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Amenability and Growth of Finitely Generated Groups

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Abstract

In this thesis, we present the notions of growth and amenability of groups and discuss the interrelation between them. We develop the theory of growth of finitely generated groups and find equivalent ways to determine the growth of a group. We give examples of classes of groups having polynomial and exponential growth, respectively. In 1968, Milnor posed the question of the existence of groups with intermediate growth between polynomial and exponential. Motivated by this problem, we will consider the Grigorchuk group Γ that Grigorchuk exhibited in 1983 and see that it does, in fact, have intermediate growth along with other interesting properties.

We also discuss the amenable groups in relation with paradoxicality and consider two specific subclasses; namely, the supramenable and the elementary amenable groups. We show that if a finitely generated group does not have exponential growth, then it is supramenable, and that finitely generated elementary amenable groups cannot have intermediate growth. In particular, Grigorchuk's group solved in the negative Day's question whether the amenable groups coincide with the elementary amenable ones. It is still an open question whether there are supramenable groups of exponential growth.

Resumé

I dette projekt vil vi præsentere begreberne vækst og amenabilitet for grupper og diskuterer, hvordan de forholder sig til hinanden. Vi udvikler teorien om vækst af endeligt frembragte grupper og finder ækvivalente måder at bestemme en gruppes vækst. Vi giver eksempler på klasser af grupper med henholdsvis polynomiel og eksponentiel vækst. I 1968 fremsatte Milnor spørgsmålet om, hvorvidt der findes grupper med mellemværende vækst mellem den polynomielle og den eksponentielle. Motiveret af dette problem vil vi betragte Grigorchuks gruppe Γ, som Grigorchuk præsenterede i 1983, og se, at den faktisk har mellemværende vækst sammen med andre interessante egenskaber.

Vi vil også diskutere de amenable grupper i relation med paradoksikalitet og betragte to specifikke underklasser; nemlig de supramenable og de elementært amenable grupper. Vi viser, at hvis en endelig frembragt gruppe ikke har eksponentiel vækst, så er den supramenabel, og at endeligt frembragte elementært amenable grupper ikke kan have mellemværende vækst. Specielt gav Grigorchuks gruppe et negativt svar på Days spørgsmål om, hvorvidt de amenable grupper er sammenfaldende med de elementært amenable. Problemet om eksistensen af supramenable grupper med eksponentiel vækst er stadig åbent.

Contents

Introduction

We wish to discuss how the notion of amenability relates to the notion of growth of finitely generated groups. For this, we shall follow mainly the presentation from [CSC10]. The interested reader is encouraged to also read the article of Rosenblatt, see [Ros74]. This thesis is divided into three chapters.

In Chapter 1, we shall develop the theory of growth of finitely generated groups. By interpreting the group elements as vertices in a Cayley graph, it is possible to define a metric on the group. The growth function then counts the (finite) number of elements in the ball of radius n. After defining a certain equivalence relation between such growth functions, we can consider the equivalence classes under this relation, the growth type of a finitely generated group, which turns out to be independent of the generating subset. We give examples of classes of groups having polynomial and exponential growth, respectively, and we shall refer to this by saying that the group has *regular* growth. Note that this term is not standard; the author has been unable to find an adequate term in the literature. If a group has neither polynomial nor exponential growth, we say that it has *intermediate* growth.

Chapter 2 is dedicated to the construction and study of the Grigorchuk group which Rostislav Grigorchuk exhibited in 1983 (see [Gri83] and [Gri84]). As already mentioned, we shall follow mainly [CSC10], but we shall follow also some of the notations and ideas from [dlH00] and denote this group by Γ. Central to the study of Grigorchuk's group are the problems of Burnside and Milnor, respectively. In 1902, William Burnside asked whether all finitely generated periodic groups are finite. Theorem 2.2.5 asserts that Γ is a finitely generated periodic group which is infinite. In 1968, John Milnor posed the question of the existence of groups with intermediate growth. Section 2.3 is devoted to showing that Γ is an example of a group with intermediate growth. In fact, this was the first example of such a group.

In Chapter 3, we first discuss the notion of amenability which was originally introduced in order to explain the Banach-Tarski paradox. We shall use the definition given by John von Neumann in 1929. A discrete group is amenable if there exists an invariant finitely additive probability measure on the power set of the group. The class of amenable groups, denoted by AG, has nice permanence properties and it coincides with the non-paradoxical groups. However, even though groups of subexponential growth are amenable (Corollary 3.2.13), there are examples of amenable groups with exponential growth (see Section 3.4.1). This motivates us to look at two subclasses of AG; namely, the supramenable groups SG, introduced by Rosenblatt in [Ros74] and the elementary amenable groups EG, introduced by Day in [Day57]. A group G is supramenable, if given a subset $A \subseteq G$ there exists a finitely additive G-invariant measure on the power set of G that normalizes A. A group is elementary amenable if it can be constructed from the finite or the abelian groups by applying the process of taking subgroups, quotients, extensions or direct unions. Using an article of Chou ([Cho80]), we show that it suffices to use only the process of taking extension or direct unions. Moreover, we show that finitely generated elementary amenable groups have regular growth. Note that the Grigorchuk group shows that $EG \neq AG$, thus answering in the negative Day's question about the coincidence of these two classes.

We follow the argument in [Wag93] to show that groups of subexponential growth are supramenable (Theorem 3.2.12), and hence amenable. The example in Section 3.4.1 shows that SG is a proper subclass of AG. It is an important open question whether subexponential growth characterizes supramenable groups.

We have also included two appendices, one on *Means and measures* and another on *Residually finiteness*. The former includes a theorem showing that a group is amenable if and only if it admits an invariant mean (Theorem A.2.1) and the Representation Theorem shows that there is a one to one correspondence between the set of invariant means and the set of invariant measures on a group (Theorem A.1.6). The latter formulates the notion of residually finiteness in terms of the residual subgroup (Theorem B.0.7).

The reader is only assumed to be familiar with basic group theory and point-set topology. However, when we discuss the elementary amenable groups, we shall also invoke the principle of transfinite induction.

A small remark on the notation used in this thesis.

$\mathcal{P}(\Omega)$	The power set of the set Ω,
$\lvert A \rvert$	The cardinality of the set A,
$A - B$	The set difference between the sets A and B,
$\coprod$	Disjoint union,
$\mathbb{N}$	The positive integers,
$\mathbb{N}_0$	The positive integers and 0,
$\mathbb{Z}$	The integers,
$\mathbb{Q}$	The rational numbers,
$\mathbb{R}$	The real numbers,
Γ	The Grigorchuk group,
$\mathscr{F}_n$	The free group on n generators, for $n \in \mathbb{N} \cup \{\infty\}$.

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Chapter 1

Growth of finitely generated groups

1.1 The word metric and the Cayley graph metric

A group G is *finitely generated* if there exists a finite subset $S \subseteq G$ such that any group element can be written as a product of elements in S. The generating subset S is said to *symmetric* if $S = S^{-1}$. If G is finitely generated by a subset $S \subseteq G$, then $S \cup S^{-1}$ is a symmetric and finite generating subset of G.

Definition 1.1.1 (S-word-length)**.** *Let G be a finitely generated group and let $S \subseteq G$ be a finite symmetric generating subset. For each fixed $g \in G$, we define the* S-word-length *as*

$$\ell_S^G(g) = \min\{n \geqslant 0 \mid g = s_1 \cdots s_n,\ s_1, \ldots, s_n \in S\} \tag{1.1}$$

When there is no fear of ambiguity we shall omit the superscript.

Note that $\ell_S^G(g) = 0$ if and only if $g = e_G$. In the following, let $S \subseteq G$ be a finite symmetric generating subset of a group G.

Proposition 1.1.2. *For all $g, h \in G$, we have*

$$\ell_S(g) = \ell_S(g^{-1}) \tag{1.2}$$

$$\ell_S(gh) \leqslant \ell_S(g) + \ell_S(h) \tag{1.3}$$

Proof. Take $g, h \in G$. If any of these is the identity element the statement is trivial, so suppose $g, h \neq e_G$. There exist natural numbers n and m such that $g = s_1 \cdots s_n$ and $h = t_1 \cdots t_m$, where $s_1, \ldots, s_n$ and $t_1, \ldots, t_m$ are elements in S. Since $g^{-1} = s_n^{-1} \cdots s_1^{-1}$, we have $\ell_S(g^{-1}) \leqslant n = \ell_S(g)$. The converse follows analogously. Also, $gh = s_1 \cdots s_n t_1 \cdots t_m$, hence $\ell_S(gh) \leqslant n + m = \ell_S(g) + \ell_S(h)$. □

This construction induces a metric on the group G.

Proposition 1.1.3 (The word metric)**.** *The map $d_S : G \times G \to \mathbb{N}_0$ given by*

$$d_S(g, h) = \ell_S(g^{-1}h), \qquad g, h \in G \tag{1.4}$$

defines a metric on the group G.

Proof. Let $g, h, k \in G$. Note that $d_S(g, h) = \ell_S(g^{-1}h) = 0$ if and only if $g^{-1}h = e_G$, i.e., $g = h$. Symmetry follows from (1.2); namely,

$$d_S(g, h) = \ell_S(g^{-1}h) = \ell_S(h^{-1}g) = d_S(h, g). \tag{1.5}$$

By (1.3) we also have the triangle inequality:

$$d_S(g, h) = \ell_S(g^{-1}h) = \ell_S(g^{-1}kk^{-1}h) \leqslant \ell_S(g^{-1}k) + \ell_S(k^{-1}h) = d_S(g, k) + d_S(k, h). \tag{1.6}$$

This proves the claim. □

The *ball of radius* $n \geqslant 0$ in G centered at $g_0 \in G$ is denoted by $B_S^G(g_0, n) = \{g \in G \mid d_S(g, g_0) \leqslant n\}$. If $g_0 = e_G$ we simply write $B_S^G(n) = \{g \in G \mid \ell_S(g) \leqslant n\}$. We shall omit the superscript when there is no ambiguity. Next, we show that the word metric is invariant under left multiplication.

Proposition 1.1.4. *The action of G on itself is isometric with respect to the word metric, i.e.,*

$$d_S(g'g, g'h) = d_S(g, h) \tag{1.7}$$

for all $g', g, h \in G$.

Proof. The following simple calculation,

$$d_S(g'g, g'h) = \ell_S(g^{-1}(g')^{-1}g'h) = \ell_S(g^{-1}h) = d_S(g, h), \tag{1.8}$$

gives the result. □

It is possible to define another metric on the finitely generated group where the distance between elements is interpreted in terms of lengths of paths. In order to do this we need the notion of a *Cayley graph.*

Definition 1.1.5 (Cayley graph)**.** *The* Cayley graph *with respect to S is the graph $\mathcal{C}_S(G) = (V, E)$, where $V = G$ is the set of vertices and the set of edges is $E = \{(g, s, gs) \mid g \in G, s \in S\} \subseteq G \times S \times G$.*

The Cayley graph has certain properties which we shall outline now. Since S is symmetric the map $s \mapsto s^{-1}$ defines an involution on S and $\mathcal{C}_S(G)$ is *edge-symmetric*, i.e., if $e = (g, s, h) \in E$ then $h = gs \in G$ and the inverse edge $e^{-1} := (h, s^{-1}, g) = (gs, s^{-1}, (gs)s^{-1}) \in E$ is well-defined. In addition, the Cayley graph is *connected*, i.e., For any $g, h \in G$ write $g^{-1}h = s_1 \cdots s_n$, for some $s_1, \ldots, s_n \in S$ and consider the finite sequence of edges $e_1 = (g, s_1, gs_1)$, ... , $e_i = (gs_1 \cdots s_{i-1}, s_i, gs_1 \cdots s_{i-1}s_i)$, ... , $e_n = (gs_1 \cdots s_{n-1}, s_n, h)$; the path $\pi = (e_1, \ldots, e_n)$ connects g and h. A *loop* is an edge of the form $(g, s, g) \in E$. It is obvious from the definition that the Cayley graph has a loop at every vertex if and only if $e_G \in S$. Finally, notice that given an edge $(g, s, h) \in E$ the element $s = g^{-1}h \in S$ is uniquely determined; we say that $\mathcal{C}_S(G)$ has no *multiple edges.*

Given a path $\pi = (e_1, \ldots, e_n)$ in $\mathcal{C}_S(G)$, the positive integer n defines the *length* of π, denoted $\ell(\pi)$. If $e_1 = (g_1, s_1, g_1s_1)$ and $e_n = (g_n, s_n, g_ns_n)$ for $g_1, g_\in G$ and $s_1, s_2 \in S$, we put

$$\pi^+ = g_ns_n, \qquad \pi^- = g_1.$$

Given two paths $\pi_1 = (e_1, \ldots, e_n)$ and $\pi_2 = (e'_1, \ldots, e'_m)$ with $\pi_1^+ = \pi_2^-$ we can define the *composite path* as

$$\pi_1\pi_2 = (e_1, \ldots, e_n, e'_1, \ldots, e'_m).$$

It follows that $\ell(\pi_1\pi_2) = \ell(\pi_1) + \ell(\pi_2)$.

Proposition 1.1.6 (The Cayley graph metric). *The map $d_{\mathcal{C}_S(G)} : G \times G \to \mathbb{N}_0$ given by*

$$d_{\mathcal{C}_S(G)}(g, h) = \min\{\ell(\pi) \mid \pi \text{ connects } g \text{ and } h\}, \qquad g, h \in G \tag{1.9}$$

defines a metric on the group G.

Proof. Since $\mathcal{C}_S(G)$ is connected, the map is well-defined. Let $g, h, k \in G$. It is obvious from the construction that $d_{\mathcal{C}_S(G)}(g, h) = 0$ if and only if $g = h$. The Cayley graph is edge-connected, so if $\pi = (e_1, \ldots, e_n)$ is a path in $\mathcal{C}_S(G)$ connecting g to h, then the inverse path $\pi^{-1} = (e_n^{-1}, \ldots, e_1^{-1})$ connects h to g; in particular, $\ell(\pi) = \ell(\pi^{-1})$ and this proves symmetry of $d_{\mathcal{C}_S(G)}$.

Let π_1 and π_2 be paths of minimal length connecting g to h and h to k, respectively. As $\pi_1^+ = \pi_2^-$, the composite path $\pi_1\pi_2$ connects g to k and

$$d_{\mathcal{C}_S(G)}(g, k) \leqslant \ell(\pi_1\pi_2) = \ell(\pi_1) + \ell(\pi_2) = d_{\mathcal{C}_S(G)}(g, h) + d_{\mathcal{C}_S(G)}(h, k).$$

This proves the triangle inequality. □

A path π connecting two group elements g and h with minimal length (i.e., $\ell(\pi) = d_{\mathcal{C}_S(G)}(g, h)$) is called a *geodesic path*.

So far we have defined two ways of measuring the distance between two elements of a finitely generated group relative to a finite symmetric generating subset. The following theorem states that these are, in fact, the same.

Theorem 1.1.7. *Let $S \subseteq G$ be a finite symmetric generating subset of a group G. Then*

$$d_S(g, h) = d_{\mathcal{C}_S(G)}(g, h), \tag{1.10}$$

for all $g, h \in G$.

Proof. Let $g, h \in G$. If $g = h$ there is nothing to prove. Let $\pi = (e_1, \ldots, e_n)$ be a geodesic path in $\mathcal{C}_S(G)$ connecting g to h. The edges are of the form $e_i = (g_i, s_i, g_1 s_1 \cdots s_i)$, for $i = 1, \ldots, n$ so we can write $g^{-1}h = s_1 \cdots s_n$. Therefore,

$$d_S(g, h) = \ell_S(g^{-1}h) = \ell(s_1 \cdots s_n) \leqslant n = \ell(\pi) = d_{\mathcal{C}_S(G)}(g, h). \tag{1.11}$$

The last equality follows from π being a geodesic path.

On the other hand, suppose $d_S(g, h) = m$. By definition, there exist $t_1, \ldots, t_m \in S$ such that $h = g t_1 \cdots t_m$. Furthermore, there is a unique path $\pi' = (e'_1, \ldots, e'_m)$ where the edges are of the form $e'_j = (h_j, t_j, h_1 t_1 \cdots t_j)$, for $j = 1, \ldots, m$ and $(\pi')^- = h_1 = g$. Consequently, $(\pi')^+ = g t_1 \cdots t_m = h$ and

$$d_{\mathcal{C}_S(G)}(g, h) \leqslant \ell(\pi') = m = d_S(g, h), \tag{1.12}$$

as claimed □

Given an edge $(g, s, gs) \in E$ with $g \neq gs$, we say that g and gs are *neighbours*. It follows from the preceding proposition that if $g, h \in G$ then g and h are neighbours if and only if $d_S(g, h) = 1$. Also, the image of the bijective map $s \mapsto gs$ for some fixed $g \in G$ is merely the set of neighbours of g with cardinality $|S|$. Defining the *valence* of a vertex g, denoted by $\delta(g)$, as the cardinality of the set of its neighbours. It follows that the valence is constant for all $g \in G$. In particular, $\delta(g) = |S|$. We say that $\mathcal{C}_S(G)$ is *regular*.

1.2 Growth functions

In this section we shall consider the notion of a growth function. We shall first consider the notion of growth functions in general.

A *growth function* is a non-decreasing map $\gamma : \mathbb{N}_0 \to [0, \infty)$. If γ and γ' are two growth functions we say that γ' *dominates* γ if there exists a positive integer c such that

$$\gamma(n) \leqslant c\gamma'(cn), \qquad n \geqslant 1, \tag{1.13}$$

in which case we write $\gamma \preceq \gamma'$. Note that the statement is trivially true for $n = 0$. If, in addition, $\gamma' \preceq \gamma$, then we write $\gamma \sim \gamma'$ and say that the two growth functions are *equivalent*.

Proposition 1.2.1. *The relation $\sim$ is an equivalence relation.*

Proof. Reflexivity and symmetry are obvious properties of $\sim$ from the definition. In order to show transitivity let $\gamma_1, \gamma_2, \gamma_3 : \mathbb{N}_0 \to [0, \infty)$ be three growth functions such that $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$. In particular, there exist positive integers c_1 and c_2 such that $\gamma_1(n) \leqslant c_1\gamma_2(c_1 n)$ and $\gamma_2(n) \leqslant c_2\gamma_3(c_2 n)$. Put $c = c_1 c_2$ and observe that

$$\gamma_1(n) \leqslant c_1\gamma_2(c_1 n) \leqslant c_1 c_2 \gamma_3(c_1 c_2 n) = c\gamma_3(cn), \qquad n \geqslant 1 \tag{1.14}$$

i.e., $\gamma_1 \preceq \gamma_3$ and the relation $\preceq$ is transitive. The converse follows analogously. □

We shall denote the equivalence class under $\sim$ of a growth function γ by $[\gamma]$. Given two growth functions γ_1 and γ_2 we write $[\gamma_1] \preceq [\gamma_2]$ if γ_2 dominates γ_1. To see that this is well-defined let $\gamma_1, \gamma_2, \gamma_1', \gamma_2'$ be growth functions such that $\gamma_1 \sim \gamma_1'$, $\gamma_2 \sim \gamma_2'$ and $\gamma_1 \preceq \gamma_2$. In particular, we have $\gamma_1' \preceq \gamma_1$, $\gamma_1 \preceq \gamma_2$ and $\gamma_2 \preceq \gamma_2'$. By virtue of the transitivity of $\preceq$ we obtain $\gamma_1' \preceq \gamma_2'$. Note that $\preceq$ now defines a partial order relation on the set of equivalence classes of growth functions. Reflexivity and anti-symmetry are obvious from the definition while transitivity was proved above.

We can define a product on the set of equivalence classes of growth functions as follows. Given two growth functions γ_1 and γ_2 set $[\gamma_1] \cdot [\gamma_2] = [\gamma_1\gamma_2]$. Let us first clarify why this is well-defined: Let $\gamma_1, \gamma_2, \gamma_1', \gamma_2' : \mathbb{N}_0 \to [0, \infty)$ be growth functions such that $\gamma_1 \sim \gamma_1'$ and $\gamma_2 \sim \gamma_2'$. Pick positive integers c_1 and c_2 such that $\gamma_1(n) \leqslant c_1\gamma_1'(c_1 n)$ and $\gamma_2 \leqslant c_2\gamma_2'(c_2 n)$, for all $n \geqslant 1$, and set $c = c_1 c_2$. The product of growth functions is itself a growth function and

$$(\gamma_1\gamma_2)(n) = \gamma_1(n)\gamma_2(n) \leqslant c_1 c_2 \gamma_1'(c_1 c_2 n)\gamma_2'(c_1 c_2 n) = c(\gamma_1'\gamma_2')(cn), \tag{1.15}$$

for all $n \geqslant 1$. The converse follows analogously, so the above product is well-defined. Thus $\gamma_1\gamma_2 \sim \gamma_1'\gamma_2'$. We shall use this in Theorem 1.4.10.

Example 1.2.2. (i) Let $a, b \in [0, \infty)$. For all $n \geqslant 1$, we have $n^a \leqslant n^b$ (resp., $n^a = b^b$) if and only if $a \leqslant b$ (resp., $a = b$). Thus $n^a \preceq n^b$ (resp., $n^a \sim n^b$) if and only if $a \leqslant b$ (resp., $a = b$).

(ii) Let the growth function $\gamma : \mathbb{N}_0 \to [0,\infty)$ be a polynomial of degree $d \geqslant 0$. Then $\gamma \sim n^d$.

(iii) Let $a, b \in (1,\infty)$ such that $a \leqslant b$. Obviously, $a^n \leqslant b^n$ for all $n \geqslant 1$, so $a^n \preceq b^n$. On the other hand, put $c = \lfloor \log_a b \rfloor + 1$ where $\lfloor \cdot \rfloor$ denotes the integer part. Note that $c > 1$. For all $n \geqslant 1$,

$$b^n = a^{(\log_a b)n} \leqslant a^{(\lfloor \log_a b \rfloor + 1)n} \leqslant c a^{cn}, \tag{1.16}$$

which shows that $b^n \preceq a^n$, hence $a^n \sim b^n$. In particular, we have $a^n \sim \exp(n)$, for all $a \in (1,\infty)$.

(iv) Let $d \geqslant 0$ be an integer. Then $n^d \preceq \exp(n)$ but $n^d \nsim \exp(n)$. Note $\lim_{n\to\infty} \frac{n^d}{\exp(n)} = 0$, so the sequence $\left(\frac{n^d}{\exp(n)}\right)_{n\in\mathbb{N}}$ is bounded. This ensures the existence of a positive integer c such that

$$\frac{n^d}{\exp(n)} \leqslant c \tag{1.17}$$

from which it follows that $n^d \leqslant c\exp(cn)$; i.e., $n^d \preceq \exp(n)$. In order to reach a contradiction, assume that $\exp(n) \preceq n^d$. Then there exists a positive integer c such that $\exp(n) \leqslant c^{d+1} n^d$, i.e.,

$$\frac{\exp(n)}{n^d} \leqslant c^{d+1}, \tag{1.18}$$

for all $n \geqslant 1$. This is in contradiction with the fact that the sequence is not bounded. Thus $n^d \nsim \exp(n)$.

We are now in position to turn our attention toward the growth function of finitely generated groups.

Definition 1.2.3. *Let G be a group and let $S \subseteq G$ be a finite symmetric generating subset. The* growth function of G relative to S *is the map $\gamma_S^G : \mathbb{N}_0 \to \mathbb{N}$ given by*

$$\gamma_S^G(n) := |B_S^G(n)| = |\{g \in G \mid \ell_S(g) \leqslant n\}|, \qquad n \geqslant 0. \tag{1.19}$$

When there is no fear of ambiguity we shall omit the superscript.

Observe that $\gamma_S(0) = |\{e_G\}| = 1$ and $\gamma_S(n) \leqslant \gamma_S(n+1)$, for all $n \geqslant 0$; thus, γ_S^G is indeed a growth function. It follows from the construction that the map $\varphi : (S \cup \{e_G\})^n \to B_S(n)$ given by $\varphi(s_1, \dots, s_n) = s_1 \cdots s_n$ with $s_i \in S \cup \{e_G\}$ is surjective (however, not injective). Hence γ_S satisfies

$$\gamma_S(n) \leqslant |S \cup \{e_G\}|^n, \qquad n \geqslant 0. \tag{1.20}$$

In particular, γ_S^G takes only finite values.

Lemma 1.2.4. *Let G be a finitely generated group. Let S and S' be two finite symmetric generating subsets of G and put $c = \max\{\ell_{S'}(s) \mid s \in S\}$. Then*

(i) $\ell_{S'}(g) \leqslant c\ell_S(g), \qquad g \in G,$

(ii) $d_{S'}(g,h) \leqslant c d_S(g,h), \qquad g,h \in G,$

(iii) $B_S(n) \subseteq B_{S'}(cn), \qquad n \geqslant 0,$

Proof. *(i)*: Fix $g \in G$ and suppose $\ell_S(g) = n$. Then there exist $s_1, \ldots, s_n \in S$ such that $g = s_1 \cdots s_n$ and

$$\ell_{S'}(g) = \ell_{S'}(s_1 \cdots s_n) \leqslant \sum_{i=1}^{n} \ell_{S'}(s_i) \leqslant cn, \tag{1.21}$$

wherein the first inequality follows from (1.3).

(ii): Fix $g,h \in G$. By (i), we have

$$d_{S'}(g,h) = \ell_{S'}(g^{-1}h) \leqslant c\ell_S(g^{-1}h) = cd_S(g,h). \tag{1.22}$$

(iii): To show the inclusion, let $g \in B_S(n)$ for some $n \geqslant 0$; i.e., $\ell_S(g) \leqslant n$. By (i) this implies that $\ell_{S'}(g) \leqslant cn$, which is equivalent to $g \in B_{S'}(cn)$. □

We say that a finitely generated group G has *exponential* (resp., *subexponential*) *growth* if $\gamma(G) \sim \exp(n)$ (resp., $\gamma(G) \nsim \exp(n)$); it has *polynomial growth* if there is an integer $d \geqslant 0$ such that $\gamma(G) \preceq n^d$. If the growth type is neither exponential nor polynomial we say that G has *intermediate* growth. Also, we say that a group has *regular* growth if it has polynomial or exponential growth.

We state the result of Example 1.2.2 (iv) and formula (1.20) as a proposition.

Proposition 1.2.5. *Polynomial growth implies subexponential growth. A finitely generated group can maximally have exponential growth.*

It is inconvenient to describe the growth of a finitely generated group *relative* to a generating subset. Therefore, we would like a notion of growth which is independent of generating subsets. First, we need an intermediate result.

Proposition 1.2.6. *Let G be a finitely generated group and let S and S' be two finite symmetric generating subsets of G. Then*

(i) The two word metrics d_S and $d_{S'}$ induce the same topology,

(ii) The growth functions γ_S and $\gamma_{S'}$ are equivalent,

Proof. *(i)*: Put $c := \max\{\ell_{S'}(s) \mid s \in S\}$ and $c' := \max\{\ell_S(s') \mid s' \in S'\}$ and take $g,h \in G$. If $c = 0$ or $c' = 0$ there is nothing to prove, so suppose $c, c' \neq 0$. By Lemma 1.2.4 (ii), we have

$$\frac{1}{c'} d_s(g,h) \leqslant d_{S'}(g,h) \leqslant cd_S(g,h), \tag{1.23}$$

which shows that the two metrics induce the same topology.

(ii): An immediate consequence of Lemma 1.2.4 (iii) is

$$\gamma_S(n) = |B_S(n)| \leqslant |B_{S'}(cn)| = \gamma_{S'}(cn) \leqslant c\gamma_{S'}(cn), \qquad n \geqslant 1 \tag{1.24}$$

Thus $\gamma_S \preceq \gamma_{S'}$. The converse follows analogously by using $c' = \max\{\ell_S(s') \mid s' \in S'\}$. □

In light of this result, we can introduce the notion of the *growth type* of a finitely generated group.

Definition 1.2.7 (Growth type)**.** *Let $S \subseteq G$ be a finite symmetric generating subset of a group G. The equivalence class $[\gamma_S]$ associated to the growth function of G relative to S is called the* growth type *of G and we write $\gamma(G)$.*

Note that the growth type is independent of a generating subset. By abuse of notation, we shall some times write $\gamma(G) \sim \exp(n)$ or $\gamma(G) \sim n^d$ for some integer d, if the group G has exponential or polynomial growth, respectively.

Example 1.2.8. (i) Consider the group $\mathbb{Z}$ and let $S = \{-1, 1\}$ be the generating subset. The ball of radius n is

$$B_S^{\mathbb{Z}}(n) = \{g \in \mathbb{Z} \mid \ell_S^{\mathbb{Z}}(g) \leqslant n\} = \{-n, -(n-1), \ldots, -1, 0, 1, \ldots, n-1, n\},$$

so $\gamma_S^{\mathbb{Z}}(n) = 2n + 1$. It follows that $\gamma(\mathbb{Z}) \sim n$; in particular, $\mathbb{Z}$ has polynomial growth.

(ii) Consider the free group on two generators $\mathscr{F}_2$ and let $S = \{a, a^{-1}, b, b^{-1}\}$ be the generating subset. For each increase element in the ball of radius $n-1$, three distinct words are included in the ball of radius n (when $n > 1$). Thus,

$$\gamma_S^{\mathscr{F}_2}(n) = 1 + 4\sum_{i=0}^{n-1} 3^i = 23^n - 1.$$

This shows that $\gamma(\mathscr{F}_2) \sim 3^n$. In particular, $\mathscr{F}_2$ has exponential growth.

The next result states that all finite groups have the same growth type.

Proposition 1.2.9. *Let G be a finitely generated group. G is finite if and only if $\gamma(G) \sim 1$.*

Proof. Let $S \subseteq G$ be a finite symmetric generating subset. Suppose G is finite. As $\gamma_S(0) = 1$, we have $1(n) \leqslant \gamma_S(n)$, for all $n \geqslant 0$; i.e., $1 \preceq \gamma_S$. Note that $\gamma_S(n) \leqslant |G| = |G|\ 1(|G|\ n)$, for all $n \geqslant 0$. That is, $\gamma_S \preceq 1$.

Conversely, suppose that $\gamma(G) \sim 1$. In particular, $\gamma_S(G) \preceq 1$, so there exists a positive integer $c \geqslant 1$ such that $\gamma_S(n) \leqslant c1(cn) = c$. It follows that $|G| \leqslant c$. □

Proposition 1.2.10. *If G is an infinite finitely generated group, then $n \preceq \gamma(G)$.*

Proof. Choose a finite symmetric generating subset $S \subseteq G$ and consider the inclusions

$$\{e_G\} = B_S(0) \subseteq B_S(1) \subseteq \cdots B_S(n) \subseteq B_S(n+1) \subseteq \cdots \tag{1.25}$$

We wish to show that these inclusions are in fact all strict. In order to do this, we shall first show that if $B_S(n) = B_S(n+1)$ for some $n \geqslant 0$, then $B_S(m) = B_S(n)$ for all $m \geqslant n$. We proceed by induction on m. The start is trivial.

Suppose $B_S(n) = B_S(n+1)$ implies $B_S(n) = B_S(n+1) = \cdots = B_S(m-1) = B_S(m)$ for some $m > n$. We are to show that $B_S(m+1) = B_S(n)$. It is clear that $B_S(n) = B_S(m) \subseteq B_S(m+1)$. Conversely, take $g \in B_S(m+1)$; there exist elements $g' \in B_S(m)$ and $s \in S$ such that $g = g's$. By hypothesis $g' \in B_S(m-1)$, so

$$g \in B_S(m-1)S \subseteq B_S(m). \tag{1.26}$$

This shows that $B_S(m+1) \subseteq B_S(m)$, hence $B_S(m+1) = B_S(m) = B_S(n)$.

Now, if it were the case that $B_S(n) = B_S(n+1)$ for some $n \geqslant 0$, then $B_S(m) = B_S(n)$ for all $m \geqslant n$ and consequently $G = B_S(n)$, which is untenable since G is assumed infinite. Thus all the inclusions must be strict and we obtain

$$n \leqslant |B_S(n)| = \gamma_S(n), \tag{1.27}$$

for all $n \geqslant 0$; hence $n \preceq \gamma_S$. This shows that $n \preceq \gamma(G)$. □

1.3 The growth rate

Lemma 1.3.1. *Let $(a_n)_{n\geqslant 1}$ be a sequence of positive real numbers satisfying $a_{n+m} \leqslant a_n a_m$ for all natural numbers m and n. Then*

$$\lim_{n\to\infty} a_n^{1/n} = \inf_{n\geqslant 1} a_n^{1/n}. \tag{1.28}$$

Proof. We shall first show that the sequence is bounded. Fix $t \in \mathbb{N}$. For each $n \in \mathbb{N}$ we use Euclid's algorithm to write $n = qt + r$ for integers $q \geqslant 0$ and $0 \leqslant r < t$. First, note that $a_n = a_{qt+r} \leqslant a_t^q a_r$ so $a_n^{1/n} \leqslant a_t^{1/n} a_r^{1/n}$. Second, observe that $\lim_{n\to\infty} r/n = 0$ (this follows from $\lim_{n\to\infty} t/n = 0$) and since $1 = (q/n)t + r/n$ this implies that $\lim_{n\to\infty} q/n = 1/t$. Consequently, $\lim_{n\to\infty} a_t^{q/n} a_r^{1/n} = a_t^{1/t}$ which shows that the sequence $(a_n^{1/n})_{n\geqslant 1}$ is bounded. From this we see that $\limsup_{n\to\infty} a_n^{1/n} \leqslant a_t^{1/t}$, and in particular (since t was arbitrarily chosen) this implies that $\limsup_{n\to\infty} a_n^{1/n} \leqslant \inf_{t\geqslant 1} a_t^{1/t}$. Obviously, $\inf_{t\geqslant 1} a_t^{1/t} \leqslant \liminf_{t\to\infty} a_t^{1/t}$. In general it holds that $\liminf_{t\to\infty} a_t^{1/t} \leqslant \limsup_{t\to\infty} a_t^{1/t}$, hence the limit of the sequence $(a_n^{1/n})_{n\geqslant 1}$ exists and equals the common value of the limes superior and limes inferior, namely $\inf_{t\geqslant 1} a_t^{1/t}$. □

Proposition 1.3.2. *Let G be finitely generated group and let $S \subseteq G$ be finite symmetric generating subset. Then the number defined by*

$$\lambda_S^G = \lim_{n\to\infty} \gamma_S(n)^{1/n} \tag{1.29}$$

is finite. In particular, it follows that $\lambda_S \in [1, \infty)$.

Proof. Fix integers $n, m \geqslant 0$ and let $g \in B_S(n+m)$. Then $g = s_1 \cdots s_n s_{n+1} \cdots s_{n+m}$ for suitable $s_1, \ldots, s_n, s_{n+1}, \ldots, s_{n+m} \in S \cup \{e_G\}$. Putting $h = s_1 \cdots s_n \in B_S(n)$ and $h' = s_{n+1} \cdots s_{n+m} \in B_S(m)$ we find that $g = hh' \in B_S(n)B_S(m)$. This shows that $B_S(n+m) \subseteq B_S(n)B_S(m)$. Hence

$$|B_S(n+m)| \leqslant |B_S(n)B_S(m)| \leqslant |B_S(n)| \cdot |B_S(m)|, \tag{1.30}$$

i.e., $\gamma_S(n+m) \leqslant \gamma_S(n)\gamma_S(m)$. By Lemma 1.3.1 we conclude that the sequence $(\gamma_S(n)^{1/n})_{n\geqslant 1}$ is convergent. Since $\gamma_S(n) \geqslant 1$, for all $n \geqslant 0$, it follows that $\lambda_S^G \in [1, \infty)$. □

This proposition renders the notion of a *growth rate* well-defined.

Definition 1.3.3. *We call the number $\lambda_S^G \in [1, \infty)$ in (1.29) the* growth rate *of G relative to S. We may omit the superscript when there is no ambiguity.*

The following proposition characterizes groups of exponential, respectively, subexponential growth in terms of the growth rate.

Proposition 1.3.4. *Let $S \subseteq G$ be a finite symmetric generating subset of a group G. Then $\lambda_S > 1$ if and only if G has exponential growth. As a consequence, $\lambda_S = 1$ if and only if G has subexponential growth.*

Proof. Using Lemma 1.3.1, we see that $\gamma_S(n)^{1/n} \geqslant \inf_{n\geqslant 1} \gamma_S(n)^{1/n} = \lambda_S$. If $\lambda_S > 1$ then $\gamma_S \geqslant (\lambda_S)^n$ which by example 1.2.2 (iii) shows that $\gamma_S \preceq \exp(n)$. It is always the case that $\gamma_S \succeq \exp(n)$ (cf. Proposition 1.2.6), hence $\gamma_S \sim \exp(n)$.

On the other hand, suppose $\gamma(G) \sim \exp(n)$. In particular, we have $\exp(n) \preceq \gamma_S$, i.e., there exists a positive integer c such that $\exp(n) \leqslant c\gamma_S(cn)$, for all $n \geqslant 1$. Since $\lim_{n\to\infty} c^{1/(cn)} = 1$ we have

$$\lambda_S = \lim_{n\to\infty} \gamma_S(cn)^{1/(cn)} = \lim_{n\to\infty} (c\gamma_S(cn))^{1/(cn)} \geqslant \lim_{n\to\infty} \exp(n)^{1/(cn)} > 1. \tag{1.31}$$

As the growth rate λ_S takes values in the interval $[1, \infty)$, we also see that $\lambda_S = 1$ if and only if G has subexponential growth. □

By Proposition 1.2.6 (ii), the characterization in terms of the growth rate is independent of the choice of generating subset.

Corollary 1.3.5. *Let S and S' be two finite symmetric generating subsets of a group G. Then $\lambda_S > 1$ (resp. $\lambda_S = 1$) if and only if $\lambda_{S'} > 1$ (resp. $\lambda_{S'} = 1$).*

1.4 Growth of subgroups and quotients

Proposition 1.4.1. *Let G be a group and let N be normal in G. If both N and the quotient G/N are finitely generated, then G is finitely generated.*

Proof. Let $\pi : G \to G/N$ be the canonical epimorphism and let $U \subseteq N$ and $T \subseteq G/N$ denote the finite symmetric generating subsets respectively. Then $S = U \cup \pi^{-1}(T)$ is a finite symmetric subset of G satisfying $U \subseteq S$ and $T \subseteq \pi(S)$. Let us show that S generates G.

Let $g \in G$. There exist $p \geqslant 0$ and $t_1, \ldots, t_p \in T$ such that $\pi(g) = t_1 \cdots t_p$. By surjectivity there exist $s_1, \ldots, s_p \in U$ such that $\pi(s_i) = t_i$ for $i = 1, \ldots, p$. Putting $h = s_1 \cdots s_p$ we see that $\pi(h) = \pi(g)$ since π is a homomorphism. Furthermore, $n = h^{-1}g \in \ker \pi = N$, since

$$\pi(n) = \pi(h)^{-1}\pi(g) = \pi(g)^{-1}\pi(g) = e_{G/N}. \tag{1.32}$$

Thus there exist $q \geqslant 0$ and $s_{p+1}, \ldots, s_{p+q} \in U$ such that $n = s_{p+1} \cdots s_{p+q}$. Therefore, we can write $g = hn = (s_1 \cdots s_p)(s_{p+1} \cdots s_{p+q})$, which is a product of elements in S. We conclude that S generates G. □

Proposition 1.4.2. *Let G be a finitely generated group and let $N \subseteq G$ be a normal subgroup. Then the quotient G/N is finitely generated and $\gamma(G/N) \preceq \gamma(G)$. If, in addition, N is finite, then $\gamma(G/N) = \gamma(G)$.*

Proof. Let $S \subseteq G$ be a finite symmetric generating subset and let $\pi : G \to G/N$ denote the canonical epimorphism. Then $S' := \pi(S) \subseteq G/N$ is a finite and symmetric subset. Let $h \in G/N$. By surjectivity there exists $g \in G$ such that $\pi(g) = h$. If we write $g = s_1 \cdots s_n$, where $s_1, \ldots, s_n \in S$, then $h = \pi(s_1 \cdots s_n) = \pi(s_1) \cdots \pi(s_n) \in S'$; thus S' generates G/N. Using surjectivity again we see that $B_{S'}^{G/N}(n) = \pi(B_S^G(n))$, for all $n \geqslant 1$. Thus $|B_S^G(n)| \leqslant |B_{S'}^{G/N}(n)|$ or equivalently, $\gamma_{S'}^{G/N}(n) \leqslant \gamma_S^G(n)$ for all $n \geqslant 1$; i.e., $\gamma(G/N) \preceq \gamma(G)$.

Suppose, in addition, that $|N|$ is finite. Since $B_{S'}^{G/N}(n) = \pi(B_S^G(n))$ we observe that $B_S^G(n) \subseteq \pi^{-1}(B_{S'}^{G/N}(n))$ and

$$|B_S^G(n)| \leqslant |\pi^{-1}(B_{S'}^{G/N}(n))| = |N||B_{S'}^{G/N}(n)| \leqslant |N||B_{S'}^{G/N}(|N|n)| = |N|\gamma_{S'}^{G/N}(|N|\, n).$$

This shows that $\gamma_S^G \preceq \gamma_{S'}^{G/N}$; consequently $\gamma(G/N) = \gamma(G)$. □

Proposition 1.4.3. *Let H be a finitely generated subgroup of a finitely generated group G. Then $\gamma(H) \preceq \gamma(G)$.*

Proof. Let S_G and S_H be finite symmetric generating subset of G and H respectively and put $S = S_H \cup S_G$. Note that S is a finite symmetric generating subset of G. For all $n \geqslant 1$, we have $B_{S_H}^H(n) \subseteq B_S^G(n)$ since $S_H \subseteq S$ and therefore $|B_{S_H}^H(n)| \leqslant |B_S^G(n)|$, i.e., $\gamma_{S_H}(n) \leqslant \gamma_S(n)$, for all $n \geqslant 1$. Thus $\gamma_{S_H} \preceq \gamma_S$ and $\gamma(H) \preceq \gamma(G)$. □

Corollary 1.4.4. *Any finitely generated group containing a finitely generated subgroup of exponential growth has exponential growth.*

In particular, any finitely generated group containing a subgroup isomorphic to the free group on two generators $\mathscr{F}_2$ has exponential growth. Notice also that a subgroup of a finitely generated group need not be finitely generated itself. An example of this is $\mathscr{F}_2$ as shown by the following proposition.

Proposition 1.4.5. *The free group $\mathscr{F}_2$ on two generators contains the free group on countably many generators, $\mathscr{F}_\infty$.*

Proof. We shall use the existence of an embedding $\varphi : \mathscr{F}_3 \to \mathscr{F}_2$ (See [CSC10, Corollary D.5.3]). Let a, b, c and x, y be the free generators of $\mathscr{F}_3$ and $\mathscr{F}_2$, respectively, and let $H \subseteq \mathscr{F}_3$ denote the subgroup generated by the elements b, c.

Define another embedding $\psi : \mathscr{F}_2 \to \mathscr{F}_3$ by $\psi(x) = b$ and $\psi(y) = c$. The composition map $\alpha : \mathscr{F}_3 \to \mathscr{F}_3$ given by $\alpha = \psi \circ \varphi$ is thus a monomorphism. Observe that $\alpha(\mathscr{F}_3) \subseteq H$. Now, suppose $a, \alpha(a), \ldots, \alpha^{n-1}(a)$ are free elements of $\mathscr{F}_3$, then $\alpha(a), \alpha^2(a), \ldots, \alpha^n(a)$ are also free, by virtue of injectivity. By the induction principle, we conclude that the elements $a, \alpha(a), \ldots, \alpha^n(a), \ldots$ are free in $\mathscr{F}_3$. Hence $\varphi(a), \varphi(\alpha(a)), \ldots, \varphi(\alpha^n(a)), \ldots$ are free in $\mathscr{F}_2$, since φ is injective. □

A sufficient condition for a subgroup of a finitely generated group to be finitely generated is that it has finite index.

Proposition 1.4.6. *Let G be a group, and let H be a subgroup of finite index. Then G is finitely generated if and only if H is finitely generated.*

Proof. Choose $R \subseteq G$ to be a complete set of representatives of right cosets of H in G containing the identity and note that $|R| = [G : H] < \infty$.

Suppose first that H is finitely generated and take a finite symmetric generating subset $S \subseteq H$. For all $g \in G$ write $g = hr$ for $h \in H$ and a unique $r \in R$. Furthermore, $h = s_1 \cdots s_n$ for $s_1, \dots, s_n \in S$ and $n \geqslant 0$. Consequently $g = s_1, \dots, s_n r$; i.e., $S \cup R \subseteq G$ is a finite generating subset of G.

Conversely, let $S \subseteq G$ be a finite symmetric generating subset and consider the finite set

$$S' = RSR^{-1} \cap H. \tag{1.33}$$

We shall show that S' generates H. Let $h \in H$ and write $h = s_1 \cdots s_m$ for some $s_1, \dots, s_m \in S$. We proceed by induction on m. Clearly, $s_1 = h_1 r_1$ for some $h_1 \in H$ and a unique $r_1 \in R$. Note that $h_1 = e_G s_1 r_1^{-1} \in S'$. Now, suppose $r_{i-1} s_i = h_i r_i$ for $i = 2, \dots, n-1$. Defining $h_n = r_{n-1} s_n$ we will show that $h_n \in S'$. First, observe that $h_n = r_{n-1} s_n e_G \in RSR^{-1}$. Since

$$h = s_1 s_2 \cdots s_n = (e_G s_1 r_1^{-1})(r_1 s_2 r_2^{-1}) \cdots (r_{n-2} s_{n-1} r_{n-1}^{-1})(r_{n-1} s_n) = h_1 h_2 \cdots h_n. \tag{1.34}$$

we also see that $h_n = h_{n-1}^{-1} \cdots h_2^{-1} h_1^{-1} h \in H$; thus $h_n \in S'$ and S' generates H. $\square$

Proposition 1.4.7. *Let G be a finitely generated group and let H be a subgroup of finite index. Then H is finitely generated and $\gamma(G) = \gamma(H)$.*

Proof. It is clear that H is finitely generated (cf. Proposition 1.4.6) with $\gamma(H) \preceq \gamma(G)$ (cf. Proposition 1.4.3). If $S \subseteq G$ is a finite symmetric generating subset (containing e_G) and R is a set of complete representatives of right cosets of H in G then we know from the proof of Proposition 1.4.6 that $S' = RSR^{-1} \cap H$ is a finite generating subset of H.

We shall now show that $B_S^G(n) \subseteq B_{S'}^H(n)R$. Let $g \in B_S^G(n)$ and write $g = s_1 \cdots s_n$ where $s_1, \dots, s_n \in S$. From the proof of Proposition 1.4.6 there exist $e_G = r_0, r_1, \dots, r_n \in R$ and $h_1, \dots, h_n \in H$ such that $h_i = r_{i-1} s_i r_i \in S'$ and

$$g = s_1 \cdots s_n = (e_G s_1 r_1^{-1})(r_1 s_2 r_2^{-1}) \cdots (r_{n-1} s_n r_n^{-1}) r_n = h_1 h_2 \cdots h_n r_n, \tag{1.35}$$

i.e., $g \in B_{S'}^H(n)R$. We are now merely a simple calculation away from the desired result:

$$|B_S^G(n)| \leqslant |B_{S'}^H(n)R| \leqslant [G : H] \gamma_{S'}^H([G : H]n), \qquad n \geqslant 1$$

or $\gamma_S^G(n) \preceq \gamma_{S'}^H(n)$. Consequently $\gamma(G) = \gamma(H)$. $\square$

Definition 1.4.8 (Commensurability)**.** *Two groups G_1 and G_2 are* commensurable *if there exist subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ of finite index such that H_1 and H_2 are isomorphic.*

If we apply the previous result to commensurable groups we immediately find the following result.

Corollary 1.4.9. *Let G_1 and G_2 be two commensurable groups. If G_1 is finitely generated, then so is G_2 and $\gamma(G_1) = \gamma(G_2)$.*

Proof. Let $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ be subgroups of finite index such that $H_1 \simeq H_2$. By Proposition 1.4.7, H_1 is finitely generated and $\gamma(H_2) = \gamma(H_1) = \gamma(G_1)$. Invoking Proposition 1.4.6 we find that G_2 is finitely generated and from Proposition 1.4.7, $\gamma(G_2) = \gamma(H_2) = \gamma(G_1)$. $\square$

We are now ready to describe the growth of a product of finitely generated groups.

Theorem 1.4.10. *Let G_1 and G_2 be two finitely generated groups and let $G_1 \times G_2$ denote their direct product. Then $\gamma(G_1 \times G_2) = \gamma(G_1)\gamma(G_2)$.*

Proof. Let $S_1 \subseteq G_1$ and $S_2 \subseteq G_2$ be finite symmetric generating subsets respectively. We shall denote e_{G_1} by e_1 and eG_2 by e_2. The set defined by

$$S = (S_1 \times \{e_2\}) \cup (\{e_1\} \times S_2) \tag{1.36}$$

is a finite symmetric subset of $G_1 \times G_2$. Let $(g_1, g_2) \in G_1 \times G_2$. There exist $s_{1,1}, s_{1,2} \dots, s_{1,p} \in S_1$ and $s_{2,1}, s_{2,2}, \dots, s_{2,q} \in S_2$ such that $g_1 = s_{1,1}s_{2,2} \cdots s_{1,p}$ and $g_2 = s_{2,1}s_{2,2} \cdots s_{2,q}$. Therefore,

$$\begin{aligned}(g_1, g_2) &= (s_{1,1}s_{2,2} \cdots s_{1,p}, s_{2,1}s_{2,2} \cdots s_{2,q}) \\ &= (s_{1,1}, e_2)(s_{1,2}, e_2) \cdots (s_{1,p}, e_2)(e_1, s_{2,1})(e_1, s_{2,2}) \cdots (e_1, s_{2,q}),\end{aligned}$$

which is a product of elements in S. This shows that S generates G.

Now, if we instead let $(g_1, g_2) \in B_S^{G_1 \times G_2}(n)$ such that $p + q \leqslant n$, then we see from the above that $B_S^{G_1 \times G_2}(n) \subseteq B_{S_1}^{G_1}(n) \times B_{S_2}^{G_2}(n)$. It follows that

$$|B_S^{G_1 \times G_2}(n)| \leqslant |B_{S_1}^{G_1}(n) \times B_{S_2}^{G_2}(n)| = |B_{S_1}^{G_1}(n)|\ |B_{S_2}^{G_2}(n)|; \tag{1.37}$$

or $\gamma_S^{G_1 \times G_2}(n) \leqslant \gamma_{S_1}^{G_1}(n)\gamma_{S_2}^{G_2}(n)$, for all $n \geqslant 1$. This shows that $\gamma(G_1 \times G_2) \preceq \gamma(G_1)\gamma(G_2)$.

Conversely, take $g_1 \in B_{S_1}^{G_1}(n)$ and $g_2 \in B_{S_2}^{G_2}(n)$. Then $(g_1, g_2) \in B_S^{G_1 \times G_2}(2n)$ showing that $B_{S_1}^{G_1}(n) \times B_{S_2}^{G_2}(n) \subseteq B_S^{G_1 \times G_2}(2n)$. Thus,

$$\gamma_{S_1}^{G_1}(n)\gamma_{S_2}^{G_2}(n) \leqslant \gamma_S^{G_1 \times G_2}(2n) \leqslant 2\gamma_S^{G_1 \times G_2}(2n), \qquad n \geqslant 1. \tag{1.38}$$

Hence $\gamma(G_1)\gamma(G_2) \preceq \gamma(G_1 \times G_2)$ and therefore $\gamma(G_1)\gamma(G_2) = \gamma(G_1 \times G_2)$. □

Corollary 1.4.11. *Let $d \in \mathbb{Z}$. Then $\gamma(\mathbb{Z}^d) \sim n^d$.*

Corollary 1.4.12. *If G is a finitely generated abelian group, then G has polynomial growth.*

Proof. If G is a finitely generated abelian group, then $G \simeq \mathbb{Z}^d \times \mathbb{Z}/(n_1\mathbb{Z}) \times \cdots \times \mathbb{Z}/(n_k\mathbb{Z})$ for integers $d, n_1, \dots, n_k$ (cf. [Hun74, Theorem 2.4]). For all $i = 1, \dots, k$, the groups $\mathbb{Z}/(n_i\mathbb{Z})$ are finite so $\gamma(\mathbb{Z}/(n_i\mathbb{Z})) \sim 1$ (cf. Proposition 1.2.9). From Theorem 1.4.10 and Corollary 1.4.11, we deduce that $\gamma(G) = \gamma(\mathbb{Z}^d)\gamma(\mathbb{Z}/(n_1\mathbb{Z})) \cdots \gamma(\mathbb{Z}/(n_k\mathbb{Z})) \sim n^d$, as wanted □

1.5 Finitely generated nilpotent groups

The goal of this section is to prove that any finitely generated almost nilpotent group has polynomial growth. The converse implication is, in fact, also true, whence a group has polynomial growth if and only if it is almost nilpotent. This is a remarkable result due to Gromov, see [Gro81]. Unfortunately, it is beyond the scope of this thesis to prove the other implication.

Lemma 1.5.1. *Let G be a group and let H and K be normal subgroups of G. If $S_H \subseteq H$ and $T_K \subseteq K$ are finite generating subsets, respectively, then $[H, K]$ equals the normal closure in G of the set $\{[s, t] \mid s \in S_H, t \in T_K\}$.*

Proof. Let N denote the normal closure in G of $\{[s,t] \mid s \in S_H, t \in T_K\}$. As $[H,K]$ is normal in G and $\{[s,t] \mid s \in S_H, t \in T_K\} \subseteq [H,K]$, we deduce that $N \subseteq [H,K]$.

Conversely, let $\pi : G \to G/N$ be the canonical epimorphism. Observe that $[\pi(s), \pi(t)] = \pi([s,t]) = e_{G/N}$, i.e., $\pi(s)$ and $\pi(t)$ commute, for all $s \in S_H$ and $t \in T_K$. However, as $S_H \subseteq H$ and $T_K \subseteq K$ are generating subsets, respectively, we have $\pi([h,k]) = [\pi(h), \pi(k)] = e_{G/N}$, for all $h \in H$ and $k \in K$. It follows that $[H,K] \subseteq N$. □

Recall that the *lower central series* is the sequence $(C^i(G))_{i \geqslant 0}$, where we put $C^0(G) = G$ and recursively define $C^{i+1}(G) = [C^i(G), G]$. Note that $C^i(G)$ is normal in G, for all $i \geqslant 0$. For elements $g_1, g_2, \ldots, g_i$ (with $i \geqslant 3$) we recursively put

$$[g_1, g_2, \ldots, g_i] = [[g_1, g_2, \ldots, g_{i-1}], g_i] \in C^{i-1}(G).$$

If $S \subseteq G$ is a subset and $i \in \mathbb{N}$, we define

$$S^{(i+1)} := \{[w,s] \mid w \in S^{(i)},\ s \in S\}, \tag{1.39}$$

where $S^{(1)} = S$. We refer to these elements of $S^{(i+1)}$ as *simple S-commutators of weight $i+1$.* Note that if $S \subseteq G$ is finite, then so is $S^{(i)}$.

Lemma 1.5.2. *Let G be any group and suppose $S \subseteq G$ is a generating subset. Then, for all $i \in \mathbb{N}$, $C^i(G)$ is the normal closure in G of the set $S^{(i+1)}$.*

Proof. We shall prove the statement by induction on i. Set $H = K = G$ and $S_H = T_K = S$ in the previous lemma. Then $C^i(G) = [G,G]$ is the normal closure in G of $S^{(2)} = \{[s,t] \mid s,t \in S\}$.

Now, suppose $C^{i-1}(G)$ is the normal closure in G of $S^{(i)}$ and let N denote the normal closure in G of $S^{(i+1)}$. As $S^{(i+1)} \subseteq C^i(G)$ and $C^i(G)$ is normal in G, we deduce that $N \subseteq C^i(G)$.

Conversely, we shall first show that $C^i(G)$ is the smallest normal group containing the set $\{[hwh^{-1}] \mid w \in S^{(i)}, h \in G, s \in S\}$, and then show that the set is contained in N. By the induction hypothesis, any element in $C^{i-1}(G)$ can be expressed in the form

$$(h_1 w h_1^{-1})(h_2 w h_2^{-1}) \cdots (h_m w h_m^{-1}),$$

for some $h_1, \ldots, h_m \in G$, $w_1, \ldots, w_m \in S^{(i)}$ and $m \in \mathbb{N}$. Setting $H = C^{i-1}(G)$, $K = G$ and $S_H = \{[hwh^{-1}] \mid w \in S^{(i)}, h \in G, s \in S\}$, $T_K = S$, we can apply Lemma 1.5.1 to find that $C^i(G) = [C^{i-1}, G]$ is the normal closure of S.

Now, let $\pi : G \to G/N$ be the canonical epimorphism and take $w \in S^{(i)}$ and $s \in S$. Observe that $[w,s] \in S^{(i+1)} \subseteq N$, so $\pi([w,s]) = [\pi(w), \pi(s)] = e_{G/N}$. As $\pi(S)$ generates G/N, this shows that $\pi(w)$ commutes with all elements in G/N. Hence, $\pi(hwh^{-1}) = \pi(w)$, for all $h \in G$. Consequently,

$$\pi([hwh^{-1}, s]) = [\pi(w), \pi(s)] = e_{G/N}.$$

It follows that $[hwh^{-1}, s] \in N$. We conclude that $C^i(G) \subseteq N$ and this finishes the proof. □

Lemma 1.5.3. *Let G be a finitely generated group. If G is nilpotent of degree $d \geqslant 0$, then the normal subgroups $C^i(G)$ are finitely generated, for all $i = 2, 3, \ldots, d-1$.*

Proof. We shall divide the proof into two steps. We show that all quotients $C^i(G)/C^{i+1}(G)$ are finitely generated, and then we prove the statement of the lemma.

Step 1: Consider the canonical epimorphism $\pi : G \to G/C^{i+1}(G)$. As $C^i(G)$ is the normal closure in G of $S^{(i+1)}$ (cf. Lemma 1.5.2), we have

$$C^i(G)/C^{i+1}(G) = \pi(C^i(G)) = \{\pi(ghg^{-1}) \mid g \in G, h \in S^{(i+1)}\}.$$

But $C^{i+1}(G) = [G, C^i(G)]$, so $\pi(g)$ and $\pi(h)$ commute, for all $g \in G$ and $h \in C^i(G)$. Since $S^{(i+1)} \subseteq C^i(G)$, we see that $\pi(ghg^{-1}) = \pi(h)$. This shows that $S^{(i+1)}$ generates $C^i(G)/C^{i+1}(G)$.

Step 2: We shall continue by finite induction on i. As G is nilpotent of degree d, we have $C^d(G) = \{e_G\}$. By step 1, we deduce that $C^{d-1}(G) \simeq C^{d-1}(G)/C^d(G)$ is finitely generated. Now, suppose $C^{i+1}(G)$ is finitely generated for some positive integer $i \leqslant d-2$. Since $C^i(G)/C^{i+1}(G)$ is finitely generated, we deduce from Proposition 1.4.1 that $C^i(G)$ is finitely generated. This finishes the proof. □

Theorem 1.5.4. *Any finitely generated nilpotent group has polynomial growth.*

Proof. Let G be a nilpotent group of degree d; we shall proceed by induction on d. If $d = 0$, then G is trivial and it has polynomial growth. Suppose $d \geqslant 1$ and that all finitely generated nilpotent groups of degree less than or equal to $d-1$ have polynomial growth.

Set $H = C^1(G)$. By Lemma 1.5.3, H is finitely generated, so from the induction hypothesis, we see that H has polynomial growth.

Let $S = \{s_1, s_2, \ldots, s_k\}$ be a finite symmetric generating subset of G. Fix $n \in \mathbb{N}$ and let $g \in B_S^G(n)$. We wish to estimate the maximal length of g. Observe that $\ell_S(g) = m \leqslant n$, so

$$g = s_{j_1} s_{j_2} \cdots s_{j_m},$$

for $1 \leqslant j_1, j_2, \ldots, j_m \leqslant k$. However, as $g_2 g_1 = g_1 g_2 [g_2^{-1}, g_1^{-1}]$ (with $[g_2^{-1}, g_1^{-1}] \in H$), for all $g_1, g_2 \in G$, we can write

$$g = s_{i_1} s_{i_2} \cdots s_{i_m} h, \tag{1.40}$$

where $1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_m \leqslant k$ and $h \in H$. Now, as the subscripts are increasing, we can express any element of the form $g' = s_{i_1} s_{i_2} \cdots s_{i_m}$ as $g' = s_1^{n_1} s_2^{n_2} \cdots s_k^{n_k}$, where $0 \leqslant n_i \leqslant n$ and $i = 1, 2 \ldots, k$. Consequently, $\ell_S(g') \leqslant (n+1)^k = c_1 n^k$, for some integer $c_1 \in \mathbb{N}$.

Let T be a finite symmetric generating subset of H. We shall now estimate the length of h in (1.40). Each exchange in the process of rewriting g to the form (1.40) produces a simple S-commutator of weight 2 and since $g \in B_S^G(n)$ there are at most n^2 such exchanges. Note that during the process, each element must exchange with all the produced simple commutators. Analogously, this produces at most n^3 simple S-commutators of weight 3; and so on. However, as G is nilpotent of degree d, any commutators of weight greater than of equal to $d+1$ are trivial. Thus, the maximal number of exchanges producing elements in H is $n^2 + n^3 + \cdots + n^d \leqslant dn^d$. Therefore, if $L := \max\{\ell_S(w) \mid w \in S^{(i)}, i = 2, 3, \ldots, d\}$, then

$$\ell_T(h) \leqslant Ldn^d. \tag{1.41}$$

The above argument proves the inclusion $B_S^G(n) \subseteq B_S^G(m) B_T^H(Ldn^d)$, for $m \leqslant n$. As H has polynomial growth, there exist integers $c_2 \geqslant 1$ and $q \geqslant 0$ such that $\gamma_T^H(n) \leqslant c_2(c_2 n)^q$. Consequently, for all $n \in \mathbb{N}$, we have

$$\gamma_S^G(n) \leqslant c_1 n^k \; c_2(c_2 Ldn^d)^q = cn^{k+dq} \leqslant c(cn)^{k+dq}, \tag{1.42}$$

where $c = c_1 c_2^{q+1}(Ld)^q$. This shows that G has polynomial growth. $\square$

Corollary 1.5.5. *Any finitely generated almost nilpotent group has polynomial growth.*

Proof. Let G be a finitely generated almost nilpotent group and let $H \subseteq G$ be a nilpotent subgroup of finite index. From Proposition 1.4.7, H and G have the same growth type. As H satisfies the hypothesis of Theorem 1.5.4, we deduce that G has polynomial growth. $\square$

We shall use this result in Corollary 3.3.6.

Chapter 2

The Grigorchuk group

2.1 Construction

Consider the alphabet $\Sigma = \{0, 1\}$. A *word* in the alphabet Σ is an element in the union

$$\Sigma^* := \bigcup_{n \geqslant 0} \Sigma^n,$$

where Σ^n denotes the cartesian product. Note that Σ^0 contains the unique element: *the empty word* ϵ. We define the *concatenation* of two words $w_1 = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \Sigma^n$ and $w_2 = (\sigma'_1, \sigma'_2, \ldots, \sigma'_m) \in \Sigma^m$ (with $\sigma_1, \ldots, \sigma_n, \sigma'_1, \ldots, \sigma'_m \in \Sigma^1 = \Sigma$) as the word

$$w_1 w_2 = (\sigma_1, \sigma_2, \ldots, \sigma_n, \sigma'_1, \sigma'_2, \ldots, \sigma'_m) \in \Sigma^{n+m}. \tag{2.1}$$

Observe that $\epsilon w_1 = w_1 \epsilon = w_1$ and $(w_1 w_2) w_3 = w_1 (w_2 w_3)$, for all $w_1, w_2, w_3 \in \Sigma^*$. Thus, if we interpret the concatenation of two words as a product on Σ^*, we find that Σ^* is a monoid with the empty word playing the role of identity element. In light of this, we shall use a product notation. Given $w = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \Sigma^n$, with $\sigma_1, \sigma_2, \ldots, \sigma_n \in \Sigma$, we write $w = \sigma_1 \sigma_2 \cdots \sigma_n$. Also, the non-negative integer $\ell(w) := n$ denotes the *length* of the word w. Clearly, $\ell(\varepsilon) = 0$ and $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$, for all $w_1, w_2 \in \Sigma^*$.

Let us now turn our attention toward the symmetry group $\mathrm{Sym}(\Sigma^*)$, i.e., the set of bijections on Σ^*. We shall first define $a : \Sigma^* \to \Sigma^*$ by $a(\epsilon) = \epsilon$ and

$$a(0w) = 1w, \qquad a(1w) = 0w, \qquad w \in \Sigma^*. \tag{2.2}$$

For all $w \in \Sigma^*$, the maps $b, c, d : \Sigma^* \to \Sigma^*$ are defined recursively by the length of w. Let $b(\epsilon) = c(\epsilon) = d(\epsilon) = \epsilon$ and

$$\begin{aligned} b(0w) &= 0a(w), & b(1w) &= 1c(w), \\ c(0w) &= 0a(w), & c(1w) &= 1d(w), \\ d(0w) &= 0w, & d(1w) &= 1b(w). \end{aligned} \tag{2.3}$$

Since any word is of finite length, the above is well-defined. Note also that $a, b, c, d \in \mathrm{Sym}(\Sigma^*)$.

Example 2.1.1. Let $w = 10101 \in \Sigma^*$. Then

$$\begin{aligned} a(w) &= a(10101) = 00101, \\ b(w) &= b(10101) = 1c(0101) = 10a(101) = 10001, \\ c(w) &= c(10101) = 1d(0101) = 10101, \\ d(w) &= d(10101) = 1b(0101) = 10a(101) = 10001. \end{aligned}$$

Definition 2.1.2 (The Grigorchuk group)**.** *The Grigorchuk group Γ is the subgroup of Sym(Σ^*) generated by the elements a, b, c and d. We shall denote the identity element in Γ by e.*

Let $u, v \in \Sigma^*$. If there exists a word $w \in \Sigma^*$ such that $uw = v$, we write $u \preceq v$. Since the product is associative, $\preceq$ defines a partial relation on Σ^*. This enables us to consider the set of bijections that preserve the order, namely,

$$(\mathrm{Sym}(\Sigma^*), \preceq) = \{g \in \mathrm{Sym}(\Sigma^*) \mid g(u) \preceq g(v),\ u \preceq v\}.$$

If $u \preceq v$, then $gh^{-1}(u) \preceq gh^{-1}(v)$, for all $g, h \in (\mathrm{Sym}(\Sigma^*), \preceq)$; hence $(\mathrm{Sym}(\Sigma^*), \preceq)$ is a subgroup of $\mathrm{Sym}(\Sigma^*)$. Equivalently, the length of a word is preserved under an element in $(\mathrm{Sym}(\Sigma^*), \preceq)$. Indeed, if $w \in \Sigma^n$, then n is the maximal integer such that $w = \sigma_1\sigma_2 \cdots \sigma_n$ for distinct $\sigma_1, \sigma_2, \ldots, \sigma_n \in \Sigma$. Since, $\epsilon \preceq \sigma_1 \preceq \sigma_1\sigma_2 \preceq \cdots \preceq \sigma_1 \cdots \sigma_n$, we have $\epsilon \preceq g(\sigma_1) \preceq g(\sigma_1\sigma_2) \preceq \cdots \preceq g(\sigma_1 \cdots \sigma_n)$, for all $g \in (\mathrm{Sym}(\Sigma^*), \preceq)$ exactly when $\ell(g(w)) = \ell(w)$. It is obvious from the definition (see (2.2) and (2.3)) that $a, b, c, d \in (\mathrm{Sym}(\Sigma^*), \preceq)$, hence

$$\Gamma \subseteq (\mathrm{Sym}(\Sigma^*), \preceq) \tag{2.4}$$

is a subgroup.

Proposition 2.1.3. *The set $S = \{a, b, c, d\}$ is a finite symmetric generating subset of the Grigorchuk group and its elements are self-inverse. Furthermore,*

$$bc = cb = d, \quad dc = cd = b, \quad db = bd = c. \tag{2.5}$$

Proof. Fix a word $w \in \Sigma^*$ with $\ell(w) = n$. Recall that $s(\epsilon) = \epsilon$, for all $s \in S$. Also, $a^2(0w) = a(1w) = 0w$ and $a^2(1w) = a(0w) = 1w$, hence $a^2 = e$. In order to show that $g^2 = e$ for $g \in \{b, c, d\}$, we proceed by induction on n. Suppose $g^2(w) = w$. Then

$$\begin{aligned} b^2(0w) &= b(0a(w)) = 0a^2(w) = 0w, & b^2(1w) &= b(1c(w)) = 1c^2(w) = 1w, \\ c^2(0w) &= c(0a(w)) = 0a^2(w) = 0w, & c^2(1w) &= c(1d(w)) = 1d^2(w) = 1w, \\ d^2(0w) &= d(0w) = 0w, & d^2(1w) &= d(1b(w)) = 1b^2(w) = 1w. \end{aligned}$$

By the induction principle, we conclude that $b^2 = c^2 = d^2 = e$; in particular, S is symmetric.

Let us now show that $ij(w) = k(w)$, for all $w \in \Sigma^*$ and distinct $i, j, k \in \{b, c, d\}$. Again, let $w \in \Sigma^*$ with $\ell(w) = n$; we proceed by induction on n. If $w = \epsilon$ there is nothing to prove. Otherwise, we have

$$\begin{aligned} bc(0w) &= b(0a(w)) = 0a^2(w) = 0w = d(0w), \\ bc(1w) &= b(1d(w)) = 1cd(w) = 1b(w) = d(1w), \\ cd(0w) &= c(0w) = 0a(w) = b(0w), \\ cd(1w) &= c(1b(w)) = 1db(w) = 1c(w) = b(1w), \\ db(0w) &= d(0a(w)) = 0a(w) = c(0w), \\ db(1w) &= d(1c(w)) = 1bc(w) = 1d(w) = c(1w). \end{aligned}$$

By the induction principle, we conclude that $bc = d$, $cd = b$ and $db = c$. Since b, c and d are self-inverse, we have $cb = cd^2b = (cd)(db) = bc$, $dc = (db)(bc) = cd$ and $bd = (dc)(cd) = db$. This proves (2.5). □

As S is a generating subset of Γ, we can express any element $g \in \Gamma$ as $g = s_1 s_2 \cdots s_n$, for some $s_1, s_2, \ldots, s_n \in S$. Let $\ell_S : \Gamma \to \mathbb{N}_0$ be the corresponding word-length function. Inspired by Proposition 2.1.3, we say that an element $g \in \Gamma$ is expressed in a *reduced form*, if every second factor is a; more precisely, if $s_i = a$ then $s_{i+1} = t$, and if $s_j = t$ then $s_{j+1} = a$, for $t \in \{b, c, d\}$ and $i, j = 1, 2, \ldots, n-1$.

Proposition 2.1.4. *For all non-negative integers n, the set $H_n = \{g \in \Gamma \mid g(w) = w,\ w \in \Sigma^n\}$ is a normal subgroup of Γ with finite index. Furthermore,*

$$\Gamma = H_0 \supsetneq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n \supseteq \cdots . \tag{2.6}$$

Proof. Since $\ell(g(w)) = \ell(w)$ for all $w \in \Sigma^n$ and $g \in \Gamma$, we have $g(w) \in \Sigma^n$. Thus, for each $n \geqslant 0$ we can consider the homomorphism $\theta_n : \Gamma \to \mathrm{Sym}(\Sigma^n)$ given by

$$\theta_n g(w) = g(w), \qquad w \in \Sigma^n.$$

Observe that $\ker \theta_n = \{g \in \Gamma \mid \theta_n g = e_{\mathrm{Sym}(\Sigma^n)}\} = \{g \in \Gamma \mid g(w) = w, w \in \Sigma^n\}$, which shows that H_n is normal in Γ, for all $n \geqslant 0$. Therefore,

$$[\Gamma : H_n] = |\Gamma / H_n| = |\theta_n(\Gamma)| \leqslant |\mathrm{Sym}(\Sigma^n)| < \infty. \tag{2.7}$$

Note that $H_0 = \{g \in \Gamma \mid g(\epsilon) = \epsilon\} = \Gamma$. In order to see that the sets H_n are nested as in (2.6), let $n \geqslant 0$, $u \in \Sigma^n$, $g \in H_{n+1}$ and put $w = u\sigma$, for some $\sigma \in \Sigma$. Then $g(u) \in \Sigma^n$ and $g(u) \preceq g(w) = g(u\sigma) = u\sigma$. This shows that $g(u) = u$, i.e., $g \in H_n$. Thus, $H_n \supseteq H_{n+1}$ for all $n \geqslant 0$. However, as $a(1) = 0$ and $a(0) = 1$, the inclusion $\Gamma \supsetneq H_1$ is strict. □

Observe that $a(1) = 0$ and $a(0) = 1$. Consequently, the first inclusion of (2.6) is strict, i.e.,

$$H_1 \subsetneq \Gamma.$$

We are now ready to state and prove the first important property of the Grigorchuk group.

Theorem 2.1.5. *The Grigorchuk group is residually finite.*

Proof. For all $n \geqslant 0$, the set H_n is a normal subgroup of Γ of finite index, cf. Proposition 2.1.4. Moreover, we have

$$\bigcap_{n \geqslant 0} H_n = \{g \in \Gamma \mid g(w) = w, w \in \Sigma^*\} = \{e\}. \tag{2.8}$$

By Proposition B.0.7, we conclude that the Grigorchuk group is residually finite. □

2.2 The Burnside problem

The Burnside problem can be stated as follows.

Are all finitely generated periodic groups finite?

Theorem 2.2.5 below shows that Γ is finitely generated, periodic and infinite, thus answering the question in the negative. In this section, the normal subgroup $H_1 = \{g \in \Gamma \mid g(\sigma) = \sigma, \sigma \in \Sigma\}$ will be of particular interest to us. Let us first characterize this subgroup to a greater detail. Recall that the *normal closure* in a group G of a subset $H \subseteq G$ is the intersection of all normal subgroups in G containing H.

Proposition 2.2.1. *The following holds*

(i) H_1 *is the set of all elements* $g \in \Gamma$ *which can be expressed by elements in* S *with an even number of occurrences of* a.

(ii) *The set* $\{b, c, d, aba, aca, ada\}$ *is a finite symmetric generating subset of* H_1,

(iii) H_1 *is the normal closure of the elements* b, c *and* d.

Proof. *(i)*: Let $\sigma \in \Sigma$. From the definition, we see that $a(\sigma) \neq \sigma$, while $t(\sigma) = \sigma$ for $t \in \{b, c, d\}$. Take $g \in \Gamma$ and write $g = s_1 s_2 \cdots s_n$, for some $s_1, s_2, \ldots, s_n \in S$. Then $g(\sigma) = \sigma$ (equivalently, $g \in H_1$) exactly when the set $\{i \mid s_i = a, i = 1, 2, \ldots, n\}$ contains an even number of elements.

(ii): Let $h \in H_1$. If $h \in \{b, c, d, aba, aca, ada\}$ the statement is trivial. Otherwise, we can write h in a reduced form with an even number of occurrences of the letter a (cf. *(i)*), i.e., in one of the following four forms

$$\begin{aligned}
h = \quad & at_1at_2 \cdots at_{2k-1}at_{2k} &&= (at_1a)t_2 \cdots (at_{2k-1}a)t_{2k}, \\
h = \quad & at_1at_2 \cdots at_{2k-1}a &&= (at_1a)t_2 \cdots (at_{2k-1}a), \\
h = t_0 \quad & at_1at_2 \cdots at_{2k-1}at_{2k} &&= t_0(at_1a)t_2 \cdots (at_{2k-1}a)t_{2k}, \\
h = t_0 \quad & at_1at_2 \cdots at_{2k-1}a &&= t_0(at_1a)t_2 \cdots (at_{2k-1}a),
\end{aligned}$$

for some $k \in \mathbb{N}$ and $t_i \in \{b, c, d\}$, $i = 0, 1, \ldots, 2k$. In each case, a occurs $2k$ times. Clearly, ata is self-inverse, for all $t \in \{b, c, d\}$. Hence $\{b, c, d, aba, aca, ada\}$ is a finite symmetric generating subset of H_1.

(iii): Denote by H the normal closure in Γ of the elements b, c, d and fix $t \in \{b, c, d\}$. By *(ii)*, we have $t \in H_1$. Note that $gtg^{-1} \in H_1$, for all $g \in \Gamma$ (since H_1 is normal). Thus $H \subseteq H_1$. Conversely, $t = ete \in H$ and $ata = ata^{-1} \in H$. This shows that all the generators of H_1 are contained in H, hence $H_1 \subseteq H$. We conclude that H_1 equals the normal closure of the elements b, c and d. □

It follows immediately from part *(i)* of the above proposition that $[G : H_1] = 2$.

Suppose $h \in H_1$, $w \in \Sigma^*$ and $\sigma \in \Sigma$. By Proposition 2.2.1, we see that $h(\sigma w) = \sigma w_\sigma$, for some unique $w_\sigma \in \Sigma^*$ with $\ell(w) = \ell(w_\sigma)$. Therefore, we can consider the bijection $h_\sigma : \Sigma^* \to \Sigma^*$ given by $h_\sigma(w) = w_\sigma$, for all $w \in \Sigma^*$. That is, $h_\sigma \in (\mathrm{Sym}(\Sigma^*), \preceq)$.

Definition 2.2.2. *For each* $\sigma \in \Sigma$, *we define the map* $\varphi_\sigma : H_1 \to (Sym(\Sigma^*), \preceq)$ *by*

$$\varphi_\sigma(h) = h_\sigma, \qquad h \in H_1, \tag{2.9}$$

and the map $\varphi : H_1 \to (Sym(\Sigma^*), \preceq) \times (Sym(\Sigma^*), \preceq)$ *given by*

$$\psi(h) = (h_0, h_1), \qquad h \in H_1. \tag{2.10}$$

Let $\sigma \in \Sigma$. With this notation, we can write $h(\sigma w) = \sigma h_\sigma(w) = \sigma\varphi_\sigma(h)(w)$, for all $w \in \Sigma^*$. By (2.3), we have

$$\begin{aligned} \psi(b) &= (a,c), & \psi(aba) &= (c,a), \\ \psi(c) &= (a,d), & \psi(aca) &= (d,a), \\ \psi(d) &= (e,b), & \psi(ada) &= (b,e). \end{aligned} \tag{2.11}$$

As H_1 is generated by $\{b, c, d, aba, aca, ada\}$, the image of H_1 under φ_σ equals Γ.

Proposition 2.2.3. *Let $\sigma \in \Sigma$. Then*

(i) The map $\varphi_\sigma : H_1 \to \Gamma$ defined in (2.9) is an epimorphism,

(ii) The map $\psi : H_1 \to \Gamma \times \Gamma$ defined in (2.10) is a monomorphism.

Proof. (i): By the above remark we note that φ_0 and φ_1 are surjections. However, as $\varphi_0(b) = \varphi_0(c) = \varphi_1(aba) = \varphi_1(aca) = a$, they are not injections. To see that φ_σ is a homomorphism, take $\sigma \in \Sigma$ and $h, h' \in H_1$. Then, for all $w \in \Sigma^*$, we have

$$\sigma\varphi_\sigma(hh')(w) = hh'(\sigma w) = h(\sigma\varphi_\sigma(h')(w)) = \sigma\varphi_\sigma(h)(\varphi_\sigma(h')(w)) = \sigma(\varphi_\sigma(h)\varphi_\sigma(h'))(w),$$

i.e., $\varphi_\sigma(hh') = \varphi_\sigma(h)\varphi_\sigma(h')$.

(ii): If $h \in \ker\psi$, then $h(\sigma w) = \sigma h_\sigma(w) = \sigma w$, for all $w \in \Sigma^*$ and $\sigma \in \Sigma$. It follows that $\ker\psi = \{e\}$. Thus, ψ is injective. □

As H_1 can be embedded in $\Gamma \times \Gamma$, we can identify any element $h \in H_1$ with its image in the product of the Grigorchuk group with itself. By abuse of notation, we will write

$$h = \psi(h) = (h_0, h_1), \qquad h \in H_1.$$

With this new machinery and notation, we are almost ready to address the theorem relating the Grigorchuk group to the Burnside problem with an answer in the negative. We first need a useful lemma.

Lemma 2.2.4. *Every element $g \in \Gamma - \{e\}$ is conjugate either to an $s \in S$ or to an element $g' \in \Gamma$ expressed in the following form*

$$g' = at_1at_2 \cdots ak_{k-1}at_k, \tag{2.12}$$

for some $t_1, t_2, \ldots, t_k \in \{b, c, d\}$. In addition, $\ell(g') \leqslant \ell(g)$.

Proof. Let $g \in \Gamma$ with $g \neq e$. If g is not conjugate to an $s \in S$, then $\ell(g) \geqslant 4$ and we can write it in reduced form,i.e., in one of the following four forms:

$$\begin{aligned} g_1 &= \quad at_1at_2 \cdots at_{k-1}at_k, \\ g_2 &= \quad at_1at_2 \cdots at_{k-1}a\ , \\ g_3 &= t_0 \quad at_1at_2 \cdots at_{k-1}at_k, \\ g_4 &= t_0 \quad at_1at_2 \cdots at_{k-1}a, \end{aligned}$$

for $t_0, t_1, \ldots, t_k \in \{b, c, d\}$ and $k \in \mathbb{N}$. Note that $g_2 = ag_3a$. By conjugating g_3 by t_0 and reducing t_kt_0 in accordance with (2.5), we get an element of the form of g_1. Finally, conjugating g_4 by a we are again left with an element of the form of g_1. It is clear that this process does not increase the length of g. □

Recall that a group element whose order is a power of two is called a *2-element*.

Theorem 2.2.5. *The Grigorchuk group is an infinite finitely generated periodic group.*

Proof. As $H_1 \subsetneq \Gamma$ and $\varphi_0(H_1) = \varphi_1(H_1) = \Gamma$, the Grigorchuk group is infinite. We shall now show that all elements in Γ are 2-elements. Let $g \in \Gamma$. We proceed by induction on $\ell_S(g)$. First, note that if $\ell_S(g) = 1$, then $g^2 = e$ (cf. Proposition 2.1.3). Also, if g is conjugate to an element in S, then it has order 2.

Suppose $g \in H_1$ with $\ell_S(g) > 1$. Since g cannot be conjugate to an element in S and also contained in H_1, Lemma 2.2.4 yields that, up to conjugacy, g is of the form (2.12) with $k \geqslant 2$ even. Let $\sigma \in \Sigma$. The image under φ_σ of each quadruple at_iat_{i+1}, for $t_i \in \{b, c, d\}$ and $i = 1, 2, \ldots, n-1$, has length smaller than or equal to two. It follows that

$$\ell_S(\varphi_\sigma(g)) = \ell_S(g_\sigma) \leqslant \ell_S(g)/2 < \ell_S(g).$$

By the induction hypothesis, we conclude that the order of both g_0 and g_1 is a power of two. Consequently, this is also the case for $g = (g_0, g_1)$.

If, instead, $g \in \Gamma - H_1$ with $\ell_S(g) > 1$, we may assume that, up to conjugacy, g is of the form (2.12) with $k \geqslant 1$ odd. We shall divide the argument into three cases.

Case 1: Suppose that d occurs in the expression, say $d = t_i$. Conjugating by $h = t_{t-1}at_i \cdots at_1a$, we obtain

$$\begin{aligned} hgh^{-1} &= (t_{i-1}at_{i-2} \cdots ada)(at_1 \cdots at_{i-1}at_i \cdots at_k)(t_{t-1}at_{i-2} \cdots at_1a)^{-1} \\ &= adat_{i+1} \cdots at_k(t_{t-1}at_{i-2} \cdots at_1a)^{-1}, \end{aligned}$$

i.e., we may assume that $d = t_1$, as conjugation preserves the length of the element. Now,

$$g^2 = (adat_2)(at_3at_4) \cdots (at_kad)(at_2at_3) \cdots (at_{k-1}at_k) \in H_1,$$

as a occurs an even number of times in the expression. Again, the image of each quadruple is smaller than or equal to two. As $\varphi_0(d) = \varphi_1(ada) = e$ (which occurs at least once), we have

$$\ell_S(\varphi_0(g^2)), \ell_S(\varphi_1(g^2)) \leqslant 2k - 1 < 2k = \ell_S(g). \tag{2.13}$$

By the induction hypothesis, we see that $g^2 = ((g^2)_0, (g^2)_1)$ is a 2-element. It follows that g is also a 2-element.

Case 2: Suppose d does not occur in the expression of g, but c does. Up to conjugacy, we may assume that $t_k = c$ and that $c = t_1$. As $\varphi_0(abat') = ca$, $\varphi_0(acat') = da$ and $\varphi_1(at'ab) = ac$, $\varphi_1(at'ac) = ad$, for $t' \in \{b, c\}$, we observe that $\varphi_0(g^2) = da \cdots a$ and $\varphi_1(g^2) = a \cdots ad$. Also,

$$\ell_S(\varphi_0(g^2)) = \ell_S(\varphi_1(g^2)) = \ell_S(g). \tag{2.14}$$

Conjugating $\varphi_0(g^2)$ and $\varphi_1(g^2)$ by a and ad, respectively, we are back in case 1. It follows that $\varphi_0(g^2)$ and $\varphi_1(g^2)$ are 2-elements, whence $g^2 = (\varphi_0(g^2), \varphi_1(g^2))$ is a 2-element. Therefore, g is a 2-element.

Case 3: Suppose neither d nor c occurs. Up to conjugacy, we have $g = (ab)^{2n+1}$, for some integer $n \geqslant 0$. As $g^2 = ((aba)b)^{2n+1}$, we have $\varphi_0(g^2) = (ca)^{2n+1}$ and $\varphi_1(g^2) = (ac)^{2n+1}$ with

$$\ell_S(\varphi_0(g^2)) = \ell_S(\varphi_1(g^2)) = \ell_S(g).$$

Conjugating $\varphi_0(g^2)$ by a, we are back in Case 2; thus, g^2 is a 2-element, and this allows us to conclude that g is a 2-element. □

The theorem just proved shows that there are, in fact, finitely generated groups that are infinite and periodic. This provides a negative answer to Burnside's problem.

2.3 The Milnor problem

Milnor's problem can be stated as follows.

Are there groups with intermediate growth?

This section is devoted to the study of the growth of the Grigorchuk group. In particular, we show that the growth of the Grigorchuk group is neither polynomial nor exponential.

Lemma 2.3.1. *Let G be an infinite finitely generated group. If G is commensurable with its own square, then it does not have polynomial growth.*

Proof. Recall that $\gamma(G \times G) = \gamma(G)\gamma(G)$ (cf. Theorem 1.4.10). Suppose G has polynomial growth. Then there exists a maximal integer $k \geqslant 1$ such that $n^k \preceq \gamma(G)$. As G and $G \times G$ are commensurable, we deduce from Corollary 1.4.9 that $\gamma(G) = \gamma(G \times G)$, so

$$n^{2k} \preceq \gamma(G)\gamma(G) = \gamma(G \times G) = \gamma(G).$$

This is a contradiction, as k was maximal. Hence, G cannot have polynomial growth. □

In order to show that the Grigorchuk group satisfies the hypothesis of the above lemma, we need two preliminary results.

Proposition 2.3.2. *The subgroup of the Grigorchuk group generated by $\{a, d\}$ is isomorphic to the dihedral group D_4 of order 8.*

Proof. The presentation of the dihedral group of order 8 given in [Hun74, Theorem I 6.13] is equivalent to the presentation we shall use here; namely, D_4 is generated by two elements x and y satisfying that both elements are of order two and that their product is of order four. As both generating elements a and d are of order 2, it suffices to show that $(ad)^4 = e$ while $(ad)^2 \neq e$. Since $\varphi_0((ada)d) = \varphi_1((ada)d) = b$, we have $(ad)^2 = (ada)d = (b, b) \neq e$. However, $(ad)^4 = (b, b)^2 = e$. Thus the subgroup of Γ generated by $\{a, d\}$ is isomorphic to D_4. □

With this in mind we can show the following result.

Proposition 2.3.3. *The Grigorchuk group is commensurable with its own square.*

Proof. The fact that $\psi : H_1 \to \Gamma \times \Gamma$ is a monomorphism (cf. Proposition 2.2.3 *(ii)*) implies that $H_1 \simeq \varphi(H_1)$. As $[\Gamma : H_1] = 2$ (cf. Proposition 2.2.1 *(ii)*), it suffices to show that $\psi(H_1)$ has finite index in $\Gamma \times \Gamma$.

Let B be the normal closure in Γ of the element b. If $\pi : \Gamma \to \Gamma/B$ is the canonical epimorphism, then the quotient Γ/B is generated by $\pi(a), \pi(b), \pi(c)$ and $\pi(d)$. However, $\pi(c) = \pi(bd) = \pi(d)$ and $b \in \ker \pi$, so the subgroup of Γ generated by a and d can be embedded into Γ/B. By the previous proposition, we have

$$[\Gamma : B] = |\Gamma/B| \leqslant 8. \tag{2.15}$$

Let $g \in \Gamma$. By surjectivity, there exist $h, h' \in H_1$ such that $\varphi_0(h) = \varphi_1(h') = g$. From Proposition 2.11, we see that both $(gbg^{-1}, e) = \psi(hadah^{-1})$ and $(e, gbg^{-1}) = \psi(h'd(h')^{-1})$ are contained in $\psi(H_1)$. As $g \in \Gamma$ was arbitrary, we conclude that both $B \times \{e\}$ and $\{e\} \times B$ are contained in $\varphi(H_1)$; consequently, this is also the case for $B \times B = (B \times \{e\})(\{e\} \times B)$.

Now, consider the epimorphism $\widetilde{\pi} : \Gamma \times \Gamma \to \Gamma/B \times \Gamma/B$ given by $(g, g') \mapsto (\pi(g), \pi(g'))$, for all $(g, g') \in \Gamma \times \Gamma$. As the kernel of $\widetilde{\pi}$ is $B \times B$, we find that

$$\Gamma \times \Gamma/(B \times B) \simeq \Gamma/B \times \Gamma/B.$$

Thus, (2.15) yields

$$[\Gamma : \psi(H_1)] \leqslant [\Gamma : B \times B] = [\Gamma : B][\Gamma : B] \leqslant 64. \tag{2.16}$$

This shows that $\psi(H_1)$ is a subgroup of finite index in $\Gamma \times \Gamma$. We conclude that the Grigorchuk group is commensurable with its own square. □

Theorem 2.3.4. *The Grigorchuk group does not have polynomial growth.*

Proof. As the Grigorchuk group is infinite (cf. Theorem 2.2.5) and commensurable with its own square (cf. Proposition 2.3.3), we can apply Lemma 2.3.1 to conclude that the Grigorchuk group does not have polynomial growth. □

Showing that the Grigorchuk group has subexponential growth requires some more substantial preliminaries. First, we need to define an important homomorphism using the following lemma.

Lemma 2.3.5. *Let $\sigma = 0, 1$. Then $\varphi_\sigma(H_{n+1}) \subseteq H_n$, for all $n \geqslant 0$.*

Proof. Fix $n \geqslant 0$. Let $\sigma \in \Sigma$, $w \in \Sigma^n$ and $g \in H_{n+1}$. Then $\sigma w = g(\sigma w) = \sigma\varphi_\sigma(g)(u)$, i.e., $u = \varphi_\sigma(g)(u)$. It follows that $\varphi_\sigma(g) \in H_n$. □

As a consequence of this lemma, we may define the homomorphism $\varphi_{\sigma_1,\sigma_2,\dots,\sigma_n} : H_n \to \Gamma$ recursively by $\varphi_{\sigma_1,\sigma_2,\dots,\sigma_n} = \varphi_{\sigma_1} \circ \varphi_{\sigma_2} \circ \cdots \circ \varphi_{\sigma_n}$. More importantly, we shall shall define a homomorphism $\psi_n : H_n \to \Gamma^{(2^n)}$ by

$$\psi_n(h) = (\varphi_{\sigma_1,\sigma_2,\dots,\sigma_n})_{\sigma_i=0,1}(h), \qquad h \in H_n. \tag{2.17}$$

Note that ψ_n is injective for all $n \geqslant 1$ and that $\psi_1 = \psi$. We shall return to this homomorphism in Theorem 2.3.9.

This next lemma provides a sufficient condition for a group's growth to be subexponential.

Lemma 2.3.6. *Let G be a finitely generated group with a finite symmetric generating subset $S \subseteq G$. Let $H \subseteq G$ be a subgroup of finite index. Suppose there exist an integer $M \geqslant 2$, two fixed constants $k \in (0, 1)$, $K \in (0, \infty)$ and a monomorphism $\vartheta : H \to G^M$ given by*

$$\vartheta(h) = (g_i)_{i=1}^M, \qquad h \in H,$$

with

$$\sum_{i=1}^{M} \ell_S(g_i) \leqslant k\ell_S(h) + K, \qquad h \in H.$$

Then G has subexponential growth.

Proof. By Proposition 1.3.4, it suffices to show that $\lambda_S^G = \lim_{n\to\infty} \gamma_S^G(n)^{1/n} = 1$. We shall divide the proof into three steps.

Step 1: Fix $\varepsilon > 0$. By convergence, there exists an $N \in \mathbb{N}$ such that $\gamma_S^G(n) < (\lambda_S^G + \varepsilon)^n$, for $n \geqslant N$ (as $\lambda_S^G \geqslant 1$). It follows that

$$\gamma_S^G(n) \leqslant \gamma_S^G(N)(\lambda_S + \varepsilon)^n, \qquad n \geqslant 1. \tag{2.18}$$

For a second approximation, fix a positive integer m and let T be a complete set of representatives of left cosets of H in G. Set $C = \max_{t\in T} \ell_S(t)$. For each $g \in B_S^G(m)$, there exist an $h \in H$ and a unique $t \in T$ such that $g = th$ and

$$\ell_S(h) \leqslant \ell_S(g) + \ell_S(t^{-1}) = \ell_S(g) + \ell_S(t) \leqslant \ell_S(g) + C \leqslant m + C.$$

It follows that $B_S^G(m) \subseteq TB_S^H(m + C)$. If $\gamma_S^H(n) = \{h \in H \mid \ell_S(h) \leqslant n\}$, we deduce that

$$\gamma_S^G(m) \leqslant |T|\gamma_S^H(m + C) = [G : T]\gamma_S^H(m + C), \qquad m \geqslant 1. \tag{2.19}$$

Step 2: Fix a positive integer n and set

$$\Lambda := \bigcup B_S^G(n_1) \times B_S^G(n_2) \times \cdots B_S^G(n_M),$$

with

$$|\Lambda| = \sum \gamma_S^G(n_1)\gamma_S^G(n_2)\cdots\gamma_S^G(n_M). \tag{2.20}$$

The union and sum is taken over all M-tuples $(n_1, n_2, \ldots, n_M) \in \mathbb{N}_0^M$ with $\sum_{i=1}^M n_i \leqslant kn + K$. Let $h \in B_S^H(n)$. By hypothesis, $\vartheta(h) = (g_i)_{i=1}^M$ is an M-tuple in G^M with

$$\sum_{i=1}^{M} \ell_S(g_i) \leqslant k\ell_S(h) + K \leqslant kn + K.$$

That is, $\vartheta(h) \in \Lambda$. It follows that $\vartheta(B_S^H(n)) \subseteq \Lambda$, for all integers $n \geqslant 0$. As ϑ is injective, we have

$$\gamma_S^H(n) \leqslant |\Lambda| = \sum \gamma_S^G(n_1)\gamma_S^G(n_2)\cdots\gamma_S^G(n_M). \tag{2.21}$$

Applying (2.18), we obtain

$$\begin{aligned}
\gamma_S^H(n) &\leqslant \gamma_S^G(N)^M \sum (\lambda_S^G + \varepsilon)^{n_1}(\lambda_S^G + \varepsilon)^{n_2}\cdots(\lambda_S^G + \varepsilon)^{n_M} \\
&\leqslant \gamma_S^G(N)^M \sum (\lambda_S^G + \varepsilon)^{kn+K} \\
&= \gamma_S^G(N)^M (\lambda_S^G + \varepsilon)^{kn+K} \sum \\
&\leqslant \left(\gamma_S^G(N)(kn + K)\right)^M (\lambda_S^G + \varepsilon)^{kn+K}.
\end{aligned}$$

Step 3: Returning to the growth function of G, we first apply (2.19) and then the above to find that

$$\begin{aligned}
\gamma_S^G(n) &\leqslant [G : H]\gamma_S^H(n + C) \\
&\leqslant [G : H]\left(\gamma_S^G(N)(k(n + C) + K)\right)^M (\lambda_S^G + \varepsilon)^{k(n+C)+K}.
\end{aligned}$$

It follows that

$$\gamma_S^G(n)^{1/n} \leqslant [G:H]^{1/n} \left(\gamma_S^G(N)(k(n+C)+K)\right)^{M/n} (\lambda_S^G + \varepsilon)^{(k(n+C)+K)/n}. \tag{2.22}$$

Observe that the right hand side of (2.22) converges to $(\lambda_S^G + \varepsilon)^k$, as n tends to infinity. That is, $\lambda_S^G \leqslant (\lambda_S^G)^k$. Since this is the case for all $k \in (0,1)$, we finally conclude that $\lambda_S^G = 1$. Hence, G has subexponential growth. □

The rest of the section is devoted to showing that the Grigorchuk group satisfies the hypothesis of the above lemma. For this, we consider a new alphabet $S = \{a, b, c, d\}$. Let S^* denote the set of words in S containing also the empty word ϵ. The concatenation of words is associative, hence S^* is a monoid. The map $s \mapsto s \in \Gamma$, for all $s \in S$, extends to a homomorphism of monoids $\varpi : S^* \to \Gamma$. Clearly, ϖ is surjective.

A word $w = \lambda_1\lambda_2 \cdots \lambda_n \in S^*$ (with $\lambda_i \in S$, $i = 1, 2, \ldots, n$) is *reduced*, if $\lambda_i = a$ implies $\lambda_{i+1} \in \{b, c, d\}$ and $\lambda_j \in \{b, c, d\}$ implies $\lambda_{j+1} = a$. The non-negative integer $\ell(w) := n$ denotes the length of w. Note that for all $g \in \Gamma$, there exists a word $w \in S^*$ such that $\varpi(w) = g$ with $\ell(w) = \ell_S(g)$. We say that w *represents* g.

We shall refer to S^*_{red} as the set of all reduced words in S. Given a word $w \in S^*_{red}$, let $|w|_\lambda$ denote the number of occurrences of the letter $\lambda \in S$. We also put $|w|_{\lambda,\lambda'} = |w|_\lambda + |w|_{\lambda'}$, for $\lambda, \lambda' \in S$. A certain subset of S^*_{red} is of special importance to us; namely,

$$\Theta_1 := \{w \in S^*_{red} \mid |w|_a \equiv 0 \ (mod\ 2)\}.$$

For each $\sigma = 0, 1$, we consider the map $\widetilde{\varphi}_\sigma : \Theta_1 \to S^*$ which satisfies $\widetilde{\varphi}_\sigma(\lambda_1 a \lambda_2 a \lambda_3 \cdots) = \widetilde{\varphi}_\sigma(\lambda_1)\widetilde{\varphi}_\sigma(a\lambda_2 a)\widetilde{\varphi}_\sigma(\lambda_3)\cdots$, for $\lambda_i \in \{b, c, d\}$ and takes the values

$$\begin{aligned}
&\widetilde{\varphi}_0(b) = a, && \widetilde{\varphi}_0(aba) = c, && \widetilde{\varphi}_1(b) = c, && \widetilde{\varphi}_1(aba) = a,\\
&\widetilde{\varphi}_0(c) = a, && \widetilde{\varphi}_0(aca) = d, && \widetilde{\varphi}_1(c) = d, && \widetilde{\varphi}_1(aca) = a,\\
&\widetilde{\varphi}_0(d) = \epsilon, && \widetilde{\varphi}_0(ada) = b, && \widetilde{\varphi}_1(d) = b, && \widetilde{\varphi}_1(ada) = \epsilon.
\end{aligned} \tag{2.23}$$

In general, the image of a reduced word under $\widetilde{\varphi}_0$ or $\widetilde{\varphi}_1$ need not be reduced. However, there is a *reduction algorithm* which reduces any word to its reduced form using a finite number of reductions. For all distinct $\lambda_1, \lambda_2, \lambda_3 \in \{b, c, d\}$, the reduction algorithm puts $\lambda_1\lambda_2 \rightsquigarrow \lambda_3$, thereby reducing the word by 1 and, for all $\lambda \in S$, it takes $\lambda\lambda \rightsquigarrow \epsilon$ thereby reducing the word by 2.

If $r(\widetilde{\varphi}_\sigma(w))$ denotes the reduced word, we shall lighten the notation by putting $w_\sigma := r(\widetilde{\varphi}_\sigma(w))$, for all $w \in \Theta_1$. Recursively, we set $w_{\sigma_1,\sigma_2,\ldots,\sigma_m,\sigma} := r(\widetilde{\varphi}_\sigma(w_{\sigma_1,\sigma_2,\ldots,\sigma_m}))$. For all $n \geqslant 1$, we can now consider the set

$$\Theta_{n+1} := \{w \in \Theta_n \mid w_0, w_1 \in \Theta_n\}.$$

In particular, $\Theta_{n+1} \subseteq \Theta_n$, for all $n \geqslant 1$.

Lemma 2.3.7. *For all $w \in \Theta_1$, we have*

$$\ell(w_0) + \ell(w_1) \leqslant \ell(w) + 1. \tag{2.24}$$

Proof. Let $w \in \Theta_1$. If $\ell(w) \leqslant 3$, there is nothing to prove. However, if $\ell(w) \geqslant 4$ we can express w in one of the following forms

$$
\begin{aligned}
w &= \quad\;\; au_1au_2\cdots au_{2k-1}au_{2k},\\
w &= \quad\;\; au_1au_2\cdots au_{2k-1}a,\\
w &= u_0 \;\; au_1au_2\cdots au_{2k-1}au_{2k},\\
w &= u_0 \;\; au_1au_2\cdots au_{2k-1}a,
\end{aligned}
$$

for some $u_0, u_1, u_2, \ldots, u_{2k} \in \{b, c, d\}$ and $k \in \mathbb{N}$. The length of w is $4k, 4k-1, 4k+1, 4k$, respectively. However, by (2.23), we see that the length of each of w_0 and w_1 is less than or equal to $2k, 2k-1, 2k+1, 2k$, respectively. This completes the proof. □

Proposition 2.3.8. *Let $w \in \Theta_3$. Then*

$$\sum_{i,j,k=0}^{1} \ell(w_{i,j,k}) \leqslant \frac{5}{6}\ell(w) + 8.$$

Proof. Let $w \in \Theta_3$. In particular, $w \in \Theta_1$, so by repeated application of (2.24), we find that

$$\sum_{i,j,k=0}^{1} \ell(w_{i,j,k}) \leqslant \sum_{i,j=0}^{1} \ell(w_{i,j}) + 4 \leqslant \sum_{i=0}^{1} \ell(w_i) + 6 \leqslant \ell(w) + 7. \tag{2.25}$$

But this is not quite good enough; we need to sharpen the inequality. As $\widetilde{\varphi}_0(d) = \widetilde{\varphi}_1(ada) = \epsilon$, each letter d in w will give rise to the empty word in one of w_0 or w_1. Thus, $\ell(w_0) + \ell(w_1) \leqslant \ell(w) + 1 - |w|_d$ and consequently

$$\sum_{i,j,k=0}^{1} \ell(w_{i,j,k}) \leqslant \ell(w) + 7 - |w|_d. \tag{2.26}$$

From the definition of $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_1$ (see (2.23)), we see that any occurrence of the letter c in w implies the occurrence of the letter d in w_0 or w_1; similarly, any b in w gives rise to the letter c in w_0 or w_1 and consequently to the letter d in one of $w_{00}, w_{01}, w_{10}, w_{11}$. This consideration shows that for $t \in \{b, c, d\}$, we have

$$\sum_{i,j,k=0}^{1} \ell(w_{i,j,k}) \leqslant \ell(w) + 7 - |w|_t \tag{2.27}$$

. In particular,

$$\sum_{i,j,k=0}^{1} \ell(w_{i,j,k}) \leqslant \min_{t\in\{b,c,d\}} \ell(w) + 7 - |w|_t = \ell(w) + 7 - \max_{t\in\{b,c,d\}} |w|_t. \tag{2.28}$$

Since w is reduced, $|w|_{b,c,d} \geqslant (\ell(w) - 1)/2$ and this yields the estimation

$$\max_{t\in\{b,c,d\}} \{|w|_t\} \geqslant \frac{\ell(w) - 1}{6} > \frac{\ell(w)}{6} - 1. \tag{2.29}$$

Using this, we finally obtain

$$
\begin{aligned}
\sum_{i,j,k=0}^{1} \ell(w_{i,j,k}) &\leqslant \ell(w) + 7 - \max_{t\in\{b,c,d\}} |w|_t\\
&\leqslant \ell(w) + 7 - \left(\frac{\ell(w)}{6} - 1\right)\\
&= \frac{5}{6}\ell(w) + 8,
\end{aligned}
$$

as desired. $\square$

We are finally ready to show that Γ does not grow exponentially.

Theorem 2.3.9. *The Grigorchuk group Γ has subexponential growth.*

Proof. Let $g \in H_3$. There exists $w \in \Theta_3$ representing g. For each $i, j, k = 0, 1$, the element $g_{i,j,k}$ is an entrance in the 8-tuple $\psi_3(g)$ with $\varpi(w_{i,j,k}) = g_{i,j,k}$. Hence $\ell(w_{i,j,k}) \leqslant \ell_S(g_{i,j,k})$. From the previous proposition, we see that

$$\sum_{i,j,k=0}^{1} \ell_S(g_{i,j,k}) \leqslant \frac{5}{6}\ell(w) + 8 = \frac{5}{6}\ell_S(g) + 8.$$

Let us put $H := H_3$ and $\vartheta := \psi_3$. By Proposition 2.1.4, $H_3 \subseteq \Gamma$ has finite index and ψ_3 is injective. If $M = 8$, $k = 5/6$ and $K = 8$, we see that the hypotheses of Lemma 2.3.6 are satisfied. Thus the Grigorchuk group has subexponential growth. $\square$

Chapter 3

Amenability and growth

3.1 Amenability and paradoxicality

One of the important results of this chapter is that finitely generated groups with subexponential growth are supramenable. However, before considering the supramenable groups, we need to define the notion of an amenable group. We shall use the definition given by John von Neumann. Recall that a measure $\mu : \mathcal{P}(G) \to [0, \infty]$ on a group G is said to be *left-invariant* (resp., *right-invariant*) if $g\mu = \mu$ (resp. $\mu g = \mu$), for all $g \in G$.

Definition 3.1.1 (Amenable)**.** *A group G is* amenable *if there exists a finitely additive left-invariant (resp., right-invariant) measure $\mu : \mathcal{P}(G) \to [0, \infty]$ such that $\mu(G) = 1$. The class of amenable groups is denoted by AG.*

We also need to consider the notion of paradoxical sets and groups. When dealing with paradoxicality, we shall some times denote the action of a group element g on a set A by $g(A)$.

Definition 3.1.2 (Paradoxical)**.** *Suppose a group G acts on a set Ω. A subset $E \subseteq G$ is* G-paradoxical*, if there exist positive integers n and m, subsets $A_1, \ldots, A_n, B_1, \ldots, B_m \subseteq E$ and group elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that*

$$E = \bigcup_{i=1}^{n} g_i(A_i) = \bigcup_{j=1}^{m} h_j(B_j). \tag{3.1}$$

If $\Omega = E = G$, we simply say that G is paradoxical.

Example 3.1.3. Consider the free group on two generators $\mathscr{F}_2$ and let a and b be the generators. Let $W(c)$ denote the set of words starting with $c = a, a^{-1}, b, b^{-1}$. Then the sets $\{e_{\mathscr{F}_2}\}$, $W(a)$, $W(a^{-1})$, $W(b)$, $W(b^{-1})$ are pairwise disjoint. Furthermore,

$$\mathscr{F}_2 = W(a) \cup aW(a^{-1}) = W(b) \cup aW(b^{-1}).$$

Indeed, if $w \notin W(a)$ is a word, then $a^{-1}w \in W(a^{-1})$, so $w = a(a^{-1}w) \in aW(a^{-1})$. That is, $\mathscr{F}_2$ is paradoxical.

In the definition of paradoxicality, we do not assume that $g_1(A_1), g_2(A_2), \ldots, g_n(A_n)$ are pairwise disjoint. However, it is not difficult to see that we may, in fact, assume this without loss of generality. Recursively, we define

$$A_1^* = A_1, \qquad A_i^* = A_i - g_i^{-1}\left(\bigcup_{k=1}^{i-1} g_k(A_k)\right). \tag{3.2}$$

Note that the sets are pairwise disjoint and that $\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n A_i^*$, i.e., $E = \coprod_{i=1}^n g_i(A_i^*)$ is a disjoint union. If $n > m$, we may define the sets $B_1^*, B_2^*, \ldots, B_n^*$ analogously and put $B_{m+1}^* = \cdots = B_n^* = \emptyset$.

It turns out that the notion of amenability provides us with a beautiful distinction between the groups that are paradoxical and those that are not. This is a derived result of Tarski's Theorem.

Theorem 3.1.4 (Tarski)**.** *Suppose a group G acts on a set Ω and let $E \subseteq \Omega$ be a subset. There exists a finitely additive, G-invariant measure $\mu : \mathcal{P}(G) \to [0, \infty]$ with $\mu(E) = 1$ if and only if A is not G-paradoxical.*

A proof can be found in [Wag93, Corollary 9.2]. If we let G act on the underlying set of G and set $\Omega = E = G$, then we see that the amenable groups coincide with the groups that are not paradoxical. As we shall see, all finitely generated groups with subexponential growth are amenable. Therefore, paradoxical groups must have exponential growth. Unfortunately, there are examples of amenable groups with exponential growth. See the group defined in (3.6).

Roughly speaking, the notion of amenability is not strong enough to distinguish between groups with subexponential and exponential growth, respectively. This is the reason for us to consider the stronger notion of supramenable groups. However, in order to prove that subexponential growth implies supramenability, we need a new equivalent characterization of paradoxicality in terms of *equidecomposability.*

Definition 3.1.5 (Equidecomposable)**.** *Suppose a group G acts on a set Ω and let $A, B \subseteq \Omega$. The subsets A and B are G-*equidecomposable*, if there exist partitions $A_1, \ldots, A_n$ of A and $B_1, \ldots, B_n$ of B, respectively, and group elements $g_1, g_2, \ldots, g_n$ of G such that*

$$g_i(A_i) = B_i.$$

and $A_i \cap A_j = B_i \cap B_j = \emptyset$, for $i \neq j$. If this is the case, we write $A \sim_G B$.

The relation $\sim_G$ is, in fact, an equivalence relation. Reflexivity and symmetry follow directly from the definition. Suppose $A \sim_G B$ and $B \sim_G C$. Then $A \sim_G C$. Indeed, let $c \in C$ and let $\{A_i\}_{i=1}^n, \{g_i\}_{i=1}^n$ and $\{B_j\}_{j=1}^m, \{h_j\}_{j=1}^m$ witness the equidecomposability of A and B, respectively. There exists a unique j such that $c \in h_j(B_j)$. Putting $b_j := h_j^{-1}(c)$, we have $b_j \in g_i(A_i) \cap B_j$, for exactly one i. This implies that $c \in h_j(g_i(A_i) \cap B_j)$, for unique i and j. Consequently,

$$C = \bigcup_{i,j} h_j(g_i(A_i) \cap B_j) = \bigcup_{i,j} (h_j g_i)(A_i \cap g_i^{-1}(B_j))$$

is a partition of C. Using a similar argument, it is not hard to see that $A = \bigcup_{i,j}(A_i \cap g_i^{-1} B_j)$ is a partition. This shows that $A \sim_G C$.

We have the following theorem.

Theorem 3.1.6. *Suppose a group G acts on a set Ω and let $E \subseteq \Omega$. The subset E is G-paradoxical if and only if there exist two disjoint subsets $A, B \subseteq E$ such that $E \sim_G A$ and $E \sim_G B$.*

Proof. Suppose E contains two disjoint subsets A and B such that $E \sim_G A$ and $E \sim_G B$. It follows from the definition that E is G-paradoxical.

Conversely, suppose E is G-paradoxical and let $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ and $g_1, \ldots, g_n$, $h_1, \ldots, h_m$ be as in the definition. Construct the disjoint union as in (3.2) and set $A := \bigcup_{i=1}^{n} A_i^*$ and $B := \bigcup_{j=1}^{m} B_j^*$. Then the new pairwise disjoint subsets form a partition of A and B, respectively, and $E \sim_G A$ and $E \sim_G B$. This finishes the proof. □

Theorem 3.1.7 (Banach-Shröder-Bernstein Theorem)**.** *Suppose a group G acts on a set Ω and let $A, B \subseteq \Omega$ be subsets. If A is equidecomposable with a subset of B and B is equidecomposable with a subset of A, then $A \sim_G B$.*

A proof can be found in [Wag93, Theorem 3.5].

Corollary 3.1.8. *A subset $E \subseteq \Omega$ is G-paradoxical if and only if there exist disjoint subsets $A, B \subseteq \Omega$ such that $E = A \cup B$ and $E \sim_G A$ and $E \sim_G B$.*

Proof. Suppose $E \subseteq \Omega$ is G-paradoxical. There exist subsets $A', B' \subseteq E$ such that $E \sim_G A'$ and $E \sim_G B'$. Note that $E \sim_G B'$ is equidecomposable with a subset of $E - A'$ (namely, B' itself) and $E - A'$ is equidecomposable with a subset of E (namely, $E - A'$ itself). By The Banach-Schöder-Bernstein Theorem, we have $E \sim_G E - A$. To finish the proof, set $A := A'$ and $B := E - A'$. □

3.2 Supramenable groups

Definition 3.2.1 (Supramenable)**.** *Let G be a group. We say that G is* supramenable *if given any non-empty subset $A \subseteq G$, there exists a finitely additive left-invariant measure $\mu : \mathcal{P}(G) \to [0, \infty]$ such that $\mu(A) = 1$.*

Clearly, the class of supramenable groups is contained in the class of amenable groups. Before we see how supramenable groups act on sets, let us first determine some permanence properties of amenable and supramenable groups.

3.2.1 Permanence properties

Proposition 3.2.2. *Finite and abelian groups are supramenable.*

Proof. The counting measure shows that any finite group is supramenable. In Theorem 3.2.12, we prove that any group of subexponential growth is supramenable. By Corollary 1.4.12, any abelian group has polynomial growth; hence abelian groups are supramenable. □

Proposition 3.2.3. *The following holds*

(i) Let G be supramenable group. If $H \subseteq G$ is a subgroup, then H is supramenable,

(ii) Let N be a normal subgroup of a group G. If G is supramenable, then the quotient G/N is supramenable.

(iii) Let G be the direct union of supramenable groups. Then G is supramenable,

(iv) Let H be subgroup of finite index in a group G. If H is supramenable, then so is G.

Proof. (i): Let $A \subseteq H$ be a subset. If μ is a measure on $\mathcal{P}(G)$ that normalizes A, then the restriction of this measure to A is a measure on $\mathcal{P}(H)$ with the desired properties.

(ii): Let $\pi : G \to G/N$ denote the canonical epimorphism and let $A \subseteq G/N$ be a subset. There exists a measure $\nu : \mathcal{P}(G) \to [0,\infty]$ such that $\nu(\pi^{-1}(A))$. Now, the image measure $\mu : \mathcal{P}(G/N) \to [0,\infty]$ given by $\mu(B) := \nu(\pi(B))$, for all $B \subseteq G/N$, shows that G/N is supramenable.

(iii): Let I be a directed set and suppose $G = \cup_{i\in I} G_i$, where G_i is supramenable for all $i \in I$. Let $A \subseteq G$ be a subset. We may assume that $G_i \cap A \neq \emptyset$, for all $i \in I$; indeed, we can remove any G_i for which $G_i \cap A = \emptyset$ as any such G_i is contained in a G_j such that $G_j \cap A \neq \emptyset$ (as I is directed).

Now, consider the topological space $[0,\infty]^{\mathcal{P}(G)} = \{f : \mathcal{P}(G) \to [0,\infty]\}$ which is compact by the Tychonoff Theorem ([Mun75, Theorem 37.3]) in the topology of point-wise convergence. Let M_i denote the set of finitely additive G_i-invariant measures that normalize the set A. As G_i is supramenable for all $i \in I$, there exists a finitely additive G_i-invariant measure μ such that $\mu_i(G_i \cap A) = 1$. Thus, the restricted measure defined by

$$\mu(B) := \mu_i(G_i \cap B), \qquad B \subseteq G, \tag{3.3}$$

is an element in M_i. That is, $M_i \neq \emptyset$ for all $i \in I$. It is straight-forward to show that M_i is closed. Note that given $i_1, i_2, \ldots, i_n \in I$ there exists a $j \in I$ such that $G_{i_1}, G_{i_2}, \ldots, G_{i_n} \subseteq G_j$; consequently, $\cap_{k=1}^{n} M_{i_k} \supset M_j \neq \emptyset$. Thus, the collection $\{M_i \mid i \in I\}$ has the finite intersection property. By compactness, we conclude that

$$M := \bigcap_{i\in I} M_i \neq \emptyset, \tag{3.4}$$

cf. [Mun75, Theorem 26.9]. Now, any measure $\mu \in M$ has the desired properties; namely, $\mu : \mathcal{P}(G) \to [0,\infty]$ is a finitely additive G-invariant measure that normalizes A. Hence, G is supramenable.

(iv): Suppose $H \subseteq G$ is a supramenable subgroup of finite index in G, say $[G : H] = m$. Let $A \subseteq G$ be a subset. Extract from each distinct right coset of H in G an element to form the set $\{g_1, g_2, \ldots, g_m\}$. Now, let H act on the set G by left multiplication. As $\cup_{i=1}^{m} g_i A \subseteq G$, there exists a finitely additive H-left-invariant measure ν such that $\nu(\cup_{i=1}^{m} g_i A) = 1$. Put $a := \sum_{i=1}^{m} \nu(g_i A)$ and note that $0 < a < \infty$.

The measure $\mu : \mathcal{P}(G) \to [0,\infty]$ defined by

$$\mu(B) := \frac{1}{a} \sum_{i=1}^{m} \nu(g_i B), \qquad B \subseteq G,$$

has the desired properties. Indeed, by construction $\mu(A) = 1$. If $B_1, B_2 \subseteq G$ are disjoint subset, then $gB_1 \cap gB_2 = \emptyset$, and we have

$$\mu(B_1 \cup B_2) = \frac{1}{a} \sum_{i=1}^{m} \nu(g_i(B_1 \cup B_2)) = \frac{1}{a} \sum_{i=1}^{m} \nu(g_i B_1) + \frac{1}{a} \sum_{i=1}^{m} \nu(g_i B_2) = \mu(B_1) + \mu(B_2).$$

Fix $g \in G$ and $B \subseteq G$ and note that the action of g on $\{g_1, g_2, \ldots, g_m\}$ by right multiplication constitutes a permutation. Thus,

$$\mu(gB) = \frac{1}{a}\sum_{i=1}^{m} \nu(g_i g B) = \frac{1}{a}\sum_{k=1}^{m} \nu(h g_k B) = \frac{1}{a}\sum_{k=1}^{m} \nu(g_k B) = \mu(B),$$

as ν is H-left-invariant. This shows that G is supramenable. □

Example 3.2.4. We saw in Example 3.1.3 that the free group on two generators $\mathscr{F}_2$ is paradoxical. By Tarski's Theorem, it is not amenable. Combined with the fact that amenable groups are stable under subgroups, we see that any group containing the free subgroup on two generators $\mathscr{F}_2$ is not amenable.

The next propositions concern amenable groups, in particular.

Proposition 3.2.5. *All solvable groups are amenable.*

Proof. Let G be a solvable group. We shall proceed by induction on the solvability degree i. If $i = 0$, then $G = \{e_G\}$ and there is nothing to prove. Suppose G has solvability degree $i + 1$. The derived subgroup $D(G)$ is solvable of degree i hence amenable by hypothesis. Also, the quotient $G/D(G)$ is abelian, hence amenable by Proposition 3.2.3 (i). By Proposition 3.2.6, we conclude that G is amenable. □

In particular, all abelian groups are amenable. During the proof of the next proposition we shall use notation and results from Appendix A.

Proposition 3.2.6. *Let G be a group and let $N \subseteq G$ be a normal subgroup. If both N and G/N are amenable, then G is amenable.*

Remark 3.2.7. *Let G be a group. Suppose*

$$e \longrightarrow H \xrightarrow{\iota} G \xrightarrow{\kappa} K \longrightarrow e$$

is a short exact sequence of groups. Then $\ker \kappa = \iota(H)$ showing that $\iota(H)$ is normal in G. As κ is surjective, there exists a unique isomorphism $\psi : G/\iota(H) \to K$. It follows that

$$e \longrightarrow \iota(H) \longrightarrow G \longrightarrow G/\iota(H) \longrightarrow e$$

is also a short exact sequence. Therefore, we may rephrase the content of the Proposition 3.2.6 by stating that the class of amenable groups is stable under taking extensions.

Proof of Proposition 3.2.6. The amenability of N ensures the existence of a N-left-invariant mean m_N. Let $\pi : G \to G/N$ denote the canonical epimorphism; for each $x \in \ell^\infty(G)$, consider $\widetilde{x} : G/N \to \mathbb{R}$ given by

$$\widetilde{x}(\pi(g)) = m_N((g^{-1}x)|_N), \qquad g \in G.$$

It may not be obvious that this is well-defined: Suppose $\pi(g_1) = \pi(g_2)$, for some $g_1, g_2 \in G$. There exists $h \in N$ such that $g_2 = g_1 h$; consequently

$$m_N((g_2^{-1}x)|_N) = m_N(([h^{-1}g_1^{-1}]x)|_N) = m_N((h^{-1}[g_1^{-1}x])|_N) = m_N((g_1^{-1}x)|_N),$$

i.e., $\widetilde{x}(\pi(g_2)) = \widetilde{x}(\pi(g_1))$. It is easy to see that $\widetilde{x} \in \ell^\infty(G/N)$: Indeed, for all $g \in G$:

$$|\widetilde{x}(\pi(g))| = |m_N((g^{-1}x)|_N)| \leqslant \sup_{h\in N} |g^{-1}x(h)| \leqslant \sup_{h\in G} |g^{-1}x(h)| = ||x||_\infty.$$

The amenability of G/N ensures the existence of a G/N-left-invariant mean $m_{G/N}$. Let $M : \ell^\infty(G) \to \mathbb{R}$ be given by

$$M(x) := m_{G/N}(\widetilde{x}), \qquad x \in \ell^\infty(G).$$

This is clearly a well-defined mean on G. Fix $g \in G$ and observe

$$\widetilde{gx}(\pi(k)) = m_N((k^{-1}[gx])|N) = \widetilde{x}(\pi(g^{-1}k)) = \widetilde{x}(\pi(g)^{-1}\pi(k)) = \pi(g)\widetilde{x}(\pi(k)),$$

for all $k \in G$. This yields

$$M(gx) = m_{G/N}(\widetilde{gx}) = m_{G/N}(\pi(g)\widetilde{x}) = m_{G/N}(\widetilde{x}) = M(x).$$

Hence, M is left-invariant and G is amenable. □

3.2.2 Growth

We are now ready to relate the notions of amenability and growth. We shall first consider the action of supramenable groups and groups of subexponential growth, respectively.

Theorem 3.2.8. *Let G be a supramenable group that acts on a set Ω. If $A \subseteq \Omega$ is a non-empty subset, then there exists a finitely additive G-left-invariant measure $\mu : \mathcal{P}(\Omega) \to [0,\infty]$ normalizing A.*

Proof. Fix $x \in A$ and let G act on Ω by left multiplication. To each subset $B \subseteq \Omega$ we assign a corresponding subset of G defined by $B_x = \{g \in G \mid gx \in B\}$. Note that $A_x \subseteq G$ is non-empty, as $e_G x = x \in A$. As G is supramenable, there exists a finitely additive G-left-invariant measure $\nu : \mathcal{P}(G) \to [0,\infty]$ normalizing A_x. Now, define a measure $\mu : \mathcal{P}(\Omega) \to [0,\infty]$ on Ω by $\mu(B) = \nu(B_x)$, for all $B \subseteq \Omega$. Clearly, $\mu(A) = 1$. If $B, B' \subseteq \Omega$ are disjoint, then $B_x \cap B'_x = \emptyset$ and

$$\mu(B \cup B') = \nu((B \cup B')_x) = \nu(B_x \cup B'_x) = \nu(B_x) + \nu(B'_x) = \mu(B) + \mu(B').$$

Fix $g \in G$ and observe that

$$(hB)_x = \{g \in G \mid gx \in hB\} = \{g \in G \mid h^{-1}g \in B_x\} = \{g \in G \mid g \in hB_x\} = hB_x.$$

The G-left-invariance of μ now follows easily from the G-left-invariance of ν

$$\mu(gB) = \nu((gB)_x) = \nu(gB_x) = \nu(B_x) = \mu(B).$$

Hence, μ has the desired properties. □

Combining Theorem 3.2.8 with Tarski's Theorem, we obtain the following useful result.

Corollary 3.2.9. *Let G be a group acting on a set Ω. If G is supramenable, then there are no non-empty G-paradoxical subsets of Ω.*

Example 3.2.10. Let G be a group and let $F_2 \subseteq G$ be a free subsemigroup on two generators σ and ρ. Set $A := \sigma F_2$ and $B := \rho F_2$ and observe that $F_2 = A \cup B$, with $A \cap B = \emptyset$. Since $\sigma^{-1}, \rho^{-1} \in G$, the fact that $F_2 = \sigma^{-1}A = \rho^{-1}B$ implies that F_2 is G-paradoxical. By Corollary 3.2.9, we conclude that G is not supramenable.

In the following theorem, we use the characterization of paradoxicality in terms of equidecomposability.

Theorem 3.2.11. *Let G be a group acting on a set Ω and let $A \subseteq \Omega$ be a non-empty subset. If G has subexponential growth, then A is not G-paradoxical.*

Proof. Suppose the subset $A \subseteq \Omega$ is G-paradoxical. Then there exists a partition $A = B \cup C$ of two subsets B and C such that $A \sim_G B$ and $A \sim_G C$. That is, there are partitions

$$A = \bigcup_{i=1}^{n} A_i = \bigcup_{j=1}^{m} A'_j, \qquad B = \bigcup_{i=1}^{n} B_i, \qquad C = \bigcup_{j=1}^{m} C_j, \tag{3.5}$$

and group elements $g_1, \ldots, g_n, h_1, \ldots, h_m \in G$ such that $g_i(A_i) = B_i$ and $h_j(A'_j) = C_j$. Let $S := \{g_1, \ldots, g_n, h_1, \ldots, h_m\}$ and define the two maps $F_1, F_2 : A \to A$ by

$$F_1(x) = \begin{cases} g_1(x), & x \in A_1 \\ g_2(x), & x \in A_2 \\ \vdots & \\ g_n(x), & x \in A_n, \end{cases} \qquad F_2(x) = \begin{cases} h_1(x), & x \in A'_1 \\ h_2(x), & x \in A'_2 \\ \vdots & \\ h_m(x), & x \in A'_m, \end{cases}$$

and observe that $F_1(A) = B$ and $F_2(A) = C$. For a fixed $x \in A$, we consider the strings $F_{i_1} \circ F_{i_2} \circ \cdots \circ F_{i_n}(x)$ of length n, where $i_1, i_2, \ldots, i_n = 1, 2$. Since F_1 and F_2 have disjoint images, each string is unique. As each string is of the form wx, where w is a words of length n expressed as a product of elements in S, there are 2^n such words of length n. Thus, G has exponential growth. This finishes the proof. □

The machinery we have developed enables us to prove the main theorem of this chapter. This combines the notions of growth and supramenability in a neat way.

Theorem 3.2.12. *Let G be a group. If G has subexponential growth, then G is supramenable.*

Proof. Let G act on the underlying set of G and let $A \subseteq G$ be a non-empty subset. By Theorem 3.2.11, A is not paradoxical. Invoking Tarski's Theorem (cf. Theorem 3.1.4), there exists a finitely additive G-left invariant measure $\mu : \mathcal{P}(G) \to [0, \infty]$ normalizing A, i.e., $\mu(A) = 1$. That is, G is supramenable. □

Corollary 3.2.13. *Subexponential growth implies amenability.*

3.3 Elementary amenable groups

Definition 3.3.1 (Elementary amenable)**.** *The* class of elementary amenable groups *is the smallest class of groups that contains the finite and the abelian groups and is stable under the four operations of taking (I) subgroups, (II) quotients, (III) extensions and (IV) direct unions. The class is denoted by EG.*

In the previous section, we saw that the class of amenable groups is stable under the four operations (I)-(IV). Since finite and abelian groups are amenable it is clear that EG is contained in AG.

Define EG_0 to be the union of all finite groups and all abelian groups. For each ordinal α, we shall use transfinite recursion to define the class EG_α as follows. If α is a limit ordinal, we set $EG_\alpha = \bigcup_{\beta<\alpha} EG_\beta$; if α is a successor ordinal, let EG_α be the groups obtainable from $EG_{\alpha-1}$ by operations (I)-(IV) once (and only once). However, the following proposition shows that is is sufficient to use operations (III) and (IV) only. We shall follow the proof of [Cho80, Proposition 2.1].

Proposition 3.3.2. *For each ordinal α, the class EG_α is stable under operations (I) and (II).*

Proof. It is clear that subgroups and quotients of finite groups (resp., abelian groups) are finite groups (resp., abelian groups). Thus, the statement is true for EG_0. Fix an ordinal $\alpha > 0$ and suppose EG_β is stable under operations (I) and (II), for all ordinals $\beta < \alpha$. Take $G \in EG_\alpha$, let $B \subseteq G$ be a subgroup and let $\varphi : G \to C$ be some surjective homomorphism of groups. It suffices to show that $B, C \in EG_\alpha$.

If α is a limit ordinal, then there exists an ordinal $\beta < \alpha$ such that $G \in EG_\beta$. By hypothesis, $B, C \in EG_\beta$. Consequently, $B, C \in EG_\alpha$.

If α is a successor ordinal, then G is obtained from $EG_{\alpha-1}$ by operations (III) or (IV). We shall divide the proof into two cases.

Case 1: Suppose $G = \bigcup_{i\in I} G_i$ is the direct union of groups $G_i \in EG_{\alpha-1}$. For all $i \in I$, $H \cap G_i \subseteq G_i$ is a subgroup, so by assumption we have $H \cap G_i \in EG_{\alpha-1}$. As I is directed, the net $(H \cap G_i)_{i\in I}$ is increasing with $H = \bigcup_{i\in I}(H \cap G_i)$. Thus, $H \in EG_\alpha$. On the other hand, for each $i \in I$ we put $C_i := \varphi(G_i)$. By assumption, we have $C_i \in EG_{\alpha-1}$. Thus,

$$C = \varphi(G) = \varphi\left(\bigcup_{i\in I} G_i\right) = \bigcup_{i\in I} \varphi(G_i) = \bigcup_{i\in I} C_i \in EG_\alpha.$$

Case 2: Suppose there exists a short exact sequence

$$e \longrightarrow N \longrightarrow G \xrightarrow{\pi} K \longrightarrow e,$$

with $N, K \in EG_{\alpha-1}$. As $H \cap N \subseteq N$ and $\pi(H) \subseteq K$ are subgroups, both $H \cap N$ and $\pi(H)$ are contained in $EG_{\alpha-1}$ (by assumption). Now, since

$$e \longrightarrow H \cap N \longrightarrow H \longrightarrow \pi(H) \longrightarrow e$$

is a short exact sequence, we conclude that $H \in EG_\alpha$. Next, let $\widetilde{\pi} : C \to C/\varphi(N)$ denote the canonical epimorphism, and define the homomorphism $\psi : K \to C/\varphi(N)$ by $\psi(\pi(g)) = \widetilde{\pi}(\varphi(g))$, for all $g \in G$. This is well-defined. Thus, the following diagram commutes

$$\begin{array}{ccccccccc} e & \longrightarrow & N & \longrightarrow & G & \xrightarrow{\pi} & K & \longrightarrow & e \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \\ e & \longrightarrow & \varphi(N) & \longrightarrow & C & \xrightarrow[\widetilde{\pi}]{} & C/\varphi(N) & \longrightarrow & e. \end{array}$$

Note that the lower sequence is exact and that $\varphi(N) \in EG_{\alpha-1}$. By a standard diagram chase, we observe that ψ is surjective. Hence, $C/\varphi(N) \in EG_{\alpha-1}$. Consequently $C \in EG_\alpha$. Invoking the principle of transfinite induction, the proof is complete. ☐

We shall now restrict out attention to the elementary amenable groups which are countable. However, this will not be reflected in the notation. As the cardinality of the set of countable groups is the continuum, we shall use the first uncountable ordinal, ω_1. We can now give a new characterization of the class of elementary amenable (countable) groups in terms of EG_α. Therefore, the theorem below is a modified version of [Cho80, Proposition 2.2(a)].

Theorem 3.3.3. *We have* $EG = \bigcup_{\alpha<\omega_1} EG_\alpha$.

Proof. Let $EG' := \bigcup_{\alpha<\omega_1} EG_\alpha$. Using an argument of minimality, we must show that EG_α is closed under operations (I)-(IV), for all ordinals $\alpha < \omega_1$. By Proposition 3.3.2, we need only consider operations (III) and (IV). Let $A, B \in EG_\alpha$. If there exists a short exact sequence

$$e \longrightarrow A \longrightarrow G \xrightarrow{\pi} B \longrightarrow e,$$

for some group G, then $G \in EG_{\alpha+1} \subseteq EG'$. On the other hand, suppose $G = \bigcup_{i\in I} G_i$ is the direct union of groups $G_i \in EG'$. For each $i \in I$, choose an ordinal α_i such that $G_i \in EG_{\alpha_i}$. If we set $\beta := \bigcup_{i\in I} \alpha_i$, then $\alpha_i \leqslant \beta$, for all $i \in I$. Note that $\beta < \omega_1$, as I is countable. We have $G \in EG_\beta$ and $G = \bigcup_{i\in I} G_i \in EG_{\beta+1} \subseteq EG'$. That is, $EG \subseteq EG'$.

Conversely, note that $EG_0 \subseteq EG$. Let $0 < \alpha < \omega_1$ be an ordinal and suppose $EG_\beta \subseteq EG$, for all $\beta < \alpha$. If α is a limit ordinal, we simply have

$$EG_\alpha = \bigcup_{\beta<\alpha} EG_\beta \subseteq EG.$$

If, instead, α is a successor ordinal, then all groups in EG_α are obtained from groups in $EG_{\alpha-1}$ by operations (III)-(IV). However, $EG_{\alpha-1} \subseteq EG$ and EG is stable under operations (I)-(IV), so $EG_\alpha \subseteq EG$. By the principle of transfinite induction, we have $EG' \subseteq EG$; consequently, $EG = EG'$. □

From this theorem, we see that (when restricting to countable groups) EG is the smallest class which contains the finite and the abelian groups and is stable under the operations of taking extensions and direct unions. In order to characterize the growth of the finitely generated elementary amenable groups, we first need a lemma. As this requires more advanced group theory, we shall omit the proof. It can be found in [Cho80, Lemma 3.1].

Lemma 3.3.4. *Let A and C be almost nilpotent groups and let B be a finitely generated group. If there exists a short exact sequence*

$$e \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow e,$$

then B is either almost nilpotent or it has exponential growth.

The next important result is Theorem 3.2 in [Cho80]. Note that the class of finitely generated groups is contained in the class of countable groups.

Theorem 3.3.5. *Let G be a finitely generated elementary amenable group. Then G is either almost nilpotent or it has exponential growth.*

Proof. We proceed by transfinite induction. Note that all groups in EG_0 are almost nilpotent. Fix an ordinal $0 < \alpha < \omega_1$ and suppose the statement is true for all groups in EG_β, with $\beta < \alpha$. Let $G \in EG_\alpha$ be a finitely generated group of subexponential growth. From the construction, we may assume that we have chosen α minimal, i.e., α is not a limit ordinal. Consequently, $\alpha - 1$ exists and $G \notin EG_{\alpha-1}$. Since G is finitely generated, it cannot be the direct union of groups in $EG_{\alpha-1}$. Indeed, if $G = \bigcup_{i \in I} G_i$ is a direct union with $G_i \in EG_{\alpha-1}$, for all $i \in I$, and $S \subseteq G$ is a generating subset, then there exists a maximal $i_0 \in I$ such that $S \subseteq G_{i_0}$.

If $A \subseteq G$ is a normal subgroup and $C \simeq G/A$, we consider the short exact sequence

$$e \longrightarrow A \longrightarrow G \longrightarrow C \longrightarrow e,$$

where $A, C \in EG_{\alpha-1}$. As G has subexponential growth, this is also the case for A and C[1]. By hypothesis, it follows that A and C are almost nilpotent. From Lemma 3.3.4, we conclude that G is almost nilpotent. □

Combining the theorem with Corollary 1.5.5, we have the following.

Corollary 3.3.6. *All finitely generated elementary amenable groups have regular growth.*

3.4 Examples

In Chapter 2 we proved that the Grigorchuk group Γ has neither polynomial nor exponential growth. Therefore, we have the following result.

Theorem 3.4.1. *The Grigorchuk group Γ is supramenable. Furthermore, it is not elementary amenable.*

In order to gain a better understanding of the interrelation between the classes EG, AG and SG, we consider two examples of interesting groups.

3.4.1 The $ax + b$-group

We have already mentioned that the class of supramenable groups is contained in the class of amenable groups. This inclusion is, in fact, strict. We shall now see an example of an amenable group which is not supramenable. Consider the set

$$G := \{g : \mathbb{R} \to \mathbb{R} \mid g(x) = ax + b,\ a \in \mathbb{Q} - \{0\}, b \in \mathbb{Q}\}. \tag{3.6}$$

Under the binary operation of composition, G has a group structure.

Proposition 3.4.2. *The group G defined in (3.6) is elementary amenable.*

Proof. We first show that G is solvable. Consider the homomorphism $\varphi : G \to \mathbb{Q} - \{0\}$ given by $\varphi(g) = a$, for all $g \in G$. Note that φ is surjective. If we denote the kernel of φ by H, it is clear that $H = \{g \in G \mid g(x) = x + b,\ b \in \mathbb{Q}\}$, whence H is normal in G and $H \simeq \mathbb{Q}$. Also, $H - \{0\} \simeq \mathbb{Q} - \{0\}$ is abelian. By the isomorphism theorem, we have $G/H \simeq \mathbb{Q} - \{0\}$ which is also abelian. It follows that G is solvable, hence amenable by Proposition 3.2.5.

Note that G is the extension of H by G/H. This shows that G is elementary amenable. In particular, as $H, G/H \in EG_0$, we have $G \in EG_1$. □

[1]It is easy to see that this is the case for C, but the normal subgroup A need not be finitely generated. However, [Wag93] defines a notion of subexponential growth for arbitrary groups which coincides with our definition, when the group is finitely generated. We shall not elaborate more on this here.

Proposition 3.4.3. *The group G defined in (3.6) is not supramenable. In particular, G has exponential growth.*

Proof. We will show that G contains a free subsemigroup on two generators. Let $\sigma(x) = 2x + 1$ and $\rho(x) = 2x$. In general, any group element $g \in G$ can be written in the form $g = g_1 \circ g_2 \circ \cdots \circ g_n$, for some $n \geqslant 1$, with $g_i = a_i x + b_i$, $i = 1, 2, \ldots, n$, such that

$$g(x) = a_1 a_2 \cdots a_n x + (b_1 + b_2 a_1 + b_3 a_1 a_2 + \cdots + b_n a_1 a_2 \cdots a_{n-1}). \tag{3.7}$$

This can easily be shown by induction. Now, a word on σ and ρ can be expressed in the form $w = w_1 \circ w_2 \circ \cdots \circ w_n$, where $w_i = a_i x + b_i$ with $a_i = 2$ and $b_i = 0, 1$, $i = 1, 2, \ldots, n$. Combining this with (3.7), we obtain

$$w(x) = 2^n x + \sum_{n=1}^{n} s^{i-1} b_i,$$

which is a unique expression. Hence, the elements σ and ρ generate a free subsemigroup. Consequently, G is not supramenable, cf. Example 3.2.10. $\square$

Remark 3.4.4. *The $ax + b$-group is the extension of an abelian group by an abelian group. This shows that the supramenable groups are not stable under extensions.*

3.4.2 The Thompson group

We have seen that if a group is finitely generated and elementary amenable, then it has either polynomial or exponential growth. However, the converse implication is not true in general. We shall now see an example of a finitely generated group of exponential growth which is not elementary amenable. This is the Thompson group F. For a good introduction to the Thompson group F, the reader is encouraged to read §§ 1-4 in [CFP96].

Definition 3.4.5 (Thompson's group)**.** Thompson's group F *is the group of piecewise linear homomorphisms from the closed unit interval* $[0, 1]$ *to itself that are differentiable except at finitely many dyadic rationals and such that on intervals of differentiability the derivatives are powers of* 2.

Consider the two maps

$$A(x) = \begin{cases} \frac{x}{2}, & x \in [0, \frac{1}{2}], \\ x - \frac{1}{4}, & x \in [\frac{1}{2}, \frac{3}{4}], \\ 2x - 1, & x \in [\frac{3}{4}, 1], \end{cases} \qquad B(x) = \begin{cases} x, & x \in [0, \frac{1}{2}], \\ \frac{x}{2} + \frac{1}{4}, & x \in [\frac{1}{2}, \frac{3}{4}], \\ x - \frac{1}{8}, & x \in [\frac{3}{4}, \frac{7}{8}], \\ 2x - 1, & x \in [\frac{7}{8}, 1]. \end{cases}$$

It is easy to see that $A, B \in F$. Furthermore, [CFP96, Corollary 2.6] states that F is, in fact, generated by A and B. Therefore, F is finitely generated. Let us now state some properties of Thompson's group. The proofs can be found in [CFP96].

Proposition 3.4.6. *The Thompson group F satisfies the following properties.*

(i) If $\varepsilon, \delta \in (0,1)$, then the subgroup

$$F' := \{f \in F \mid f|_{[0,\varepsilon]} = id,\ f|_{[\delta,1]} = id\}$$

is simple and $F' = [F,F]$,

(ii) If $a, b \in \mathbb{Z}[1/2]$ with $0 \leqslant a < b \leqslant 1$, then

$$\{f \in F \mid f|_{[0,a]} = id,\ f|_{[b,1]} = id\} \simeq F,$$

(iii) F has exponential growth.

From Proposition 3.4.6, we deduce that F is infinite and that any non-trivial normal subgroup of F is contained in $[F,F]$. Also, F cannot be abelian, since this would imply polynomial growth. Furthermore, observe that $[F,F]$ contains a copy of F.

Theorem 3.4.7. *The Thompson group F is not elementary amenable.*

Proof. The Thompson group F is not finite nor abelian. Let $0 < \alpha < \omega_1$ be an ordinal and suppose $F \notin EG_\beta$, for all ordinals $\beta < \alpha$. If α is a limit ordinal, then

$$F \notin \bigcup_{\beta < \alpha} EG_\beta = EG_\alpha.$$

If, instead, α is a successor ordinal, then $\alpha - 1$ exists. Since F is finitely generated, it cannot be the direct union of groups in $EG_{\alpha-1}$. Let us consider extensions. Suppose in order to reach a contradiction that

$$e \longrightarrow N \longrightarrow F \longrightarrow K \longrightarrow e,$$

with $N, K \in EG_{\alpha-1}$ is exact. If N is non-trivial, then $[F,F] \subseteq N$ and $[F,F]$ contains a copy of F, by Proposition 3.4.6. Since $EG_{\alpha-1}$ is stable under subgroups, we have $F \in EG_{\alpha-1}$, i.e., $N = \{e_F\}$. But then $F \simeq K \in EG_{\alpha-1}$. Each case is in contradiction with the hypothesis. By the principle of transfinite induction, we conclude that $F \notin EG$. □

It is still unknown whether the Thompson group F is amenable or not. However, it is known that F is not supramenable (see e.g. [KMR13]) and we have seen that it is not elementary amenable.

3.5 Conclusion

Most of this chapter has been concerned with the interrelation between the notion of growth of finitely generated groups and that of amenability. We have defined the notions of amenability and the stronger one of supramenability, whilst showing some permanence properties of these classes of groups. We have seen some interesting examples of groups with different kinds of growth that are either amenable, supramenable or not amenable. In particular, the Grigorchuk group gave us an example of a group with intermediate growth implying the fact that it is supramenble but not elementary amenable. Conversely, the $ax + b$-group showed that there exist elementary amenable groups that are not supramenable.

By Chou's result (Theorem 3.3.5), all finitely generated elementary amenable groups have regular growth, i.e., either polynomial or exponential growth. Thompson's group F showed, however, the existence of groups of exponential growth that are not elementary amenable. It is known to be non-supramenable (see e.g. [KMR13]) but its amenability is still an open question.

Therefore, we have the following implications.

$$\begin{array}{ccccc}
\text{Amenable} & \Longleftarrow & \text{Supramenable} & \Longleftarrow & \text{Subexponential growth} \\
\Uparrow & & & & \\
\text{Elementary amenable} & \Longrightarrow & \text{Regular growth} & &
\end{array}$$

Note that the diagram is complete in the sense that all implications not drawn, except for one, are false. It is still unknown whether all supramenable groups have subexponential growth. This question is very interesting. An answer in the affirmative would provide us with a new and simple characterization of groups with subexponential growth.

Appendix A

Means and measures

A.1 The Representation Theorem

Definition A.1.1. *Let Ω be a set. A map $\mu : \mathcal{P}(G) \to [0,1]$ is a* finitely additive probability measure *on Ω if*

$$\mu(\Omega) = 1, \quad \textit{and} \quad \mu(A \cup B) = \mu(A) + \mu(B). \tag{A.1}$$

for all disjoint subsets $A, B \subseteq \Omega$. By $\mathcal{PM}(\Omega)$ we shall denote the set of all finitely additive probability measures on Ω.

Recall that the Banach space $\ell^\infty(\Omega)$ is the space of all real maps defined on Ω that are bounded with respect to the norm $||x||_\infty := \sup_{\omega\in\Omega} |x(\omega)|$. Let us also equip the space with a partial ordering given by $x \leqslant y$ if and only if $x(\omega) \leqslant y(\omega)$, for all $\omega \in \Omega$. We shall refer to the constant map $x = c \in \mathbb{R}$ simply as c.

Definition A.1.2. *Let Ω be a set. A linear map $x : \ell^\infty(\Omega) \to \mathbb{R}$ is a* mean *on Ω provided*

$$m(1) = 1, \quad \textit{and} \quad m(x) \geqslant 0, \quad x \geqslant 0 \tag{A.2}$$

We shall refer to $\mathcal{M}(G)$ as the set of all means on Ω.

It turns out that there is a bijection between the set of means to the set of finitely additive probability measures on the set Ω. This allows us to interpret all $\mu \in \mathcal{PM}(\Omega)$ as points in the dual space $(\ell^\infty(\Omega))^*$.

Let $m \in \mathcal{M}(\Omega)$. We can define the map $\widehat{m} : \mathcal{P}(\Omega) \to \mathbb{R}$ by

$$\widehat{m}(A) = m(1_A), \qquad A \in \mathcal{P}(\Omega), \tag{A.3}$$

where $1_A : \Omega \to \{0,1\}$ is the characteristic map. Note that $\widehat{m} \in \mathcal{PM}(\Omega)$. Now, consider the set of *simple maps* $\mathcal{E}(\Omega)$, i.e., of maps $x : \Omega \to \mathbb{R}$ taking only finitely many values. Clearly each $x \in \mathcal{E}(\Omega)$ is a linear combination of characteristic maps and $\mathcal{E}(\Omega) \subseteq \ell^\infty(\Omega)$ is a vector subspace. We shall first show that each $x \in \ell^\infty(\Omega)$ is the pointwise limit of simple functions.

Lemma A.1.3. *The vector subspace $\mathcal{E}(\Omega)$ is dense in $\ell^\infty(\Omega)$.*

Proof. Let $x \in \ell^\infty(\Omega)$ and $\varepsilon > 0$ be given. Put $\alpha = \inf_\Omega x$ and $\beta = \sup_\Omega x$ and choose $n \in \mathbb{N}$ such that $(\beta - \alpha)/n < \varepsilon$. Define $\lambda_i := \alpha + i(\beta - \alpha)/n$ for $i = 1, \ldots, n$. Consider the map $y : \Omega \to \mathbb{R}$ given by $y(a) = \min\{\lambda_i \mid x(a) \leqslant \lambda_i\}$ taking only finitely many values. Since $\lambda_n = \beta$ this is well-defined. Now we have

$$||x - y||_\infty = \sup_{a \in \Omega} |x(a) - y(a)| \leqslant \frac{\beta - \alpha}{n} < \varepsilon \tag{A.4}$$

which shows that $\mathcal{E}(\Omega)$ is dense in $\ell^\infty(\Omega)$. □

Take $\mu \in \mathcal{PM}(\Omega)$. We shall now define the map $\overline{\mu} : \mathcal{E}(\Omega) \to \mathbb{R}$ by

$$\overline{\mu}(x) = \sum_{\lambda \in \mathbb{R}} \mu(x^{-1}(\lambda))\lambda, \qquad x \in \mathcal{E}(\Omega). \tag{A.5}$$

Since x is simple, only finitely many $\lambda \in \mathbb{R}$ have non-empty pre-image under x, so $\overline{\mu}$ is well-defined. Note that $\overline{\mu}(x) \geqslant 0$, whenever $x \geqslant 0$ and that $\overline{\mu}(1_A) = \mu(A)$. Hence $\overline{\mu} \in \mathcal{M}(\Omega)$. Furthermore, by the finite additivity $\sum_{\lambda \in \mathbb{R}} \mu(x^{-1}(\lambda)) = 1$ which implies that $\overline{\mu}(x) \leqslant \sup_\Omega x$.

Now, let $x \in \mathcal{E}(\Omega)$ be a simple map. Suppose there exists a finite partition $(A_i)_{i \in \Lambda}$ of Ω such that $x|_{A_i} = \alpha_i \in \mathbb{R}$, for all $a \in A_i$ and each $i \in \Lambda$. Then we can rewrite (A.5) to

$$\overline{\mu}(x) = \sum_{i \in \Lambda} \mu(A_i)\alpha_i. \tag{A.6}$$

Indeed, since $x^{-1}(\alpha_i) = A_i$ and $x^{-1}(\lambda) = \emptyset$ whenever $\lambda \neq \alpha_i$, we obtain, by the finite additivity,

$$\sum_{i \in \Lambda} \mu(A_i)\alpha_i = \sum_{\lambda \in \mathbb{R}} \left(\sum_{\alpha_i = \lambda} \mu(x^{-1}(\alpha_i)) \right) \lambda = \sum_{\lambda \in \mathbb{R}} \mu(x^{-1}(\lambda))\lambda = \overline{\mu}(x). \tag{A.7}$$

This allows us to proof further properties of $\overline{\mu}$.

Lemma A.1.4. *The map $\overline{\mu} : \mathcal{E}(\Omega) \to \mathbb{R}$ is linear and continuous.*

Proof. Take $x, y \in \mathcal{E}(\Omega)$ and $\xi, \eta \in \mathbb{R}$. Let $V \subseteq \Omega$ and $W \subseteq \Omega$ denote the set of values of x and y, respectively. The sets $x^{-1}(\alpha) \cap y^{-1}(\beta)$, for all $(\alpha, \beta) \in V \times W$, form a partition of Ω. Therefore, the linearity of $\overline{\mu}$ follows from an application of (A.6)

$$\begin{aligned} \overline{\mu}(\xi x + \eta y) &= \sum_{(\alpha,\beta) \in V \times W} \mu(x^{-1}(\alpha) \cap y^{-1}(\beta))(\xi\alpha + \eta\beta) \\ &= \xi \sum_{(\alpha,\beta) \in V \times W} \mu(x^{-1}(\alpha) \cap y^{-1}(\beta))\alpha + \eta \sum_{(\alpha,\beta) \in V \times W} \mu(x^{-1}(\alpha) \cap y^{-1}(\beta))\beta \\ &= \xi\overline{\mu}(x) + \eta\overline{\mu}(y). \end{aligned}$$

For each $x \in \mathcal{E}(\Omega)$, observe that $|\overline{\mu}(x)| \leqslant \sup_\Omega |x| = ||x||_\infty$, since $\mu(x) \leqslant \sup_\Omega x$. This shows that $\overline{\mu}$ is also continuous. □

We need one more result before we can prove the main theorem of this section

Lemma A.1.5. *Let $\mathcal{X}$ be a normed vector space and $\mathcal{Y} \subseteq \mathcal{X}$ a dense vector subspace. If there is a continuous and linear map $\varphi : \mathcal{Y} \to \mathbb{R}$, then there exists a continuous and linear map $\widetilde{\varphi} : \mathcal{X} \to \mathbb{R}$ extending φ.*

Proof. Fix $x \in \mathcal{X}$. There exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathcal{Y}$ converging to x. As i, j tends to infinity the expression

$$|\varphi(y_i) - \varphi(y_j)| = |\varphi(y_i - y_j)| \leqslant ||\varphi|| \; |y_i - y_j| \tag{A.8}$$

tends to zero. Thus, the sequence $(\varphi(x_n))_{n\in\mathbb{N}}$ is Cauchy. We can now put $\widetilde{\varphi}(x) := \lim_{n\to\infty} \varphi(y_n)$ which is well-defined. Linearity of $\widetilde{\varphi}$ follows from taking limits and the continuity and linearity of φ. Likewise, we have

$$||\widetilde{\varphi}|| = ||\lim_{n\to\infty} \varphi(x_n)|| \leqslant \lim_{n\to\infty} ||\varphi|| \; ||x_n|| = ||\varphi|| \; ||x|| \tag{A.9}$$

which shows that $\widetilde{\varphi}$ is continuous. Finally, take $y \in \mathcal{Y}$ and put $y_n = y$, for all n. Then $\widetilde{\varphi}(x) = \varphi(x)$ and $\widetilde{\varphi}$ extends φ. □

We are now ready to prove the theorem allowing us to interpret the finitely additive probability measures as points in the dual space $(\ell^\infty(\Omega))^*$:

Theorem A.1.6. *The map $\Phi : \mathcal{M}(\Omega) \to \mathcal{MP}(\Omega)$ given by $m \mapsto \widehat{m}$ is a bijection.*

Proof. Fix $\mu \in \mathcal{MP}(\Omega)$. Since $\overline{\mu}$ is continuous and linear and $\mathcal{E}(\Omega)$ is dense in $\ell^\infty(\Omega)$, we can invoke the above Lemma A.1.5 to see that there exists a continuous and linear extension $\widetilde{\mu} : \ell^\infty(\Omega) \to \mathbb{R}$. Let us first show that $\widetilde{\mu} \in \mathcal{M}(\Omega)$.

Since $\widetilde{\mu}$ extends $\overline{\mu}$ it is clear that $\widetilde{\mu}(1) = 1$ and $\widetilde{\mu}(y) \geqslant 0$, whenever $y \geqslant 0$ (for $y \in \mathcal{E}(\Omega)$). Now, take $x \in \ell^\infty(\Omega)$ and choose a sequence $(y_n)_{n\in\mathbb{N}}$ in $\mathcal{E}(\Omega)$ converging to x. Define $z_n := |y_n|$, for each n, and note that $\widetilde{\mu}(z_n) \geqslant 0$. Furthermore, $||x - z_n||_\infty \leqslant ||x - y_n||_\infty$ so $\lim_{n\to\infty} z_n = x$. This shows that $\widetilde{\mu}(x) = \lim_{n\to\infty} \overline{\mu}(z_n) \geqslant 0$ and $\widetilde{\mu} \in \mathcal{M}(\Omega)$.

Surjectivity of Φ now follows from a simple computation. Given $A \subseteq \Omega$, we have

$$\widehat{\widetilde{\mu}}(A) = \widetilde{\mu}(1_A) = \overline{\mu}(1_A) = \mu(A), \tag{A.10}$$

i.e., $\Phi(\widetilde{\mu}) = \mu$.

In order to show injectivity let us take $m_1, m_2 \in \mathcal{M}(\Omega)$ such that $\Phi(m_1) = \Phi(m_2)$, i.e., $m_1(1_A) = m_2(1_A)$, for all $A \subseteq \Omega$. Linearity of the means show that they agree on the simple functions. By continuity, we obtain $m_1 = m_2$. □

A.2 Additional group structure

Take $\mu \in \mathcal{PM}(G)$. Define the map $g\mu : \mathcal{P}(G) \to [0,1]$ (resp., $\mu g : \mathcal{P}(G) \to [0,1]$) by

$$g\mu(A) = \mu(g^{-1}A) \quad (\text{resp. } \mu(A)g = \mu(Ag^{-1})), \qquad A \in \mathcal{P}(G). \tag{A.11}$$

Note that $g\mu(G) = \mu(g^{-1}G) = 1$ and $g\mu(A \cup B) = \mu(g^{-1}(A \cup B)) = g\mu(A) + g\mu(B)$ for two disjoint subsets $A, B \subseteq G$. Thus $g\mu \in \mathcal{PM}(G)$ (analogously $\mu g \in \mathcal{PM}(G)$; moreover (A.11) defines a left (resp. right) action of G on $\mathcal{PM}(G)$.

$$g_2(g_1\mu(A)) = g_2\mu(g_1^{-1}A) = \mu(g_2^{-1}(g_1^{-1}A)) = \mu((g_1g_2)^{-1}A) = (g_1g_2)\mu(A) \tag{A.12}$$

for $g_1, g_2 \in G$ (analogously $(\mu g_1)g_2 = \mu(g_1g_2)$). In the following, we shall only make explicit the left action; the corresponding right action is analogous.

Take $x \in \ell^\infty(G)$ and define $gx : G \to \mathbb{R}$ by

$$gx(h) = x(g^{-1}h), \qquad h \in G. \tag{A.13}$$

Clearly $gx \in \ell^\infty(\Omega)$. The (left) action of G on $\ell^\infty(G)$ is linear and preserves norm: $||gx||_\infty = \sup_G |gx| = \sup_G |x| = ||x||_\infty$. This shows that the action is continuous.

For $u \in (\ell^\infty(\Omega))^*$ we can consider the (left) action $gu : \ell^\infty(\Omega) \to \mathbb{R}$ given by

$$gu(x) = u(g^{-1}x), \tag{A.14}$$

and $gu \in (\ell^\infty(\Omega))^*$, for all $x \in \ell^\infty(\Omega)$.

We can now state an equivalent condition of amenability.

Theorem A.2.1. *A group G is amenable if and only if there exists a left-invariant mean $m : \ell^\infty(G) \to \mathbb{R}$ on G.*

Proof. Let $m \in \mathcal{P}(G)$. The mean m is left-invariant if and only if the associated finitely additive probability measure $\widehat{m} \in \mathcal{PM}(G)$ is left-invariant. Indeed, if $A \in \mathcal{P}(G)$, then

$$\widehat{gm}(A) = gm(1_A) = m(g^{-1}1_A) = m(1_{g^{-1}A}) = \widehat{m}(g^{-1}A) = g\widehat{m}(A), \tag{A.15}$$

i.e., $\widehat{gm} = g\widehat{m}$. □

We shall make use of the evaluation map $\psi_{x_0} : (\ell^\infty(\Omega))^* \to \mathbb{R}$ at $x_0 \in \ell^\infty(\Omega)$, where $(\ell^\infty(\Omega))^*$ denotes the topological dual space. Recall that the *weak-$*$ topology* is the coarsest topology on $(\ell^\infty(\Omega))^*$ that renders the evaluation map continuous, for all $x \in \ell^\infty(\Omega)$.

Appendix B

Residually finiteness

The goal of this section is to characterize a residually finite group in terms of the residual subgroup.

Definition B.0.2 (Residually finite). *Let G be a group. We say that G is* residually finite *if for all $g \in G$ with $g \neq e_G$ there exist a finite group F and a homomorphism $\varphi : G \to F$ such that $\varphi(g) \neq e_F$.*

An alternative definition is given in the following.

Proposition B.0.3. *Let G be a group. The following are equivalent*

(i) G is residually finite,

(ii) For all $g, h \in G$ with $g \neq h$ there exist a finite group F and a homomorphism $\varphi : G \to F$ such that $\varphi(g) \neq \varphi(h)$.

Proof. *(ii)* $\implies$ *(i)*: The implication follows from the definition by putting $h = e_G$.

(i) $\implies$ *(ii)*: Let $g, h \in G$ such that $g^{-1}h \neq e_G$. There exist a finite group F and a homomorphism $\varphi : G \to F$ such that $\varphi(g^{-1}h) \neq e_F$, i.e., $\varphi(g) \neq \varphi(h)$. □

Lemma B.0.4. *Let G be a group, and let $H \subseteq G$ be a subgroup. If $K = \cap_{g \in G} gHG^{-1}$ then $K \subseteq H$ and K is a normal subgroup of G. If, in addition, H is of finite index in G, then K is of finite index in G.*

Proof. Note that $K = H \cap (\bigcap_{g \neq e_G} gHg^{-1}) \subseteq H$. Let G/H denote the set of all left cosets of H in G and consider the action of G on G/H given by left multiplication. For each $gH \in G/H$ the stabilizer is $G_{gH} = \{h \in G \mid h(gH) = gH\} = gHg^{-1}$. Thus, the kernel of the homomorphism $\psi : G \to \mathrm{Sym}(G/H)$ is the intersection of all the stabilizers, i.e., $\ker \psi = K$ showing that K is a normal subgroup of G.

Suppose H is a subgroup of finite index. Then G/H is finite and also $\mathrm{Sym}(G/H)$ is finite. As G/K is isomorphic to $\psi(G) \subseteq \mathrm{Sym}(G/H)$ we find that G/K is finite. In particular, K is a normal subgroup of G with finite index. □

Definition B.0.5 (Residual subgroup). *Let G be a group. The* residual subgroup *of G is the intersection of all subgroups of finite index.*

Proposition B.0.6. *Let G be a group and let N denote the residual subgroup of G. Then*

(i) N is the intersection of all normal subgroups of G of finite index,

(ii) N is a normal subgroup of G,

Proof. (i): Let N' denote the intersection of all normal subgroups of G with finite index. Clearly, $N \subseteq N'$. Conversely, let $H \subseteq G$ be a subgroup of finite index. By Lemma B.0.4, $K = \bigcap_{g \in G} gHg^{-1} \subseteq G$ is a normal subgroup of finite index contained in H. Thus $N' \subseteq K \subseteq H$ showing that $N' \subseteq N$.

(ii): The (arbitrary) intersection of normal subgroups of G is a normal subgroup of G. □

Theorem B.0.7. *Let G be a group and let N denote the residual subgroup of G. Then G is residually finite if and only if $N = \{e_G\}$.*

Proof. Suppose G is residually finite. Assume that there exists an element $g \in N$ such that $g \neq e_G$. Then there exist a finite group F and a homomorphism $\varphi : G \to F$ such that $g \neq e_F$, i.e., $g \notin \ker\varphi$. As $G/(\ker\varphi)$ is isomorphic to $\varphi(G) \subseteq F$ we deduce that $\ker\varphi$ is a normal subgroup of G with finite index. Thus $N \subseteq \ker\varphi$ which contradicts the assumption that $g \in N$. Hence $N = \{e_G\}$.

On the other hand, suppose $N = \{e_G\}$. Take $g \in G$ such that $g \neq e_G$. Applying (i) there exists a normal subgroup $K \subseteq G$ of finite index such that $g \notin K$. Note that G/K is finite and if $\pi : G \to G/K$ is the canonical epimorphism, then $g \notin \ker\pi$, i.e., $\pi(g) \neq e_{G/K}$. Thus G is residually finite. □

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