$ is a von Neumann algebra and τ is a normal faithful tracial state, then $(\mathcal{A})_1$ is complete with respect to $\|\cdot\|_2$.

Proof. (i): Let $\tilde{\tau}$ be the state on $B(H_\tau)$ implemented by ξ_τ, i.e., $\tilde{\tau}(y) = \langle y\xi_\tau\,, \xi_\tau\rangle$, for all $y \in B(H)$. Then $\tilde{\tau}(\pi_\tau(x)) = \langle \pi_\tau(x)\xi_\tau\,, \xi_\tau\rangle = \tau(x)$, for all $x \in \mathcal{A}$. Clearly $\tilde{\tau}$ is a faithful tracial state on $\pi_\tau(\mathcal{A})$ because τ is faithful on $\mathcal{A}$. Moreover, $\pi_\tau : \mathcal{A} \to B(H_\tau)$ is an isometric isomorphism between the normed spaces $(\mathcal{A}, \|\cdot\|_{\tau,2})$ and $(\pi_\tau(\mathcal{A}), \|\cdot\|_{\tilde{\tau},2})$. Hence $((\mathcal{A})_1, \|\cdot\|_{\tau,2})$ is complete if and only if $((\pi_\tau(\mathcal{A}))_1, \|\cdot\|_{\tilde{\tau},2})$ is complete. Since $\xi_\tau \in H_\tau$ is cyclic for $\pi_\tau(\mathcal{A})$ it follows from Theorem 3.17 that $((\pi_\tau(\mathcal{A}))_1, \|\cdot\|_{\tilde{\tau},2})$ is complete if and only if $\pi_\tau(\mathcal{A})$ is a von Neumann algebra. In this case we get by Lemma 3.11 that $\tilde{\tau}$ is a normal state on $\pi_\tau(\mathcal{A})$. Because π_τ is a homomorphism, it follows that τ is a normal state as well.

(ii): It follows from [16, Proposition 7.1.15] that $\pi_\tau(\mathcal{A})$ is a von Neumann algebra. Let, as before, $\tilde{\tau}$ be the state on $\pi_\tau(\mathcal{A})$ implemented by the cyclic vector ξ_τ. This is a normal faithful tracial state because τ is. Hence, $(\pi_\tau(\mathcal{A}))_1$ is complete with respect to $\|\cdot\|_2$ by Theorem 3.17, and therefore so is $(\mathcal{A})_1$. □

3.3 Abelian von Neumann algebras

The function space $L_\infty([0,1])$ is an example of a von Neumann algebra possessing a normal faithful tracial state. This is shown in a slightly more general setting in Proposition 3.22.

Definition 3.11. Let (X, μ) be a measure space, and let f be a measurable function on X. We define the *essential supremum* of f by

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in X} |f(x)| = \inf\{a \geq 0 : \mu(\{x \in X : |f(x)| > a\}) = 0\}.$$

Proposition 3.22. *Let (X, μ) be a finite measure space. Then $L_\infty(X, \mu)$ is a von Neumann algebra, and if $\mu(X) = 1$ then the linear functional $\chi : L_\infty(X, \mu) \to \mathbb{C}$ given by $\chi(f) = \int_X f\,\mathrm{d}\mu$ is a normal faithful tracial state on $L_\infty(X, \mu)$.*

Proof. The space $L_\infty(X, \mu)$ is clearly a C^*-algebra equipped with the norm defined by the essential supremum. Consider the Hilbert space $L_2(X, \mu)$ with inner product $\langle f\,, g\rangle = \int_X f\,\overline{g}\,\mathrm{d}\mu$. Consider for a function $f \in L_\infty(X, \mu)$ multiplication operator M_f defined by

$$M_f g = fg, \qquad \text{for all } g \in L_2(X, \mu).$$

It is well-known from functional analysis, that this is a well-defined bounded linear operator on $L_2(X, \mu)$. Let $\mathcal{A} := \{M_f : f \in L_\infty(X, \mu)\}$. It is clear that $\mathcal{A} \subseteq B(L_2(X, \mu))$ is a self-adjoint subalgebra. Moreover, the map $L_\infty(X, \mu) \to \mathcal{A}$ given by $f \mapsto M_f$ is an isometric *-isomorphism onto $\mathcal{A}$. We may therefore identify $L_\infty(X, \mu)$ with $\mathcal{A}$ and the task is thus to show, that $\mathcal{A}$ is a von Neumann algebra. We show the stronger statement $\mathcal{A} = \mathcal{A}'$.

Since $\mathcal{A}$ is commutative we have $\mathcal{A} \subseteq \mathcal{A}'$. For the other inclusion, let $T \in \mathcal{A}'$, i.e., $TM_f = M_f T$, for all $f \in L_\infty(X, \mu)$. We need to find $h \in L_\infty(X, \mu)$ such that $T = M_h$.

Let $\Psi : L_2(X, \mu) \to \mathbb{C}$ be the linear functional given by $\Psi g = \int_X Tg \, \mathrm{d}\mu$, for all $g \in L_2(X, \mu)$. Since (X, μ) is a finite measure space, we have by Hölder's inequality that $\|g\|_1 \leq \sqrt{\mu(X)} \, \|g\|_2 < \infty$, for all $g \in L_2(X, \mu)$. Hence

$$|\Psi g| \leq \int_X |Tg| \, \mathrm{d}\mu = \|Tg\|_1 \leq \sqrt{\mu(X)} \, \|Tg\|_2 \leq \sqrt{\mu(X)} \, \|T\| \, \|g\|_2 < \infty.$$

So Ψ is bounded because T is bounded. By the Riesz representation theorem for Hilbert spaces [18, Theorem 5.25] we can find an element $h \in L_2(x, \mu)$ such that $\Psi g = \langle g \, , \overline{h} \rangle$, for all $g \in L_2(X, \mu)$. Then

$$\int_X (Tg - hg) \, \mathrm{d}\mu = \int_X (T - M_h) g \, \mathrm{d}\mu = 0, \qquad \text{for all } g \in L_2(X, \mu). \tag{3.5}$$

Recall that two μ-measurable functions h_1 and h_2 in $L_2(X, \mu)$ are considered equal if $\mu(\{h_1 \neq h_2\}) = 0$. The condition $\int_A (h_1 - h_2) \, \mathrm{d}\mu = 0$, for all $A \subseteq X$ measurable, is equivalent by [19, Corollary 10.14]. Fix for a moment $g \in L_2(X, \mu)$ and let $A \subseteq X$ be any measurable set. Then $g\mathbf{1}_A \in L_2(X, \mu)$, and since $\mathbf{1}_A \in L_\infty(X, \mu)$ and $T \in \mathcal{A}'$ we have $T(g\mathbf{1}_A) = TM_{\mathbf{1}_A} g = M_{\mathbf{1}_A} Tg = \mathbf{1}_A Tg$. From this it follows that

$$\int_A (Tg - hg) \, \mathrm{d}\mu = \int_X (\mathbf{1}_A Tg - \mathbf{1}_A hg) \, \mathrm{d}\mu = \int_X (T(g\mathbf{1}_A) - hg\mathbf{1}_A) \, \mathrm{d}\mu = 0,$$

where we use equation (3.5) in the last step. It follows that $Tg = hg = M_h g$, and since g was arbitrary we conclude that $T = M_h$. Since T is bounded and $\|h\|_\infty = \|M_h\|$ we infer that $h \in L_\infty(X, \mu)$. Hence $T \in \mathcal{A}$. We have now established that $\mathcal{A} = \mathcal{A}'$ which implies that $\mathcal{A} = \mathcal{A}''$, and we conclude by von Neumanns double commutant theorem (Theorem 3.4) that $\mathcal{A}$ is a von Neumann algebra.

Consider the map $\chi : L_\infty(X, \mu) \to \mathbb{C}$ given by $\chi(f) = \int_X f \, \mathrm{d}\mu$. This is a well-defined map since (X, μ) is a finite measure space; indeed $\left|\int_X f \, \mathrm{d}\mu\right| \leq \int_X |f| \, \mathrm{d}\mu \leq \mu(X) \, \|f\|_\infty$. Moreover, χ is clearly a linear functional, and since $L_\infty(X, \mu)$ is abelian, χ satisfies the trace property. Further, $\chi(\overline{f} f) = \int_X |f|^2 \, \mathrm{d}\mu = 0$ if and only if $f = 0$ μ-almost everywhere (see [19, Theorem 10.9]). This shows that χ is faithful. If we suppose that $\mu(X) = 1$, we have that $\|\chi\| = \chi(\mathbf{1}_X) = 1$, and hence that χ is a state by Remark 3.8. In particular, the boundedness of χ implies that χ is continuous. In this case we conclude that χ is a faithful tracial state. It remains to show that χ is normal. Suppose that (f_i) is an increasing net of self-adjoint operators in $L_\infty(X, \mu)$ which is bounded from above, and let $f = \sup_i f_i$. This supremum exists by Theorem 3.10. The net $(\chi(f_i))$ is increasing in $\mathbb{R}$ by linearity of χ, hence the limit equals $\sup_i \chi(f_i)$. It follows directly by continuity of χ that $\sup_i \chi(f_i) = \chi(\sup_i f_i)$, and we conclude that χ is normal. □

Theorem 3.23. *Every abelian von Neumann algebra acting on a separable Hilbert space is C^*-isomorphic to $L_\infty(K, \mu)$, where K is a compact Hausdorff space and μ is a finite positive Borel measure on K with* $\operatorname{supp} \mu = K$.

The proof of the above theorem can be found in [14, Theorem 22.6]. It is quite involved, and outside the scope of this thesis, and we shall therefore omit it here. We prove instead the following similar, but weaker, result.

Theorem 3.24. *Let N be an abelian von Neumann algebra acting on a Hilbert space H, and let $\xi \in H$ be a unit vector that is cyclic for N. There exists a compact Hausdorff space K, a positive finite Borel measure μ on K with* $\operatorname{supp} \mu = K$ *and a unitary map $U : H \to L_2(K, \mu)$ such that UNU^* is the algebra of multiplication operators on $L_2(K, \mu)$ by functions in $L_\infty(K, \mu)$.*

Proof. It is well known that $\widehat{N}$, the character space on N, is a compact Hausdorff space in the weak*-topology. Now N is in particular a commutative C^*-algebra, so the Gelfand transform $\Gamma : N \to C(\widehat{N})$ is a C^*-isomorphism. The map $f \mapsto \langle \Gamma^{-1}(f)\xi\,,\xi\rangle$ is a linear functional on $C(\widehat{N})$, and it is bounded with norm less than or equal to 1 because Γ is an isometry and ξ is a unit vector. By the Riesz representation theorem, there exists a unique Radon measure μ on $\widehat{N}$ such that

$$\langle \Gamma^{-1}(f)\xi\,,\xi\rangle = \int_{\widehat{N}} f \,\mathrm{d}\mu, \qquad \text{for all } f \in C(\widehat{N}).$$

When $f \in C(\widehat{N})$ is positive then so is $\Gamma^{-1}(f)$. This follows since Γ^{-1} is a C^*-homomorphism and from the definition of positivity. In this case the inner product $\langle \Gamma^{-1}(f)\xi\,,\xi\rangle$ is clearly positive. It follows that μ is a positive measure.

Suppose now that the support of μ is not all of $\widehat{N}$. Then we can find a non-empty open set $U \subseteq \widehat{N}$ with $\mu(U) = 0$. By Urysohn's lemma, [18, Theorem 4.32], we can find a non-zero positive continuous function which is zero outside of a compact subset of U. Then $\left\|\Gamma^{-1}(\sqrt{f})\xi\right\|^2 = \int_{\widehat{N}} f \,\mathrm{d}\mu = 0$ implying that $\Gamma^{-1}(\sqrt{f})\xi = 0$. Since ξ is cyclic and N is commutative, ξ is separating in N by Proposition 3.16. Hence $\Gamma^{-1}(\sqrt{f}) = 0$ and we arrive at the contradiction $\sqrt{f} = 0$ by injectivity of Γ^{-1}. We conclude hereby that $\operatorname{supp}\mu = \widehat{N}$.

Consider the bijective map from $N\xi$ to $C(\widehat{N})$ given by $x\xi \mapsto \Gamma(x)$, for all $x \in N$. This map is an isometry. Indeed,

$$\int_{\widehat{N}} |\Gamma(x)|^2 \,\mathrm{d}\mu = \langle \Gamma^{-1}(\Gamma(x^*)\Gamma(x))\xi\,,\xi\rangle = \|x\xi\|^2 .$$

Since $N\xi$ is dense in H, we can extend by continuity to a unitary map, U, from H onto $L_2(\widehat{N}, \mu)$. For any $x \in N$, UxU^* is an operator on $L_2(K, \mu)$, and for any $y \in N$ we have

$$UxU^*(\Gamma(y)) = Uxy\xi = \Gamma(xy) = \Gamma(x)\Gamma(y) = M_{\Gamma(x)}\Gamma(y).$$

Here M_f is the multiplication operator by f. Since $\Gamma(N) = C(K)$ is dense in $L_2(K, \mu)$ (see [18, Proposition 7.9]) we conclude that $UxU^* = M_{\Gamma(x)} \in \{M_f : f \in C(K)\}$. The map $x \mapsto UxU^*$ is clearly a C^*-isomorphism between N and UNU^*. Moreover, it follows easily from the identity $\|UxU^*f\| = \|x(U^*f)\|$ together with the bijectivity of U that the map $x \mapsto UxU^*$ is a SOT homeomorphism between $B(H)$ and $B(L_2(K, \mu))$. Hence, UNU^* is a von Neumann algebra. It is exactly the SOT closure of $\{M_f : f \in C(K)\}$ in $B(L_2(K, \mu))$. As $C(K)$ is dense in $L_\infty(K, \mu)$ with respect to $\|\cdot\|_\infty$, it is straightforward to show that the SOT closure is the set $\{M_f : f \in L_\infty(K, \mu)\}$. This concludes the proof. □

3.4 Ultraproducts of tracial von Neumann algebras

An introduction to ultrafilters and ultraproducts is offered in Appendix **??**. We shall in this section develop an ultraproduct construction for families of von Neumann algebras possessing normal faithful tracial states.

Let $(M_\alpha)_{\alpha\in\Lambda}$ be a family of von Neumann algebras each possessing a normal faithful tracial state. We denote by $\mathbf{1}_\alpha$ the identity element of M_α and by τ_α the tracial state of M_α. Recall from equation (3.2) that a faithful trace induces an inner product and a norm. We denote the inner product on M_α defined through τ_α by $\langle\,\cdot\,,\,\cdot\,\rangle_\alpha$, and with a slight abuse of notation, we shall omit the subscript and denote the norm induced by this inner product by $\|\cdot\|_2$. We shall also omit the subscript and denote the operator norm on M_α by $\|\cdot\|$.

Just as in the Banach space ultraproduct construction we shall consider the space

$$\ell_\infty(\Lambda, M_\alpha) := \left\{(x_\alpha)_{\alpha\in\Lambda} : x_\alpha \in M_\alpha, \left\|(x_\alpha)_{\alpha\in\Lambda}\right\|_\infty < \infty\right\}. \tag{3.6}$$

Let $\mathcal{U}$ be an ultrafilter on the index set Λ and define

$$I_{\mathcal{U}} := \left\{ (x_\alpha)_{\alpha\in\Lambda} \in \ell_\infty(\Lambda, M_\alpha) : \lim_{\mathcal{U}} \|x_\alpha\|_2 = 0 \right\}. \tag{3.7}$$

Remark 3.25. $I_{\mathcal{U}}$ is a closed subspace of $\ell_\infty(\Lambda, M_\alpha)$ with respect to the norm $\|\cdot\|_\infty$. This makes $\ell_\infty(\Lambda, M_\alpha)/I_{\mathcal{U}}$ into a Banach space with the quotient norm $\|[\,\cdot\,]\|_{\mathcal{U}}$ (see Proposition B.16). The quotient norm is given by

$$\left\|[(x_\alpha)_{\alpha\in\Lambda}]\right\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_\alpha\|, \qquad \text{for all } [(x_\alpha)_{\alpha\in\Lambda}] \in \ell_\infty(\Lambda, M_\alpha)/I_{\mathcal{U}}.$$

Moreover, $I_{\mathcal{U}}$ is a two-sided ideal in $\ell_\infty(\Lambda, X_\alpha)$. Hence the multiplication on $\ell_\infty(\Lambda, X_\alpha)/I_{\mathcal{U}}$ given by $[(x_i)_{i\in I}][(y_i)_{i\in I}] := [(x_i y_i)_{i\in I}]$ is well-defined. We can also, just as in the construction of the Banach space ultraproduct, define an involution and show that $\ell_\infty(\Lambda, M_\alpha)/I_{\mathcal{U}}$ is a C^*-algebra. The proofs of all these facts are, with the use of Lemma 3.12, completely analogous to the proofs of the similar statements in the construction of the Banach space ultraproduct. We shall therefore omit them here and refer to Section B.2.2 for details.

Definition 3.12. Let $(M_\alpha)_{\alpha\in\Lambda}$ be a family of von Neumann algebras each possessing a normal faithful tracial state. The quotient space $\ell_\infty(\Lambda, M_\alpha)/I_{\mathcal{U}}$ is called the *tracial ultraproduct* of $(M_\alpha)_{\alpha\in\Lambda}$ with respect to $\mathcal{U}$, and it is denoted by $\prod_{\mathcal{U}} M_\alpha$.

Remark 3.26. Note that this definition makes sense without the assumption of τ_α being normal for each $\alpha \in \Lambda$. In fact, it is meaningful to define the tracial ultraproduct of a family of unital C^*-algebras each possessing a faithful tracial state. When the index set, Λ, is countable, one can even show that the tracial ultraproduct of such a family of C^*-algebras is a von Neumann algebra (see Remark 3.30).

As already remarked $\prod_{\mathcal{U}} M_\alpha$ is a C^*-algebra, and it is clearly unital with identity $[(\mathbf{1}_\alpha)_{\alpha\in\Lambda}]$. Our aim is now to show that the tracial ultraproduct possesses a faithful tracial state, and then to use Corollary 3.21.

Define the linear functional $f_{\mathcal{U}} \in \ell_\infty(\Lambda, M_\alpha)^*$ by

$$f_{\mathcal{U}}(x) = \lim_{\mathcal{U}} \tau_\alpha(x_\alpha), \qquad \text{for all } x = (x_\alpha)_{\alpha\in\Lambda} \in \ell_\infty(\Lambda, M_\alpha).$$

Let $q : \ell_\infty(\Lambda, M_\alpha) \to \ell_\infty(\Lambda, M_\alpha)/I_{\mathcal{U}}$ be the quotient map. The linear functional $f_{\mathcal{U}}$ induces a linear functional $\tau_{\mathcal{U}}$ on $\ell_\infty(\Lambda, M_\alpha)/I_{\mathcal{U}}$ satisfying

$$\tau_{\mathcal{U}}(q(x)) = f_{\mathcal{U}}(x), \qquad \text{for all } x = (x_\alpha)_{\alpha\in\Lambda} \in \ell_\infty(\Lambda, M_\alpha). \tag{3.8}$$

Proposition 3.27. *The linear functional $\tau_{\mathcal{U}}$ defined in equation (3.8) is a faithful tracial state on $\ell_\infty(\Lambda, M_\alpha)/I_{\mathcal{U}}$.*

Proof. It is clear that $f_{\mathcal{U}}$ satisfies the trace property, because τ_α does for all α and because $\mathbb{C}$ is Hausdorff. Therefore so does $\tau_{\mathcal{U}}$. Moreover, since $\tau_\alpha(\mathbf{1}_\alpha) = 1$, for all α it is clear that $\tau_{\mathcal{U}}([(\mathbf{1}_\alpha)_{\alpha\in\Lambda}]) = \lim_{\mathcal{U}} \tau_\alpha(\mathbf{1}_\alpha) = 1$. Hence $\tau_{\mathcal{U}}$ is a tracial state.

Suppose that $\tau_{\mathcal{U}}([(x_\alpha^*)_{\alpha\in\Lambda}][(x_\alpha)_{\alpha\in\Lambda}]) = \tau_{\mathcal{U}}([(x_\alpha^* x_\alpha)_{\alpha\in\Lambda}]) = 0$. By definition of $\tau_{\mathcal{U}}$ this means that $\lim_{\mathcal{U}} \tau_\alpha(x_\alpha^* x_\alpha) = 0$. It follows directly from Proposition B.13 that $\lim_{\mathcal{U}} \|x_\alpha^* x_\alpha\|_2 = \lim_{\mathcal{U}} \sqrt{\tau_\alpha(x_\alpha^* x_\alpha)} = \sqrt{\lim_{\mathcal{U}} \tau_\alpha(x_\alpha^* x_\alpha)} = 0$. We conclude that $[(x_\alpha^* x_\alpha)_{\alpha\in\Lambda}] = [0]$ and hereby that $\tau_{\mathcal{U}}$ is faithful. $\square$

Before we proceed to proof that the construction in Definition 3.12 is a von Neumann algebra, we need the following result which is taken from [20, 2.2.10].

Proposition 3.28. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a surjective C^*-homomorphism.*

(i) For every self-adjoint element $b \in \mathcal{B}$ there exists a self-adjoint element $a \in \mathcal{A}$ such that $\varphi(a) = b$ and $\|a\| = \|b\|$.

(ii) For every element $b \in \mathcal{B}$ there exists an element $a \in \mathcal{A}$ such that $\varphi(a) = b$ and $\|a\| = \|b\|$.

Proof. (i): Let $x \in \mathcal{A}$ be any element such that $\varphi(x) = b$, and set $a_0 = (x + x^*)/2$. Then a_0 is self-adjoint, and since b is self-adjoint and φ is a C^*-homomorphism, $\varphi(a_0) = b$. Define a continuous function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} -\|b\|, & \text{if } t \leq -\|b\| \\ t, & \text{if } -\|b\| \leq t \leq \|b\| \\ \|b\|, & \text{if } t \geq \|b\| \end{cases}.$$

Then $a = f(a_0)$ is a self-adjoint element of $\mathcal{A}$ by the continuous functional calculus, [14, Theorem 10.3]. Since $f(t) = t$ for all t in the spectrum of b, we get

$$\varphi(a) = \varphi(f(a_0)) = f(\varphi(a_0)) = f(b) = b.$$

Moreover, the spectrum of a, $\sigma(a)$, equals $f(\sigma(a_0))$ which is included in the interval $[-\|b\|, \|b\|]$. We infer that $\|a\| \leq \|b\|$ by the spectral radius formula for normal elements, [14, Theorem 8.1]. All C^*-homomorphisms are contractive by [14, Corollary 9.2], so as φ is a C^*-homomorphism, we conclude that $\|a\| = \|b\|$.

(ii): Let b_0 be the self-adjoint element in the C^*-algebra $M_2(\mathcal{B})$ given by

$$b_0 = \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}.$$

Observe that $b_0^* b_0 = bb^* \oplus b^*b$. It then follows from Proposition 1.2 and Remark 1.3 that $\|b_0\|^2 = \|b_0^* b_0\| = \max\{\|bb^*\|, \|b^*b\|\} = \|b\|^2$. By part (i) we can find an element $a_0 \in M_2(\mathcal{A})$ such that $\|a_0\| = \|b_0\|$. Let $a = a_0(1,2)$. Then $\varphi(a) = b$ and $\|a\| = \|b\|$. □

Theorem 3.29. *Let $(M_\alpha)_{\alpha\in\Lambda}$ be a family of von Neumann algebras with normal faithful tracial states $(\tau_\alpha)_{\alpha\in\Lambda}$, and let $\mathcal{U}$ be an ultrafilter on Λ. Then the tracial ultraproduct $\prod_{\mathcal{U}} M_\alpha$ is a von Neumann algebra with a normal faithful tracial state.*

Proof. By Corollary 3.21(i) it suffices to show that the closed unit ball of $\prod_{\mathcal{U}} M_\alpha$ is complete in the trace norm. So let $(x_n)_{n\geq 1}$ be a sequence in the closed unit ball, i.e., $\|x_n\| = \lim_{\mathcal{U}} \|x_n^{(\alpha)}\| \leq 1$, for all $n \geq 1$. For each $n \geq 1$ we have $x_n = \left[(x_n^{(\alpha)})_{\alpha\in\Lambda}\right]$. Since the quotient map is a surjective C^*-homomorphism, Proposition 3.28 allows us to choose the representative of x_n such that $\|x_n^{(\alpha)}\| \leq 1$, for all $\alpha \in \Lambda$. Suppose $(x_n)_{n\geq 1}$ is Cauchy with respect to the trace norm. We may assume that $\|x_n - x_{n+1}\|_2 < 2^{-n}$, for all $n \geq 1$; every Cauchy sequence has a subsequence satisfying this, and if this subsequence converges, the original sequence converges to the same point.

Consider for each $n \geq 1$ the set

$$F_n := \left\{\alpha \in \Lambda : \|x_k^{(\alpha)} - x_{k+1}^{(\alpha)}\|_2 < 2^{-k}, k = 1, 2, \ldots, n\right\}.$$

By assumption $F_n \in \mathcal{U}$, for all $n \geq 1$. Let $F = \bigcap_{k=0}^{\infty} F_k$. It is clear that we have the inclusions

$$\Lambda \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_n \supseteq \cdots \supseteq F.$$

Each $\alpha \in \Lambda$ is then either in $\Lambda \backslash F_1$, in $F_k \backslash F_{k+1}$ for some $k \geq 1$ or in F. If $\alpha \in \Lambda \backslash F_1$, set $x^{(\alpha)} = x_1^{(\alpha)}$, and if $\alpha \in F_k \backslash F_{k+1}$ for some $k \geq 1$, set $x^{(\alpha)} = x_k^{(\alpha)}$. If $\alpha \in F$ we have

$\|x_n^{(\alpha)} - x_{n+1}^{(\alpha)}\|_2 < 2^{-n}$, for all $n \geq 1$, and thus that $\big(x_n^{(\alpha)}\big)_{n\geq 1}$ is a Cauchy sequence in $(M_\alpha)_1$. Then $\big(x_n^{(\alpha)}\big)_{n\geq 1}$ is convergent by Corollary 3.21(ii), because τ_α is assumed to be normal. Let $x^{(\alpha)} \in (M_\alpha)_1$ be the limit point. By construction, $\|x^{(\alpha)}\| \leq 1$, for all $\alpha \in \Lambda$, hence $x = [(x^{(\alpha)})]$ lies in the closed unit ball. This is our candidate for the limit point of $(x_n)_{n\geq 1}$.

Fix for a moment $n \geq 1$ and let $\alpha \in F_n$. In the case where $\alpha \in F$ we have

$$\|x_n^{(\alpha)} - x^{(\alpha)}\|_2 = \Big\|\sum_{j=n}^{\infty}(x_j^{(\alpha)} - x_{j+1}^{(\alpha)})\Big\|_2 \leq \sum_{j=n}^{\infty}\|x_j^{(\alpha)} - x_{j+1}^{(\alpha)}\|_2 < \sum_{j=n}^{\infty} 2^{-j} = 2^{1-n}.$$

In the case where $\alpha \notin F$ there is a $k \geq n$ such that $\alpha \in F_k \backslash F_{k+1}$. If $k > n$ then

$$\|x_n^{(\alpha)} - x^{(\alpha)}\|_2 = \|x_n^{(\alpha)} - x_k^{(\alpha)}\|_2 = \Big\|\sum_{j=n}^{k-1}(x_j^{(\alpha)} - x_{j+1}^{(\alpha)})\Big\|_2 \leq \sum_{j=n}^{k-1}\|x_j^{(\alpha)} - x_{j+1}^{(\alpha)}\|_2 < \sum_{j=n}^{k-1} 2^{-j} < 2^{1-n},$$

and if $k = n$ then $\|x_n^{(\alpha)} - x^{(\alpha)}\|_2 = \|x_n^{(\alpha)} - x_n^{(\alpha)}\|_2 = 0$.

Given $\varepsilon > 0$ choose $n_\varepsilon \geq 1$ such that $2^{1-n} < \varepsilon$, for all $n \geq n_\varepsilon$. For each $n \geq n_\varepsilon$ let A_n be the set of indices in Λ such that $\|x_n^{(\alpha)} - x^{(\alpha)}\|_2 < \varepsilon$. By the above estimates we see that $F_n \subseteq A_n$ and hence that $A_n \in \mathcal{U}$, for all $n \geq n_\varepsilon$. It follows that

$$\|x_n - x\|_2 = \lim_{\mathcal{U}}\|x_n^{(\alpha)} - x^{(\alpha)}\|_2 < \varepsilon,$$

for all $n \geq n_\varepsilon$. This shows that $\|x_n - x\|_2 \to 0$, as $n \to \infty$. Hence the closed unit ball of $\prod_{\mathcal{U}} M_\alpha$ is complete in the trace norm, and we conclude by Corollary 3.21(i) that $\prod_{\mathcal{U}} M_\alpha$ is isometrically isomorphic to a von Neumann algebra and $\tau_{\mathcal{U}}$ is normal. □

Remark 3.30. Suppose that Λ is a countable set, and that $(\mathcal{A}_\alpha)_{\alpha\in\Lambda}$ is a family of C^*-algebras each possessing a faithful tracial state. Let $(A_n)_{n\geq 1}$ be a family of finite sets subsets of Λ such that $A_n \subseteq A_m$ whenever $n \leq m$, and such that $\cup_{n=1}^{\infty} A_n = \Lambda$. For each $n \in \mathbb{N}$, replace the set F_n in the proof of Theorem 3.29 by $F_n \backslash A_n$. The set $F_n \backslash A_n$ is still in $\mathcal{U}$ because A_n is finite and $\mathcal{U}$ is an ultrafilter. This construction ensures that F is empty. The assumption that we had a family of von Neumann algebras can now be dropped, as it were only needed when dealing with the indices in F.

3.5 Literature

Section 3.1 and 3.3 are based on [14]. Section 3.2 and 3.4 are mainly based on discussions with my supervisor Magdalena Musat.

Chapter 4

Factorizable maps

The notion of factorizable completely positive maps was introduced by Claire Anantharaman-Delaroche in [7, Definition 6.2] in the setting of von Neumann algebras. For the purpose of this thesis, we only consider factorizability of maps in UCPT_n.

4.1 Factorizability of completely positive maps on $M_n(\mathbb{C})$

Following Uffe Haagerup and Magdalena Musat [8, Definition 3.1] we introduce the following definition of *exact factorization* for maps in UCPT_n.

Definition 4.1. We say that a map $T \in \mathrm{UCPT}_n$ admits an *exact factorization* through $M_n(N)$, for some von Neumann algebra N with a normal faithful tracial state τ_N if there exist a unitary $u \in M_n(\mathbb{C}) \otimes N$ such that

$$Tx = (\iota_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes \mathbf{1}_N)u), \qquad \text{for all } x \in M_n(\mathbb{C}). \tag{4.1}$$

Remark 4.1. Suppose that $T \in \mathrm{UCPT}_n$ admits an exact factorization through $M_n(N)$ for some von Neumann algebra N with a normal faithful tracial state τ_N. If M is another von Neumann algebra with a normal faithful tracial state τ_M, then T admits an exact factorization through $M_n(N \otimes M)$ as well. Indeed, since T admits an exact factorization through $M_n(N)$, there exists a unitary u in $M_n(N)$ such that

$$Tx = (\iota_n \otimes \tau_N)(u^*(x \otimes \mathbf{1}_N)u), \qquad \text{for all } x \in M_n(\mathbb{C}),$$

Define $w = u \otimes \mathbf{1}_M$, which is a unitary in $M_n(N \otimes M)$ and

$$\begin{aligned}(\iota_n \otimes \tau_{N\otimes M})(w^*(x \otimes \mathbf{1}_{N\otimes M})w) &= (\iota_n \otimes \tau_N \otimes \tau_M)(u^* \otimes \mathbf{1}_M(x \otimes \mathbf{1}_N \otimes \mathbf{1}_M)u \otimes \mathbf{1}_M) \\ &= \tau_M(\mathbf{1}_M)(\iota_n \otimes \tau_N)(u^*(x \otimes \mathbf{1}_N)u) = Tx,\end{aligned}$$

for all $x \in M_n(\mathbb{C})$. The assertion follows.

Remark 4.2. By [3, Theorem 2.2] it turns out that for a map in UCPT_n, having an exact factorization through $M_n(N)$ as in the above definition, where N is a finite von Neumann algebra, is equivalent to being factorizable in the sense of the original definition of Claire Anantharaman-Delaroche.

We denote the set of factorizable maps on $M_n(\mathbb{C})$ by $\mathfrak{F}(M_n(\mathbb{C}))$ and sometimes by $\mathfrak{F}_n$.

Recall that by Choi's result (Theorem 1.8) any map $T \in \mathrm{UCPT}_n$ can be written as $Tx = \sum_{i=1}^d a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$, where $a_1, a_2, \ldots, a_d \in M_n(\mathbb{C})$ satisfy $\sum_{i=1}^d a_i^* a_i = \sum_{i=1}^d a_i a_i^* = \mathbf{1}_n$, and can be chosen to be linearly independent. The following is a standard result. We include the proof for completeness.

Lemma 4.3. *The map $\sum_{i=1}^r a_i \otimes b_i \mapsto \sum_{i=1}^r L_{a_i} \otimes R_{b_i}$ is a vector space isomorphism of $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ onto $\mathcal{L}(M_n(\mathbb{C}))$.*

Proof. It is clear that this map is linear. We show that it is a bijection. Let $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map on $M_n(\mathbb{C})$. The Choi matrix associated to T is given by

$$C_T = \iota_n \otimes T([e_{ij}]_{i,j}) = [T(e_{ij})]_{i,j} = \sum_{i,j=1}^{n} e_{ij} \otimes T(e_{ij}),$$

where $\{e_{ij}\}$ denote, as before, the standard matrix units in $M_n(\mathbb{C})$. Let $C_T = USV$ be the singular value decomposition of C_T. Note that U and V are unitary $n^2 \times n^2$ matrices and S is a diagonal $n^2 \times n^2$ matrix with non-negative entries. Let $v_1, v_2, \ldots, v_{n^2}$ be the rows of V, and let $u_1, u_2, \ldots, u_{n^2}$ be the columns of U. Then

$$C_T = \sum_{i=1}^{n^2} s_i u_i v_i^*.$$

For each $1 \leq i \leq n^2$, we can view the $n^2 \times 1$ matrices u_i and v_i as $n \times 1$ block matrices with $n \times 1$ matrices as entries. Let $\widetilde{A}_i$ be the $n \times n$ matrix built from u_i by using the blocks as columns, and let $\widetilde{B}_i$ be the $n \times n$-matrix built from v_i. Then $u_i v_i^* = [\widetilde{A}_i e_{jk} \widetilde{B}_i^*]_{j,k}$. Let $A_i = \sqrt{s_i}\widetilde{A}_i$ and $B_i = \sqrt{s_i}\widetilde{B}_i$. We get the following expression for the Choi matrix C_T of T

$$C_T = \left[\sum_{i=1}^{n^2} A_i e_{jk} B_i^* \right]_{j,k}.$$

We infer that $T(e_{jk}) = \sum_{i=1}^{n^2} A_i e_{jk} B_i^*$, for all $1 \leq j, k \leq n$. By linearity, $Tx = \sum_{i=1}^{n^2} A_i x B_i^*$, for $x \in M_n(\mathbb{C})$. This shows surjectivity of the considered map.

To show injectivity, let $\sum_{k=1}^r a_k \otimes b_k$ and $\sum_{l=1}^s c_l \otimes d_l$ be elements of $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ for some $r, s \in \mathbb{N}$, and suppose that

$$\sum_{k=1}^{r} a_k \otimes b_k \neq \sum_{l=1}^{s} c_l \otimes d_l.$$

Then we can pick an entry where these two $n^2 \times n^2$ matrices differ. Suppose, e.g., that they differ in the $(n(i-1)+p, n(j-1)+q)$'th entry, i.e.,

$$\sum_{k=1}^{r} a_k(i,j) b_k(p,q) \neq \sum_{l=1}^{s} c_l(i,j) d_l(p,q).$$

The left-hand side of the above is exactly the (i,q)'th entry of the matrix $\sum_{k=1}^r a_k e_{jp} b_k$, while the right-hand side is the (i,q)'th entry of $\sum_{l=1}^s c_l e_{jp} d_l$. Thus the maps $\sum_{k=1}^r L_{a_k} \otimes R_{b_k}$ and $\sum_{l=1}^s L_{c_l} \otimes R_{d_l}$ disagree on e_{jp}, and therefore they cannot be equal. □

The following result [3, Theorem 2.2] gives a characterization of factorizability for maps in UCPT_n which will be useful for establishing a criteria for non-factorizability.

Theorem 4.4. *Let T be the completely positive unital and τ_n-preserving map given by $Tx = \sum_{i=1}^d a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$, where $a_1, a_2, \ldots, a_d \in M_n(\mathbb{C})$ are chosen to be linearly independent. Then T is factorizable if and only if there exists a finite von Neumann algebra equipped with a normal faithful tracial state τ_N and operators $v_1, v_2, \ldots, v_d \in N$ such that $u := \sum_{i=1}^d a_i \otimes v_i$ is a unitary operator on $M_n(N)$ and $\tau_N(v_i^* v_j) = \delta_{ij}$, $1 \leq i, j \leq d$.*

Proof. Assume that T is factorizable. Then T has an exact factorization through a finite von Neumann algebra N with a normal faithful tracial state τ_N. Hence, there exists a unitary u in $M_n(N)$ such that equation (4.1) holds. We can extend the linearly independent set $\{a_i : 1 \leq i \leq d\}$ to a basis $\{a_i : 1 \leq i \leq n^2\}$ for $M_n(\mathbb{C})$. Then u has a representation of the form $\sum_{i=1}^{n^2} a_i \otimes v_i$, for some $v_1, v_2, \dots, v_{n^2} \in N$. The idea is now to show that $v_i = 0$ for $i > d$, and it will then follow that $u = \sum_{i=1}^{d} a_i \otimes v_i$ is unitary. For all $x \in M_n(\mathbb{C})$,

$$\begin{aligned} Tx = \sum_{i=1}^{d} a_i^* x a_i &= (\iota_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes \mathbf{1}_N)u) \\ &= (\iota_n \otimes \tau_N)\left(\sum_{i,j=1}^{n^2} (a_i^* \otimes v_i^*)(x \otimes \mathbf{1}_N)(a_j \otimes v_j) \right) \\ &= (\iota_n \otimes \tau_N)\left(\sum_{i,j=1}^{n^2} a_i^* x a_j \otimes v_i^* v_j \right) \\ &= \sum_{i,j=1}^{n^2} a_i^* x a_j \tau_N(v_i^* v_j). \end{aligned}$$

With the use of Lemma 4.3, this equality can be rewritten as

$$\sum_{i=1}^{d} a_i^* \otimes a_i = \sum_{i,j=1}^{n^2} \tau_N(v_i^* v_j) a_i^* \otimes a_j.$$

Now the set $\{a_i^* \otimes a_j : 1 \leq i, j \leq n^2\}$ is an algebraic basis for $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, so by linear independence it follows that

$$\tau_N(v_i^* v_j) = \begin{cases} \delta_{ij}, & \text{for } 1 \leq i, j \leq d \\ 0, & \text{otherwise} \end{cases}.$$

This is exactly the stated condition on $\tau_N(v_i^* v_j)$, for $1 \leq i, j \leq d$. Moreover, we have that $\tau_N(v_i^* v_i) = 0$ for $i > d$, implying that $v_i = 0$ for $i > d$ by faithfulness of τ_N. Hence $u = \sum_{i=1}^{d} a_i \otimes v_i$ is unitary.

Assume there exists a finite von Neumann algebra N with a normal faithful tracial state τ_N, and operators $v_1, v_2, \dots, v_d \in N$ be such that $u = \sum_{i=1}^{d} a_i \otimes v_i$ is a unitary in $M_n(N)$ and $\tau_N(v_i^* v_j) = \delta_{ij}$, for all $1 \leq i, j \leq d$. It is a straightforward calculation to see that

$$Tx = \sum_{i=1}^{d} a_i^* x a_i = \sum_{i,j=1}^{d} \tau_N(v_i^* v_j) a_i^* x a_j = (\iota_n \otimes \tau_N)(u^*(x \otimes \mathbf{1}_N)u),$$

for all $x \in M_n(\mathbb{C})$. □

We are now ready to establish a criteria for a map in UCPT_n to be non-factorizable. It shall come in handy when we later provide a concrete example of a non-factorizable map.

Corollary 4.5. *Let T be the map in UCPT_n which is given by $Tx = \sum_{i=1}^{d} a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$, where $a_1, a_2, \dots, a_d \in M_n(\mathbb{C})$ are chosen to be linearly independent. If $d \geq 2$ and the set $\{a_i^* a_j : 1 \leq i, j \leq d\}$ is linearly independent, then T is not factorizable.*

Proof. Suppose that T is factorizable. By Theorem 4.4 there exists a finite von Neumann algebra N with a normal faithful tracial state τ_N and operators $v_1, v_2, \dots, v_d \in N$ such that

$u := \sum_{i=1}^{d} a_i \otimes v_i \in M_n(N)$ is unitary and $\tau_N(v_i^* v_j) = \delta_{ij}$, for $1 \le i, j \le d$. Then

$$\sum_{i,j=1}^{d} a_i^* a_j \otimes (v_i^* v_j - \delta_{ij} \mathbf{1}_N) = u^* u - \sum_{i=1}^{d} a_i^* a_i \otimes \mathbf{1}_N = \mathbf{1}_{M_n(N)} - T(\mathbf{1}_n) \otimes \mathbf{1}_N$$
$$= \mathbf{1}_{M_n(N)} - \mathbf{1}_n \otimes \mathbf{1}_N = \mathbf{0}_{M_n(N)}.$$

Let φ be any bounded linear functional on N. It follows from the above equality that

$$\iota_n \otimes \varphi \left(\sum_{i,j=1}^{d} a_i^* a_j \otimes (v_i^* v_j - \delta_{ij} \mathbf{1}_N) \right) = \sum_{i,j=1}^{d} \varphi(v_i^* v_j - \delta_{ij} \mathbf{1}_N) a_i^* a_j = \mathbf{0}_{M_n(N)}.$$

By linear independence of the set $\{a_i^* a_j : 1 \le i, j \le d\}$, we get that $\varphi(v_i^* v_j - \delta_{ij} \mathbf{1}_N) = 0$, for all $1 \le i, j \le d$, and it follows that $v_i^* v_j = \delta_{ij} \mathbf{1}_N$. Since $d \ge 2$ by assumption, we infer in particular that $v_1^* v_2 = \mathbf{0}_N$ and $v_1^* v_1 = v_2^* v_2 = \mathbf{1}_N$. The latter equality implies by Proposition 3.5 that v_1 and v_2 are unitaries because N is assumed to be finite. Multiplying the former equality by v_1 from the left and v_2^* from the right we get the contradiction $\mathbf{1}_N = v_1 v_1^* v_2 v_2^* = \mathbf{0}_N$. We conclude that T cannot be factorizable. □

Remark 4.6. Notice that maps in UCPT_n satisfying the conditions of the above corollary are extreme points in UCPT_n by Theorem 1.14.

We shall now consider the special class of factorizable maps, namely the special class of factorizable maps that admit an exact factorization through abelian von Neumann algebras.

Proposition 4.7. *Let T be a completely positive unital and τ_n-preserving map. Then T lies in the convex hull of automorphisms on $M_n(\mathbb{C})$ if and only if T admits an exact factorization through an abelian von Neumann algebra.*

Proof. Assume that $T \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$. Then there exist unitaries $u_1, u_2, \ldots, u_d \in M_n(\mathbb{C})$ and positive real numbers $c_1, c_2, \ldots, c_d$ which sum to one such that

$$Tx = \sum_{i=1}^{d} c_i u_i^* x u_i, \qquad \text{for all } x \in M_n(\mathbb{C}).$$

Consider the abelian von Neumann algebra $N := \ell_\infty(\{1, 2, \ldots, d\})$ with normal faithful tracial state τ_N given by

$$\tau_N(a) := \sum_{i=1}^{d} c_i a_i, \qquad \text{for all } a = (a_1, a_2, \ldots, a_d) \in N.$$

It is easy to check that this is indeed a normal faithful tracial state on N. We can think of the space $M_n(N)$ as either d-tuples with $n \times n$ matrices as entries, or as $n \times n$ matrices with d-tuples as entries. We shall use the former for notational brevity. Let $u := (u_1, u_2, \ldots, u_d)$. It is clear that u is a unitary element of $M_n(N)$. Moreover, for $x \in M_n(\mathbb{C})$

$$(\iota_n \otimes \tau_N)(u^*(x \otimes \mathbf{1}_N)u) = (\iota_n \otimes \tau_N)(u_1^* x u_1, u_2^* x u_2, \ldots, u_d^* x u_d)$$
$$= \sum_{i=1}^{d} c_i u_i^* x u_i = Tx.$$

Hence T admits an exact factorization through N.

Assume on the other hand that T admits an exact factorization through an abelian von Neumann algebra N with a normal faithful tracial state τ_N. Then there exists a unitary $u \in M_n(N)$ such that

$$Tx = (\iota_n \otimes \tau_N)(u^*(x \otimes \mathbf{1}_N)u), \qquad \text{for all } x \in M_n(\mathbb{C}).$$

The set $\widehat{N}$ of characters on N is weak*-compact since it is a weak*-closed subset of the closed unit ball in the dual space of N, which is weak*-compact by the Alaoglu theorem. Because N is commutative, N isometrically isomorphic to $C(\widehat{N})$, the set of continuous functions on $\widehat{N}$, via the Gelfand transform Γ. Then $\widetilde{u} := (\iota_n \otimes \Gamma)(u) \in M_n(C(\widehat{N})) = C(\widehat{N}, M_n(\mathbb{C}))$ is a unitary operator, and $\widetilde{\tau}_N := \tau_N \circ \Gamma^{-1}$ is a normal faithful tracial state on $C(\widehat{N})$. In particular, $\widetilde{\tau}_N$ is a linear functional, and moreover, it is positive. By the Riesz representation theorem [18, Theorem 7.2], there is a unique Radon measure μ on $\widehat{N}$ such that $\widetilde{\tau}_N(f) = \int_{\widehat{N}} f \, \mathrm{d}\mu$, for all $f \in C(\widehat{N})$. Note that $\mu(\widehat{N}) = 1$ since $\tilde{\tau}_N$ is a state.

$$\begin{aligned}
Tx &= ((\iota_n \otimes \tau_N) \circ (\iota_n \otimes \Gamma^{-1}) \circ (\iota_n \otimes \Gamma))(u^*(x \otimes \mathbf{1}_N)u) \\
&= (\iota_n \otimes (\tau_N \circ \Gamma^{-1}))((\iota_n \otimes \Gamma)(u^*)(\iota_n \otimes \Gamma)(x \otimes \mathbf{1}_N)(\iota_n \otimes \Gamma)(u)) \\
&= (\iota_n \otimes \widetilde{\tau}_N)\left(\widetilde{u}^*\left(x \otimes \mathbf{1}_{C(\widehat{N})}\right)\widetilde{u}\right) \\
&= \int_{\widehat{N}} \widetilde{u}^*(t)\left(x \otimes \mathbf{1}_{C(\widehat{N})}\right)\widetilde{u}(t) \, \mathrm{d}\mu(t),
\end{aligned}$$

The above integral belongs to the norm-closure of $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$. It is clear from Theorem 2.6 combined with the result of Choi in Theorem 1.8 that $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$ is contained in UCPT_n. Moreover, we know from Remark 2.8 that $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$ is a norm-closed subset of $\mathcal{L}(M_n(\mathbb{C}))$. Hence, $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$ is also closed in the subspace topology on UCPT_n, and we conclude that $T \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$. □

Remark 4.8. It follows from this proposition that $\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))) \subseteq \mathfrak{F}_n$.

Corollary 4.9. *Let $T \in \mathrm{UCPT}_n$ be the map given by $Tx = \sum_{i=1}^d a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$. If $a_1, a_2, \ldots, a_d \in M_n(\mathbb{C})$ are self-adjoint and $a_i a_j = a_j a_i$, for all $1 \leq i, j \leq d$, then the following hold:*

(i) T is factorizable,

(ii) If $d \geq 3$ and the set $\{a_i a_j : 1 \leq i \leq j \leq d\}$ is linearly independent, then $T \notin \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$.

Proof. (i) Let

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

be the Pauli matrices in $M_2(\mathbb{C})$. These are all idempotent. Recall also in particular the anti-commutation relation $\sigma_x\sigma_z + \sigma_z\sigma_x = 0$. Let N be the finite von Neumann algebra $M_{2^d}(\mathbb{C})$ and define $v_1, v_2, \ldots, v_d \in N$ by $v_i = \sigma_x \otimes \cdots \otimes \sigma_x \otimes \sigma_z \otimes \sigma_x \otimes \cdots \otimes \sigma_x$, where this is the tensor product of d Pauli matrices with σ_z in the i'th position. It is easy to check that $v_1, v_2, \ldots, v_d$ form a set of anti-commuting and self-adjoint unitaries. Let τ_N be the normalized trace on $M_{2^d}(\mathbb{C})$. For all $1 \leq i, j \leq d$,

$$\tau_N(v_i^* v_j) = \tau(v_i v_j) = \tfrac{1}{2}(\tau_N(v_i v_j + v_j v_i)) = \delta_{ij}.$$

Set $u := \sum_{i=1}^d a_i \otimes v_i \in M_n(N)$. Then

$$\begin{aligned}
u^* u &= \sum_{i,j=1}^d a_i a_j \otimes v_i v_j = \frac{1}{2}\sum_{i,j=1}^d (a_i a_j + a_j a_i) \otimes v_i v_j \\
&= \frac{1}{2}\sum_{i,j=1}^d a_i a_j \otimes (v_i v_j + v_j v_i) = \sum_{i=1}^d a_i^2 \otimes \mathbf{1}_N = \mathbf{1}_n.
\end{aligned}$$

Notice that we have used here that $a_1, a_2, \ldots, a_d$ commute. Similarly, $uu^* = \mathbf{1}_N$, showing that u is unitary. It follows by Theorem 4.4 that T is factorizable.

(ii) Assume that $d \geq 3$ and that $\{a_i a_j : 1 \leq i \leq j \leq d\}$ is a linearly independent set. We deduce in particular that $\{a_i : 1 \leq i \leq j \leq d\}$ is a linearly independent set. Assume for contradiction that $T \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$. By Proposition 4.7 and Theorem 4.4, there is an abelian von Neumann algebra N with a normal faithful tracial state τ_N and operators $v_1, v_2, \ldots, v_d \in N$ such that $u := \sum_{i=1}^d a_i \otimes v_i$ is a unitary in $M_n(N)$ and $\tau_N(v_i^* v_j) = \delta_{ij}$, $1 \leq i, j \leq d$. Since $a_1, a_2, \ldots, a_d$ are self-adjoint and mutually commuting we conclude that

$$\mathbf{1}_n \otimes \mathbf{1}_N = u^* u = \sum_{i=1}^d a_i^* a_j \otimes v_i^* v_j = \sum_{i=1}^d a_i^2 \otimes v_i^* v_i + \sum_{1 \leq i < j \leq d} a_i a_j \otimes (v_i^* v_j + v_j^* v_i).$$

Since $\sum_{i=1}^d a_i^2 = \mathbf{1}_n$, we get that $\mathbf{0}_n \otimes \mathbf{0}_N = \sum_{i=1}^d a_i^2 \otimes (v_i^* v_i - \mathbf{1}_N) + \sum_{1 \leq i < j \leq d} a_i a_j \otimes (v_i^* v_j + v_j^* v_i)$. Let φ be a state on N. Then

$$\mathbf{0}_n = (\iota_n \otimes \varphi)(\mathbf{0}_n \otimes \mathbf{0}_N) = \sum_{i=1}^d \varphi(v_i^* v_i - \mathbf{1}_N) a_i^2 + \sum_{1 \leq i < j \leq d} \varphi(v_i^* v_j + v_j^* v_i) a_i a_j.$$

By the linear independence of the set $\{a_i a_j : 1 \leq i \leq j \leq d\}$ it follows that $\varphi(v_i^* v_i - \mathbf{1}_N) = 0$, for all $1 \leq i \leq d$ and that $\varphi(v_i^* v_j + v_j^* v_i) = 0$, for all $1 \leq i < j \leq d$. This is true for all states, and hence $v_i^* v_i = \mathbf{1}_N$, for all $1 \leq i \leq d$ and $v_i^* v_j + v_j^* v_i = \mathbf{0}_N$, for all $1 \leq i < j \leq d$.

Since N is commutative, the set of characters $\widehat{N}$ is non-empty, so take $\xi \in \widehat{N}$. From the above we deduce that

$$\begin{aligned} |\xi(v_i)|^2 = \xi(v_i^* v_i) = \xi(\mathbf{1}_N) = 1, &\qquad 1 \leq i \leq d, \\ \mathrm{Re}(\overline{\xi(v_i)}\xi(v_j)) = \xi(v_i^* v_j + v_j^* v_i) = 0, &\qquad 1 \leq i \neq j \leq d. \end{aligned}$$

As $d \geq 3$, we have in particular that $\overline{\xi(v_1)}\xi(v_2)$, $\overline{\xi(v_2)}\xi(v_3)$ and $\overline{\xi(v_3)}\xi(v_1)$ are purely imaginary complex numbers, and hence the product of all three of these is purely imaginary. But as $|\xi(v_i)|^2 = 1$, for all $1 \leq i \leq d$, we have $|\xi(v_1)|^2 |\xi(v_2)|^2 |\xi(v_3)|^2 = 1$, which is a contradiction. We conclude that $T \notin \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$. □

4.2 Permanence properties of factorizable maps

Lemma 4.10. *The tensor product of factorizable maps is factorizable.*

Proof. Let T and S be a factorizable maps on $M_n(\mathbb{C})$ and $M_m(\mathbb{C})$ respectively. Then T has an exact factorization through $M_n(N)$ by Remark 4.2, where N is a von Neumann algebra with a normal faithful tracial state τ_N. Likewise S has an exact factorization through $M_n(M)$ for some von Neumann algebra M with a normal faithful tracial state τ_M. Hence, there exist unitary elements $u \in M_n(N)$ and $v \in M_m(M)$ such that

$$\begin{aligned} Tx = (\iota_n \otimes \tau_N)(u^*(x \otimes \mathbf{1}_N)u), &\qquad \text{for all } x \in M_n(\mathbb{C}), \\ Sy = (\iota_m \otimes \tau_M)(v^*(y \otimes \mathbf{1}_M)v), &\qquad \text{for all } y \in M_m(\mathbb{C}). \end{aligned}$$

Let $w \in M_{nm}(N \otimes M)$ be the canonical shuffle of $u \otimes v \in M_n(N) \otimes M_m(M)$. Clearly w is unitary. Define the map R on $M_{nm}(\mathbb{C})$ by

$$Rz = (\iota_{nm} \otimes \tau_{N \otimes M})(w^*(z \otimes \mathbf{1}_{N \otimes M})w), \qquad \text{for all } z \in M_{nm}(\mathbb{C}).$$

For all $x \in M_n(\mathbb{C})$ and all $y \in M_m(\mathbb{C})$ we have that

$$\begin{aligned} R(x \otimes y) &= (\iota_{nm} \otimes \tau_{N \otimes M})(w^*(x \otimes y \otimes \mathbf{1}_{N \otimes M})w) \\ &= (\iota_n \otimes \tau_N \otimes \iota_m \otimes \tau_M)((u^* \otimes v^*)(x \otimes \mathbf{1}_N \otimes y \otimes \mathbf{1}_M)(u \otimes v)) \\ &= [(\iota_n \otimes \tau_N)(u^*(x \otimes \mathbf{1}_N)u)] \otimes [(\iota_m \otimes \tau_M)(v^*(y \otimes \mathbf{1}_M)v)] \\ &= Tx \otimes Sy. \end{aligned}$$

The tensor product map $T \otimes S$ is the unique map satisfying this equality, hence $T \otimes S = R$ showing that $T \otimes S$ is indeed factorizable. □

Proposition 4.11. *The set $\mathfrak{F}_n$ of factorizable maps is convex.*

Proof. Let $T, S \in \mathfrak{F}_n$ and let N be a von Neumann algebra equipped with a normal faithful tracial state τ_N such that both T and S has an exact factorization through N. Such a von Neumann algebra exists by Lemma 4.1. Let $0 < t < 1$. We wish to show that $tT + (1-t)S$ is a map in $\mathfrak{F}_n$ as well. Let $u, v \in M_n(N)$ be unitary operators such that

$$\begin{aligned} Tx &= (\iota_n \otimes \tau_N)(u^*(x \otimes \mathbf{1}_N)u), \qquad x \in M_n(\mathbb{C}), \\ Sx &= (\iota_n \otimes \tau_N)(v^*(x \otimes \mathbf{1}_N)v), \qquad x \in M_n(\mathbb{C}). \end{aligned}$$

The space $L_\infty([0,1])$ is a von Neumann algebra (see Proposition 3.22), and the linear functional $\chi : L_\infty([0,1]) \to \mathbb{C}$ given by $\chi(f) = \int_{[0,1]} f \, \mathrm{d}m$ is a normal faithful tracial state. Denote by $\mathbf{1}_{[0,1]}$ the identity in $L_\infty([0,1])$. Let $p \in L_\infty([0,1])$ be the projection $p = \mathbf{1}_{[0,t]}$. Then $\mathbf{1}_{[0,1]} - p = \mathbf{1}_{(t,1]}$ is also a projection in $L_\infty([0,1])$, and $p(\mathbf{1}_{[0,1]} - p) = p - p^2 = 0$. Moreover, we have $\chi(p) = t$ and $\chi(\mathbf{1}_{[0,1]} - p) = 1 - t$. Let $w \in N \otimes L_\infty([0,1])$ be the operator $w = u \otimes p + v \otimes (\mathbf{1}_{[0,1]} - p)$. We have

$$\begin{aligned} w^*w &= (u^* \otimes p + v^* \otimes (\mathbf{1}_{[0,1]} - p))(u \otimes p + v \otimes (\mathbf{1}_{[0,1]} - p)) \\ &= \mathbf{1}_N \otimes p + \mathbf{1}_N \otimes (\mathbf{1}_{[0,1]} - p) = \mathbf{1}_N \otimes \mathbf{1}_{[0,1]}. \\ ww^* &= (u \otimes p + v \otimes (\mathbf{1}_{[0,1]} - p))(u^* \otimes p + v^* \otimes (\mathbf{1}_{[0,1]} - p)) \\ &= \mathbf{1}_N \otimes p + \mathbf{1}_N \otimes (\mathbf{1}_{[0,1]} - p) = \mathbf{1}_N \otimes \mathbf{1}_{[0,1]}. \end{aligned}$$

This shows that w is unitary. For all $x \in M_n(\mathbb{C})$, we have

$$\begin{aligned} &(\iota_n \otimes \tau_N \otimes \chi)(w^*(x \otimes \mathbf{1}_N \otimes \mathbf{1}_{[0,1]})w) \\ &\quad = (\iota_n \otimes \tau_N \otimes \chi)((u^*(x \otimes \mathbf{1}_N)u) \otimes p + (v^*(x \otimes \mathbf{1}_N)v) \otimes (\mathbf{1}_{[0,1]} - p)) \\ &\quad = \chi(p)(\iota_n \otimes \tau_N)(u^*(x \otimes \mathbf{1}_N)u) + \chi(\mathbf{1}_{[0,1]} - p)(\iota_n \otimes \tau_N)(v^*(x \otimes \mathbf{1}_N)v) \\ &\quad = tTx + (1-t)Sx, \end{aligned}$$

This shows that $tT + (1-t)S$ admits an exact factorization through $M_n(N \otimes L_\infty([0,1]))$. □

The following proposition will be crucial when disproving the Asymptotic Quantum Birkhoff Conjecture in Section 4.3. It is in the proof of this proposition that we shall need that the tracial ultraproduct of a family of tracial von Neumann algebras is again a tracial von Neumann algebra (see Theorem 3.29).

Proposition 4.12. *The set $\mathfrak{F}_n$ of factorizable maps is closed with respect to the completely bounded norm.*

Proof. Note that it is enough to show that $\mathfrak{F}_n$ is sequentially closed. So let $(T_k)_{k \geq 1}$ be a sequence in $\mathfrak{F}_n$ and suppose that $(T_k)_{k \geq 1}$ converges in norm to $T \in \mathfrak{F}_n$, i.e., $\|T_k - T\|_{\mathrm{cb}} \to 0$

as $k \to \infty$. For each $k \in \mathbb{N}$, there is a von Neumann algebra N_k equipped with a normal faithful tracial state τ_k and a unitary operator $u_k \in M_n(N_k)$ such that

$$T_k x = (\iota_n \otimes \tau_k)(u_k^*(x \otimes \mathbf{1}_{N_k})u_k), \qquad \text{for all } x \in M_n(\mathbb{C}).$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let $N = \prod_{\mathcal{U}} N_k$ be the tracial ultraproduct of the sequence of von Neumann algebras. Then N is a von Neumann algebra by Theorem 3.29. Let $u = \prod_{\mathcal{U}} u_k$ be the ultraproduct of the sequence of unitaries in the sense of Definition B.6. It is straightforward to check that this is a unitary operator on the ambient Hilbert space on which N acts. The elements of N are equivalence classes of sequences, e.g. $[(v_k)_{k\geq 1}]$ where $v_k \in N_k$, for all $k \in \mathbb{N}$. The identity of N is the element $\mathbf{1}_N = [(\mathbf{1}_{N_k})_{k\geq 1}]$. Moreover, N is equipped with the normal faithful tracial state τ given by $\tau([(v_k)_{k\geq 1}]) = \lim_{\mathcal{U}} \tau_k(v_k)$, for all $[(v_k)_{k\geq 1}] \in N$. For any matrix $x \in M_n(\mathbb{C})$ and $v = [(v_k)_{k\geq 1}] \in N$ the tensor product $x \otimes v \in M_n(N)$ can be thought of as an $n \times n$ matrix with entries in N, or equivalently, as the equivalence class $x \otimes v = [(x \otimes v_k)_{k\geq 1}]$, where $(x \otimes v_k)_{k\geq 1} \in \prod_{k\geq 1} M_n(N_k)$.

Define the map T' on $M_n(\mathbb{C})$ by

$$T'x = (\iota_n \otimes \tau)(u^*(x \otimes \mathbf{1}_N)u), \qquad \text{for all } x \in M_n(\mathbb{C}).$$

With the above clarification of the setup it is clear that

$$\begin{aligned} T'x &= (\iota_n \otimes \tau)(u^*(x \otimes \mathbf{1}_N)u) = (\iota_n \otimes \tau)([(u_k^*(x \otimes \mathbf{1}_{N_k})u_k)_{k\geq 1}]) \\ &= \lim_{\mathcal{U}} (\iota_n \otimes \tau_k)(u_k^*(x \otimes \mathbf{1}_{N_k})u_k) = \lim_{\mathcal{U}} T_k x, \end{aligned}$$

for all $x \in M_n(\mathbb{C})$. We wish to show that $(T_k)_{k\geq 1}$ converges in the completely bounded norm to T'. It will then follow that $T' = T$ by the uniqueness of limits in Hausdorff spaces, from which we can infer that $T \in \mathfrak{F}_n$.

Fix $\ell \in \mathbb{N}$ and let $y \in M_n(\mathbb{C}) \otimes M_\ell(\mathbb{C})$. Recall that the limit of a sequence along a free ultrafilter agrees with the usual limit. For a fixed $m \geq 1$, it is clear that the limit along $\mathcal{U}$ of the constant sequence $((T_m \otimes \iota_\ell)(y))_{k\geq 1}$ is $(T_m \otimes \iota_\ell)(y)$. By linearity we get:

$$T_m \otimes \iota_\ell(y) - T' \otimes \iota_\ell(y) = T_m \otimes \iota_\ell(y) - \lim_{\mathcal{U}} T_k \otimes \iota_\ell(y) = \lim_{\mathcal{U},k} (T_m - T_k) \otimes \iota_\ell(y).$$

To avoid confusion we have indicated the index over which the limit is taken explicitly. The operator norm $\|\cdot\| : M_n(\mathbb{C}) \otimes M_\ell(\mathbb{C}) \to \mathbb{R}$ is a continuous function since $M_n(\mathbb{C}) \otimes M_\ell(\mathbb{C})$ is equipped with the metric topology. It follows from Proposition B.13 that

$$\left\|(T_m - T') \otimes \iota_\ell(y)\right\| = \lim_{\mathcal{U},k} \|(T_m - T_k) \otimes \iota_\ell(y)\| = \lim_{k\to\infty} \|(T_m - T_k) \otimes \iota_\ell(y)\|,$$

where we have again used that the limit of a sequence along a free ultrafilter agrees with the usual limit. We can now use the properties of the operator norm and the completely bounded norm together with the triangle-inequality to get the following estimate:

$$\begin{aligned} \left\|(T_m - T') \otimes \iota_\ell(y)\right\| &\leq \|y\| \lim_{k\to\infty} \|(T_m - T_k) \otimes \iota_\ell\| \leq \|y\| \lim_{k\to\infty} \|T_m - T_k\|_{cb} \\ &\leq \|y\| \lim_{k\to\infty} (\|T_m - T\|_{cb} + \|T - T_k\|_{cb}) = \|y\| \, \|T_m - T\|_{cb} \end{aligned}$$

Since $\ell \in \mathbb{N}$ was arbitrary, it follows that $\|T_m - T'\|_{cb} \leq \|T_m - T\|_{cb}$, and hence that the sequence $(T_k)_{k\geq 1}$ converges in the completely bounded norm to T' as required. □

4.3 Disproving the Asymptotic Quantum Birkhoff Conjecture

Theorem 4.13. *Let $n, m \geq 3$. For any maps $T \in \mathrm{UCPT}_n$ and $S \in \mathrm{UCPT}_m$, it holds that*

$$d_{cb}(T \otimes S, \mathfrak{F}(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))) \geq d_{cb}(T, \mathfrak{F}(M_n(\mathbb{C}))).$$

Proof. Define $\zeta : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ by $\zeta(x) := x \otimes \mathbf{1}_m$, for all $x \in M_n(\mathbb{C})$. It is clear that ζ is a completely positive, unital and τ_n-preserving linear map. Recall that the matrix algebra $M_k(\mathbb{C})$ is a Hilbert space with the Hilbert-Schmidt inner product $\langle \cdot\, , \cdot \rangle_k$. Let $\zeta^* : M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \to M_n(\mathbb{C})$ be the Hilbert space adjoint of ζ, i.e., the unique map satisfying $\langle \zeta(x)\, , y\rangle_{nm} = \langle x\, , \zeta^*(y)\rangle_n$, for all $x \in M_n(\mathbb{C})$ and $y \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$. Let $x \in M_n(\mathbb{C})$ and $y \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$. Then

$$\begin{aligned}\langle \zeta(x)\, , y\rangle_{nm} &= \tau_{nm}(y^*(x \otimes \mathbf{1}_m)) = \tau_n \circ (\iota_n \otimes \tau_m)(y^*(x \otimes \mathbf{1}_m)) = \tau_n((\iota_n \otimes \tau_m(y))^* x), \\ \langle x\, , \zeta^*(y)\rangle_n &= \tau_n((\zeta^*(y))^* x),\end{aligned}$$

and we derive that $\zeta^* = \iota_n \otimes \tau_m$. For all $x \in M_n(\mathbb{C})$ we have

$$\zeta^*(T \otimes S)\zeta(x) = \zeta^* \circ (T \otimes S)(x \otimes \mathbf{1}_m) = T(x).$$

Hence $\zeta^*(T \otimes S)\zeta = T$. The same is true if we replace S by the identity map ι_m.

Let $k \geq 1$ and consider the maps $\iota_k \otimes \zeta$ and $\iota_k \otimes \zeta^* = \iota_{kn} \otimes \tau_m$. We have already noted that the former is positive linear and unital. The latter is commonly referred to as the (normalized) *partial trace* in Quantum Information Theory. One can check that this is, indeed, a positive map by computing its Choi decomposition or its associated Choi matrix. Moreover, the partial trace is linear and unital. Hence, the norm of both of these maps equals 1 by Theorem 1.17. As this holds for all $k \geq 1$, we conclude that $\|\zeta\|_{cb} = \|\zeta^*\|_{cb} = 1$.

Let $R \in \mathfrak{F}(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))$ and consider the distance in the completely bounded norm between the maps $\zeta^* R \zeta$ and T:

$$\|T - \zeta^* R\zeta\|_{cb} = \|\zeta^*(T \otimes S - R)\zeta\|_{cb} \leq \|T \otimes S - R\|_{cb}\,.$$

From this we obtain the following bound on the completely bounded distance from the map $T \otimes S$ to the set of factorizable maps on $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$:

$$d_{cb}(T \otimes S, \mathfrak{F}(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))) \geq d_{cb}(T, \zeta^* \mathfrak{F}(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))\zeta).$$

If $Q \in \mathfrak{F}(M_n(\mathbb{C}))$, then $Q \otimes \iota_m$ is a factorizable map on $M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$ by Lemma 4.10. Moreover, $\zeta^*(Q \otimes \iota_m)\zeta = Q$ and therefore $Q \in \zeta^* \mathfrak{F}(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))\zeta$. Hence we have the inclusion $\mathfrak{F}(M_n(\mathbb{C})) \subseteq \zeta^* \mathfrak{F}(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))\zeta$, and it follows that

$$d_{cb}(T \otimes S, \mathfrak{F}(M_n(\mathbb{C}) \otimes M_m(\mathbb{C}))) \geq d_{cb}(T, \mathfrak{F}(M_n(\mathbb{C}))),$$

as claimed. □

Let T be a map in UCPT_n, and for each $k \geq 1$, let $\alpha_k \geq 0$ and $\beta_k \geq 0$ be the distances

$$\alpha_k = d_{cb}\left(\bigotimes_{i=1}^{k} T,\ \mathfrak{F}\left(\bigotimes_{i=1}^{k} M_n(\mathbb{C})\right)\right), \quad \beta_k = d_{cb}\left(\bigotimes_{i=1}^{k} T,\ \mathrm{conv}\left(\mathrm{Aut}\left(\bigotimes_{i=1}^{k} M_n(\mathbb{C})\right)\right)\right).$$

It follows as a special case of Theorem 4.13 above that $\alpha_k \geq d_{cb}(T, \mathfrak{F}_n)$ for all $k \geq 1$. Suppose further that T is not factorizable. Then $d_{cb}(T, \mathfrak{F}_n) > 0$ because the set $\mathfrak{F}_n$ of factorizable maps is closed with respect to the completely bounded norm (see Proposition 4.12). Since the sequence $(\alpha_k)_{k\geq 1}$ is bounded from below by $d_{cb}(T, \mathfrak{F}_n)$, we conclude that $(\alpha_k)_{k\geq 1}$ will not converge to 0. Recall from Remark 4.8 the inclusion $\mathrm{conv}(\mathrm{Aut}(M_m(\mathbb{C}))) \subseteq \mathfrak{F}_m$, which holds for all $m \geq 1$. It follows that $\alpha_k \leq \beta_k$ for all $k \geq 1$, and hence, the sequence $(\beta_k)_{k\geq 1}$ will not converge to 0. We present in the following an example of a non-factorizable map on $M_3(\mathbb{C})$. This will by the above discussion serve as a counterexample to the Asymptotic Quantum Birkhoff Conjecture.

Example 4.1 (A non-factorizable UCPT$_3$-map)**.** Define a_1, a_2 and a_3 in $M_3(\mathbb{C})$ as follows.

$$a_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad a_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad a_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is a simple calculation to see that

$$a_1^* a_1 = a_1 a_1^* = \tfrac{1}{2}(e_{22} + e_{33}), \quad a_2^* a_2 = a_2 a_2^* = \tfrac{1}{2}(e_{11} + e_{33}), \quad a_3^* a_3 = a_3 a_3^* = \tfrac{1}{2}(e_{11} + e_{22}).$$

It follows that $\sum_{i=1}^3 a_i^* a_i = \sum_{i=1}^3 a_i a_i^* = \mathbf{1}_3$. Hence, the operator T on $M_3(\mathbb{C})$ given by $Tx = \sum_{i=1}^3 a_i^* x a_i$, for all $x \in M_3(\mathbb{C})$, is in UCPT$_3$. One can check by straightforward calculation that the conditions of Corollary 4.5 are satisfied. Hence T is not factorizable. For an alternative proof, suppose that T is factorizable. Then there would exist a finite von Neumann algebra N with a normal faithful tracial state τ_N and elements $v_1, v_2, v_3 \in N$ such that the operator $u \in M_3(N)$ given by

$$u := \sum_{i=1}^3 a_i \otimes v_i = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

is unitary. We get the following equalities:

(1) $v_1^* v_1 + v_2^* v_2 = v_2^* v_2 + v_3^* v_3 = v_3^* v_3 + v_1^* v_1 = 2\,\mathbf{1}_N$,

(2) $v_1^* v_2 = v_2^* v_3 = v_3^* v_1 = \mathbf{0}_N$.

Equality (1) implies that $v_1^* v_1 = v_2^* v_2 = v_3^* v_3 = \mathbf{1}_N$ and hence that v_1, v_2 and v_3 are unitaries by Proposition 3.5 because N is finite. But this clearly contradicts equality (2), and we conclude that T is not factorizable.

People in Quantum Information Theory will recognize T in the above example as being the so-called Holevo-Werner channel W_3^-. Concrete examples of maps in UCPT$_n$, for all $n \geq 4$, which are not factorizable can be found in [3, Remark 4.8].

A natural further question to ask at this point, is whether all factorizable maps satisfy the Asymptotic Quantum Birkhoff property. Before showing that this is *not* the case, we introduce a class of linear maps on $M_n(\mathbb{C})$ referred to as *Schur multipliers.* Note also that the examples of [3, Remark 4.8] are Schur multipliers.

4.4 Schur multipliers

Definition 4.2. For a matrix $B = (b_{ij})_{i,j=1}^n \in M_n(\mathbb{C})$, its corresponding *Schur multiplier* $T_B : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is defined by

$$T_B(x) = (b_{ij} x_{ij})_{i,j=1}^n, \qquad \text{for all } x = (x_{ij})_{i,j=1}^n \in M_n(\mathbb{C}).$$

Proposition 4.14. *Let $B = (b_{ij})_{i,j=1}^n \in M_n(\mathbb{C})$ and let $T_B : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be its corresponding Schur multiplier. The following are equivalent:*

(i) T_B is positive.

(ii) T_B is completely positive.

(iii) There exist diagonal matrices $a_1, a_2, \ldots, a_d \in M_n(\mathbb{C})$ such that

$$T_B(x) = \sum_{i=1}^d a_i^* x a_i, \qquad \text{for all } x \in M_n(\mathbb{C}). \tag{4.2}$$

(iv) There exist linearly independent diagonal matrices $a_1, a_2, \ldots, a_d \in M_n(\mathbb{C})$ such that equation (4.2) hold.

(v) B is a positive in $M_n(\mathbb{C})$.

Proof. The implications (iv)⇒(iii) and (ii)⇒(i) are trivial and (iii)⇒(ii) follows from Theorem 1.8 by Choi.
(i)⇒(v): Assume that T_B is a positive map. Let $x_0 \in M_n(\mathbb{C})$ be the matrix with all entries equal to 1. Observe that if $v \in M_{1,n}(\mathbb{C})$ is the vector with all entries equal to 1, then $x_0 = v^*v$ showing that x_0 is positive. Then $B = T_B(x_0)$ is positive.
(v)⇒(iv): Assume that B is positive. Then B is in particular self-adjoint and can therefore, by the (finite dimensional) spectral theorem, be diagonalized by a unitary matrix. Hence we can write $B = U^*DU$ where $U = [u_{ij}]$ is a unitary matrix and $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Here $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of B, repeated according to multiplicity, and since B is positive we have $\lambda_i \geq 0$, for all $1 \leq i \leq n$. Let $d = \operatorname{rank} B = \operatorname{rank} D$. We may assume that $\lambda_1, \lambda_2, \ldots, \lambda_d > 0$ and $\lambda_{d+1} = \ldots = \lambda_n = 0$. Define for each $1 \leq i \leq d$ the diagonal matrix $a_i = \sqrt{\lambda_i}\operatorname{diag}(u_{i1}, u_{i2}, \ldots, u_{in})$. Here $(u_{i1}, u_{i2}, \ldots, u_{in})$ is the i'th row of U, and since U is a unitary matrix, it follows that the $\{a_i : 1 \leq i \leq d\}$ is a linearly independent set. We have:

$$\sum_{i=1}^{d} a_i^* x a_i = \sum_{i=1}^{d} \lambda_i [\overline{u_{ij}} u_{ik} x_{jk}]_{j,k} = \left[\sum_{i=1}^{d} \lambda_i \overline{u_{ij}} u_{ik} x_{jk}\right]_{j,k}.$$

Note that the (i,j)'th entry of B is exactly $\sum_{i=1}^{d} \lambda_i \overline{u_{ij}} u_{ik}$. Hence $T_B(x) = \sum_{i=1}^{d} a_i^* x a_i$, proving (iv). □

Remark 4.15. Suppose that B is a positive element of $M_n(\mathbb{C})$. If B is also real, then there exist linearly independent real diagonal matrices $a_1, a_2, \ldots, a_d$ such that equation (4.2) holds. Indeed in this case B has eigendecomposition $B = C^T DC$ where C is an orthogonal matrix. This ensures that the diagonal matrices constructed in the proof of (v)⇒(iv) in the above theorem are real.

The following proposition recovers a result proved by Eric Ricard [21].

Proposition 4.16. *Suppose that B is a positive element of $M_n(\mathbb{C})$. If B is a real matrix whose diagonal entries are all equal to 1, then the associated Schur multiplier is factorizable.*

Proof. The Schur multiplier T_B associated with B is completely positive by Proposition 4.14 and unital since all diagonal entries are equal to 1. By Remark 4.15 we can write

$$T_B(x) = \sum_{i=1}^{d} a_i x a_i, \qquad \text{for all } x \in M_n(\mathbb{C}). \tag{4.3}$$

where $a_1, a_2, \ldots, a_d$ are real and diagonal matrices. In particular they are self-adjoint and pairwise commuting. Moreover, $\sum_{i=1}^{d} a_i^2 = \mathbf{1}_n$ as T_B is unital. It follows from Corollary 4.9 that T_B is factorizable. □

We shall need the following result, which is standard in the theory of C^*-algebra. We include a proof for completeness.

Lemma 4.17. *Let $\mathcal{A}$ be a C^*-algebra, and let $x, y \in \mathcal{A}$. If $x^*x + y^*y = 0$, then $x = y = 0$.*

Proof. We have by definition that x^*x and y^*y are positive, and therefore that $\sigma(x^*x) \subseteq \mathbb{R}^+$ and $\sigma(y^*y) \subseteq \mathbb{R}^+$. Since $x^*x = -y^*y$ we have moreover, by the spectral mapping theorem, that $\sigma(x^*x) \subseteq \mathbb{R}^-$. Hence $\sigma(x^*x) = \{0\}$. Since x^*x is normal the spectral radius formula implies that $\|x\|^2 = \|x^*x\| = 0$, and hence $x = 0$. Then $y^*y = 0$ and hence $y = 0$ as well. □

Proposition 4.18. *Let $T \in \mathrm{UCPT}_n$ be a Schur-multiplier. Then the following hold:*

(i) If $Tx = \sum_{i=1}^d a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$, for some $a_1, a_2, \dots, a_d \in M_n(\mathbb{C})$, then $a_1, a_2, \dots, a_d$ are diagonal.

(ii) If $T \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$, i.e., if $Tx = \sum_{i=1}^d c_i u_i^* x u_i$, for all $x \in M_n(\mathbb{C})$, where u_i is a unitary matrix $M_n(\mathbb{C})$ is unitary and $c_i > 0$ for $1 \le i \le d$ and $\sum_{i=1}^d c_i = 1$, then $u_1, u_2, \dots, u_d$ are diagonal.

Proof. Observe that (ii) follows from (i) by setting $a_i = \sqrt{c_i} u_i$ for $1 \le i \le d$. Suppose $Tx = \sum_{i=1}^d a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$, for some $a_1, a_2, \dots, a_d \in M_n(\mathbb{C})$. Let $D_n(\mathbb{C})$ denote the set of $n \times n$ diagonal matrices with complex entries, and let $p \in D_n(\mathbb{C})$ be a projection. Since T is a unital Schur-multiplier and p is diagonal, we must have $Tp = p$. Therefore

$$\sum_{i=1}^d (pa_i(\mathbf{1}_n - p))^*(pa_i(\mathbf{1}_n - p)) = \sum_{i=1}^d (\mathbf{1}_n - p)a_i^* p a_i (\mathbf{1}_n - p) = (\mathbf{1}_n - p)(Tp)(\mathbf{1}_n - p) = 0.$$

It follows by Lemma 4.17, which applies even for $d > 2$ since the set of positive elements is a cone, that $pa_i(\mathbf{1}_n - p) = 0$, for all $1 \le i \le d$. Suppose that the (j,k)'th entry of a_i is non-zero for $j \neq k$ and for some $1 \le i \le d$. Let p_j be the diagonal projection with the (j,j)'th entry equal to 1 and with zeros elsewhere. Then the (j,k)'th entry of $pa_i(\mathbf{1}_n - p)$ equals the (j,k)'th entry of a_i which is non-zero. The contradicts that $pa_i(\mathbf{1}_n - p) = 0$. Hence $a_1, a_2, \dots, a_d$ are diagonal matrices. □

4.5 A factorizable map not satisfying the AQB property

We start this section by presenting an example of a factorizable UCPT_6-map that is not in the convex hull of automorphisms.

Example 4.2 (A Schur multiplier in $\mathfrak{F}_6 \setminus \mathrm{conv}(\mathrm{Aut}(M_6(\mathbb{C})))$)**.** Let $\beta = 1/\sqrt{5}$ and define

$$x_1 = (1, \beta, \beta, \beta, \beta, \beta), \quad x_2 = \sqrt{2/5}\,(0, 1, e^{i2\pi/5}, e^{i4\pi/5}, e^{i6\pi/5}, e^{i8\pi/5}), \quad x_3 = \overline{x}_2.$$

The matrix $B = x_1^* x_1 + x_2^* x_2 + x_3^* x_3$ is a positive element of $M_6(\mathbb{C})$ since the set of positive elements is a cone. Moreover, B is real since $x_1^* x_1$ is real and $x_3^* x_3 = \overline{x_2^* x_2}$. Hence T_B is factorizable by Proposition 4.16. Define $b_1 = \mathrm{diag}(\overline{x_1})$, $b_2 = \mathrm{diag}(\overline{x_2})$ and $b_3 = \mathrm{diag}(x_3)$. Observe that the (i,j)'th entry of B is $\overline{x_1(i)} x_1(j) + \overline{x_2(i)} x_2(j) + \overline{x_3(i)} x_3(j)$. With this in mind it is easy to see that $T_B(y) = \sum_{i=1}^3 b_i^* y b_i$. Set $a_1 = b_1$, $a_2 = (b_2 + b_3)/\sqrt{2}$ and $a_3 = (b_2 - b_3)/(i\sqrt{2})$. We see that a_1, a_2 and a_3 are self-adjoint and that $\sum_{i=1}^3 a_i y a_i = \sum_{i=1}^3 b_i^* y b_i$. Furthermore, a_1, a_2 and a_3 are mutually commuting since they are diagonal matrices. It is a straightforward computation to check that $\{a_i a_j : 1 \le i \le j \le 3\}$ is a linearly independent set, and it then follows by Corollary 4.9 that $T_B \notin \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$.

First of all, the above example shows that the inclusion of Remark 4.8 is strict. Hence, asking whether all factorizable maps satisfies the Asymptotic Quantum Birkhoff property is a non-trivial question.

Proposition 4.19. *Let $T \in \mathrm{UCPT}_n$ and $S \in \mathrm{UCPT}_k$ for $n, k \in \mathbb{N}$ be Schur-multipliers. The following are equivalent:*

(i) $T \otimes S \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}) \otimes M_k(\mathbb{C})))$.

(ii) $T \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$ and $S \in \mathrm{conv}(\mathrm{Aut}(M_k(\mathbb{C})))$.

Proof. Suppose $T \otimes S \in \operatorname{conv}(\operatorname{Aut}(M_n(\mathbb{C}) \otimes M_k(\mathbb{C})))$. Take $u_1, u_2, \ldots, u_d \in M_{nk}(\mathbb{C})$ unitary and $c_i > 0$ for $1 \le i \le d$ satisfying $\sum_{i=1}^d c_i = 1$ such that

$$T \otimes Sx = \sum_{i=1}^d c_i u_i^* x u_i, \qquad \text{for all } x \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C}).$$

By Proposition 4.18 the matrix u_i is diagonal for each $1 \le i \le d$. Let e_{ij} for $1 \le i, j \le n$ and f_{ij} for $1 \le i, j \le k$ be the matrix units of $M_n(\mathbb{C})$, respectively $M_k(\mathbb{C})$. Observe that

$$(\mathbf{1}_n \otimes f_{11})(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}))(\mathbf{1}_n \otimes f_{11}) \cong M_n(\mathbb{C}) \otimes f_{11}.$$

For each $1 \le i \le d$, define $v_i := u_i(\mathbf{1}_n \otimes f_{11})$. Note that the u_i commute with $(\mathbf{1}_n \otimes f_{11})$ because u_i is diagonal. It is then an easy computation to see that v_i is unitary in $M_n(\mathbb{C}) \otimes f_{11}$ for $1 \le i \le d$. Hence there exist unitaries $w_i \in M_n(\mathbb{C})$, $1 \le i \le d$, such that $v_i = w_i \otimes f_{11}$. Now $S(f_{11}) = f_{11}$ because S is a unital Schur multiplier. Hence $T \otimes S(x \otimes f_{11}) = T(x) \otimes S(f_{11}) = T(x) \otimes f_{11}$ is an element of $M_n(\mathbb{C}) \otimes f_{11}$, for all $x \in M_n(\mathbb{C})$. Then

$$\begin{aligned}(T \otimes S)(x \otimes f_{11}) &= (\mathbf{1}_n \otimes f_{11})((T \otimes S)(x \otimes f_{11}))(\mathbf{1}_n \otimes f_{11}) \\ &= \sum_{i=1}^d c_i (\mathbf{1}_n \otimes f_{11}) u_i^* (x \otimes f_{11}) u_i (\mathbf{1}_n \otimes f_{11}) \\ &= \left(\sum_{i=1}^d c_i w_i^* x w_i \right) \otimes f_{11},\end{aligned}$$

for all $x \in M_n(\mathbb{C})$. We infer that $Tx = \sum_{i=1}^d c_i w_i^* x w_i$, for all $x \in M_n(\mathbb{C})$, hence $T \in \operatorname{conv}(\operatorname{Aut}(M_n(\mathbb{C})))$. The proof that $S \in \operatorname{conv}(\operatorname{Aut}(M_k(\mathbb{C})))$ is similar.

Conversely, if $T \in \operatorname{conv}(\operatorname{Aut}(M_n(\mathbb{C})))$ and $S \in \operatorname{conv}(\operatorname{Aut}(M_k(\mathbb{C})))$, we may write $Tx = \sum_{i=1}^d c_i u_i^* x u_i$, for all $x \in M_n(\mathbb{C})$ and $Sy = \sum_{j=1}^{d'} b_i v_i^* y v_i$, for all $y \in M_k(\mathbb{C})$. Consider the map R on $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$ given by

$$R(x \otimes y) = \sum_{i,j} c_i b_j (u_i^* \otimes v_j)^* (x \otimes y)(u_i \otimes v_j) = \left(\sum_{i=1}^d c_i u_i^* x u_i \right) \otimes \left(\sum_{j=1}^{d'} b_j v_i^* y v_i \right),$$

for all $x \in M_n(\mathbb{C})$ and $y \in M_k(\mathbb{C})$. This is clearly a map in $\operatorname{conv}(\operatorname{Aut}(M_n(\mathbb{C}) \otimes M_k(\mathbb{C})))$ satisfying $R(x \otimes y) = T(x) \otimes S(y)$, for all $x \in M_n(\mathbb{C})$ and $y \in M_k(\mathbb{C})$. Since $T \otimes S$ is the unique map with this property, we conclude that $T \otimes S = R \in \operatorname{conv}(\operatorname{Aut}(M_n(\mathbb{C}) \otimes M_k(\mathbb{C})))$. □

Theorem 4.20. *Let $T \in \mathrm{UCPT}_n$ and $S \in \mathrm{UCPT}_k$ for $n, k \in \mathbb{N}$ be Schur-multipliers. Then*

$$d_{cb}(T \otimes S, \operatorname{conv}(\operatorname{Aut}(M_{nk}(\mathbb{C})))) \ge \frac{1}{2} d_{cb}(T, \operatorname{conv}(\operatorname{Aut}(M_n(\mathbb{C})))).$$

Proof. Let $\alpha = d_{cb}(T \otimes S, \operatorname{conv}(\operatorname{Aut}(M_n(\mathbb{C}))))$. Given $\varepsilon > 0$ take $\Phi \in \operatorname{conv}(\operatorname{Aut}(M_{nk}(\mathbb{C})))$ such that $\|T \otimes S - \Phi\|_{\mathrm{cb}} \le \alpha + \varepsilon$, and write $\Phi(x) = \sum_{i=1}^d c_i u_i^* x u_i$, for all $x \in M_{nk}(\mathbb{C})$.

Let $\{f_{st}\}_{s,t=1}^k$ be the matrix units of $M_k(\mathbb{C})$. For each $1 \le i \le d$, take $a_i \in M_n(\mathbb{C})$ such that $a_i \otimes f_{11} = (\mathbf{1}_n \otimes f_{11}) u_i (\mathbf{1}_n \otimes f_{11})$ and define $R : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $R(x) = \sum_{i=1}^d c_i a_i^* x a_i$, for all $x \in M_n(\mathbb{C})$. The map R is a completely positive map by Choi (see Theorem 1.8), and it relates to Φ in the following way:

$$\begin{aligned}R(x) \otimes f_{11} &= \left(\sum_{i=1}^d c_i a_i^* x a_i \right) \otimes f_{11} = \sum_{i=1}^d c_i (a_i \otimes f_{11})^* (x \otimes f_{11})(a_i \otimes f_{11}) \\ &= (\mathbf{1}_n \otimes f_{11}) \Phi(x \otimes f_{11})(\mathbf{1}_n \otimes f_{11}),\end{aligned}$$

for all $x \in M_n(\mathbb{C})$. Let $m \geq 1$ be an integer, let $x \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$, and let $p_1^{(k)}$ be the projection on $M_k(\mathbb{C})$ given by $p_1^{(k)}(y) = f_{11} y f_{11}$, for all $y \in M_k(\mathbb{C})$. Suppose $x \in M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) \otimes M_m(\mathbb{C})$ satisfies $x = (\mathbf{1}_n \otimes f_{11} \otimes \mathbf{1}_n) x (\mathbf{1}_n \otimes f_{11} \otimes \mathbf{1}_n)$. Following the same line of thinking as before, we derive a slightly more general relation between R and Φ:

$$\begin{aligned}(R \otimes p_1^{(k)} \otimes \iota_m)(x) &= \sum_{i=1}^{d} c_i (a_i^* \otimes f_{11} \otimes \mathbf{1}_m) x (a_i \otimes f_{11} \otimes \mathbf{1}_m) \\ &= (\mathbf{1}_n \otimes f_{11} \otimes \mathbf{1}_m)(\Phi \otimes \iota_m)(x)(\mathbf{1}_n \otimes f_{11} \otimes \mathbf{1}_m).\end{aligned}$$

Let $x \in M_n(\mathbb{C}) \otimes M_m(\mathbb{C})$. Recall that $M_n(\mathbb{C}) \otimes M_m(\mathbb{C}) \otimes M_k(\mathbb{C})$ is isometrically isomorphic to $M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) \otimes M_m(\mathbb{C})$ (via the canonical shuffle), and let $\tilde{x}$ be the element corresponding to $x \otimes f_{11}$ under this shuffling. Clearly $\tilde{x}$ satisfies $\tilde{x} = (\mathbf{1}_n \otimes f_{11} \otimes \mathbf{1}_m)\tilde{x}(\mathbf{1}_n \otimes f_{11} \otimes \mathbf{1}_m)$. Observe further that $f_{11} = S(f_{11}) = p_1^{(k)}(f_{11})$.

$$\begin{aligned}\|(T-R)_m(x)\| &= \|(T-R)_m(x) \otimes f_{11}\| = \left\|(T \otimes \iota_m \otimes S)(x \otimes f_{11}) - (R \otimes \iota_m \otimes p_1^{(k)})(x \otimes f_{11})\right\| \\ &= \left\|(T \otimes S \otimes \iota_m)(\tilde{x}) - (R \otimes p_1^{(k)} \otimes \iota_m)(\tilde{x})\right\| \\ &= \|(\mathbf{1}_n \otimes f_{11} \otimes \mathbf{1}_m)((T \otimes S \otimes \iota_m)(\tilde{x}) - \Phi \otimes \iota_m(\tilde{x}))(\mathbf{1}_n \otimes f_{11} \otimes \mathbf{1}_m)\| \\ &\leq \|(T \otimes S \otimes \iota_m)(\tilde{x}) - (\Phi \otimes \iota_m)(\tilde{x})\| \leq \|T \otimes S - \Phi\|_{\mathrm{cb}} \|x\| \leq (\alpha + \varepsilon)\|x\|.\end{aligned}$$

As $m \geq 1$ was arbitrary, we conclude that $\|T - R\|_{\mathrm{cb}} \leq \alpha + \varepsilon$.

Since $\|a_i\| = \|a_i \otimes f_{11}\| = \|(\mathbf{1}_n \otimes f_{11}) u_i (\mathbf{1}_n \otimes f_{11})\| \leq \|u_i\| = 1$, we can by Lemma 1.5 find unitary matrices $v_i, w_i \in M_n(\mathbb{C})$ for each $1 \leq i \leq d$ such that $a_i = (v_i + w_i)/2$. Define the linear map $\widetilde{T} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\widetilde{T}x = \frac{1}{2}\sum_{i=1}^{d} c_i (v_i^* x v_i + w_i^* x w_i), \qquad \text{for all } x \in M_n(\mathbb{C}).$$

Clearly $\widetilde{T} \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$. We see that

$$\begin{aligned}(\widetilde{T} - R)(x) &= \frac{1}{2}\sum_{i=1}^{d} c_i (v_i^* x v_i + w_i^* x w_i) - \frac{1}{4}\sum_{i=1}^{d} c_i (v_i + w_i)^* x (v_i + w_i) \\ &= \frac{1}{2}\sum_{i=1}^{d} c_i (v_i^* x v_i + w_i^* x w_i) - \frac{1}{4}\sum_{i=1}^{d} c_i (v_i^* x v_i + w_i^* x w_i + v_i^* x w_i + w_i^* x v_i) \\ &= \frac{1}{4}\sum_{i=1}^{d} c_i (v_i^* x v_i + w_i^* x w_i - v_i^* x w_i - w_i^* x v_i) \\ &= \frac{1}{4}\sum_{i=1}^{d} c_i (v_i - w_i)^* x (v_i - w_i).\end{aligned}$$

Hence $\widetilde{T} - R$ is completely positive. Let $m \geq 1$. By Theorem 1.17 we get that

$$\begin{aligned}\left\|(\widetilde{T} - R) \otimes \iota_m\right\| &= \left\|(\widetilde{T} - R) \otimes \iota_m(\mathbf{1}_{nm})\right\| = \left\|(\widetilde{T} - R)(\mathbf{1}_n) \otimes \iota_m(\mathbf{1}_m)\right\| \\ &= \left\|(\widetilde{T} - R)(\mathbf{1}_n)\right\| = \left\|\widetilde{T}(\mathbf{1}_n) - R(\mathbf{1}_n)\right\| = \left\|\mathbf{1}_n - R(\mathbf{1}_n)\right\| \\ &= \left\|T(\mathbf{1}_n) - R(\mathbf{1}_n)\right\| \leq \|T - R\| \leq \|T - R\|_{cb} \leq \alpha + \varepsilon.\end{aligned}$$

Hence $\left\|\widetilde{T} - R\right\|_{cb} \leq \alpha + \varepsilon$. Using the triangle inequality, we can now derive that $\left\|T - \widetilde{T}\right\|_{cb} \leq \|T - R\|_{cb} + \left\|R - \widetilde{T}\right\|_{cb} = 2(\alpha + \varepsilon)$. As $\widetilde{T} \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$, this implies that

$$d_{cb}(T, \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))) \leq \left\|T - \widetilde{T}\right\|_{cb} \leq 2(\alpha + \varepsilon).$$

This holds for all $\varepsilon > 0$, hence $d_{cb}(T, \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))) \leq 2\alpha$ and the assertion follows. □

Recall that the convex hull of automorphisms on $M_n(\mathbb{C})$ is a closed set (see Remark 2.8). By the same line of reasoning as in Section 4.3, it will be a consequence of Theorem 4.20 that any Schur multiplier in UCPT_n, which is not in the convex hull of automorphisms on $M_n(\mathbb{C})$, will not satisfy the Asymptotic Quantum Birkhoff property. We can then conclude by Example 4.2 that there exists a map in UCPT_6 that is factorizable, and that does not satisfy the Asymptotic Quantum Birkhoff property.

4.6 Literature

This chapter is based on the work of Uffe Haagerup and Magdalena Musat in [3] and [8].

Appendix A

Convexity

In this appendix we collect the necessary convexity results that will be used in this thesis. We begin by recalling some basic definitions.

Definition A.1. Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A non-empty subset $K \subseteq X$ is said to be *convex* if $tx + (1-t)y \in K$, for all $x, y \in K$ and $t \in [0,1]$.

Definition A.2. Let X be a vector space over $\mathbb{K}$ and let $K \subseteq X$ be a convex subset. A point $x \in K$ is said to be an *extreme point* of K if $x = tx_1 + (1-t)x_2$, for some $x_1, x_2 \in K$ and some $0 < t < 1$, implies that $x = x_1 = x_2$. We denote by $\mathrm{Ext}(K)$ the set of all extreme points of K.

Definition A.3. Let X be a vector space over $\mathbb{K}$ and let $A \in X$ be a non-empty subset. The *convex hull* of A, denoted by $\mathrm{conv}(A)$, is the smallest convex subset of X containing A.

More explicitly, the convex hull of a subset A of X is the set

$$\mathrm{conv}(A) = \left\{ \sum_{i=1}^{n} \alpha_i x_i : x_i \in A, \alpha_i > 0, \sum_{i=1}^{n} \alpha_i = 1, n \in \mathbb{N} \right\}.$$

We state next a very useful theorem which is due to Constantin Carathéodory. For a proof, see [22, Proposition 1.3.1].

Theorem A.1. *Let K be a non-empty subset of $\mathbb{R}^n$. Then every element of the convex hull of K is a convex combination of at most $n+1$ points from K.*

This theorem of Carathéodory allows us to prove the following result.

Proposition A.2. *The convex hull of a compact set in $\mathbb{R}^n$, $n \geq 1$, is compact.*

Proof. Let $C \subseteq \mathbb{R}^n$ be compact. For $k \geq 1$, define the set

$$T^{(k)} := \left\{ (t_1, \ldots, t_k) \in \mathbb{R}^k : t_1, \ldots, t_k \geq 0, \sum_{i=1}^{k} t_i = 1 \right\},$$

Note that $T^{(k)} = T_1^{(k)} \cap \ldots \cap T_{k+1}^{(k)}$, where $T_i^{(k)} = \{(t_1, \ldots, t_k) \in \mathbb{R}^k : t_i \geq 0\}$ for $1 \leq i \leq k$ and $T_{k+1}^{(k)} = \{(t_1, \ldots, t_k) \in \mathbb{R}^k : \sum_{i=1}^{k} t_i = 1\}$. Let $\pi_i : \mathbb{R}^k \to \mathbb{R}$ be the i'th projection mapping. This is a continuous map, and so $T_i^{(k)} = \pi_i^{-1}([0,\infty))$ is closed, for $1 \leq i \leq k$. Moreover, $T_{k+1}^{(k)}$ is the preimage of the closed set $\{1\}$ under the continuous map $(t_1, \ldots, t_k) \mapsto \sum_{i=1}^{k} t_i$, and is therefore also closed. Intersections of closed sets are closed, and therefore $T^{(k)}$ is closed. As $T^{(k)} \subseteq [0,1]^k$, we infer that $T^{(k)}$ is also bounded and therefore compact by the Heine-Borel theorem.

Recall that in metric spaces compactness is equivalent to sequential compactness. It therefore suffices to show that the convex hull of C is sequentially compact. Let $(v_i)_{i\geq 1}$ be a sequence in $\mathrm{conv}(C)$. By Theorem A.1, each v_i can be written as a convex combination of at most $n+1$ points, say,

$$v_i = \sum_{j=1}^{n+1} t_{i,j} x_{i,j}$$

This defines a sequence $(t_i)_{i\geq 1}$ in $T^{(n+1)}$ and sequences $(x_{i,j})_{i\geq 1}$ in C, for $1 \leq j \leq n+1$. Since $T^{(n+1)}$ is compact, we can find a convergent subsequence $(t_{i_k})_{k\geq 1}$ of $(t_i)_{i\geq 1}$. Since C is compact, the corresponding subsequence $(x_{i_k,1})_{k\geq 1}$ in C has a further converging subsequence. We can then continue taking subsequences of subsequences finitely many times. For notational convenience, let $(t_{i_k})_{k\geq 1}$ and $(x_{i_k,j})_{k\geq 1}$, for $1 \leq j \leq n+1$, denote the final convergent subsequences. Let $t \in \mathbb{R}^{n+1}$ be the limit of $(t_{i_k})_{k\geq 1}$, and let $x_j \in C$ be the limit of $(x_{i_k,j})_{k\geq 1}$, for $1 \leq j \leq n+1$. Set $v = \sum_{j=1}^{n+1} t_j x_j$. As all norms are equivalent in finite dimensions, we do not specify the choice of norm in the following. We see that

$$\|v_{i_k} - v\| \leq \sum_{j=1}^{n+1} \|t_{i_k,j} x_{i_k,j} - t_j x_j\| ,$$

and that

$$\begin{aligned} \|t_{i_k,j} x_{i_k,j} - t_j x_j\| &\leq \|t_{i_k,j} x_{i_k,j} - t_{i_k,j} x_j\| + \|t_{i_k,j} x_j - t_j x_j\| \\ &= |t_{i_k,j}| \, \|x_{i_k,j} - x_j\| + |t_{i_k,j} - t_j| \, \|x_j\| . \end{aligned}$$

It now only takes a standard argument to see that $(v_{i_k})_{k\geq 1}$ indeed converges to v, which lies in $\mathrm{conv}(C)$. □

We introduce next the notion of convex polytope, which will be used in connection with the classical Birkhoff theorem in Chapter 2. A closed *half-space* in $\mathbb{R}^n$, is a subset of the form $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b\}$, for some $a_1, a_2, \ldots, a_n, b \in \mathbb{R}$.

Definition A.4. A *convex polytope* is an intersection of finitely many half-spaces.

We end the appendix by stating the powerful Krein-Milman theorem in the general context of locally convex Hausdorff topological vector spaces.

Theorem A.3 (Krein-Milman)**.** *Let K be a non-empty, compact and convex subset of a locally convex and Hausdorff topological vector space (X, τ).*

(a) $\mathrm{Ext}(K) \neq \emptyset$.

(b) $K = \overline{\mathrm{conv}(\mathrm{Ext}(K))}^{\tau}$.

Proof. See [23, Theorem 3.23]. □

Appendix B

Ultrafilters and ultraproducts

B.1 Filters and ultrafilters

Definition B.1. Let I be a non-empty set. A *filter* on I is a non-empty collection $\mathcal{F}$ of subsets of I satisfying the following conditions:

(i) $\emptyset \notin \mathcal{F}$.

(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$.

(iii) $A \in \mathcal{F}, A \subseteq B \Rightarrow B \in \mathcal{F}$.

Note that since $\mathcal{F}$ is non-empty, the third condition implies that $I \in \mathcal{F}$.

Example B.1. Let $I \neq \emptyset$ and consider the following examples of filters.

(a) (Principal filter) Let $M \subseteq I$ be a finite and non-empty subset of I. Then

$$\mathcal{F}_M := \{A \subseteq I : M \subseteq A\}$$

is a filter on I. It is clearly non-empty as $M \in \mathcal{F}_M$. If $A, B \in \mathcal{F}_M$, then $M \subseteq A \cap B$ and thus $A \cap B \in \mathcal{F}_M$. In addition, if $A \in \mathcal{F}$ and $A \subseteq B$, then $M \subseteq A \subseteq B$ implying that $B \in \mathcal{F}_M$. The filter $\mathcal{F}_M$ is called the principal filter on I generated by M.

(b) (Fréchet filter) Suppose I is an infinite set, and let

$$\mathcal{F} := \{A \subseteq I : I \backslash A \text{ is finite}\}.$$

This is a non-empty set as $I \in \mathcal{F}$. If $A, B \in \mathcal{F}$ the set $I \backslash (A \cap B) = I \cap (A^{\complement} \cup B^{\complement}) = (I \backslash A) \cup (I \backslash B)$ is finite, hence $A \cap B \in \mathcal{F}$. And if $A \in \mathcal{F}$ and $A \subseteq B$, the set $I \backslash B \subseteq I \backslash A$ is finite, so $B \in \mathcal{F}$. The filter $\mathcal{F}$ is called the *Fréchet filter* or the *cofinite* filter on I.

Definition B.2. A filter $\mathcal{F}$ is called *free* if the intersection of all of its elements is empty.

Example B.2. Let $I \neq \emptyset$.

(a) (Principal filter) Let $M \subseteq I$ be a finite and non-empty subset of I. The principal filter $\mathcal{F}_M$ is not free, as $\bigcap_{A \in \mathcal{F}_M} A = M$.

(b) (Fréchet filter) Suppose I is an infinite set. The Fréchet filter $\mathcal{F}$ on I is free. Indeed $\bigcap_{A \in \mathcal{F}} A \subseteq \bigcap_{i \in I} I \backslash \{i\} = \emptyset$.

(c) If I is a finite set, then no filter on I can be free. This is because there are only finitely many different subsets of a finite set, and finite intersections of sets in a filter is again in the filter.

The set of all filters on a non-empty set I is naturally partially ordered by inclusion. With this in mind, we are ready to define the notion of an ultrafilter.

Definition B.3. An *ultrafilter* on I is a maximal filter on I with respect to the partial order on the set of filters on I given by inclusion. That is, $\mathcal{U}$ is an ultrafilter on I if $\mathcal{U}$ satisfies the following conditions:

(i) $\mathcal{U}$ is a filter on I,

(ii) If $\mathcal{V}$ is a filter on I such that $\mathcal{U} \subseteq \mathcal{U}$, then $\mathcal{V} = \mathcal{U}$.

Example B.3 (The trivial ultrafilter)**.** If $M = \{a\}$, for some $a \in I$, then the filter $\mathcal{F}_M$ of Example B.1(a) is an ultrafilter. This ultrafilter is usually denoted by $\mathcal{U}_a = \{A \in I : a \in A\}$, and is called the *trivial ultrafilter.* Notice that $\{a\} \in \mathcal{U}_a$ and that $\bigcap_{A \in \mathcal{U}_a} A = \{a\}$.
To see that $\mathcal{U}_a$ is an ultrafilter, suppose that $\mathcal{V}$ is another filter on I containing $\mathcal{U}_a$. Then there is a set $V \in \mathcal{V} \backslash \mathcal{U}_a$. Now, $a \notin V$, since otherwise V would be in $\mathcal{U}_a$. We also see that $\{a\} \in \mathcal{U}_a \subseteq \mathcal{V}$. But then $\emptyset = \{a\} \cap V \in \mathcal{V}$, contradicting that $\mathcal{V}$ is a filter. So $\mathcal{U}_a$ must be maximal. Notice that if M consists of more than one element, then $\mathcal{F}_M$ is not an ultrafilter. Just take any $a \in M$, and it is easy to see that $\mathcal{F}_M \subsetneq \mathcal{U}_a$.

A *free ultrafilter* is naturally an ultrafilter which is free in the sense of Definition B.2. It is easy to see that an ultrafilter is free if and only if it is nontrivial. We showed in Example B.2(b) that the Fréchet filter is a free filter. With this in mind, the following proposition shows in particular that free ultrafilters always exist when the index set I is infinite.

Proposition B.1. *Every filter $\mathcal{F}$ on I is contained in an ultrafilter on I. In particular, every free filter is contained in a free ultrafilter.*

Proof. Let $\mathcal{F}$ be a filter on I, and let $\mathscr{F}$ be the set of all filters on I containing $\mathcal{F}$. This is a non-empty set as $\mathcal{F}$ contains itself. Moreover, $\mathscr{F}$ is partially ordered by inclusion. Let $\mathscr{C}$ be a chain in $\mathscr{F}$. We claim that $\bigcup \mathscr{C} := \bigcup_{\mathcal{C} \in \mathscr{C}} C$, which is clearly an upper bound for $\mathscr{C}$, is a filter on I containing $\mathcal{F}$. Indeed, the empty set is not an element of $\bigcup \mathscr{C}$, as it is not an element of any $\mathcal{C} \in \mathscr{C}$. If $A, B \in \bigcup \mathscr{C}$, then $A \in \mathcal{C}_\infty$ and $B \in \mathcal{C}_2$, for some $\mathcal{C}_1, \mathcal{C}_2 \in \mathscr{C}$. Because $\mathscr{C}$ is linearly ordered, one of these filters contains the other; we may assume without loss of generality that $\mathcal{C}_1 \subseteq \mathcal{C}_2$, so that $A, B \in \mathcal{C}_2$. Using the properties of filters, we infer that $A \cap B \in \mathcal{C}_2 \subseteq \bigcup \mathscr{C}$. Finally, if $A \in \bigcup \mathscr{C}$ and $A \subseteq B$, then $A \in \mathcal{C}$ for some filter $\mathcal{C} \in \mathscr{C}$, and then $B \in \mathcal{C} \subseteq \bigcup \mathscr{C}$. This shows that $\bigcup \mathscr{C}$ is a filter on I. That $\mathcal{F} \subseteq \bigcup \mathscr{C}$ is automatic, as $\mathscr{C}$ is a chain in $\mathscr{F}$, and all filters in $\mathscr{F}$ contain $\mathcal{F}$. We conclude that $\bigcup \mathscr{C}$ is an upper bound for $\mathscr{C}$ in $\mathscr{F}$. It follows by Zorn's lemma that there exists a maximal element $\mathcal{U}$ in $\mathscr{F}$. Note that $\mathcal{U}$ will also be maximal in the set of all filters on I, since any filter containing $\mathcal{U}$ will also contain $\mathcal{F}$, and hence be an element of $\mathscr{F}$. In other words, $\mathcal{F}$ is contained in an ultrafilter.

Now if $\mathcal{F}$ is a free filter contained in the ultrafilter $\mathcal{U}$, then $\bigcap_{A \in \mathcal{U}} A \subseteq \bigcap_{A \in \mathcal{F}} A = \emptyset$, showing that $\mathcal{U}$ is also free. □

It follows in particular from the above proposition that if I is an infinite set, then the Fréchet filter on I is contained in a free ultrafilter.

Proposition B.2. *A filter $\mathcal{F}$ on I is an ultrafilter if and only if every $A \subseteq I$ satisfies that either $A \in \mathcal{F}$, or $I \backslash A \in \mathcal{F}$.*

Proof. Suppose that $\mathcal{F}$ is not an ultrafilter. Then $\mathcal{F}$ is contained in an ultrafilter $\mathcal{U}$ by Proposition B.1. Let $A \in \mathcal{U} \backslash \mathcal{F}$. Since $\mathcal{U}$ is a filter, we cannot have both A and $I \backslash A$ in $\mathcal{U}$, as these intersect to the empty set. So $A \backslash I \notin \mathcal{F}$.

On the other hand, suppose that $\mathcal{F}$ is an ultrafilter and assume for contradiction that $A \subseteq I$ is such that neither A, nor $I \backslash A$ is in $\mathcal{F}$. Define

$$\mathcal{G} := \{G \subseteq I : F \cap A \subseteq G \text{ for some } F \in \mathcal{F}\}.$$

We claim that this is a filter containing $\mathcal{F}$. If $\emptyset \in \mathcal{G}$ there would be an $F \in \mathcal{F}$ such that $F \cap A = \emptyset$. But then $F \subseteq I \backslash A$, implying that $I \backslash A \in \mathcal{F}$ and contradicting our assumption. If $G_1, G_2 \in \mathcal{G}$ we can find $F_1, F_2 \in \mathcal{F}$ such that $F_1 \cap A \subseteq G_1$ and $F_2 \cap A \subseteq G_2$. Then $F_1 \cap F_2 \in \mathcal{F}$ and $(F_1 \cap F_2) \cap A \subseteq G_1 \cap G_2$. If $G \in \mathcal{G}$ and $G \subseteq H$ then there is an $F \in \mathcal{F}$ such that $F \cap A \subseteq G \subseteq H$. This shows that $\mathcal{G}$ is a filter. Now, $A \in \mathcal{G}$, since $F \cap A \subseteq A$, for any $F \in \mathcal{F}$, and if $F \in \mathcal{F}$, then $F \cap A \subseteq F$. Hence $\mathcal{F} \subsetneq \mathcal{G}$. But this contradicts that F is maximal. ☐

Remark B.3. Suppose that $\mathcal{U}$ is a free ultrafilter on I. It follows by Proposition B.2 that no finite subset of I can be an element of $\mathcal{U}$. To see this, suppose $M \in \mathcal{U}$ is a finite subset of I. Since $\mathcal{U}$ is free by assumption, the intersection of all its members is empty. Hence for each $m \in M$ there is a set $A_m \in \mathcal{U}$ such that $m \notin A_m$. Because M is finite there are only finitely many A_m, and so $\bigcap_{m \in M} A_m \in \mathcal{U}$. But clearly $M \cap A = \emptyset$, so this leads to a contradiction.

Remark B.4. The Fréchet filter is not necessarily an ultrafilter. Consider for example the Fréchet filter on the natural numbers. Since there are infinitely many even numbers and infinitely many odds, neither set is a member of the Fréchet filter on $\mathbb{N}$. However by Proposition B.2 combined with the above remark, it is true that any free ultrafilter contains the Fréchet filter.

B.1.1 Convergence along filters

Definition B.4 (Convergence along a filter)**.** Let I be a nonempty set, and let $\mathcal{F}$ be a filter on I. Let X be a topological space, and let $\{x_i\}_{i \in I} \subseteq X$. We say that $\{x_i\}_{i \in I}$ converges along the filter $\mathcal{F}$ to $x \in X$ if for all open neighborhoods of O_x of x, $\{i \in I : x_i \in O_x\} \in \mathcal{F}$. We then write $\lim_{\mathcal{F}} x_i = x$ and refer to this limit as the $\mathcal{F}$-limit of $\{x_i\}_{i \in I}$.

Proposition B.5. *Let $(x_n)_{n \geq 1}$ be a sequence in a topological space X, and suppose that $(x_n)_{n \geq 1}$ converges to x in the usual sense. If $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ then the $\mathcal{U}$-limit of $(x_n)_{n \geq 1}$ agrees with the usual limit.*

Proof. Let O_x be an open neighborhood of x. Since $(x_n)_{n \geq 1}$ converges to x there exists an $N \in \mathbb{N}$ such that $x_n \in O_x$ for all $n \geq N$. We then have the inclusion $\{n \geq N\} \subseteq \{n \in \mathbb{N} : x_n \in O_x\}$. Since $\mathcal{U}$ is assumed to be an ultrafilter, $\mathcal{U}$ contains the Fréchet filter. The set $\{n \geq N\}$ is in the Fréchet filter, hence $\{n \in \mathbb{N} : x_n \in O_x\} \in \mathcal{U}$, by the properties of a filter. As O_x was arbitrary it follows that $\lim_{\mathcal{U}} x_i = x$. ☐

As with any type of limit we are interested in considering existence and uniqueness. We shall see that if X is compact and Hausdorff, and if $\mathcal{F}$ is an ultrafilter on I, then any subset $\{x_i\}_{i \in I} \subseteq X$ has a unique $\mathcal{F}$-limit.

Theorem B.6. *Let $\mathcal{F}$ be a filter on the set I. Let X be a Hausdorff topological space, and let $\{x_i\}_{i \in I} \subseteq X$. If the limit $\lim_{\mathcal{F}} x_i$ exists, then it is unique.*

Proof. Suppose for contradiction that $\lim_{\mathcal{F}} x_i = x$ and $\lim_{\mathcal{F}} x_i = y$, for some $x \neq y$. Since X is Hausdorff we can find open neighborhoods O_x and O_y of x and y, respectively, such that $O_x \cap O_y = \emptyset$. By definition of convergence along a filter, the sets $\{i \in I : x_i \in O_x\}$ and $\{i \in I : x_i \in O_y\}$ are both in $\mathcal{F}$, hence, the intersection must be as well. But $\{i \in I : x_i \in O_x\} \cap \{i \in I : x_i \in O_y\} = \{i \in I : x_i \in O_x \cap O_y\} = \emptyset$, contradicting that $\mathcal{F}$ is a filter. ☐

Lemma B.7. *Let $\mathcal{U}$ be a filter on the index set I. Let X be a topological space, and let $\{x_i\}_{i\in I} \subseteq X$. Define for every $S \in \mathcal{U}$ the set $X_S := \{x_i : i \in S\}$. If S is any member of $\mathcal{U}$, then all $\mathcal{U}$-limits of $\{x_i\}_{i\in I}$ are contained in $\overline{X}_S$. Conversely, if $\mathcal{U}$ is an ultrafilter, then every point of $\bigcap_{S\in\mathcal{U}} \overline{X}_S$ is a $\mathcal{U}$-limit of $\{x_i\}_{i\in I}$.*

Proof. Let $S \in \mathcal{U}$ and suppose for contradiction that $x \notin \overline{X}_S$ is a limit of $\{x_i\}_{i\in I}$ along $\mathcal{U}$. Then $\left\{i \in I : x_i \in \overline{X}_S^{\complement}\right\} \in \mathcal{U}$ by definition of convergence along a filter. Let $S' := \left\{i \in I : x_i \in \overline{X}_S\right\}$. Clearly $S \subseteq S'$ and hence $S' \in \mathcal{U}$. But this implies that the empty set is an element of $\mathcal{U}$, which is a contradiction. Hence x is not a limit of $\{x_i\}_{i\in I}$ along $\mathcal{U}$.

Suppose now that $\mathcal{U}$ is an ultrafilter, and let $x \in \bigcap_{S\in\mathcal{U}} \overline{X}_S$ (if this intersection is empty the statement is trivial, so assume that it is not). Let O_x be an open neighborhood of x. We want to show that the set $S_x := \{i \in I : x_i \in O_x\}$ is a member of $\mathcal{U}$. If it was not, then the complement would be by Proposition B.2. But then we would have by construction that

$$x \in \overline{X}_{S_x^{\complement}} = \overline{\{x_i : i \notin S_x\}} = \overline{\{x_i : x_i \notin O_x\}} = O_x^{\complement}.$$

This is clearly a contradiction. □

Theorem B.8. *Let $\mathcal{U}$ be an ultrafilter on the index set I. Let X be a compact topological space, then any I-indexed subset $\{x_i\}_{i\in I} \subseteq X$ converges along $\mathcal{U}$ to at least one point in X.*

Proof. Define for every $S \in \mathcal{U}$ the set $X_S := \{x_i : i \in S\}$. Clearly X_S is non-empty unless S is the empty set, which is not an element of $\mathcal{U}$. We show first that the collection $\{\overline{X}_S : S \in \mathcal{U}\}$ has the finite intersection property. Let $A_1, A_2, \ldots, A_n \in \mathcal{U}$ be any finite collection of sets from $\mathcal{U}$, then the finite intersection $A := \bigcap_{i=1}^n A_i$ is in $\mathcal{U}$. Since the empty set is not an element of $\mathcal{U}$, we infer that A is non-empty. Then

$$\bigcap_{i=1}^n X_{A_i} = \bigcap_{i=1}^n \{x_i : i \in A_i\} = \{x_i : i \in A\} = X_A \neq \emptyset.$$

As $\bigcap_{i=1}^n X_{A_i} \subseteq \bigcap_{i=1}^n \overline{X}_{A_i}$ it follows that $\{\overline{X}_S : S \in \mathcal{U}\}$ has the finite intersection property. Since X is compact this implies that $\bigcap_{S\in\mathcal{U}} \overline{X}_S \neq \emptyset$. By Lemma B.7 every point in $\bigcap_{S\in\mathcal{U}} \overline{X}_S$ is a $\mathcal{U}$-limit of $\{x_i\}_{i\in I}$, hence $\{x_i\}_{i\in I} \subseteq X$ converges along $\mathcal{U}$ to at least one point in X. □

B.1.2 Useful properties of the ultralimit map

Proposition B.9. *Let I be an infinite set, let $\mathcal{U}$ be an ultrafilter on I and let X be a Hausdorff topological vector space over $\mathbb{K}$. Let $\widetilde{X}$ be the set of all I-indexed subsets $\{x_i\}_{i\in I}$ of X such that $\lim_{\mathcal{U}} x_i$ exists. $\widetilde{X}$ is a linear subspace and the map $\{x_i\}_{i\in I} \mapsto \lim_{\mathcal{U}} x_i$ defined on $\widetilde{X}$ is linear. Furthermore, if X is a topological algebra over $\mathbb{K}$, then $\{x_i\}_{i\in I} \mapsto \lim_{\mathcal{U}} x_i$ is also multiplicative.*

Proof. Note first that the assumption that X is Hausdorff ensures us that the limit along $\mathcal{U}$ is unique whenever it exists. Let $\{x_i\}_{i\in I}, \{y_i\}_{i\in I} \in \widetilde{X}$, and let $x := \lim_{\mathcal{U}} x_i$ and $y := \lim_{\mathcal{U}} y_i$. Adding I-indexed subsets of X is done pointwise and so is multiplication by a scalar. We will show that $x+y$ is the limit of $\{x_i + y_i\}_{i\in I}$ along $\mathcal{U}$. Let O_{x+y} be an open neighborhood of $x+y$. There exist open neighborhoods O_x and O_y of x and y respectively such that $O_x + O_y \subseteq O_{x+y}$. Indeed as X is a topological vector space open sets stay open under translations and scalings. The sets $\frac{1}{2}(O_{x+y} - (x+y)) + x$ and $\frac{1}{2}(O_{x+y} - (x+y)) + y$ are therefore open neighborhoods of x and y respectively, and $\frac{1}{2}(O_{x+y} - (x+y)) + x + \frac{1}{2}(O_{x+y} - (x+y)) + y = O_{x+y}$. As $\{x_i\}_{i\in I}$ converges along $\mathcal{U}$ to x and $\{y_i\}_{i\in I}$ along $\mathcal{U}$ to y, we have that $\{i \in I : x_i \in O_x\} \in \mathcal{U}$ and $\{i \in I : y_i \in O_y\} \in \mathcal{U}$. Now

$$\begin{aligned}\{i \in I : x_i \in O_x\} \cap \{i \in I : y_i \in O_y\} &\subseteq \{i \in I : x_i + y_i \in O_x + O_y\} \\ &\subseteq \{i \in I : x_i + y_i \in O_{x+y}\}.\end{aligned}$$

Hence $\{i \in I : x_i + y_i \in O_{x+y}\} \in \mathcal{U}$ by the properties of filters. We conclude that

$$\lim_{\mathcal{U}}(x_i + y_i) = x + y = \lim_{\mathcal{U}} x_i + \lim_{\mathcal{U}} y_i.$$

Next let $\alpha \in \mathbb{K}, \alpha \neq 0$, and let $O_{\alpha x}$ be an open neighborhood of αx. Then $O_x := O_{\alpha x}/\alpha = (O_{\alpha x} - \alpha x)/\alpha + x$ is an open neighborhood of x. Since $\{x_i\}_{i\in I}$ converges along $\mathcal{U}$ to x, it follows that $\{i \in I : \alpha x_i \in O_{\alpha x}\} = \{i \in I : x_i \in O_{\alpha x}/\alpha\} = \{i \in I : x_i \in O_x\} \in \mathcal{U}$. Hence

$$\lim_{\mathcal{U}} \alpha x_i = \alpha x = \alpha \lim_{\mathcal{U}} x_i.$$

For $\alpha = 0$ we have $\{i \in I : 0 \cdot x_i \in O_0\} = I \in \mathcal{U}$.

Suppose now that X is a topological algebra, i.e., that X besides being a topological vector space is also equipped with a continuous, bilinear multiplication map, $\cdot : X \times X \to X$, where $X \times X$ is equipped with the product topology. Note that the topology on X is assumed to be such that multiplication is jointly continuous. As with adding, multiplying I-indexed subsets is done pointwise. Let O_{xy} be an open neighborhood of xy, and let $B := \{(x', y') \in X \times X : x'y' \in O_{xy}\}$ be the inverse image of O_{xy} under the multiplication map. B is an open set in $X \times X$, hence we can find an open neighborhood O_x of x and an open neighborhood O_y of y such that $O_x \times O_y \subseteq B$. By construction O_x and O_y are such that $O_x O_y \subseteq O_{xy}$. Since $x := \lim_{\mathcal{U}} x_i$ and $y := \lim_{\mathcal{U}} y_i$ the sets $\{i \in I : x_i \in O_x\}$ and $\{i \in I : y_i \in O_y\}$ are in $\mathcal{U}$.

$$\{i \in I : x_i \in O_x\} \cap \{i \in I : y_i \in O_y\} \subseteq \{i \in I : x_i y_i \in O_x O_y\} \subseteq \{i \in I : x_i y_i \in O_{xy}\}.$$

It follows by the properties of filters that $\{i \in I : x_i y_i \in O_{xy}\} \in \mathcal{U}$ and hence that

$$\lim_{\mathcal{U}} x_i y_i = xy = \lim_{\mathcal{U}} x_i \lim_{\mathcal{U}} y_i,$$

which was what we wanted to show. □

Remark B.10. Note that if X is an algebra equipped with a norm, then X is a topological algebra with the topology induced by the norm. To see this, we need to show that addition, multiplication and scalar multiplication are continuous maps in the norm topology. This is completely standard, so we shall only show here that multiplication is continuous. Let $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ be two sequences in X converging to x respectively to y for elements $x, y \in X$. These sequences are bounded in norm and $\|x_n y_n - xy\| \leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\| \leq \|x\|_\infty \|y_n - y\| + \|x_n - x\| \|y\|$. So given $\varepsilon > 0$ choose $n_\varepsilon \geq 1$ such that $\|x_n - x\| < \varepsilon/(2\|y\|)$ and $\|y_n - y\| < \varepsilon/(2\|x\|_\infty)$ for all $n \geq n_\varepsilon$. This n_ε works.

Proposition B.11. *Let $\mathcal{U}$ be an ultrafilter on an infinite index set I, let X be a C^*-algebra and let $\{x_i\}_{i\in I} \subseteq X$. If $\lim_{\mathcal{U}} x_i$ exists, then $\lim_{\mathcal{U}} x_i^* = (\lim_{\mathcal{U}} x_i)^*$.*

Proof. Suppose $x = \lim_{\mathcal{U}} x_i$, and let O_{x^*} be an open neighborhood of x^*. Note that the involution map is continuous, and since it is it's own inverse, it is also open. Then $O_{x^*}^*$ is an open neighborhood of x. Moreover, $\{i \in I : x_i^* \in O_{x^*}\} = \{i \in I : x_i \in O_{x^*}^*\}$ which is in $\mathcal{U}$ by definition of convergence along a filter. Hence

$$\lim_{\mathcal{U}} x_i^* = x^* = \left(\lim_{\mathcal{U}} x_i\right)^*.$$

This shows that the ultralimit map is *-preserving as claimed. □

Proposition B.12. *If $X = \mathbb{R}$, the map $\{x_i\}_{i\in I} \mapsto \lim_{\mathcal{U}} x_i$ is positive and increasing.*

Proof. Suppose that $\{x_i\}_{i\in I}$ is positive, i.e., $x_i \geq 0$ for all $i \in I$, and suppose for contradiction that $\lim_{\mathcal{U}} x_i = x$ with $x < 0$. Take $\varepsilon > 0$ such that $x \in (-2\varepsilon, \varepsilon)$. By definition of convergence along a filter $\{i \in I : x_i \in (-2\varepsilon, \varepsilon)\} \in \mathcal{U}$. But since $x_i \geq 0$ for all $i \in I$ this set is empty, so this is a contradiction. This shows that the map $\{x_i\}_{i\in I} \mapsto \lim_{\mathcal{U}} x_i$ is positive.

Now suppose $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ is such that $y_i \geq x_i$ for all $i \in I$ and let $x := \lim_{\mathcal{U}} x_i$ and $y := \lim_{\mathcal{U}} y_i$. By linearity, and by the above result, we obtain $y - x = \lim_{\mathcal{U}}(y_i - x_i) \geq 0$. Hence the map $\{x_i\}_{i\in I} \mapsto \lim_{\mathcal{U}} x_i$ is increasing. □

Proposition B.13. *Let X and Y be Hausdorff topological spaces, let $f : X \to \mathbb{R}$ be a continuous map, and let $\mathcal{U}$ be an ultrafilter on the index set I. If the limit of $\{x_i\}_{i\in I} \subseteq X$ along $\mathcal{U}$ exists, then $\lim_{\mathcal{U}} f(x_i) = f(\lim_{\mathcal{U}} x_i)$.*

Proof. Let $x = \lim_{\mathcal{U}} x_i$ and let $O_{f(x)}$ be an open neighborhood around $f(x)$ in Y. Then $O_x := f^{-1}O_{f(x)}$ is an open neighborhood of x in X, and so $\{i \in I : x_i \in O_x\} \in \mathcal{U}$. Since $\mathcal{U}$ is a filter and $\{i \in I : x_i \in O_x\} \subseteq \{i \in I : f(x_i) \in O_{f(x)}\}$, it follows that the latter set is in $\mathcal{U}$ as well, and hence that $\lim_{\mathcal{U}} f(x_i) = f(x)$. □

B.2 Ultraproducts

B.2.1 Quotients of Banach spaces

Before we begin to construct ultraproducts of Banach spaces, we shall need to discuss some preliminaries on quotient spaces. Let $(X, \|\cdot\|)$ be a Banach space over $\mathbb{K}$(=$\mathbb{R}$ of $\mathbb{C}$) and let M be a closed subspace of X. Consider the quotient space X/M defined as

$$X/M := \{[x] : x \in X\},$$

where $[x] = x + M = \{x' \in X : x' - x \in M\}$. We define operations on X/M by

- $[x] + [y] = [x + y]$ for all $x, y \in X$.
- $\alpha[x] = [\alpha x]$ for all $x \in X$, $\alpha \in \mathbb{K}$.

These operations are well-defined and make X/M into a vector space. Define, for every $x \in X$,

$$\|[x]\| := \inf\{\|x + y\| : y \in M\}. \tag{B.1}$$

Lemma B.14. *Let X be a normed space and let M be a subspace. Then the functional $\|\cdot\| : X/M \to [0, \infty)$ given in equation (B.1) is a (well-defined) seminorm on X/M. Moreover, if M is closed, then $\|\cdot\|$ is a norm on X/M.*

Proof. Let $x_1, x_2 \in X$ be two different representatives of the same coset, i.e., $[x_1] = [x_2]$, or, equivalently, $x_2 - x_1 \in M$. Then

$$\begin{aligned}\|[x_1]\| &= \inf\{\|x_1 + y\| : y \in M\} = \inf\{\|x_1 + ((x_2 - x_1) + y)\| : y \in M\} \\ &= \inf\{\|x_2 + y\| : y \in M\} = \|[x_2]\|.\end{aligned}$$

In the second equality we use that M is a subspace. This shows that $\|\cdot\|$ is well-defined.

It is clear that $\|[x]\| \geq 0$ for all $x \in X$, and that $\|\alpha x\| = |\alpha| \, \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$, again using that M is a subspace. The triangle inequality is also easy to show.

$$\begin{aligned}\|[x + z]\| &= \inf\{\|(x + z) + y\| : y \in M\} = \inf\{\|(x + z) + 2y\| : y \in M\} \\ &\leq \inf\{\|x + y\| + \|z + y\| : y \in M\} \\ &\leq \inf\{\|x + y\| : y \in M\} + \inf\{\|z + y\| : y \in M\} \\ &= \|[x]\| + \|[z]\|.\end{aligned}$$

Finally, we show that if M is closed, then $\|[x]\| = 0$ if and only if $x \in M$. If $\inf\{\|x+y\| : y \in M\} = 0$ we can find a sequence $(y_n)_{n\geq 1}$ in M such that $\|x + y_n\| \to 0$ as $n \to \infty$, or in other words, such that y_n converges in norm to $-x$. Since M is a closed subspace we infer that $x \in M$. Conversely, if $x \in M$ then $\|[x]\| = \|[0]\| = \inf\{\|y\| : y \in M\} = 0$ since $0 \in M$. □

Remark B.15. The quotient map $\pi : X \to X/M$ defined by $\pi(x) = [x]$ for all $x \in X$ is linear and surjective. Moreover it is a contraction, i.e., $\|\pi(x)\| = \|[x]\| \leq \|x\|$ for all $x \in X$. This is because $0 \in M$. In particular it is a continuous map.

Proposition B.16. *If X is a Banach space and M is a closed subspace, then X/M equipped with the norm defined in equation (B.1) is a Banach space. We call this Banach space the quotient space of X by M.*

Proof. We have already shown in Lemma B.14 that X/M is a normed space. To show that it is complete, it is enough to prove that every absolutely convergent series in X/M converges. Suppose that $([x]_n)_{n\geq 1}$ is a sequence in X/M satisfying $\sum_{n=1}^{\infty} \|[x_n]\| < \infty$. We need to show that there is an $[x] \in X/M$ such that $[x] = \sum_{n=1}^{\infty}[x_n]$. Recall that $\|[x]\| = \inf\{\|x+y\| : y \in M\}$. For each $n \geq 1$, by definition of the infimum we can take $y_n \in M$ such that $\|x_n + y_n\| \leq \|[x_n]\| + 2^{-n}$. Then $\sum_{n=1}^{N} \|x_n + y_n\| \leq \sum_{n=1}^{\infty} \|[x_n]\| + 1$, for all $N \in \mathbb{N}$. Hence $\sum_{n=1}^{\infty}(x_n + y_n)$ is an absolutely convergent series in X. As X is a Banach space, there is an element $x \in X$ such that $x = \sum_{n=1}^{\infty}(x_n + y_n) = \lim_{N\to\infty} \sum_{n=1}^{N}(x_n + y_n)$. It follows by continuity and linearity of the quotient map π that

$$[x] = \pi(x) = \pi\left(\lim_{N\to\infty}\sum_{n=1}^{N}(x_n + y_n)\right) = \lim_{N\to\infty}\sum_{n=1}^{N}\pi(x_n + y_n) = \sum_{n=1}^{\infty}[x_n].$$

We conclude that X/M is a Banach space. □

B.2.2 Ultraproducts of Banach spaces

Let I be an index set, and let $(X_i, \|\cdot\|)_{i\in I}$ be a family of Banach spaces over $\mathbb{K}$ indexed by I. The norms on the Banach spaces need not be the same, but we shall omit this subtlety for notational convenience. Consider the space

$$\ell_\infty(I, X_i) := \left\{(x_i)_{i\in I} : x_i \in X_i, \left\|(x_i)_{i\in I}\right\|_\infty < \infty\right\}, \tag{B.2}$$

where $\left\|(x_i)_{i\in I}\right\|_\infty := \sup_{i\in I}\|x_i\|$. We may take I to be a directed set such that the elements $(x_i)_{i\in I}$ of $\ell_\infty(I, X_i)$ are nets.

Proposition B.17. *The space $(\ell_\infty(I, X_i), \|\cdot\|_\infty)$ is a Banach space.*

Proof. It is clear that $\ell_\infty(I, X_i)$ is a linear space over $\mathbb{K}$ with pointwise operations. Moreover it is easy to check that $\|\cdot\|_\infty$ does in fact define a norm on $\ell_\infty(I, X_i)$. So $(\ell_\infty(I, X_i), \|\cdot\|_\infty)$ is a normed space. What is left to show is completeness.

Let $(x^n)_{n\geq 1}$, where $x^n = (x_i^n)_{i\in I}$ for all $n \geq 1$, be a Cauchy sequence in $\ell_\infty(I, X_i)$. Let $j \in I$ and note that

$$\left\|x_j^n - x_j^m\right\| \leq \sup_{i\in I}\|x_i^n - x_i^m\| = \|x^n - x^m\|_\infty .$$

So the sequence $(x_j^n)_{n\geq 1}$ in the Banach space X_j is Cauchy and hence convergent. For each $i \in I$, let $x_i \in X_j$ be the limit of $(x_i^n)_{n\geq 1}$. We claim that $x := (x_i)_{i\in I}$ is the limit of $(x^n)_{n\geq 1}$. Given $\varepsilon > 0$, we can choose $N \geq 1$ such that $\|x^n - x^m\|_\infty < \varepsilon/3$, for all $n, m \geq N$. Hence for all $i \in I$, and $n, m \geq N$, we have $\|x_i^n - x_i^m\| < \varepsilon/3$. Now for all $i \in I$ and all $n \geq N$

we can find an $m \geq n$ such that $\|x_i^m - x_i\| < \varepsilon/3$. It follows by the triangle inequality that $\|x_i^n - x_i\| \leq \|x_i^n - x_i^m\| + \|x_i^m - x_i\| < 2\varepsilon/3$. Because this holds for all $i \in I$ we can take the supremum over I on each side, and obtain

$$\|x^n - x\|_\infty = \sup_{i \in I} \|x_i^n - x_i\| \leq 2\varepsilon/3 < \varepsilon.$$

This shows that $(x^n)_{n \geq 1}$ converges to x as claimed. Note that x is an element of $\ell_\infty(I, X_i)$ since $\|x\|_\infty \leq \|x - x^n\|_\infty + \|x^n\|_\infty$ and Cauchy sequences are bounded. □

Lemma B.18. *If $\mathcal{U}$ is an ultrafilter on I and $(x_i)_{i \in I}$ is any element of $\ell_\infty(I, X_i)$, then $\lim_{\mathcal{U}} \|x_i\|$ exists and is unique.*

Proof. Let $K := \|(x_i)_{i \in I}\|_\infty = \sup_{i \in I} \|x_i\|$. Then $(\|x_i\|)_{i \in I}$ is a net in the compact Hausdorff space $[0, K]$. Existence and uniqueness of the limit $\lim_{\mathcal{U}} \|x_i\|$ along the ultrafilter $\mathcal{U}$ follows from Theorem B.6 and Theorem B.8. □

Let $\mathcal{U}$ be an ultrafilter on the index set I and define

$$N_{\mathcal{U}} := \left\{ (x_i)_{i \in I} \in \ell_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0 \right\}. \tag{B.3}$$

Proposition B.19. *$N_{\mathcal{U}}$ is a closed linear subspace of $\ell_\infty(I, X_i)$.*

Proof. Let $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ be elements of $N_{\mathcal{U}}$ and let $\alpha \in \mathcal{K}$. The nets $(x_i + y_i)_{i \in I}$ and $(\alpha x_i)_{i \in I}$ are elements of $\ell_\infty(I, X_i)$. Note that the limits $\lim_{\mathcal{U}} \|x_i + y_i\|$ and $\lim_{\mathcal{U}} \|\alpha x_i\|$ exist and are unique by Lemma B.18. By Proposition B.9 and B.12 we know that the map $(z_i)_{i \in I} \mapsto \lim_{\mathcal{U}} \|z_i\|$ is linear, positive and increasing.

$$\begin{aligned} 0 \leq \lim_{\mathcal{U}} \|x_i + y_i\| &\leq \lim_{\mathcal{U}} (\|x_i\| + \|y_i\|) = \lim_{\mathcal{U}} \|x_i\| + \lim_{\mathcal{U}} \|y_i\| = 0, \\ \lim_{\mathcal{U}} \|\alpha x_i\| &= \lim_{\mathcal{U}} |\alpha| \, \|x_i\| = |\alpha| \lim_{\mathcal{U}} \|x_i\| = 0. \end{aligned}$$

Hence $(x_i + y_i)_{i \in I}$ and $(\alpha x_i)_{i \in I}$ are elements of $N_{\mathcal{U}}$, showing that $N_{\mathcal{U}}$ is a linear subspace.

To show that $N_{\mathcal{U}}$ is closed, let $(x^n)_{n \geq 1}$ be a sequence in $N_{\mathcal{U}}$, i.e., $x^n = (x_i^n)_{i \in I}$ where $\lim_{\mathcal{U}} \|x_i^n\| = 0$ for all $n \geq 1$, and suppose that $(x^n)_{n \geq 1}$ converges to some $x = (x_i)_{i \in I}$ in $\ell_\infty(I, X_i)$. By Lemma B.18 we know that $\lim_{\mathcal{U}} \|x_i\|$ exists and is unique. We want to show that $\lim_{\mathcal{U}} \|x_i\| = 0$. Let O is any open neighborhood in $\mathbb{R}$ of 0, and choose $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq O$. Choose $n_\varepsilon \geq 1$ such that $\|x^n - x\|_\infty = \sup_{i \in I} \|x_i^n - x_i\| < \varepsilon/2$ for all $n \geq n_\varepsilon$. Then $\|x_i - x_i^{n_\varepsilon}\| < \varepsilon/2$ for all $i \in I$. Moreover $\|x_i\| \leq \|x_i - x_i^{n_\varepsilon}\| + \|x_i^{n_\varepsilon}\|$, so

$$\{i \in I : \|x_i^{n_\varepsilon}\| < \varepsilon/2\} \subseteq \{i \in I : \|x_i\| < \varepsilon\} \subseteq \{i \in I : \|x_i\| \in O\}.$$

Since $\lim_{\mathcal{U}} \|x_i^{n_\varepsilon}\| = 0$ we have $\{i \in I : \|x_i^{n_\varepsilon}\| < \varepsilon/2\} \in \mathcal{U}$ and hence, the properties of filters, $\{i \in I : \|x_i\| \in O\} \in \mathcal{U}$ as well. This shows that $\lim_{\mathcal{U}} \|x_i\| = 0$ and therefore that $x \in N_{\mathcal{U}}$. □

Now that we know that $(\ell_\infty(I, X_i), \|\cdot\|_\infty)$ is a Banach space, and that $N_{\mathcal{U}} \subseteq \ell_\infty(I, X_i)$ is a closed subspace, it follows from Proposition B.16 that $\ell_\infty(I, X_i)/N_{\mathcal{U}}$ is a Banach space with the quotient norm

$$\|[(x_i)_{i \in I}]\|_{\mathcal{U}} = \inf \left\{ \|(x_i)_{i \in I} + (y_i)_{i \in I}\|_\infty : (y_i)_{i \in I} \in N_{\mathcal{U}} \right\}. \tag{B.4}$$

The next proposition shows that there is a particularly nice expression for the quotient norm.

Proposition B.20. *The quotient norm on $\ell_\infty(I, X_i)/N_{\mathcal{U}}$ is given by*

$$\|[(x_i)_{i \in I}]\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_i\|, \qquad \text{for all } [(x_i)_{i \in I}] \in \ell_\infty(I, X_i)/N_{\mathcal{U}}.$$

Proof. Take $[(x_i)_{i\in I}]$ in $\ell_\infty(I, X_i)/N_{\mathcal{U}}$. For any $(y_i)_{i\in I} \in N_{\mathcal{U}}$ we have the equality

$$\lim_{\mathcal{U}} \|x_i + y_i\| = \lim_{\mathcal{U}} \|x_i\| .$$

This can be easily checked using the triangle inequality, together with the definition of $N_{\mathcal{U}}$ and the fact that $N_{\mathcal{U}}$ is a subspace.

Given any $\varepsilon > 0$ we can take $(y_i)_{i\in I} \in N_{\mathbb{U}}$ such that $\left\|(x_u + y_u)_{u\in I}\right\|_\infty \leq \left\|[(x_i)_{i\in I}]\right\|_{\mathcal{U}} + \varepsilon$. So for all $j \in I$ we have $\|x_j + y_j\| \leq \left\|(x_u + y_u)_{u\in I}\right\|_\infty \leq \left\|[(x_i)_{i\in I}]\right\|_{\mathcal{U}} + \varepsilon$, and by Proposition B.12 this inequality holds true when taking the limit along $\mathcal{U}$ on either side. Hence

$$\lim_{\mathcal{U}} \|x_i\| = \lim_{\mathcal{U}} \|x_i + y_i\| \leq \left\|[(x_i)_{i\in I}]\right\|_{\mathcal{U}} + \varepsilon.$$

Since $\varepsilon > 0$ were arbitrary, it follows that $\lim_{\mathcal{U}} \|x_i\| \leq \left\|[(x_i)_{i\in I}]\right\|_{\mathcal{U}}$.

On the other hand, suppose for contradiction that this inequality is strict. Then we can find an $r > 0$ such that

$$\lim_{\mathcal{U}} \|x_i\| < r < \left\|[(x_i)_{i\in I}]\right\|_{\mathcal{U}} .$$

Let $A_r := \{i \in I : \|x_i\| < r\}$. By assumption on $\lim_{\mathcal{U}} \|x_i\|$ the set A_r is in $\mathcal{U}$. Define the net $(y_i)_{i\in I}$ in $\ell_\infty(I, X_i)$ by $y_i = 0$ for all $i \in A_r$ and $y_i = -x_i$ otherwise. Given any $\varepsilon > 0$, we have by definition of this net that $A_r \subseteq \{i \in I : \|y_i\| < \varepsilon\}$, showing that $\lim_{\mathcal{U}} \|y_i\| = 0$. In other words, $(y_i)_{i\in I} \in N_{\mathcal{U}}$. Now observe that $x_i + y_i = x_i$ for $i \in A_r$ and 0 otherwise. Then

$$\sup_{i\in I} \|x_i + y_i\| = \sup_{i\in A_r} \|x_i\| \leq r < \left\|[(x_i)_{i\in I}]\right\|_{\mathcal{U}} \leq \left\|(x_i + y_i)_{i\in I}\right\|_\infty = \sup_{i\in I} \|x_i + y_i\| ,$$

which is a contradiction. We conclude that $\left\|[(x_i)_{i\in I}]\right\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_i\|$ as wanted. □

Definition B.5. The Banach space $\ell_\infty(I, X_i)/N_{\mathcal{U}}$ is called the *ultraproduct* of $(X_i)_{i\in I}$ with respect to $\mathcal{U}$ and is denoted by $(\prod_{i\in I} X_i)/\mathcal{U}$ or by $\prod_{\mathcal{U}} X_i$. If $X_i = X$ for all $i \in I$ then it is called the *ultrapower* of X with respect to $\mathcal{U}$, and is usually denoted by $X_{\mathcal{U}}$.

B.2.3 Ultraproducts of operators

Let $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ be two families of Banach spaces indexed over I, and suppose for each $i \in I$ that $T_i : X_i \to Y_i$ is a bounded linear map. We aim to define the ultraproduct of the family $(T_i)_{i\in I}$ as a map from $\prod_{i\in I} X_i/\mathcal{U}$ to $\prod_{i\in I} Y_i/\mathcal{U}$.

Remark B.21. Let $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ be two families of Banach spaces and suppose we have for each $i \in I$ a bounded linear map $T_i : X_i \to Y_i$. Suppose further that $\sup_{i\in I} \|T_i\| < \infty$. The equivalence class $[(T_i(x_i))_{i\in I}]$ in $\prod_{i\in I} Y_i/\mathcal{U}$ is independent of the representative of $[(x_i)_{i\in I}]$ in $\prod_{i\in I} X_i/\mathcal{U}$.

Proof. Suppose that $(x_i)_{i\in I}$ and $(x_i')_{i\in I}$ are representatives of the same equivalence class in $\prod_{i\in I} X_i/\mathcal{U}$. Then

$$\lim_{\mathcal{U}} \left\|T_i(x_i - x_i')\right\| \leq \lim_{\mathcal{U}} \|T_i\| \left\|x_i - x_i'\right\| \leq \sup_{i\in I} \|T_i\| \lim_{\mathcal{U}} \left\|x_i - x_i'\right\| = 0.$$

Hence $(T_i(x_i))_{i\in I}$ and $(T_i(x_i'))_{i\in I}$ represent the same equivalence class in $\prod_{i\in I} Y_i/\mathcal{U}$. □

Definition B.6. Let $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ be two families of Banach spaces and suppose we have for each $i \in I$ a bounded linear map $T_i : X_i \to Y_i$. Suppose further that $\sup_{i\in I} \|T_i\| < \infty$. The map $(T_i)_{\mathcal{U}} : \prod_{i\in I} X_i/\mathcal{U} \to \prod_{i\in I} Y_i/\mathcal{U}$ given by

$$(T_i)_{\mathcal{U}}([(x_i)_{i\in I}]) = [(T_i(x_i))_{i\in I}], \qquad \text{for all } [(x_i)_{i\in I}] \in \prod_{i\in I} X_i/\mathcal{U},$$

is called the ultraproduct of the family of operators $(T_i)_{i\in I}$.

Note that Remark B.21 secures that $(T_i)_{\mathcal{U}}$ is a well-defined operator.

Proposition B.22. *The operator norm of the ultraproduct $(T_i)_{\mathcal{U}}$ of a family of operators $(T_i)_{i\in I}$, where $T_i : X_i \to Y_i$, for each $i \in I$, is given by*

$$\|(T_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|T_i\| .$$

Proof. Let $[(x_i)_{i\in I}] \in \prod_{i\in I} X_i/\mathcal{U}$ and recall from Proposition B.9 that the ultralimit map is multiplicative. We have that

$$\left\|(T_i)_{\mathcal{U}}([(x_i)_{i\in I}])\right\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|T_i(x_i)\| \leq \lim_{\mathcal{U}} \|T_i\| \|x_i\| = \lim_{\mathcal{U}} \|T_i\| \lim_{\mathcal{U}} \|x_i\| = \left\|[(x_i)_{i\in I}]\right\|_{\mathcal{U}} \lim_{\mathcal{U}} \|T_i\| .$$

Hence $\|(T_i)_{\mathcal{U}}\| \leq \lim_{\mathcal{U}} \|T_i\|$. On the other hand, given $\varepsilon > 0$ pick for each $i \in I$ an element $x_i \in X_i$ such that $\|T_i(x_i)\| \geq \|T_i\| - \varepsilon$. We get by linearity of the ultralimit map that

$$\|(T_i)_{\mathcal{U}}\| \geq \left\|(T_i)_{\mathcal{U}}([(x_i)_{i\in I}])\right\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|T_i(x_i)\| \geq \lim_{\mathcal{U}}(\|T_i\| - \varepsilon) = \lim_{\mathcal{U}} \|T_i\| - \varepsilon.$$

Since this holds for all $\varepsilon > 0$ we conclude that $\|(T_i)_{\mathcal{U}}\| \geq \lim_{\mathcal{U}} \|T_i\|$. □

B.2.4 Ultraproducts of Hilbert spaces

We will show in this section that if $(H_i)_{i\in I}$ is a family of Hilbert spaces then the Banach space ultraproduct $\prod_{i\in I} H_i/\mathcal{U}$ is again a Hilbert space.

Proposition B.23. *Let $\mathcal{U}$ be a free ultrafilter on I and let $(H_i)_{i\in I}$ be a family of Hilbert spaces. Then $\prod_{i\in I} H_i/\mathcal{U}$ is a Hilbert space.*

Proof. It is a well-known fact that a norm comes from an inner product if and only if it satisfies the parallelogram law. Using this fact together with Proposition B.20 and Proposition B.9 we see that

$$\begin{aligned}
&\left\|[(x_i)_{i\in I}] + [(y_i)_{i\in I}]\right\|_{\mathcal{U}}^2 + \left\|[(x_i)_{i\in I}] - [(y_i)_{i\in I}]\right\|_{\mathcal{U}}^2 \\
&\qquad = \lim_{\mathcal{U}} \|x_i + y_i\|^2 + \lim_{\mathcal{U}} \|x_i - y_i\|^2 = \lim_{\mathcal{U}}(\|x_i + y_i\|^2 + \|x_i - y_i\|^2) \\
&\qquad = \lim_{\mathcal{U}}(2\|x_i\|^2 + 2\|y_i\|^2) = 2\lim_{\mathcal{U}} \|x_i\|^2 + 2\lim_{\mathcal{U}} \|y_i\|^2 \\
&\qquad = 2\left\|[(x_i)_{i\in I}]\right\|_{\mathcal{U}}^2 + 2\left\|[(y_i)_{i\in I}]\right\|_{\mathcal{U}}^2 .
\end{aligned}$$

We conclude by the aforementioned fact that the norm on the Banach space ultraproduct $\prod_{i\in I} H_i/\mathcal{U}$ comes from an inner product, and hence that $\prod_{i\in I} H_i/\mathcal{U}$ is a Hilbert space. □

B.2.5 Ultraproducts of Banach algebras and C^*-algebras

Let $(X_i)_{i\in I}$ be a family of Banach algebras over $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$. We will show that the Banach space ultraproduct $\prod_{i\in I} X_i/\mathcal{U}$ constructed in the previous section is also a Banach algebra.

When all X_i are Banach algebras, a multiplication on $\ell_\infty(I, X_i)$ is naturally defined pointwise. Moreover, it is a straightforward consequence of Proposition B.9 and B.12 that the quotient norm $\lim_{\mathcal{U}} \|x_i\|$ satisfies the inequality

$$\lim_{\mathcal{U}} \|x_i y_i\| \leq \lim_{\mathcal{U}} \|x_i\| \lim_{\mathcal{U}} \|x_i\| . \tag{B.5}$$

Proposition B.24. *$N_{\mathcal{U}}$ is a two-sided ideal in $\ell_\infty(I, X_i)$.*

Proof. Let $(x_i)_{i\in I} \in N_{\mathcal{U}}$ and let $(y_i)_{i\in I} \in \ell_\infty(I, X_i)$. Then

$$0 \leq \lim_{\mathcal{U}} \|x_i y_i\| \leq \lim_{\mathcal{U}} \|x_i\| \lim_{\mathcal{U}} \|y_i\| = 0 \cdot \lim_{\mathcal{U}} \|y_i\| = 0,$$
$$0 \leq \lim_{\mathcal{U}} \|y_i x_i\| \leq \lim_{\mathcal{U}} \|y_i\| \lim_{\mathcal{U}} \|x_i\| = \lim_{\mathcal{U}} \|y_i\| \cdot 0 = 0,$$

showing that $(x_i y_i)_{i\in I} \in N_{\mathcal{U}}$ and $(y_i x_i)_{i\in I} \in N_{\mathcal{U}}$. □

Define a multiplication on $\prod_{i\in I} X_i/\mathcal{U}$ as follows:

$$[(x_i)_{i\in I}][(y_i)_{i\in I}] := [(x_i y_i)_{i\in I}]. \tag{B.6}$$

This is a well-defined operation by Proposition B.24. Indeed if $[(x_i)_{i\in I}] = [(x_i')_{i\in I}]$ and $[(y_i)_{i\in I}] = [(y_i')_{i\in I}]$, i.e., if $(x_i - x_i')_{i\in I}, (y_i - y_i')_{i\in I} \in N_{\mathcal{U}}$, we have that

$$[(x_i y_i)_{i\in I}] - [(x_i' y_i')_{i\in I}] = [\big((x_i - x_i')y_i\big)_{i\in I}] + [\big(x_i'(y_i - y_i')\big)_{i\in I}] = [0],$$

or equivalently $[(x_i y_i)_{i\in I}] = [(x_i' y_i')_{i\in I}]$. We conclude that the multiplication defined in equation (B.6) is independent of the choice of representative of the equivalence class. As $\lim_{\mathcal{U}} \|\cdot\|$ is the norm on the quotient space, and we have already noted that equation (B.5) is satisfied, we conclude that $\prod_{i\in I} X_i/\mathcal{U}$ is a Banach algebra.

Now if all X_i's are unital, it is straightforward to check that $\prod_{i\in I} X_i/\mathcal{U}$ is unital as well with $[(\mathbf{1}_i)_{i\in I}]$ as the multiplicative identity. Moreover $\lim_{\mathcal{U}} \|\mathbf{1}_i\| = 1$ as required in a unital Banach algebra.

Suppose next that $(X_i)_{i\in I}$ is a family of C^*-algebras over $\mathbb{K}$. We will show that $\prod_{i\in I} X_i/\mathcal{U}$ is then also a C^*-algebra. Define an involution on $(X_i, \|\cdot\|)_{i\in I}$ as follows:

$$[(x_i)_{i\in I}]^* := [(x_i^*)_{i\in I}]. \tag{B.7}$$

That this is a well-defined operation is easily checked using Proposition B.11. Moreover it is immediately clear that

- $([(x_i)_{i\in I}]^*)^* = [(x_i)_{i\in I}]$,
- $\big(\alpha[(x_i)_{i\in I}] + \beta[(y_i)_{i\in I}]\big)^* = \overline{\alpha}[(x_i)_{i\in I}]^* + \overline{\beta}[(y_i)_{i\in I}]^*$,
- $\big([(x_i)_{i\in I}][(y_i)_{i\in I}]\big)^* = [(y_i)_{i\in I}]^*[(x_i)_{i\in I}]^*$,

for all $\alpha, \beta \in \mathbb{C}$, $[(x_i)_{i\in I}], [(y_i)_{i\in I}] \in \prod_{i\in I} X_i/\mathcal{U}$. So the map defined in equation (B.7) is indeed an involution. Finally it is a straightforward calculation using Proposition B.9 to show that the quotient norm $\lim_{\mathcal{U}} \|x_i\|$ satisfy the C^*-identity.

$$\lim_{\mathcal{U}} \|x_i^* x_i\| = \lim_{\mathcal{U}} \|x_i\|^2 = \left(\lim_{\mathcal{U}} \|x_i\|\right)^2.$$

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