Random walks on groups:
Amenability and convolution operators

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Abstract

In this project we study the concept of amenability of locally compact groups, and how this can be described using convolution operators arising from probability measures on the group. We will use the first part of the project to introduce the notion of an amenable locally compact group and from there on discuss a variety of equivalent statements concerning such groups. This initial part will follow Greenleaf's book [5] on invariant means.

Once we have familiarised ourselves with amenable groups in the locally compact setting, we will approach this concept in a more probabilistic manner. This will follow an article of Furstenberg from 1972 [4] where the notion of μ-boundaries of a locally compact group G is introduced, and the following conjecture was formulated:

> *G possesses a measure m whose support is all of G and for which no nontrivial μ-boundaries exists iff G is amenable.*

Furstenberg proved in [4] that existence of a measure μ with these properties is sufficient to ensure amenability, and conjectured that is also necessary. We will go through his proof.

The final part of the project concerns the proof of the necessity condition in Furstenberg's conjecture, as provided by Rosenblatt in 1981 in [9]. Rosenblatt's proof links amenability of G to the existence of an ergodic probability measure on the group with some additional properties. This ergodic measure will then be used to construct a measure with full support and no nontrivial μ-boundaries, thus proving the conjecture.

Contents

1 Introduction

In this paper we will introduce the notion of amenability for locally compact groups and investigate various equivalent characterisations of this concept. The initial part will follow Greenleaf's book [5] on *invariant means*. After introducing the concept of amenability, the main goal will be to link the amenability of a group to the behavior of convolutions operators arising from distributions of certain random walks on the groups, and properties of associated μ-boundaries as defined by Furstenberg [4]. Therein he formulated the following conjecture:

> *G possesses a measure μ whose support is all of G and for which no nontrivial μ-boundaries exists iff G is amenable.*

The sufficient condition (that is, existence of a probability measure μ on the group with these properites implies amenability of the group) was proved by Furstenberg, and we will discuss his proof. We will devote the remaining part of the project to the proof provided by Rosenblatt in [9] that this condition is also necessary In the process of going through Rosenblatt's paper we will also introduce the concepts of a measure being *ergodic by convolutions* and *mixing by convolutions*.

The reader is assumed to have some knowledge of functional analysis and of the Haar measure on a locally compact group. These topics can be found in [3].

Throughout this paper we will use the following notation

- G is a locally compact group with identity $e \in G$.
- $\mathcal{B}(G)$ will be the Borel sets on G.
- λ is a fixed left Haar measure on a group G. If there is ambiguity, λ_G will be used.
- $\Delta \colon G \to \mathbb{C}$ will denote the modular function associated to λ.
- $L^1(G)$ will denote the set of λ-integrable functions from G to $\mathbb{C}$.
- If $f \colon G \to \mathbb{C}$ is some map and $x \in G$ is given, we will let ${}_xf$ and f_x denote the maps

$$ {}_xf(t) = f(x^{-1}t) \quad \text{and} \quad f_x(t) = f(tx) \quad \text{for } t \in G. $$

- If $f\colon G \to \mathbb{C}$ is some map, we will let $\tilde{f}$ denote the map $\tilde{f}(x) = f(x^{-1})$, for $x \in G$.

When working with the left Haar measure on a group G, we should notice that whenever $f \in L^1(G)$, then

$$\int f(t)\,\mathrm{d}\lambda(t) = \int f(ts)\Delta(s)\,\mathrm{d}\lambda(t),$$
$$\int f(t)\,\mathrm{d}\lambda(t) = \int f(st)\,\mathrm{d}\lambda(t),$$
$$\int f(t)\,\mathrm{d}\lambda(t) = \int f(t^{-1})\Delta(t^{-1})\,\mathrm{d}\lambda(t),$$

for all $s \in G$. These identities are very useful for integrating against the Haar measure, and can be obtained from integrating simple functions and then extending to all of $L^1(G)$. Combining these properties with the uniqueness of the left Haar measure, we get the following useful statement.

Proposition 1.1. *The following conditions are equivalent for a positive linear functional φ on $L^1(G)$:*

1. $\varphi(f) = c\int f\,\mathrm{d}\lambda$ *for some* $c > 0$*, for all* $f \in L^1(G)$.
2. $\varphi({}_xf) = \varphi(f)$*, for all* $f \in L^1(G)$ *and* $x \in G$.
3. $\varphi(f_x) = \Delta(x^{-1})\varphi(f)$*, for all* $f \in L^1(G)$ *and* $x \in G$.

Proof. The implications *1.* $\implies$ *2.* and *1.* $\implies$ *3.* follow directly from the properties of the Haar measure. The implication *2.* $\implies$ *1.* is uniqueness of the left Haar measure. For the final implication, let positive $\varphi \in (L^1(G))^*$ be given satisfying *3.* Let now μ be the positive, regular Borel measure associated to φ via the Riesz Representation Theorem. Then for any f in $C_0(G)$,

$$\begin{aligned}\int f_x(t)\Delta(t^{-1})\,\mathrm{d}\mu(t) &= \Delta(x)\int f(tx)\Delta((tx)^{-1})\,\mathrm{d}\mu(t) = \Delta(x)\varphi((f\cdot\tilde{\Delta})_x)\\ &= \varphi(f\cdot\tilde{\Delta}) = \int f(t)\Delta(t^{-1})\,\mathrm{d}\mu(t),\end{aligned}$$

so integrating against $\Delta(t^{-1})\,\mathrm{d}\mu(t)$ becomes a right invariant linear functional on $C_0(G)$. By uniqueness of right Haar measure, there exists $c > 0$ such that $c\cdot\Delta(t^{-1})\,\mathrm{d}\mu(t) = \mathrm{d}\lambda(t^{-1})$. This behaviour resembles that of λ itself and in fact

$$c\cdot\Delta(t^{-1})\,\mathrm{d}\mu(t) = \mathrm{d}\lambda(t^{-1}) = \Delta(t^{-1})\,\mathrm{d}\lambda(t).$$

By integrating functions in $C_c(G)$ against these measures we obtain $c \cdot \mu = \lambda$ and thus

$$\varphi(f) = c \int f \, \mathrm{d}\lambda \quad \text{for all } f \in L^1(G).$$

□

One of our primary tools when working with bounded regular Borel measures on G is the *convolution* operation connected with the multiplication in G.

Definition 1.2. For two bounded regular Borel measures μ, ν on G we define the *convolution* $\mu * \nu$ as the measure on G given by

$$\mu * \nu(A) = \int 1_A(st) \, \mathrm{d}(\mu \times \nu)(s,t), \text{ for } A \in \mathcal{B}(G).$$

We will from here on let $\mu^{(n)}$ denote the convolution of n consecutive copies of μ. Since every $f \in L^1(G)$ can be identified with a bounded regular Borel measure on G, we can extend Definition 1.2 to construct new elements in $L^1(G)$ using a bounded regular Borel measure μ on G, and $f, \varphi \in L^1(G)$ in the following way:

- $\mu * f(s) = \int f(t^{-1}s) \, \mathrm{d}\mu(t)$, for $s \in G$.
- $f * \mu(s) = \int f(st^{-1}) \Delta(t^{-1}) \, \mathrm{d}\mu(t)$, for $s \in G$.
- $f * \varphi(s) = \int f(t)\varphi(t^{-1}s) \, \mathrm{d}\lambda(t)$, for $s \in G$.

Remark 1.3. If a measure μ is absolutely continuous with respect to λ with density $\varphi \in L^1(G)$, then $f * \mu = f * \varphi$. This can be seen through straightforward calculations.

In each of the above cases we could let either f be an element in some $L^p(G)$ for some $1 \leq p < \infty$, and obtain new elements in $L^p(G)$. In the case of $p = \infty$, we can construct $\mu * f$ and $f * \varphi$, but $f * \mu$ need not make sense. All of these considerations follow from the proposition below.

Proposition 1.4. *Let μ be a bounded, regular Borel measure on G and let $f \in L^p(G)$, where $1 \leq p \leq \infty$. Then $\|\mu * f\|_p \leq \|\mu\| \cdot \|f\|_p$.*

Proof. In the case of $p \in (1, \infty)$, the proof will be an application of Hölder's inequality, but we should first recall the following inequality:

$$\left| \int f \, \mathrm{d}\mu \right| \leq \int |f| \, \mathrm{d}|\mu|.$$

Here $|\mu|$ denotes the total variation of μ and recall that $|\mu|(G) = \|\mu\|$. We will now let q be the dual exponent of p, i.e., $1/p + 1/q = 1$ and hence

$$\begin{aligned}\int |f| \,\mathrm{d}|\mu| &= \int |f| \cdot 1 \;\; \mathrm{d}|\mu| \\ &\leq \left(\int |f|^p \,\mathrm{d}|\mu|\right)^{1/p} \left(\int 1 \,\mathrm{d}|\mu|\right)^{1/q} \\ &= \left(\int |f|^p \,\mathrm{d}\; |\mu|\right)^{1/p} \cdot \|\mu\|^{1/q},\end{aligned}$$

for any $f \in L^p(G)$ by Hölder's inequality. From here we can now conclude

$$\begin{aligned}\|\mu * f\|_p^p &= \int |(\mu * f)(s)|^p \,\mathrm{d}\lambda(s) \\ &= \int \left| \int f(t^{-1}s) \,\mathrm{d}\mu(t) \right|^p \mathrm{d}\lambda(s) \\ &\leq \int \left(\int |f(t^{-1}s)| \,\mathrm{d}|\mu|(t) \right)^p \mathrm{d}\lambda(s) \\ &\leq \|\mu\|^{p/q} \int \left(\int |f(t^{-1}s)|^p \,\mathrm{d}|\mu|(t) \right) \mathrm{d}\lambda(s) \\ &= \|\mu\|^{p/q} \int \|f\|_p^p \,\mathrm{d}|\mu|(t) = \|f\|_p^p \cdot \|\mu\|^p.\end{aligned}$$

Hence $\|\mu * f\|_p \leq \|\mu\| \cdot \|f\|_p$, as we intended to show. In the case of $p = 1$ we can not use Hölder in the same way, but instead we see

$$\begin{aligned}\int |\mu * f(s)| \,\mathrm{d}\lambda(s) &\leq \int \int |f(t^{-1}s)| \,\mathrm{d}|\mu|(t) \,\mathrm{d}\lambda(s) \\ &= \int \int |f(s)| \,\mathrm{d}\lambda(s) \,\mathrm{d}|\mu|(t) \\ &= \int \|f\|_1 \,\mathrm{d}|\mu|(t) = \|f\|_1 \cdot \|\mu\|.\end{aligned}$$

Finally, in the case of $p = \infty$, any $s \in G$ will give us

$$|\mu * f(s)| \leq \int |f(t^{-1}s)| \,\mathrm{d}|\mu|(s) \leq \|f\|_\infty \cdot \|\mu\|,$$

and hence also $\|\mu * f\|_\infty \leq \|f\|_\infty \cdot \|\mu\|$. Note that for any $p < \infty$, the inequality $\|f * \mu\|_p \leq \|\mu\| \cdot \|f\|_p$ is true, as well. The arguments for this are analogous, as long as we handle the modular function with care. □

Some useful examples of convolutions arises from one point measures as $x \in G$ and $f \in L^p(G)$ gives us

$$f * \delta_x = \Delta(x^{-1}) f_{x^{-1}} \quad \text{and} \quad \delta_x * f = {}_x f.$$

These identities follow since integration agains a one point measure is simply evaluation of the function in the given point.

2 Amenability of locally compact groups

Our chosen approach to defining amenability of locally compact groups will be based on studying certain functions spaces on the group G. The first four of these spaces will be

i) $L^\infty(G)$: The essentially bounded Borel functions on G. In this space we will identify functions which only differ on a locally nullset and equip the space with the norm

$$\|f\|_\infty = \text{ ess sup } \{|f(t)| : t \in G\}.$$

ii) $CB(G)$: The bounded continuous functions on G equipped with the usual supremum norm.

iii) $UCB_r(G)$: The bounded right uniformly continuous functions on G. Here we say that a function $f : G \to \mathbb{C}$ is *right uniformly continuous*, if for any $\varepsilon > 0$ we can find an open neighbourhood $U(\varepsilon)$ of $e \in G$ such that

$$|f(x) - f(yx)| < \varepsilon,$$

for all $x \in G$ and $y \in U(\varepsilon)$. We will in a completely similar manner define $UCB_\ell(G)$ as the left uniformly continuous functions, where the *left uniform continuity* is linked to the approximation

$$|f(x) - f(xy)| < \varepsilon.$$

iv) $UCB(G)$: The bounded uniformly continuous functions on G. This space is simply the intersection of $UCB_r(G)$ and $UCB_\ell(G)$.

The first desirable property of these spaces is that they are nested

$$UCB(G) \subset UCB_r(G) \subset CB(G) \subset L^\infty(G)$$

as closed subspaces with respect to the supremum norm. Furthermore theses spaces are closed under translations, i.e., if f is in one of the spaces, then both ${}_xf$ and f_x will be in the space, as well. The following proposition will also give us an interesting family of uniformly continuous functions:

Proposition 2.1. *Any function f in $C_c(G)$ or $C_0(G)$ is both left and right uniformly continuous.*

Proof. Assume first that $f \in C_c(G)$. We will show right uniform continuity, and the proof of left uniform continuity will be analogous. Let K denote the support of f and let $\varepsilon > 0$ be given. For each $x \in G$ pick an open neighbourhood U^x of $e \in G$, such that

$$|f(x) - f(y)| < \varepsilon/2$$

for any $y \in (U^x)x$, or, in other terms $|f(x) - f(yx)| < \varepsilon/2$, for $y \in U^x$. For each such U^x, pick an open symmetric neighbourhood V^x of $e \in G$, such that $V^x V^x \subset U^x$. By compactness of K, we can find $x_1, \dots, x_n$ such that

$$K \subset \bigcup_{i=1}^{n} (V^{x_i}) x_i$$

and we will use these to define $V = \bigcap_{i=1}^{n} V^{x_i}$. Our claim is now that this V is the open neighbourhood we are searching for, so let $x \in K$ be given with $x \in (V^{x_i})x_i \subset (U^{x_i})x_i$. Then $xx_i^{-1} \in V^{x_i}$, so $yx = y(xx_i^{-1})x_i \in (U^{x_i})x_i$, for any $y \in V \subset V^{x_i}$. By our choice of U^{x_i} this allows us to conclude that

$$|f(yx) - f(x)| \leq |f(yx) - f(x_i)| + |f(x_i) - f(x)| < \varepsilon.$$

So f is right uniformly continuous on K. To show right uniform continuity on all of G, let $x \in G \backslash K$ be given and $y \in V$. Our problem can now be split up in two cases. One where $yx \in G \backslash K$, and respectively where $yx \in K$. In the first case both $f(x)$ and $f(yx)$ will be equal to zero, so we will have $|f(yx) - f(x)| = 0 < \varepsilon$. In the second case we recall that $y^{-1} \in V$ and hence

$$|f(yx) - f(x)| = |f(yx) - f(y^{-1}(yx))| < \varepsilon,$$

by right uniform continuity on K. In conclusion we have shown that f is right uniformly continuous on G. For left uniform continuity the left translates in the definition of U^x should be replaced with right translates.

To obtain the results for $f \in C_0(G)$ we will repeat the argument with the compact set $K = f^{-1}(B(0, \varepsilon/2)^c)$, such that the cases of both $x, xy \in G \backslash K$ will give us

$$|f(x) - f(yx)| \leq |f(x)| + f(yx)| < \varepsilon,$$

as we set out to show. □

The study of these spaces will center upon the investigation of invariant means on these spaces, so let us start out with a definition:

Definition 2.2. Let X be any of the above four spaces. A linear functional $m \in X^*$ is said to be a mean on X if $m(f) \geq 0$, for any $f \geq 0$ and $m(1) = 1$. Furthermore, m is said to be

- *a left invariant mean* if $m({}_xf) = m(f)$, for all $f \in X$ and $x \in G$.
- *a topological left invariant mean* if $m(\varphi * f) = m(f)$, for all $f \in X$ and $\varphi \in L^1(G)$ such that $\varphi \geq 0$ and $\int \varphi \ \mathrm{d}\lambda = 1$.

To ease notation further on we will let $\mathrm{Prob}(G)$ denote the subspace in $L^1(G)$ of positive functions f with 1-norm equal to one, i.e., $\int f(t)\,\mathrm{d}\lambda(t) = 1$ (Note that this notation is different from the one in [5]). In this way a topological left invariant mean from Definition 2.2 satisfies

$$m(\varphi * f) = m(f),$$

for all $f \in X$ and $\varphi \in \mathrm{Prob}(G)$. The motivation for the name of the space is that $\mathrm{Prob}(G)$ consists of the probability measures on G having density with respect to λ. Let us now turn to a result regarding these two types of left invariance.

Proposition 2.3. *If m is a topological left invariant mean on $L^\infty(G)$, then m is also a left invariant mean on $L^\infty(G)$.*

Proof. Let m be a topological left invariant mean on $L^\infty(G)$ and consider $\varphi \in \mathrm{Prob}(G)$. Then

$$\varphi * ({}_xf) = \varphi * \delta_x * f = \Delta(x^{-1})\varphi_{x^{-1}} * f.$$

Furthermore by handling the modular function carefully

$$\begin{aligned} \int \Delta(x^{-1})\varphi_{x^{-1}}(t)\,\mathrm{d}\lambda(t) &= \Delta(x^{-1}) \int \varphi(tx^{-1})\,\mathrm{d}\lambda(t) \\ &= \Delta(x^{-1}) \int \varphi(t)\Delta(x)\,\mathrm{d}\lambda(t) = 1, \end{aligned}$$

so $\Delta(x^{-1})\varphi_{x^{-1}} \in \mathrm{Prob}(G)$, for any $x \in G$. By topological left invariance of m we get

$$m({}_xf) = m(\varphi * ({}_xf)) = m((\Delta(x^{-1})\varphi_{x^{-1}}) * f) = m(f),$$

concluding our proof. ∎

Note that this result also holds true if we replace $L^\infty(G)$ with any of the other three function spaces defined above, or if we substitute left invariance with right invariance. Another useful property of the functions in $\mathrm{Prob}(G)$ can be found in the following lemma which will be useful later.

Lemma 2.4. *If $f \in L^\infty(G)$ and $\varphi \in Prob(G)$, then $\varphi * f \in UCB_r(G)$ and $f * \tilde{\varphi} \in UCB_\ell(G)$. Furthermore if $g \in UCB_r(G)$, then $g * \tilde{\varphi} \in UCB(G)$ and if $g \in UCB_\ell(G)$ then $\varphi * g \in UCB(G)$.*

Proof. The proofs of these four implications are very similar, so we will only proof the case of $\varphi * f \in UCB_r(G)$ whenever $f \in L^\infty(G)$ and $\varphi \in \mathrm{Prob}(G)$. For this let such $f \in L^\infty(G)$ and $\varphi \in \mathrm{Prob}(G)$ be given along with $x, y \in G$. Then

$$\begin{aligned}
|(\varphi * f)(x) - (\varphi * f)(yx)| &= \left| \int \varphi(t) f(t^{-1}x) \,\mathrm{d}\lambda(t) - \int \varphi(t) f(t^{-1}yx) \,\mathrm{d}\lambda(t) \right| \\
&= \left| \int \varphi(t) f(t^{-1}x) \,\mathrm{d}\lambda(t) - \int \varphi(yt) f(t^{-1}x) \,\mathrm{d}\lambda(t) \right| \\
&\leq \int |f(t^{-1}x)| \cdot |\varphi(t) - \varphi(yt)| \,\mathrm{d}\lambda(t) \\
&\leq \|f\|_\infty \int |\varphi(t) - \varphi(yt)| \,\mathrm{d}\lambda(t)
\end{aligned}$$

By hypothesis of f it remains to show that we can minimise the final integral above by picking y sufficiently close to the identity $e \in G$. Let $\varepsilon > 0$ be given and recall that $C_c(G)$ is a dense subset of $L^1(G)$. This latter fact allows us to pick $g \in C_c(G)$ satisfying $\|\varphi - g\|_1 < \frac{\varepsilon}{3\|f\|_\infty}$. Applying the triangle inequality and left invariance of the Haar measure to the integral in question, we see that

$$\|f\|_\infty \int |\varphi(t) - \varphi(yt)| \,\mathrm{d}\lambda(t) \leq \|f\|_\infty \left(\frac{2\varepsilon}{3\|f\|_\infty} + \int |g(t) - g(yt)| \,\mathrm{d}\lambda(t) \right)$$

leaving us with the task of minimising the latter integral above. If we let K denote the support of g, we must necessarily obtain

$$\mathrm{supp}(|g - ({}_{y^{-1}}g)|) \subset K \cup (y^{-1}K).$$

From here on we will let K' denote a compact neighborhood containing $K \cup (y^{-1}K)$. By proposition 2.1 we can pick an open neighborhood U of $e \in G$, s.t.,

$$|g(s) - g(ys)| \leq \frac{\varepsilon}{3\|f\|_\infty \lambda(K')}$$

for all $s \in G$ and $y \in U$. This inequality will extend to

$$\int |g(t) - g(yt)| \, \mathrm{d}\lambda(t) < \frac{\varepsilon}{3\|f\|_\infty},$$

finally giving us $|(\varphi * f)(x) - (\varphi * f)(yx)| < \varepsilon$ for all $x \in G$ and $y \in U$, and hence $\varphi * f \in UCB_r(G)$. □

We will now turn to the first major result that will allow us to formulate the definition of an amenable locally compact group:

Theorem 2.5. *For a locally compact group G the following are equivalent:*

1. *There is a topological left invariant mean on $L^\infty(G)$,*
2. *There is a left invariant mean on $L^\infty(G)$,*
3. *There is a left invariant mean on $CB(G)$,*
4. *There is a left invariant mean on $UCB_r(G)$,*
5. *There is a left invariant mean on $UCB(G)$.*

Proof. The implication 1. $\implies$ 2. follows from Proposition 2.3 and the implications 2. $\implies$ 3. $\implies$ 4. $\implies$ 5. are clear by restricting the mean to smaller closed subspaces. We are hence left with showing that a left invariant mean on $UCB(G)$ gives us a toplogical left invariant mean on $L^\infty(G)$. For this we will introduce a lemma that we will prove afterwards:

Lemma 2.6. *If m is a left invariant mean on $UCB(G)$, then m is a topological left invariant mean on $UCB(G)$.*

Proof of Theorem 2.5 continued. Let m be a topological left invariant mean on $UCB(G)$ and let E be a compact symmetric neighbourhood of the unit in G. By letting $\varphi_E \in \mathrm{Prob}(G)$ be the normalised characteristic function on E, we see that $\varphi_E * f * \varphi_E \in UCB(G)$ for any $f \in L^\infty(G)$. This allows us to define a map $\overline{m}\colon L^\infty(G) \to \mathbb{C}$ by

$$\overline{m}(f) = m(\varphi_E * f * \varphi_E), \text{ for } f \in L^\infty(G).$$

It is not difficult to see that $\overline{m}$ is a mean on $L^\infty(G)$, so let us show that it is a topological left invariant mean, as well. In order to do this we will pick an approximate unit $\{e_j\}$ in $L^1(G)$ such that $e_j \in \mathrm{Prob}(G)$ for each

j, and let $\lambda_1, \lambda_2 \in \mathrm{Prob}(G)$ be given. For $g \in UCB_\ell(G)$ we will then have $e_j * g \in UCB(G)$, and using that $\{e_j\}$ is an approximate unit we get

$$\|\lambda_i * e_j * g - \lambda_i * g\|_\infty \to 0,$$

for $i = 1, 2$. As m is a topological left invariant mean this allows to see

$$m(e_j * g) = m(\lambda_1 * e_j * g) \to m(\lambda_1 * g),$$
$$m(e_j * g) = m(\lambda_2 * e_j * g) \to m(\lambda_2 * g),$$

and hence $m(\lambda_1 * g) = m(\lambda_2 * g)$. As $\lambda_1, \lambda_2 \in \mathrm{Prob}(G)$ and $g \in UCB_\ell(G)$ were arbitrary, this allows us to conclude that

$$\overline{m}(\varphi * f) = m((\varphi_E * \varphi) * (f * \varphi_E)) = m(\varphi_E * f * \varphi_E) = \overline{m}(f),$$

for all $\varphi \in \mathrm{Prob}(G)$ and $f \in L^\infty(G)$, since $\varphi_E * \varphi, \varphi_E \in \mathrm{Prob}(G)$ and $f * \varphi_E \in UCB_\ell(G)$. In conclusion $\overline{m}$ is a topological left invariant mean on $L^\infty(G)$. □

Proof of Lemma 2.6. Let $f \in UCB(G)$ and $\varphi \in L^1(G)$ be given. Then

$$m(({}_x\varphi) * f) = m({}_x(\varphi * f)) = m(\varphi * f)$$

by left invariance of m and since $\varphi * f \in UCB(G)$. Then $\varphi \mapsto m(\varphi * f)$ is a left invariant mean on $L^1(G)$, and hence there exists $k\colon UCB(G) \to \mathbb{C}$ such that

$$m(\varphi * f) = k(f) \int \varphi \, \mathrm{d}\lambda(t).$$

This map k is obtained by using the Riesz Representation Theorem on the bounded linear functional

$$\varphi \mapsto m(\varphi * f)$$

restricted to $C_c(G)$, and then extending by continuity. Notice furthermore that if $\varphi \in \mathrm{Prob}(G)$, then $m(\varphi * f)$ is simply equal to $k(f)$. To finish the proof, let $\{U_j\}$ be an open neighbourhood basis for $e \in G$ with $\lambda(U_j) < \infty$, considered as a net when ordered by reverse inclusion, and let $e_j \in \mathrm{Prob}(G)$ be the normalised characteristic function for U_j. If we now consider f in $UCB(G)$ and $t \in G$, then

$$\begin{aligned} |e_j * f(t) - f(t)| &= \left| \int_G e_j(s) f(s^{-1}t) \, \mathrm{d}\lambda(s) - f(t) \right| \\ &\le \frac{1}{m(U_j)} \int_{U_j} |f(s^{-1}t) - f(t)| \, \mathrm{d}\lambda(s) \\ &\le \sup_{s \in U_j} |f(s^{-1}t) - f(t)|. \end{aligned}$$

Looking at all $t \in G$ then gives us

$$\|e_j * f - f\|_\infty \leq \sup_{s \in U_j} \sup_{t \in G} |f(s^{-1}t) - f(t)|$$

which lets us conclude that $\|e_j * f - f\|_\infty \to 0$ when the U_j's get smaller by right uniform continuity of f. As m is continuous, this allows us to see that $k(f) = m((\varphi * e_j) * f) = m(e_j * f)$ converges to both $m(\varphi * f)$ and $m(f)$, and hence m is a topological left invariant mean on $UCB(G)$. □

This Theorem 2.5 will be the core result in defining the notion of amenability for locally amenable groups, more precisely,

Definition 2.7. A locally compact group G is said to be amenable if one and hence all of the conditions in Theorem 2.5 are satisfied.

We could also have chosen to define amenability using right invariant means. The reason for not worrying about this ambiguity is that the existence of a left invariant mean on any of our spaces gives both right invariant and two-sided invariant means on the space, as well. For the first claim notice that if m is a left invariant mean on any of our spaces then we can define a new mean $\overline{m}$ on the same space by

$$\overline{m}(f) = m(\tilde{f}), \text{ for } f \text{ in the given space.}$$

In this way $\overline{m}$ becomes a right invariant mean. To get a two-sided invariant mean, let m_ℓ be a left invariant mean on $UCB(G)$ and m_r be a right invariant mean on $CB(G)$. Given $f \in UCB(G)$, define $F\colon G \to \mathbb{C}$ by $F(x) = \langle m_\ell, f_x \rangle$. By uniform continuity of f, we see that $x_j \to x_0 \in G$ gives us $\|f_{x_j} - f_{x_0}\|_\infty$ converging to 0, and thus F is continuous on G. F is also bounded, so we can now define $m\colon UCB(G) \to \mathbb{C}$ by $m(f) = \langle m_r, F \rangle$ for $f \in UCB(G)$. Then m is a two-sided invariant mean on $UCB(G)$. By arguments similar to those of the proofs above, we can use this m to obtain a two-sided version of Theorem 2.5.

Before venturing further into the subject, we should at this point mention that amenable locally compact groups do exist. For this let us consider a series of theorems regarding types of amenable groups and how to construct new amenable groups from known ones.

Theorem 2.8. *The following are true for a locally compact group G*

1. *If G is compact, then G is amenable.*
2. *If G is abelian, then G is amenable.*

3. *If G is amenable and $\pi\colon G \to H$ is a continuous homomorphism onto a locally compact group H, then H is amenable.*

4. *If G is amenable and H is closed subgroup of G, then H is amenable.*

5. *If N is a closed normal subgroup in G, such that both N and G/N are amenable, then G is amenable.*

6. *If $\{H_\alpha\}$ is a directed system of closed subgroups in G where each H_α is amenable, then G is amenable.*

Here a directed system is a family $\{H_\alpha\}_{\alpha \in A}$ of subsets in G such that G is equal to the union of all these sets, and for each $\alpha, \beta \in A$ there exists $\gamma \in A$ with $H_\alpha \cup H_\beta \subset H_\gamma$.

3 Alternative characterisations of amenability

When working with amenability of locally compact groups and delving into the many properties of these groups, one discovers that a lot of these properties are equivalent to our chosen definition of amenability. In other words, we could define amenability of a locally compact groups in a number of different ways. This section will investigate several of these characterisations.

3.1 The Method of Day

The first characterisation we will discuss is a result, which Greenleaf describes as *The celebrated method of Day* concerning the existence of certain nets in $\mathrm{Prob}(G)$. To get more specific, we will start out with a definition:

Definition 3.1. Let (φ_j) be a net in $\mathrm{Prob}(G)$. We say that

- φ_j is *weakly convergent to left invariance*, if $({}_x\varphi_j - \varphi_j) \to 0$ weakly in $L^1(G)$,
- φ_j is *weakly convergent to topological left invariance*, if $(\varphi * \varphi_j - \varphi_j)$ converges weakly to 0 for all $\varphi \in \mathrm{Prob}(G)$,
- φ_j is *strongly convergent to left invariance*, if $\|{}_x\varphi_j - \varphi_j\|_1 \to 0$,
- φ_j is *strongly convergent to topological left invariance*, if

  $$\|\varphi * \varphi_j - \varphi_j\|_1 \to 0$$

  for all $\varphi \in \mathrm{Prob}(G)$.

From this definition it is not that difficult to see that strong convergence in either way implies the corresponding weak convergence, hence justifying the names. What might be more surprising is that we have a result giving a variant of the converse implication.

Theorem 3.2. *The following statements are equivalent for any locally compact group G:*

1. *There is a net in $Prob(G)$ weakly convergent to left invariance.*
2. *There is a net in $Prob(G)$ strongly convergent to left invariance.*

The statements are also equivalent if we replace left invariance with topological left invariance.

Proof. By the remark above it suffices to show that the existence of a net (φ_j) weakly convergent to left invariance also gives us a net (ψ_j) strongly convergent to left invariance. For this let (φ_j) be a net in $\mathrm{Prob}(G)$ weakly convergent to left invariance, and let $E = \prod_{x\in G} L^1(G)$ be equipped with the weak topology. Consider now the map $T\colon L^1(G) \to E$ given by

$$Tf(x) = {}_xf - f, \qquad x \in G, f \in L^1(G)$$

As the weak topology on E coincides with the product of the weak topologies on each $L^1(G)$ ([7], 17.13), the weak convergence of (φ_j) tells us that 0 is in the weak closure of $T(\mathrm{Prob}(G)) \subset E$. With $T(\mathrm{Prob}(G))$ being a a convex subset of a locally convex space, the weak and strong closures are equal, and hence there is a net (ψ_j) in $\mathrm{Prob}(G)$ strongly convergent to left invariance.

In the proof of the topological version we will instead let E be the product over all $\varphi \in \mathrm{Prob}(G)$ (instead of $g \in G$). We will also alter T to be

$$Tf(\varphi) = \varphi * f - f, \text{ for } f \in L^1(G), \varphi \in \mathrm{Prob}(G),$$

but from there on the proof is analogous. □

The main result in relation to the Method of Day is that there is a strong correlation between amenability of a locally compact group G and these convergent nets. This correlation follows from the next theorem.

Theorem 3.3. *A locally compact group G is amenable if and only if there exists a net (φ_j) weakly convergent to either topological left invariance or left invariance.*

To prove this main theorem we will need the embedding of $L^1(G)$ into $(L^\infty(G))^*$, and in particular the following lemma for which we will omit the proof.

Lemma 3.4. *The set Prob(G) can be identified with a w^*-dense subset of the w^*-compact convex set of all means on $L^\infty(G)$.*

Proof of Theorem 3.3. We will start out by proving the *non-topological* version. Assume first that (φ_j) is some net in $\mathrm{Prob}(G)$ weakly convergent to left invariance. Then (φ_j) is contained in the w^*-compact set of all means on $L^\infty(G)$. By compactness, there exists a subnet (φ_i) and a mean m on $L^\infty(G)$, such that $\varphi_i \xrightarrow{w^*} m$. To see that this mean must necessarily be left invariant, we notice that for any $f \in L^\infty(G)$ and $x \in G$,

$$\varphi_j({}_xf) - \varphi_j(f) \to m({}_xf) - m(f).$$

At the same time weak convergence of (φ_j) will also give us

$$\begin{aligned}\varphi_j({}_xf) - \varphi_j(f) &= \int (f(x^{-1}t) - f(t))\varphi_j(t)\,\mathrm{d}\lambda(t) \\ &= \int f(t)({}_{x^{-1}}\varphi_j(t) - \varphi_j(t))\,\mathrm{d}\lambda(t) \to 0.\end{aligned}$$

By uniqueness of limits we have $m({}_xf) = m(f)$, for all $f \in L^\infty(G)$ and $x \in G$. Then m is a left invariant mean on $L^\infty(G)$ and hence G is amenable. For the converse implication assume that G is amenable and let m be a left invariant mean on $L^\infty(G)$. From Lemma 3.4 we obtain a net (φ_j) in $\mathrm{Prob}(G)$ w^*-convergent to m.

Left invariance of m will give us $m({}_{x^{-1}}f) = m(f)$ for any $f \in L^\infty(G)$ and $x \in G$, and hence

$$\begin{aligned}\langle {}_x\varphi_j - \varphi_j, f\rangle &= \langle {}_x\varphi_j, f\rangle - \langle \varphi_j, f\rangle = \langle \varphi_j, {}_{x^{-1}}f\rangle - \langle \varphi_j, f\rangle \\ &= \langle \varphi_j, {}_{x^{-1}}f\rangle - \langle m, {}_{x^{-1}}f\rangle + \langle m, f\rangle - \langle \varphi_j, f\rangle \to 0,\end{aligned}$$

by w^*-convergence of (φ_j). As this is true for all $L^\infty(G)$ and $x \in G$ we can conclude that ${}_x\varphi_j - \varphi_j \to 0$ weakly by identifying $(L^1(G))^*$ with $L^\infty(G)$.

The proof of the *topological* version is analogous using topological left invariant means on $L^\infty(G)$ instead of left invariant means. □

Combining the results in this section tells us that amenability of G and the existence of any type of net from Definition 3.1 are equivalent. In particular a net converging in one these ways will give nets converging in the other three.

3.2 Reiter's condition

The next characterisation of amenability is Reiter's condition which is a tool for construction functions with a certain level of left invariance. The definition in question is as follows:

Definition 3.5. A locally compact group G is said to satisfy Reiter's condition (RC), if for any compact $K \subset G$ and $\varepsilon > 0$, there exists $\varphi \in \mathrm{Prob}(G)$ such that $\|{}_x\varphi - \varphi\|_1 < \varepsilon$, for all $x \in K$.

Staying true to the purpose of this section our main priority should be to prove the following theorem:

Theorem 3.6. *A locally compact group G is amenable if and only if it satisfies Reiter's condition.*

Proof. Assume first that G satisfies Reiter's condition and consider the set

$$J = \{(K, \varepsilon) \in P(G) \times (0, \infty) : K \subset G \text{ compact}\},$$

equipped J with the ordering $\prec$ defined by $(K, \varepsilon) \prec (K', \varepsilon')$ if any only if $K \subset K'$ and $\varepsilon' < \varepsilon$. Using Reiter's condition to pick $\varphi_j \in \text{Prob}(G)$ for each $j \in J$ satisfying

$$\|_x\varphi_j - \varphi_j\|_1 < \varepsilon_j \text{ for all } x \in K_j,$$

will then give us a net strongly convergent to left invariance, and hence G is amenable. For the other implication assume now that G is amenable and pick a net (φ_j) in $\text{Prob}(G)$ strongly convergent to topological left invariance. To show that G satisfies Reiter's condition let compact $K \subset G$ and $\varepsilon > 0$ be given, and let furthermore $\beta \in \text{Prob}(G)$ be some fixed function. To continue from here we will need an approximation lemma that we will prove afterwards:

Lemma 3.7. *For any $\varphi \in Prob(G)$ and $\varepsilon > 0$ there exists a compact neighbourhood E of the unit in G such that*

$$\|\varphi_E * \varphi - \varphi\|_1 < \varepsilon \text{ and } \|_x\varphi - \varphi\|_1 < \varepsilon,$$

where φ_E is the normalised characteristic function on E and $x \in E$.

Applying the above lemma, we can now find a compact neighbourhood E of the unit in G such that

$$\|\varphi_E * \beta - \beta\|_1 < \varepsilon/5 \text{ and } \|_x\beta - \beta\|_1 < \varepsilon/5, \text{ for all } x \in E$$

for all $x \in E$. By compactness of K there exists $x_1, \ldots, x_N$ in G such that

$$K \subset \bigcup_{k=1}^{N} x_k E.$$

For $k \in \{1, \ldots, N\}$, let ψ_k denote the normalised characteristic function on $x_k E$. By strong convergence of (φ_j) to topological left invariance, there exists φ_j such that $\|\beta * \varphi_j - \varphi_j\|_1 < \varepsilon/5$, and moreover

$$\|\psi_k * \varphi_j - \varphi_j\|_1 < \varepsilon/5 \text{ for } k = 1, \ldots, N.$$

Our claim is then that $\varphi = \beta * \varphi_j \in \text{Prob}(G)$ is our desired function. In order to prove this we will need to apply all of our previous estimates, but let us first make two intermediate computations. First we notice that for $t \in E$ and $i = 1, \ldots, N$,

$$\begin{aligned}\|\varphi_{x_iE} * \varphi - {}_{x_it}\varphi\|_1 &= \|\varphi_E * \varphi - {}_t\varphi\|_1 \leq \|\varphi_E * \varphi - \varphi\|_1 + \|\varphi - {}_t\varphi\|_1 \\ &\leq \|\varphi_E * \beta - \beta\|_1 + \|\beta - {}_t\beta\|_1 < (2\varepsilon)/5.\end{aligned}$$

Secondly for $i = 1, \ldots, N$ we have

$$\begin{aligned}\|\varphi_{x_iE} * \varphi - \varphi\|_1 &= \|\varphi_{x_iE} * \beta * \varphi_j - \beta * \varphi_j\|_1 \\ &\leq \|\varphi_{x_iE} * \beta * \varphi_j - \varphi_{x_iE} * \varphi_j\|_1 + \|\varphi_{x_iE} * \varphi_j - \varphi_j\|_1 \\ &\quad + \|\varphi_j - \beta * \varphi_j\|_1 \\ &\leq \|\beta * \varphi_j - \varphi_j\|_1 + \|\varphi_{x_iE} * \varphi_j - \varphi_j\|_1 + \|\varphi_j - \beta * \varphi_j\|_1 \\ &< (3\varepsilon)/5.\end{aligned}$$

Combining the two inequalities above gives us

$$\|{}_{x_it}\varphi - \varphi\|_1 \leq \|{}_{x_it}\varphi - \varphi_{x_iE} * \varphi\|_1 + \|\varphi_{x_iE} * \varphi - \varphi\|_1 < \varepsilon,$$

for any $t \in E$ and $i = 1, \ldots, N$. By the choice of $x_1, \ldots, x_N$ this will in particular give us $\|{}_x\varphi - \varphi\|_1 < \varepsilon$, for any $x \in K$. □

Proof of Lemma 3.7. Notice first that it suffices to show that there exists a compact neighbourhood E of the unit in G such that

$$\|{}_x\varphi - \varphi\|_1 < \varepsilon,$$

for all $x \in E$. The other condition will then follow since

$$(\varphi_E * \varphi - \varphi)(s) = \frac{1}{\lambda(E)} \int_E \varphi(t^{-1}s) - \varphi(s)\,\mathrm{d}\lambda(t).$$

So by integrating over all of G gives us

$$\begin{aligned}\|\varphi_E * \varphi - \varphi\|_1 &= \frac{1}{\lambda(E)} \int_G \left| \int_E \varphi(t^{-1}s) - \varphi(s)\,\mathrm{d}\lambda(t) \right| \mathrm{d}\lambda(s) \\ &\leq \frac{1}{\lambda(E)} \int_G \int_E |\varphi(t^{-1}s) - \varphi(s)|\,\mathrm{d}\lambda(t)\,\mathrm{d}\lambda(s) \\ &= \frac{1}{\lambda(E)} \int_E \int_G |{}_t\varphi(s) - \varphi(s)|\,\mathrm{d}\lambda(s)\,\mathrm{d}\lambda(t) \\ &= \frac{1}{\lambda(E)} \int_E \|{}_t\varphi - \varphi\|_1\,\mathrm{d}\lambda(t) < \varepsilon.\end{aligned}$$

Returning to the matter of hand, we should prove that we can find this compact neighbourhood E with $\|_x\varphi - \varphi\|_1 < \varepsilon$, for all $x \in E$. Using the density of $C_c(G)$ in $L^1(G)$ we let $\psi \in C_c(G)$ be given such that $\|\varphi - \psi\|_1$ is stricly less than $\varepsilon/3$ and hence

$$\begin{aligned}\|_x\varphi - \varphi\|_1 &\leq \|_x\varphi -_x \psi\|_1 + \|_x\psi - \psi\|_1 + \|\varphi - \psi\|_1 \\ &\leq 2\|\varphi - \psi\|_1 + \|_x\psi - \psi\|_1 < \frac{2\varepsilon}{3} + \|_x\psi - \psi\|_1.\end{aligned}$$

By this approximation we only need to worry about the behaviour of ψ. Let K denote the support of ψ and let L be a compact neighbourhood of the unit in G. Then $\operatorname{supp}(_x\psi) \subset xK$ and hence $\operatorname{supp}(_x\psi) \subset LK$ for any $x \in L$. Using compactness of LK and uniform continuity of ψ there exists a compact neighbourhood $E \subset L$ of the unit in G such that

$$\|_x\psi - \psi\|_\infty < \frac{\varepsilon}{3 \cdot \lambda(KL)}, \text{ for all } x \in E.$$

For any $x \in E$ we then have $\operatorname{supp}(_x\psi - \psi) \subset LK$ and hence

$$\begin{aligned}\|_x\psi - \psi\|_1 &= \int_G |_x\psi(t) - \psi(t)| \,\mathrm{d}\lambda(t) = \int_{LK} |_x\psi(t) - \psi(t)| \,\mathrm{d}\lambda(t) \\ &\leq \int_{LK} \|_x\psi - \psi\|_\infty \,\mathrm{d}\lambda(t) < \frac{\varepsilon}{3}.\end{aligned}$$

We then have the desired inequality $\|_x\varphi - \varphi\|_1 < \varepsilon$ for all $x \in E$. □

There is a natural generalisation of Reiter's condition discussed above to all L^q spaces. More precisely wee will for each $1 \leq q \leq \infty$ introduce:

(R_q) For any compact $K \subset G$ and $\varepsilon > 0$ there exists $\varphi \in L^q(G)$ such that $\varphi \geq 0, \|\varphi\|_q = 1$ and $\|_x\varphi - \varphi\|_q < \varepsilon$ for any $x \in K$.

In this way Reiter's condition is simply (R_1). It is clear that (R_∞) is trivially satisfied for any group G by picking $\varphi \equiv 1$, but what might be more surprising is the following result.

Proposition 3.8. *For any $1 \leq q < \infty$, Reiter's condition (R_1) is equivalent to (R_q).*

Proof. The strategy of this proof is to show that $(R_q) \implies (R_r)$ whenever $q \leq r$, and then show $(R_{2q}) \implies (R_q)$, which combined will give the desired equivalence of all (R_q)'s. Let first $q \leq r$ be given along with a compact set $K \subset G$ and $\varepsilon > 0$. If we now assume that (R_q) is true, we can pick

$\varphi \in L^q(G)$ with φ positive, $\|\varphi\|_q = 1$ and $\|_x\varphi - \varphi\|_q < \varepsilon^{r/q}$, for any $x \in K$. Then $\psi = \varphi^{q/r} \in L^r(G)$, with $\psi \geq 0$. Furthermore

$$\|\psi\|_r^{r/q} = \left(\int |\varphi^{q/r}|^r \,\mathrm{d}\lambda\right)^{1/q} = \left(\int |\varphi|^q \,\mathrm{d}\lambda\right)^{1/q} = 1,$$

so $\|\psi\|_r = 1$. Finally for any $x \in K$

$$\|_x\psi - \psi\|_r^{r/q} = \left(\int |_x\varphi^{q/r} - \varphi^{q/r}|^r \,\mathrm{d}\lambda\right)^{1/q} \leq \left(\int |_x\varphi - \varphi|^q \,\mathrm{d}\lambda\right)^{1/q} < \varepsilon^{r/q},$$

and hence $\|_x\psi - \psi\|_r < \varepsilon$. The first inequality above is due to the inequality $|\alpha - \beta|^t \leq |\alpha^t - \beta^t|$ for $\alpha, \beta \geq 0$ and $t \geq 1$. To prove $(R_{2q}) \implies (R_q)$ let $1 \leq q < \infty$ be given along with a compact set $K \subset G$ and $\varepsilon > 0$. Use (R_{2q}) to pick $\varphi \in L^{2q}(G)$ with $\varphi \geq 0, \|\varphi\|_{2q} = 1$ and $\|_x\varphi - \varphi\|_{2q} < \varepsilon/2$ for $x \in K$. As before, put $\psi = \varphi^2$ such that $\psi \in L^q(G), \psi \geq 0$ and $\|\psi\|_q = 1$. To finish the proof we can apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
\|_x\psi - \psi\|_q &= \left(\int |_x\varphi^2 - \varphi^2|^q \,\mathrm{d}\lambda\right)^{1/q} \\
&\leq \left(\left(\int |_x\varphi + \varphi|^{2q} \,\mathrm{d}\lambda\right)^{1/2} \left(\int |_x\varphi - \varphi|^{2q} \,\mathrm{d}\lambda\right)^{1/2}\right)^{1/q} \\
&= \left(\int |_x\varphi + \varphi|^{2q} \,\mathrm{d}\lambda\right)^{1/2q} \left(\int |_x\varphi - \varphi|^{2q} \,\mathrm{d}\lambda\right)^{1/2q} \\
&= \|_x\varphi + \varphi\|_{2q} \cdot \|_x\varphi - \varphi\|_{2q} \leq 2\|\varphi\|_{2r} \cdot \|_x\varphi - \varphi\|_{2r} < \varepsilon.
\end{aligned}$$

Hence ψ satisfies the requirements for (R_q). □

3.3 Følner's condition

The third characterisation of amenability we will investigate in this section is Følner's condition, which is a description on how compact subsets behave under translations. To be more specific,

Definition 3.9. A locally compact group G is said to satisfy Følner's condition (FC) if for every compact $K \subset G$ and $\varepsilon > 0$ there exists a Borel set U such that $0 < \lambda(U) < \infty$ and $\lambda((xU)\Delta U) < \varepsilon\lambda(U)$, for all $x \in K$.

The first thing we should notice is that (FC) is equivalent to the existence of a net (U_j) of Borel sets such that $0 < \lambda(U_j) < \infty$ and

$$\frac{\lambda(xU_j\Delta U_j)}{\lambda(U_j)} \to 0$$

for all $x \in G$. The proof of this follows the idea of the proof of Theorem 3.6. This description has the desirable property that we are allowed to translate by every element in G instead of being restricted to some compact set. Claiming that Følner's condition is an alternative characterisation of amenability, we should prove the following theorem:

Theorem 3.10. *Let G be a locally compact group. Then G is amenable if and only if it satisfies Følner's condition, as follows:*

Proof. To prove this theorem we will introduce the *weak Følner's condition* (FC^*):

- For any compact set $K \subset G$ and $\varepsilon, \delta > 0$ there exist Borel sets $U \subset G$ and $N \subset K$ such that $0 < \lambda(U) < \infty$, $\lambda(N) < \delta$ and

$$\lambda((xU)\Delta U) < \varepsilon\lambda(U), \quad \text{for all } x \in K\backslash N.$$

Our goal will the be to prove the following implications:

$$(FC) \implies \text{amenability} \implies (FC^*) \implies (FC).$$

Assume now that G satisfies (FC) and let (U_j) be a net of Borel sets in G such that $0 < \lambda(U_j) < \infty$ and

$$\frac{\lambda((xU_j)\Delta U_j)}{\lambda(U_j)} \to 0$$

for all $x \in G$. Letting $\varphi_j \in \mathrm{Prob}(G)$ denote the normalised characteristic function for U_j we obtain

$$\|{}_x\varphi_j - \varphi_j\|_1 = \frac{\lambda((xU_j)\Delta U_j)}{\lambda(U_j)} \to 0 \quad \text{for all } x \in G$$

This is simply the statement that (φ_j) is strongly convergent to left invariance, so by Section 3.1 G is amenable. For the next implication assume that G is amenable and let a compact set $K \subset G$ and $\varepsilon, \delta > 0$ be given. First we should notice that we can assume $\lambda(K) > 0$, since we could otherwise pick $N = K$ and be done. By Reiter's condition for amenability, let $\varphi \in \mathrm{Prob}(G)$ be given such that $\|{}_x\varphi - \varphi\|_1 < (\delta\varepsilon)/\lambda(K)$.

As the simple functions in $L^1(G)$ form a dense subset of $L^1(G)$ we can assume that φ is simple, i.e. there exist Borel sets $A_1 \supset A_2 \supset \cdots \supset A_N$ with $0 < \lambda(A_i) < \infty$ and strictly positive constants $\lambda_1, \ldots, \lambda_N$ such that

$$\sum_{k=1}^{N} \lambda_k = 1 \quad \text{and} \quad \varphi = \sum_{k=1}^{N} \frac{\lambda_k}{\lambda(A_k)} 1_{A_k}.$$

The nesting of the A_i's ensures that

$$\sum_{k=1}^{N} \lambda_k \cdot \frac{\lambda((xA_k)\Delta A_k)}{\lambda(A_k)} = \|_x\varphi - \varphi\|_1,$$

and hence this sum is stricly smaller that $(\delta\varepsilon)/\lambda(K)$ for all $x \in K$. Integrating over all of K we will then obtain

$$\sum_{k=1}^{N} \lambda_k \int_K \frac{\lambda((xA_k)\Delta A_k)}{\lambda(A_k)} \, \mathrm{d}\lambda(x) < \delta\varepsilon.$$

This means that at least one $U = A_k$ satisfy.

$$\int_K \frac{\lambda((xU)\Delta U)}{\lambda(U)} \, \mathrm{d}\lambda(x) < \delta\varepsilon.$$

Finally we can put $N = \{x \in K : \lambda((xU)\Delta U)/\lambda(U) \geq \varepsilon\}$ such that $\lambda(N) < \delta$ and hence satisfying (FC^*). For the final implication assume that G satisfies (FC^*) and let compact $K \subset G$ and $\varepsilon > 0$ be given. Once again we will assume $\lambda(K) > 0$, since we could otherwise replace K with a compact neighbourhood of itself. Let now $A = K \cup KK$ such that $\lambda(kA \cap A) \geq \lambda(K)$ for all $k \in K$ and let $\delta = \lambda(K)/2$. Any Borel set $N \subset A$ satisfying $\lambda(A \backslash N) < \delta$ will for $k \in K$ satisfy

$$\begin{aligned} 2\delta \leq \lambda(K) \leq \lambda(kA \cap A) &\leq \lambda(kN \cap N) + \lambda(A\backslash N) + \lambda(k(A\backslash N)), \\ &< \lambda(kN \cap N) + 2\delta, \end{aligned}$$

and hence $\lambda(kN \cap N) > 0$. In particular, $kN \cap N$ is a non-empty set. Rearranging things a bit will then tell us that $K \subset NN^{-1}$ whenever $\lambda(A\backslash N) < \delta$. We will now apply (FC^*) to pick $U \subset G$ and $N \subset A$ such that $\lambda(U) \in (0, \infty)$, $\lambda(A\backslash N) < \delta$ and $\lambda(nU\Delta U) < (\varepsilon\lambda(U))/2$, for all $n \in N$. To conclude the proof we notice that for any $n_1, n_2 \in N$,

$$\begin{aligned} \lambda((n_1 n_2^{-1} U)\Delta U) &\leq \lambda(n_2^{-1} U \Delta U) + \lambda(U \Delta n_1^{-1} U) \\ &= \lambda(n_1 U \Delta U) + \lambda(n_2 U \Delta U) < \varepsilon\lambda(U), \end{aligned}$$

so in particular $\lambda(xU\Delta U) < \varepsilon\lambda(U)$, for any $x \in K \subset NN^{-1}$. □

Remark 3.11. Extending this result it turns out that the Borel set U picked with the use of Følner's condition can be chosen to be compact and symmetric. This was shown by Emerson [2].

3.4 The Fixed Point Property

Let G be a locally compact group and X be a compact convex subset of a locally convex space E. We say that G acts affinely on X if there exists a continuous map $T: G \times X \to X$, such that $T_{xy} = T_x \circ T_y$ and

$$T_x(\lambda s_0 + (1-\lambda)s_1) = \lambda T_x(s_0) + (1-\lambda)T_x(s_1)$$

for all $x, y \in G$ and $s_0, s_1 \in X$. Here T_x denotes the map $s \mapsto T(x, s)$. We say that a group G has the *fixed point property* if any such affine action gives rise to an element $s_0 \in X$, such that $T_x(s_0) = s_0$ for all $x \in G$. We will state and prove three lemmas before proving the equivalence of amenability and the fixed point property.

Lemma 3.12. *The convex hull of the point masses on G forms a dense subset in the set of all means on $CB(G)$. Such means will be called finite means.*

Proof. Let Σ be the set of all means on $CB(G)$ and let

$$A = \left\{ \sum_{i=1}^{n} \lambda_i \delta_{x_i} \;\middle|\; x_i \in G, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Assume in order to reach a contradiction that A is not dense in Σ. By the Hahn-Banach Separation Theorem, there exists a mean m in Σ a real number λ and $f \in CB(G)$ such that $\mathrm{Re}(m(f)) > \lambda \geq \mathrm{Re}(\varphi(f))$, for all φ in the closure of A. As the point masses on G belong to A, this in particular gives us $m(\mathrm{Re}(f)) = \mathrm{Re}(m(f)) > \lambda \geq \mathrm{Re}(f(s))$, for all $s \in G$. Here we notice that $m(\mathrm{Re}(f)) = \mathrm{Re}(m(f))$, for any $f \in CB(G)$. Let now $c = \|\mathrm{Re}(f)\|_\infty$ and notice that this gives us

$$m(\mathrm{Re}(f) - c) = m(\mathrm{Re}(f)) - m(c) = m(\mathrm{Re}(f)) - c > 0,$$

contradicting the fact that m is positive. In conclusion A must be dense in the set of all means. □

Lemma 3.13. *Let X be a compact convex subset of a locally convex set E and let m be a mean on $CB(G)$. If there exists $g: G \to X$ such that $\varphi \circ g \in CB(G)$ for every $\varphi \in E^*$, then there exists a unique $x_0 \in X$ such that $\varphi(x_0) = m(\varphi \circ g)$, for all $\varphi \in E^*$. In the affirmative case we will say that g is integrable and will let $\int g \,\mathrm{d}m$ denote this unique element.*

Proof. Let $g\colon G \to X$ satisfy $\varphi \circ g \in CB(G)$, for all $\varphi \in E^*$. By Lemma 3.12 we will first assume that m is a finite mean, i.e., $m = \sum_{i=1}^n \lambda_i \delta_{x_i}$, for some $x_1, \dots, x_n \in G$ and $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$. We will then let

$$x = \sum_{i=1}^n \lambda_i g(x_i) \in X.$$

Hence $\varphi(x) = m(\varphi \circ g)$, for all $\varphi \in E^*$. In the case where m is not necessarily a finite mean we can pick a net (m_j) of finite means w^*-convergent to m. For each m_j we define $x_j \in X$ in the same way as above. As X is a compact set there exists a convergent subnet (x_i) of (x_j) with some limit $x_0 \in X$. Then any $\varphi \in E^*$ will satisfy

$$m(\varphi \circ g) = \lim m_i(\varphi \circ g) = \lim \varphi(x_i) = \varphi(x),$$

by w^*-convergence of the subnet (m_i) and continuity of φ. The uniqueness of x_0, follows, since E^* separates points in E. □

Lemma 3.14. *Let $\beta\colon X \to X$ be some affine, continuous map, let $g\colon G \to X$ be integrable and let m be a mean on $CB(G)$. If $\beta \circ g$ is also integrable, then*

$$\int (\beta \circ g)\,\mathrm{d}m = \beta\left(\int g\,\mathrm{d}m\right).$$

Proof. Let us begin with the case where $m = \sum_{i=1}^n \lambda_i \delta_{x_i}$, for $x_1, \dots, x_n \in G$ and $\lambda_1, \dots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$. Then

$$\beta\left(\int g\,\mathrm{d}m\right) = \beta\left(\sum_{i=1}^n \lambda_i g(x_i)\right) = \sum_{i=1}^n \lambda_i (\beta \circ g)(x_i) = \int (\beta \circ g)\,\mathrm{d}m,$$

by the arguments in the proof of the preceding lemma. In the general case where m is not necessarily a finite mean, we can at least pick a net (m_j) of finite means w^*-convergent to m.

Let now τ denote the coarsest topology on X, making all $\varphi|_X$ continuous where $\varphi \in E^*$. This topology will then satisfy $x_j \xrightarrow{\tau} x \in X$ if and only if $\varphi(x_j) \to \varphi(x)$, for all $\varphi \in E^*$, and hence $\int g\ \mathrm{d}m_j \to \int g\ \mathrm{d}m \in (X, \tau)$. Since the linear functionals on E also separate points in X, this new topological space (X, τ) is Hausdorff and the identity from X to (X, τ) is continuous. Recall now that a bijective, continuous map from a compact space to a Hausdorff space is in fact a homeomorphism and thus

$$\int g\ \mathrm{d}m_j \to \int g\ \mathrm{d}m,$$

in the ordinary topology on X. This will allow us to conclude

$$\begin{aligned}\varphi\left(\int(\beta\circ g)\,\mathrm{d}m\right) &= m(\varphi\circ\beta\circ g) = \lim m_j(\varphi\circ\beta\circ g)\\ &= \lim\varphi\left(\int(\beta\circ g)\,\mathrm{d}m_j\right) = \lim(\varphi\circ\beta)\left(\int g\ \mathrm{d}m_j\right)\\ &= (\varphi\circ\beta)\left(\int g\ \mathrm{d}m\right) = \varphi\left(\beta\left(\int g\ \mathrm{d}m\right)\right),\end{aligned}$$

for all $\varphi \in E^*$, and hence $\int(\beta\circ g)\,\mathrm{d}m = \beta(\int g\,\mathrm{d}m)$. □

Theorem 3.15. *For a locally compact group G the following are equivalent:*

1. *G is amenable,*
2. *G has the fixed point property.*

Proof: Assume first that G has the fixed point property and let Σ the set of all means on $UCB_r(G)$. Then Σ is a compact convex set of the locally convex space $(UCB_r(G))^*$ equipped with the w^*-topology. Define now an action $T\colon G\times\Sigma\to\Sigma$ by

$$\langle T(x,m),f\rangle = \langle m,{}_{x^{-1}}f\rangle.$$

We want to show that this is a continuous affine action. To show continuity of T let $(x_j,m_j)_{j\in J}$ be a net in $G\times\Sigma$ with limit $(x,m)\in G\times\Sigma$. We now wish to show that $T_{x_j}(m_j)\to T_x(m)$ in the w^*-topology on the dual space of $UCB_r(G)$, so let $\varepsilon>0$ and $f_1,\dots,f_n\in UCB_r(G)$ be given. By the right uniform continuity of each f_k, there exists $i_0\in J$ such that

$$\|{}_{x_i^{-1}}f_k - {}_{x^{-1}}f_k\|_\infty < \varepsilon/6$$

for each $k=1,\dots,n$ and $i\succ i_0$, and hence $\|{}_{x_i^{-1}}f_k - {}_{x_{i_0}^{-1}}f_k\|_\infty < \varepsilon/3$. By the convergence $m_j\to m$ we can also find $i_1\succ i_0$ such that

$$\Big|\langle T_{x_{i_0}}(m_i),f_k\rangle - \langle T_{x_{i_0}}(m),f_k\rangle\Big| < \varepsilon/3.$$

for all $k=1,\dots,n$ and $i\succ i_1$. Combining these inequalities will let us conclude that

$$|\langle T_{x_i}(m_i),f_k\rangle - \langle T_x(m),f_k\rangle| < \varepsilon,$$

for all $k = 1, \ldots, n$ and $i \succ i_1$, and hence T is continuous. Furthermore T is clearly affine, so by the fixed point property of G there exists $m_0 \in \Sigma$ such that $T_x(m) = m$ for all $x \in G$, i.e.,

$$\langle m, {}_{x^{-1}}f \rangle = \langle m, f \rangle.$$

Then m is a left invariant mean on $UCB_r(G)$, so G is amenable. For the converse implication, assume that G is amenable and let m be a left invariant mean on $CB(G)$. Assume now that $T \colon G \times X \to X$ is some affine action, where X is a compact, convex subset of some locally convex set E. Fix $x \in X$ and let $\psi_x \colon G \to X$ denote that map $\psi_x(t) = T_t(x)$.

For any $\varphi \in E^*$ the map $\varphi \circ \psi_x$ is continuous, and since $(\varphi \circ \psi_x)(G) \subset \varphi(X)$ is a compact set, the map is also bounded. This tells us that ψ_x is integrable and hence gives the existence of the element $\int \psi_x \, \mathrm{d}m \in X$. By the same argument, any map $T_g \circ \psi_x$ is also integrable and hence

$$T_g \left(\int \psi_x \, \mathrm{d}m \right) = \int (T_g \circ \psi_x) \, \mathrm{d}m$$

for all $g \in G$. For any $\varphi \in E^*$ and $g \in G$, we see that $\varphi \circ T_g \circ \psi_x = {}_{g^{-1}}(\varphi \circ \psi_x)$ and hence

$$\begin{aligned}(\varphi \circ T_g) \left(\int \psi_x \, \mathrm{d}m \right) &= \varphi \left(\int (T_g \circ \psi_x) \, \mathrm{d}m \right) = m(\varphi \circ T_g \circ \psi_x) \\ &= m({}_{g^{-1}}(\varphi \circ \psi_x)) = m(\varphi \circ \psi_x) = \varphi \left(\int \psi_x \, \mathrm{d}m \right).\end{aligned}$$

As this holds true for all $\varphi \in E^*$, we can conclude $T_g \left(\int \psi_x \, \mathrm{d}m \right) = \left(\int \psi_x \, \mathrm{d}m \right)$ and thus $\int \psi_x \, \mathrm{d}m$ is a fixed point for the action T.

3.5 Norms of convolution operators on $L^p(G)$

The final thing we will study in this section is how amenability of the group is linked to the norm of the convolution operators on different L^p spaces.

Definition 3.16. Let G be a locally compact group. For any bounded regular Borel measure μ on G and $1 \le p \le \infty$, we define $\lambda_{\mu,p}$ as the linear operator on $L^p(G)$ defined by $\lambda_{\mu,p}(f) = \mu * f$.

These convolution operators will satisfy $\|\lambda_{\mu,p}\| \le \|\mu\|$, regardless of the choice of $1 \le p \le \infty$. This follows from the inequality $\|\mu * f\|_p \le \|\mu\| \cdot \|f\|_p$ where μ is a bounded regular Borel measure on G and $f \in L^p(G)$. In details,

$$\begin{aligned}\|\lambda_{\mu,p}\| &= \sup\{\|\mu * f\|_p : \|f\|_p = 1\} \\ &\le \sup\{\|\mu\| \cdot \|f\|_p : \|f\|_p = 1\} = \|\mu\|.\end{aligned}$$

Proposition 3.17. *For a positive bounded regular Borel measure μ we have*

$$\|\lambda_{\mu,1}\| = \|\mu\| \quad \text{and} \quad \|\lambda_{\mu,\infty}\| = \|\mu\|.$$

Proof. By the remark above, it suffices to show that $\|\lambda_{\mu,1}\| \geq \|\mu\|$, so let $\varepsilon > 0$ be given. As μ is inner regular, there exists a compact set $K \subset G$, such that

$$\mu(K) \geq \mu(G) - \varepsilon = \|\mu\| - \varepsilon.$$

We can then apply Urysohn's lemma to pick $f \in C_c(G)$ such that $1_K \leq f \leq 1_G$, and hence $\int f \, \mathrm{d}\mu \geq \mu(K) \geq \|\mu\| - \varepsilon$. Let now (U_j) be an open neighbourhood base of $e \in G$, with $\lambda(U_j) < \infty$, ordered by reverse inclusion and let (φ_j) be the corresponding net of normalised characteristic functions. We will now for each j define the map $g_j \colon G \to \mathbb{C}$ by

$$g_j(t) = \int \varphi_j(x) f(tx) \, \mathrm{d}\lambda(x), \text{ for } t \in G.$$

These functions might seem strange at first, but they appear natural once we consider the convolution $\mu * \varphi_j$ as an element in $(C_0(G))^*$. We have

$$\begin{aligned}
\langle \mu * \varphi_j, f \rangle &= \int \mu * \varphi_j(x) f(x) \, \mathrm{d}\lambda(x) \\
&= \int \int \varphi_j(t^{-1}x) f(x) \, \mathrm{d}\mu(t) \, \mathrm{d}\lambda(x) \\
&= \int \int \varphi_j(t^{-1}x) f(x) \, \mathrm{d}\lambda(x) \, \mathrm{d}\mu(t) \\
&= \int \int \varphi_j(x) f(tx) \, \mathrm{d}\lambda(x) \, \mathrm{d}\mu(t) \\
&= \int g_j(t) \, \mathrm{d}\mu(t).
\end{aligned}$$

By left uniform continuity of f, pick j_0 such that $|f(tx) - f(t)| < \varepsilon$, for all $t \in G$ and $x \in U_{j_o}$. Then

$$\begin{aligned}
|g_{j_0}(t) - f(t)| &= \left| \int \varphi_{j_0}(x) f(tx) \, \mathrm{d}\lambda(x) - f(t) \right| \\
&\leq \frac{1}{\lambda(U_{j_0})} \int_{U_{j_0}} |f(tx) - f(t)| \, \mathrm{d}\lambda(x) < \varepsilon.
\end{aligned}$$

By integrating over all of G with respect to μ we get

$$\left| \langle \mu * \varphi_{j_0}, f \rangle - \int f \, \mathrm{d}\mu \right| \leq \int |g_{j_0}(t) - f(t)| \, \mathrm{d}\mu(t) < \varepsilon \|\mu\|.$$

From this, we can apply the triangle inequality to obtain a lower bound on $|\langle \mu * \varphi_{j_0}, f\rangle|$ given by $\|\mu\| - \varepsilon(\|\mu\| + 1)$. To finish the argument, we will recall Hölder's inequality $|\langle \mu * \varphi_{j_0}, f\rangle| \leq \|\mu * \varphi_{j_0}\|_1 \cdot \|f\|_\infty \leq \|\mu * \varphi_{j_0}\|_1$, and hence

$$\|\lambda_{\mu,1}\| \geq \|\mu * \varphi_{j_0}\|_1 \geq |\langle \mu * \varphi_{j_0}, f\rangle| \geq \|\mu\| - \varepsilon(\|\mu\| + 1).$$

As $\varepsilon > 0$ was arbitrary, we can conclude that $\|\lambda_{\mu,1}\| \geq \|\mu\|$.

In the case of $p = \infty$, let $f \in C_0(G)$ be given. We will first show that $\mu * f$ is continuous, so let $s \in G$ and $\varepsilon > 0$ be given. As $C_0(G) \subset UCB(G)$ we can pick an open neighbourhood U of $e \in G$ such that $|f(x) - f(xy)| < \varepsilon/\|\mu\|$ for all $x \in G$ and $y \in U$. For every $t \in sU$ we will have $s^{-1}t \in U$, and thus

$$\begin{aligned} |\mu * f(s) - \mu * f(t)| &\leq \int |f(x^{-1}s) - f(x^{-1}t)| \,\mathrm{d}\mu(x) \\ &= \int |f(x^{-1}s) - f(x^{-1}s(s^{-1}t))| \,\mathrm{d}\mu(x) \\ &< \int \varepsilon/\|\mu\| \,\mathrm{d}\mu(x) \leq \varepsilon. \end{aligned}$$

This lets us conclude that $\mu * f$ is continuous in s, and since s was arbitrary, $\mu * f$ is continuous on all of G. We can now use continuity of $\mu * f$ to obtain

$$\|\mu * f\|_\infty \geq |\mu * f(e)| = \left| \int f(t^{-1}) \,\mathrm{d}\mu(t) \right| = | < \mu, \tilde{f} > |.$$

Notice that $\|f\|_\infty = \|\tilde{f}\|_\infty$, for $f \in C_0(G)$, and thus

$$\begin{aligned} \|\lambda_{\mu,\infty}\| &= \sup\{\|\mu * f\|_\infty \mid f \in L^\infty(G), \|f\|_\infty = 1\} \\ &\geq \sup\{\|\mu * f\|_\infty \mid f \in C_0(G), \|f\|_\infty = 1\} \\ &\geq \sup\{|\langle \mu, f\rangle| \mid f \in C_0(G), \|f\|_\infty = 1\} = \|\mu\|, \end{aligned}$$

the inequality we were looking for. □

Theorem 3.18. *Let G be an amenable locally compact group. Then $\|\lambda_{\mu,p}\|$ is equal to $\|\mu\|$ for all positive bounded regular Borel measure on G and $1 \leq p \leq \infty$.*

Proof. Assume first that G is amenable and let positive $\mu \in (C_0(G))^*$ and $p \in (1, \infty)$ be given. We can assume that $\|\mu\| = 1$ and furthermore that μ has compact support, since these measures are dense in these of regular Borel measures on G.

Let K denote the support of μ and let $\varepsilon > 0$ be given. By the generalised version of Reiter's condition, we can pick positive $\varphi \in L^p(G)$ with norm 1 and satisfying $\|_x\varphi - \varphi\|_p < \varepsilon$, for all $x \in K$. Identify now $L^p(G)$ with $(L^q(G))^*$ for $q > 1$ such that $1/p + 1/q$ and let $g \in L^q(G)$ be given. Then

$$\begin{aligned}
|\langle \mu * \varphi - \varphi, g\rangle| &= |\langle \mu * \varphi, g\rangle - \langle \varphi, g\rangle| \\
&= \left| \int \int \varphi(t^{-1}x) g(x)\, \mathrm{d}x \ \ \mathrm{d}\mu(t) - \int \int \varphi(x) g(x)\, \mathrm{d}\lambda(x)\, \mathrm{d}\mu(t) \right| \\
&\leq \int \int |\varphi(t^{-1}x) - \varphi(x)| \cdot |g(x)|\, \mathrm{d}\lambda(x)\, \mathrm{d}\mu(t) \\
&= \int \int |_t\varphi(x) - \varphi(x)| \cdot |g(x)|\, \mathrm{d}\lambda(x)\, \mathrm{d}\mu(t) \\
&\leq \int_K \|_x\varphi - \varphi\|_p \cdot \|g\|_q\, \mathrm{d}\mu(t) < \varepsilon \cdot \|g\|_q.
\end{aligned}$$

Taking the supremum over all $g \in L^q(G)$ with $\|g\|_q = 1$ in the above inequality will then give us $\|\mu * \varphi - \varphi\|_p < \varepsilon$. By the triangle inequality, this tells us $\|\lambda_{\mu,p}\| \geq \|\mu * \varphi\|_p \geq 1 - \varepsilon$. As $\varepsilon > 0$ was arbitrary, we can conclude $\|\lambda_{\mu,p}\| \geq 1 = \|\mu\|$, and hence amenability of G gives us $\|\lambda_{\mu,p}\| = \|\mu\|$, for all $1 \leq p \leq \infty$. □

The converse result is also true. That is, if $\|\lambda_{\mu,p}\| = \|\mu\|$, for all $1 \leq p \leq \infty$ and positive bounded regular Borel measures μ, then G is amenable. The proof of this can be found in [8]. Notice also that by Riesz convexity theorem ([1], VI.10.8), the map $\alpha \mapsto \|\lambda_{\mu,1/\alpha}\|$ is convex for $\alpha \in (0, 1)$, and hence it suffices to show $\|\lambda_{\mu,p}\| = \|\mu\|$ for one $1 < p < \infty$.

4 Furstenberg's conjecture

After discussing different characterisations of amenability for a while, we will now turn our attention to the characterisation that Furstenberg conjectured in [4]:

G possesses a measure μ whose support is all of G and for which no nontrivial μ-boundary exists iff G is amenable.

We will use this section to develop the machinery for understanding the above conjecture and also prove that if G is not amenable, then all measures on G with full support will give rise to a nontrivial μ-boundary.

4.1 The world of Furstenberg

In the following we will only consider σ-compact locally compact groups G and investigate how such groups act on topological spaces. Recall that a group is σ-compact, if we can find compact sets $K_1, K_2, \dots$ in G such that $G = \bigcup_{n=1}^{\infty} K_n$.

Definition 4.1. A locally compact space X is said to be a G-space if G acts on X in a continuous manner.

As in Section 3.4 we will commonly use $T\colon G \times X \to X$ to denote such an action and for each $g \in G$ let $T_g\colon X \to X$ denote the map $T_g(x) = T(g, x)$. We will also use the notation $g.x$ for $T_g(x)$ from time to time. Given two G-spaces X and X', we say that a continuous map $\varphi\colon X \to X'$ is equivariant if

$$T'_g(\varphi(x)) = \varphi(T_g(x)),$$

i.e., φ respects the actions on X and X'. For a G-space X we let $\mathcal{P}(X)$ denote the set of regular Borel probability measures on X, such that the action on X induces an action on $\mathcal{P}(X)$, given by

$$T_g(\mu)(A) = \mu(gA).$$

Here we will view $\mathcal{P}(X)$ as a compact convex subset of means in $(C_0(X))^*$, and when endowed with the w^*-topology, the action above will be continuous by arguments similar to those of Theorem 3.15. We are now ready for the first result of this section.

Proposition 4.2. *Let G be a σ-compact locally compact group. Then G is amenable if and only if any compact G-space X admits a G-invariant measure $\mu \in \mathcal{P}(X)$.*

Proof. Assume first that G is amenable and let X be a compact G-space. By the remarks above, this gives us a continuous action of G on $\mathcal{P}(X)$. This action is affine as well, and since $\mathcal{P}(X)$ is a compact convex subset of a locally convex vector space, the fixed point property of G will give us a G-invariant measure in $\mathcal{P}(X)$.

For the other implication, assume now that any compact G-space X admits a G-invariant measure in $\mathcal{P}(X)$. Let $\beta(G)$ denote the Stone-Čech compactification of G. Hence $CB(G) = C(\beta(G))$. Left multiplication in G can be viewed as a continuous function from G to $G \subset \beta(G)$ and hence it extends to a continuous action on $\beta(G)$. By assumption this allows us to find a G-invariant measure $\mu \in \mathcal{P}(\beta(G))$. Integrating with respect to this μ will then be a G-invariant mean on $C(\beta(G))$, which when viewed as a mean on $CB(G)$ will simply be a left invariant mean, and hence G is amenable. □

When dealing with these G-spaces we can extend our definition of the convolution of two measures. If X is a G-space and $\mu \in \mathcal{P}(G), \nu \in \mathcal{P}(X)$, we let $\mu * \nu \in \mathcal{P}(X)$ denote the measure

$$\mu * \nu(A) = \int 1_A(g.x)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(x), \text{ for } A \in \mathcal{B}(X),$$

or, when viewed as a linear functional on $C_0(X)$,

$$\int_X f(x)\,\mathrm{d}\mu * \nu(x) = \int_X \int_G f(g.y)\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(y).$$

Notice that this definition is consistent with our previous definition when viewing G itself as a G-space by left multiplication.

4.2 A probabilistic approach

With the notion of G-spaces in place, we are now ready to approach the notion of a μ-boundary, and for this we will delve into the world of probability theory. A random variable on a G-space X will be a function f from some background probability space $(\Omega, \mathcal{F}, P)$ into X, such that f is measurable with respect to the Borel sets on X. The measurability of f will then let us define a probability measure $f(P)$ on X by

$$f(P)(A) = P(f \in A), \qquad A \in \mathcal{B}(X),$$

where $(f \in A)$ denotes the preimage $f^{-1}(A)$. We say that $f(P)$ is the distribution of f. The next important concept with random variables is that of independence.

Definition 4.3. We say that a family of random variables (f_α) with values in X_α are independent, if any finite subfamily $f_{\alpha_1}, \dots, f_{\alpha_n}$ and $A_k \in X_{\alpha_k}$ satisfies

$$P\left(\bigcap_{k=1}^{n} (f_{\alpha_k} \in A_k)\right) = \prod_{k=1}^{n} P(f_{\alpha_k} \in A_k).$$

Such random variables have a very strong connection with our convolution operation as seen in the following propositions.

Proposition 4.4. *If f, h are independent random variables from a common probability space into measurable spaces X and Y, respectively, then $(f, h)(P)$ and $f(P) \times g(P)$ agrees as measures on $X \times Y$.*

Proof. This results follows from the fact that the family

$$\mathcal{A} = \{A \times B \mid A \in \mathcal{B}(X), B \in \mathcal{B}(Y)\}$$

is an intersection-stable generator for $\mathcal{B}(X \times Y)$. □

Proposition 4.5. *Let X be a G-space and let f, h be independent random variables from a common background space with values in G and X, respectively. Then $f.h$ is a random variable with values in X and*

$$(f.h)(P) = f(P) * h(P).$$

Proof. For any Borel set A in X,

$$\begin{aligned}(f.h)(P)(A) &= \int 1_A(g.x)\, \mathrm{d}(f.h)(P)(g, x) = \int 1_A(g.x)\, \mathrm{d}f(P)(g)\, \mathrm{d}h(P)(x) \\ &= \int 1_A(y)\, \mathrm{d}f(P) * h(P)(y) = f(P) * h(P)(A),\end{aligned}$$

proving the statement □

The point of considering such random variables is to describe different probability measures only through the language of distributions of random variables. The fact that this is even possible follows from this important theorem in probability theory, for which we will omit the proof.

Theorem 4.6. *Let (X_α) be a family of topological spaces with corresponding measures $\mu_\alpha \in \mathcal{P}(X_\alpha)$, then there exists a probability space $(\Omega, \mathcal{F}, P)$ and a family of independent random variables (f_α), such that $f_\alpha \colon \Omega \to X_\alpha$ and $f_\alpha(P) = \mu_\alpha$.*

Our next goal will now be to construct a so-called μ-process on X, where X is a G-space and $\mu \in \mathcal{P}(G)$. The first thing to do is to let $(f_n)_{n\geq 1}$ be a sequence of independent random variables with values in G and common distribution μ. By independence, the partial products $f_1 \cdots f_n$ will have distribution $\mu^{(n)}$. If now $(\Omega, \mathcal{F}, P)$ is the background space of $(f_n)_{n\geq 1}$ we can define $\Omega' = M \times \Omega$ and equip this space with the measure $P' = \nu \times P$, where ν is some measure in $\mathcal{P}(M)$.

We can now let $h_0 \colon \Omega \to X$ be the projection onto X, such that h_0 is a random variable independent of $(f_n)_{n\geq 1}$ and $h_0(P') = \nu$. Finally we will define a sequence $(h_n)_{n\geq 1}$ of random variables from Ω to X by

$$h_n(x, \omega) = f_n(\omega) f_{n-1}(\omega) \cdots f_1(\omega).h_0(x, \omega),$$

with $h_n(P') = \mu^{(n)} * \nu$. This constuction justifies the following definition:

Definition 4.7. A μ-process on a G-space X is a sequence of random variables $(f_{n+1}, h_n)_{n\geq 1}$ from $(\Omega, \mathcal{F}, P)$ with values in $G \times X$ such that

1. $f_n(P) = \mu$,
2. $h_{n+1} = f_{n+1} h_n$,
3. f_{n+1} is independent of $\{f_n, \ldots, f_1, h_n, \ldots, h_1\}$.

As an example of such a μ-process, let's look at the case where G itself is considered as a G-space by left multiplication and $\nu = \delta_e$. By letting $(f_n)_{n\geq 1}$ be a sequence of independent and identically distributed random variables with values in G and common distribution μ we obtain a sequence $(h_n)_{n\geq 1}$ of random variables on G defined as

$$h_n = f_n \cdots f_1,$$

such that $h_n(P) = \mu^{(n)}$. We will call $(h_n)_{n\geq 1}$ the left-handed random walk associated to μ and similarly let

$$h'_n = f_1 \cdots f_n$$

be the right-handed random walk associated to μ. The next thing we will investigate are the notions of a stationary measure on a G-space and a (G, μ)-space.

Definition 4.8. Let $\mu \in \mathcal{P}(G)$ be given and X be a G-space. We say that $\nu \in \mathcal{P}(X)$ is stationary with respect to μ, if $\mu * \nu = \nu$. If the measure μ in question is clear from the context, we will simply say that ν is stationary.

Definition 4.9. Let $\mu \in \mathcal{P}(G)$ be given. A G-space X equipped with a measure $\nu \in \mathcal{P}(X)$ is said to be a (G, μ)-space if ν is stationary with respect to μ.

It might not be clear that such (G, μ)-spaces exist for all $\mu \in \mathcal{P}(G)$, but the following proposition gives us a setup that allows us to construct (G, μ)-spaces.

Proposition 4.10. *Let X be a compact G-space and $\mu \in \mathcal{P}(G)$ be given. Then there exists $\nu \in \mathcal{P}(X)$ such that (X, ν) is a (G, μ)-space.*

Proof. To prove this proposition we should find a stationary $\nu \in \mathcal{P}(X)$. For this consider the map $T \colon \mathcal{P}(X) \to \mathcal{P}(X)$ given by

$$T(\nu) = \mu * \nu.$$

Then T will be a continuous affine map on a compact convex subset of the locally convex vector space $(C_0(X))^*$. Thus, T will have a fixed point $\nu \in \mathcal{P}(X)$, that will be our stationary measure on X. □

We are now ready to introduce the notion of a μ-boundary as defined by Furstenberg.

Definition 4.11. Let (X, ν) be a (G, μ)-space. We say that (M, ν) is a μ-boundary if $f_1 \cdots f_n.\nu$ converges to a point measure with probability one, where $(f_n)_{n \geq 1}$ is a sequence of independent random variables with values in G and common distribution μ.

To prove the stated connection between μ-boundaries and amenability of G, we will look at the right-handed random walk $(h_n)_{n \geq 1}$ and say that $(h_n)_{n \geq 1}$ is transient if $h_n \to \infty$ as $n \to \infty$ with probability one. Here $h_n \to \infty$ is to be interpreted as $(h_n)_{n \geq 1}$ leaving any compact set at some point. Connected with this concept are the following results:

Theorem 4.12. *If (G, μ) has a μ-boundary (M, ν) such that ν is not a one point measure, then (h_n) is transient.*

Proof. Assume that (M, ν) is a μ-boundary, i.e., $h_n.\nu$ converges to a point measure with probability one. Assume now that (h_n) is not transient, such that it has a subsequence (h_{n_k}) contained in some compact K with a strictly positive probability. This gives us a further subsequence $(h'_m) \subset (h_{n_k})$ converging to some random variable h on more than a null-set, and hence $h'_n.\nu \to h.\nu$. By our initial convergence of $h_n.\nu$, we can conclude that $h.\nu$ is a one point measure on more than a null-set, which happens if and only if ν is a one point measure. □

Theorem 4.13. *If (X,ν) is a (G,μ)-space such that there exists a g in the support of μ with $g.\nu \neq \nu$, then (h_n) transient.*

This is Theorem 9.2 of [4], for which we will omit the proof.

Corollary 4.14. *Assume that G is not amenable, and let $\mu \in \mathcal{P}(G)$ be given such that supp(μ) generates all of G. Then (h_n) is transient*

Proof. If G is not amenable, let X be a compact G-space with no G-invariant measures in $\mathcal{P}(X)$. By Proposition 4.10 we can still find $\nu \in \mathcal{P}(X)$ such that ν is stationary with respect to μ. Then (X,ν) is a (G,μ)-space and since ν is not G-invariant, we can by Theorem 4.13 conclude that the right-handed random walk (h_n) is transient. □

Theorem 4.15. *Assume that G is not amenable and let $\mu \in \mathcal{P}(G)$ be a measure will full support. Then there exists a nontrivial μ-boundary.*

Proof. First we should notice that by Corollary 4.14, the right-handed random walk (h_n) associated to μ is transient, i.e., $h_n \to \infty$ with probability one. Let now $G' = G \cup \{\infty\}$ denote the one-point compactification of G. Then both $h_n \to \infty \in G'$ and $h_n^{-1} \to \infty \in G'$ almost surely. We can now define an action of G on G' by left multiplication with the special case of $g \cdot \infty = \infty$, for all $g \in G$. Then G' becomes a compact G-space, with $h_n.t, h_n^{-1}.t \to \infty$ almost surely, for all $t \in G'$. By compactness we can find a stationary $\nu \in \mathcal{P}(G')$ making (G',ν) a (G,μ)-space.

Our claim is now that (G',ν) is a μ-boundary. This will follow from the convergence $h_n.\nu \to \delta_\infty$, which is a one-point measure on G'. To see this let $f \in C_0(G')$ be given. Then

$$\langle h_n.\nu, f\rangle = \int_{G'} f(h_n^{-1}.t)\,\mathrm{d}\nu(t) \to \int_{G'} f(\infty)\,\mathrm{d}\nu(t) = f(\infty) = \langle \delta_\infty, f\rangle$$

on the set where $h_n \to \infty$, and thus with probability one. The above convergence can be obtain by dominated convergence, where $\|f\|_\infty$ is an integrable upper bound. This proves that (G',ν) is a μ-boundary. □

5 Rosenblatt's approach

In this section we aim to prove the remaining implication of Furstenberg's conjecture, and for this we will go through the proof that Joseph Rosenblatt provided in [9]. This proof introduces the notions of a measure $\mu \in P(G)$ being *ergodic* and *mixing by convolutions* To understand these concepts we will introduce $L_0^1(G)$ as the functions f in $L^1(G)$ such that $\int f \, \mathrm{d}\lambda = 0$.

Definition 5.1. Let μ be a regular Borel probability measure on G. We say that

- μ is ergodic by convolutions if

$$\left\| f * \frac{1}{N} \sum_{n=1}^{N} \mu^{(n)} \right\|_1 \to 0, \text{ as } n \to \infty, \text{ for all } f \in L_0^1(G).$$

- μ is mixing by convolutions if $\|f * \mu^{(n)}\| \to 0$, as $n \to \infty$, for all $f \in L_0^1(G)$.

After this definition we will discuss a couple of results concerning these definitions.

Lemma 5.2. *Let μ be a regular Borel probability measure on G. If μ is mixing by convolutions then μ is also ergodic by convolutions.*

Proof. Assume that μ is mixing by convolutions and let $\varepsilon > 0, f \in L_0^1(G)$ be given. Pick $N \in \mathbb{N}$ such that $\|f * \mu^n\| < \varepsilon/2$ for all $n \geq N$. For any $n \geq N$ we have

$$\begin{aligned} \left\| f * \left(\frac{1}{n} \sum_{k=1}^{n} \mu^{(k)} \right) \right\| &\leq \frac{1}{n} \left(\sum_{k=1}^{n} \|f * \mu^{(k)}\| \right) \\ &< \frac{1}{n} \left(\sum_{k=1}^{N-1} \|f * \mu^{(k)}\| + \sum_{k=N}^{n} \frac{\varepsilon}{2} \right) \\ &= \frac{1}{n} \sum_{k=1}^{N-1} \|f * \mu^{(k)}\| + \frac{\varepsilon}{2}. \end{aligned}$$

Picking $n \geq N$ large enough will give us $\left\| f * \left(\frac{1}{n} \sum_{k=1}^{n} \mu^{(k)} \right) \right\| < \varepsilon$, and hence μ is ergodic by convolutions. $\square$

Proposition 5.3. *Let μ be a regular Borel probability measure on G and A be the convex hull of $\{\mu^{(k)} : k \geq 1\}$. The following are equivalent:*

1. *μ is ergodic by convolutions.*

2. *there exists a sequence $(\nu_n)_{n\geq 1}$ in A such that $\|f * \nu_n\| \to 0$, for all f in $\mathcal{L}_0^1(G)$.*

3. *for all $f \in L_0^1(G)$, there exists a sequence $(\nu_n)_{n\geq 1}$ in A such that $\|f * \nu_n\|$ converges to zero, as $n \to \infty$.*

4. *for all $f \in L_0^1(G)$ and $H \in L^\infty(G)$, there exists a sequence $(\nu_n)_{n\geq 1}$ in A such that $\int (f * \nu_n) H \,\mathrm{d}\lambda \to 0$.*

5. *if $\mu * H = H$ for some $H \in L^\infty(G)$, then H is constant λ almost surely.*

6. *if $\mu * H = H$ for some continuous $H \in L^\infty(G)$, then H is constant.*

Proof. The implications *1.* $\implies$ *2.*, *2.* $\implies$ *3.*, *3.* $\implies$ *4.* and *5.* $\implies$ *6.* are clear. For *4.* $\implies$ *5.* assume that $H \in L^\infty(G)$ satisfies $\mu * H = H$. For a fixed $f \in L_0^1(G)$, pick (ν_n) in A such that $\int (f * \nu_n)\tilde{H} \,\mathrm{d}\lambda \to 0$. Then

$$\begin{aligned}\int (f * \nu_n)\tilde{H} \,\mathrm{d}\lambda &= (f * \nu_n) * H(e) = f * (\nu_n * H)(e) \\ &= f * H(e) = \int f\tilde{H} \,\mathrm{d}\lambda,\end{aligned}$$

and hence $\int f\tilde{H} \,\mathrm{d}\lambda = 0$. As $f \in L_0^1(G)$ was arbitrary we can conclude that $\tilde{H}$ is constant λ almost surely and so is H. We will omit the proof of *6.* $\implies$ *1.* in this project. $\square$

We will now turn to the connection between ergodic measures and the conjecture in question. Following the previous section, we need to show that if G is amenable, then there exists a measure with full support and only trivial μ-boundaries. For this let us consider the relation between amenability of a group and the existence of an ergodic measure, as established by Rosenblatt ([9], Theorem 1.10). This is the main result of the section.

Theorem 5.4. *Let G be a σ-compact amenable locally compact group. Then there exists an probability measure μ on G which is mixing by convolutions. Furthermore μ is absolutely continuous with respect to λ, and the corresponding density is symmetric.*

Before proving this theorem we introduce a small lemma. This lemma will give us a way of characterising whether a measure is mixing by convolutions or not. We will say that a measure μ is *spread-out* if there exists $n \geq 1$ such that $\mu^{(n)}$ and λ are not mutually singular.

Lemma 5.5. *If μ is a regular spread-out Borel probability measure on G, then μ is mixing by convolutions if and only if*

$$\|\delta_g * \mu^{(n)} - \mu^{(n)}\|_1 \to 0 \quad \text{as } n \to \infty \quad \text{for all } g \in G$$

Proof. Assume that $\|\delta_g * \mu^{(n)} - \mu^{(n)}\|_1 \to 0$ as $n \to \infty$, for all $g \in G$. If $f \in L^1(G)$ and $h = f * \delta_g - f$ for some $g \in G$, then

$$\|h * \mu^{(n)}\|_1 \leq \|f\|_1 \cdot \|\delta_g * \mu^{(n)} - \mu^{(n)}\|_1,$$

so by assumption we can conclude $\|h * \mu^{(n)}\|_1 \to 0$ as $n \to \infty$. To prove that μ is mixing by convolutions, it suffices to show that

$$A = \{f * \delta_g - f \mid f \in L^1(G), g \in G\}$$

spans a dense subset of $L^1_0(G)$. Notice first that $H := \text{span}\{A\} \subset L^1_0(G)$ and assume now in order to reach a contradiction that there exists $f_0 \in L^1_0(G)$, but not in the closure of H. By the Hahn-Banach Theorem there exists positive bounded linear functional $\Phi : L^1(G) \to \mathbb{C}$ such that $\Phi(f_0) = 1$ and Φ is constantly zero on H. This will in particular give us $\Phi(f) = \Delta(x^{-1})\Phi(f_x)$, for all $f \in L^1(G)$ and $x \in G$. By Proposition 1.1, there exists $c > 0$, such that

$$\Phi(f) = c \cdot \int f \, \mathrm{d}\lambda,$$

for all $f \in L^1(G)$, contradicting $\Phi(f_0) = 1$. In conclusion, H is dense in $L^1_0(G)$ and thus μ is mixing by convolutions.

For the converse implication assume that μ is mixing by convolutions and let $\varepsilon > 0$ and $g \in G$ be given. For each $n \in \mathbb{N}$ let $\mu^{(n)} = \alpha_n \cdot \lambda + \beta_n$ be the Lebesgue decomposition of $\mu^{(n)}$. As μ is spread-out, we can find $n \geq 1$ such that $\mu^{(n)}$ and λ are not mutually singular. From here on, we see that $\|\beta_{mn}\| \to 0$ as $m \to \infty$, and hence also $\|\beta_n\| \to 0$ as $n \to \infty$. Let now $n_0 \in \mathbb{N}$ be given such that $\|\mu^{(n)} - \alpha_n\|_1 < \varepsilon/3$ for $n \geq n_0$ and let $n_1 \geq n_0$ be given such that $\|(\delta_g * \alpha_{n_0} - \alpha_{n_0}) * \mu^{(nn_0)}\|_1 < \varepsilon/3$ for $n \geq n_1$. For $n \geq n_1$ we now have

$$\begin{aligned}
&\|\delta_g * \mu^{(nn_0+n_0)} - \mu^{(nn_0+n_0)}\|_1 \\
&\quad = \|(\delta_g * (\alpha_{n_0} + \mu^{(n_0)} - \alpha_{n_0}) - (\alpha_{n_0} + \mu^{(n_0)} - \alpha_{n_0})) * \mu^{(nn_0)}\|_1 \\
&\quad \leq \|(\delta_g * \alpha_{n_0} - \alpha_{n_0}) * \mu^{(nn_0)}\|_1 + 2\|\mu^{(n_0)} - \alpha_{n_0}\|_1 \|\mu^{(nn_0)}\| \\
&\quad < \varepsilon.
\end{aligned}$$

For any $k \geq (n_1+1)n_0$ we then obtain $\|\delta_g * \mu^{(k)} - \mu^{(k)}\|_1 < \varepsilon$, and thus

$$\|\delta_g * \mu^{(n)} - \mu^{(n)}\|_1 \to 0 \text{ as } n \to \infty.$$

As $g \in G$ was arbitrary, this lets us conclude the desired result. □

Proof of Theorem 5.4. The strategy is to construct a symmetric positive function $f \in L^1(G)$ such that $\int f \, \mathrm{d}\lambda = 1$ and $\|\delta_g * f^{(n)} - f^{(n)}\|_1 \to 0$ as $n \to \infty$ for all $g \in G$. The proof will include some technical details, both in the construction of f and in the proof of the convergence.

Preliminary considerations: Before we start on the construction of f, let us first investigate some consequences of the amenability of G. Since G is amenable, we know that G satisfies Følner's condition, that is, for any $\varepsilon > 0$ and compact set $K \subset G$, there exists a symmetric compact set $S \subset G$ with strictly positive measure such that $\lambda(gS\Delta S) < \lambda(S)\varepsilon$, for all $g \in K$. That S can be picked to be compact and symmetric follows from Remark 3.11.

Consider now $\alpha, \varepsilon > 0$ and $K \subset G$ compact. As explained above choose a compact and symmetric set $S \subset G$, such that $\lambda(gS\Delta S) < (\lambda(S) \cdot \varepsilon)/\alpha$, for all $g \in K$ and define

$$\mu_k = \alpha 1_K/\lambda(K), \quad \mu_S = 1_S/\lambda(S).$$

Calculating the norm of $\mu_K * \mu_S - \alpha\mu_S$ will then give us

$$\begin{aligned}
&\|\mu_K * \mu_S - \alpha\mu_S\|_1 \\
&\qquad = \int \left| \int \frac{\alpha}{\lambda(K)\lambda(S)} 1_S(t^{-1}s) 1_K(t) \, \mathrm{d}\lambda(t) - \frac{\alpha}{\lambda(S)} 1_S(s) \right| \mathrm{d}\lambda(s) \\
&\qquad = \int \left| \int_K \frac{\alpha}{\lambda(K)\lambda(S)} 1_S(t^{-1}s) - 1_S(s) \, \mathrm{d}\lambda(t) \right| \mathrm{d}\lambda(s) \\
&\qquad \leq \int \int_K \frac{\alpha}{\lambda(K)\lambda(S)} |1_{tS}(s) - 1_S(s)| \, \mathrm{d}\lambda(t) \, \mathrm{d}\lambda(s) \\
&\qquad = \int_K \int \frac{\alpha}{\lambda(K)\lambda(S)} 1_{tS\Delta S}(s) \, \mathrm{d}\lambda(s) \, \mathrm{d}\lambda(t) \\
&\qquad < \int_K \frac{\varepsilon}{\lambda(K)} \, \mathrm{d}t = \varepsilon.
\end{aligned}$$

A similar argument can be applied to any number of compact sets $K_1, \ldots, K_n$ and $\alpha_1, \ldots, \alpha_n, \varepsilon > 0$, i.e., there exists a compact set $S \subset G$ with

$$\|\mu_{K_1} * \cdots * \mu_{K_n} * \mu_S - \alpha_1 \ldots \alpha_n \mu_S\|_1 < \varepsilon,$$

where μ_{K_i} and μ_S are defined as before. To do this apply Følner's condition to the compact set $K_1 K_2 \cdots K_n$, with $\varepsilon/(\alpha_1 \cdots \alpha_n)$ as the *bound*. That this suffices comes from the inequality

$$\begin{aligned}&\|\mu_{K_1} * \cdots * \mu_{K_n} * \mu_S - \alpha_1 \cdots a_n \mu_S\| \\ &\leq \prod_{i=1}^{n} \frac{\alpha_i}{\lambda(K_i)} \int_G \int_{K_1} \cdots \int_{K_n} |\mu(g_n^{-1} \cdots g_1^{-1} s) - \mu(s)| \, \mathrm{d}\lambda(g_n) \cdots \mathrm{d}\lambda(g_1) \, \mathrm{d}\lambda(s),\end{aligned}$$

so if $\int |\mu(t^{-1}s) - \mu(s)| \, \mathrm{d}\lambda(s) < \varepsilon/(\alpha_1 \cdots \alpha_n)$, for all $t \in K_1 \cdots K_n$, then the integral above is less that ε.

The construction of the function: For the construction, let $(\varepsilon_n), (\gamma_n)$ be sequences of strictly positive real numbers such that $\sum_{n=1}^{\infty} \varepsilon_n = 1$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$ and use σ-compactness of G to pick compact subsets

$$F_1 \subset F_2 \subset F_3 \subset \ldots$$

of G with $\bigcup F_n = G$. Notice that for any $k \geq 1$ we have $\sum_{n=1}^{k} \varepsilon_n < 1$, so we are able to pick a strictly increasing sequence (p_m) in $\mathbb{N}$ such that

$$\sum_{k=2}^{m-1} \left(\sum_{n=1}^{k} \varepsilon_l \right)^{p_m} \to 0 \quad \text{as} \quad m \to \infty.$$

Our next step will be to choose a sequence (S_m) of symmetric, compact subsets in G with strictly positive measure satisfying some desirable properties. To describe these properties we will for a short while pretend that these sets are chosen and then return to the process of choice later on. For each $m \in \mathbb{N}$ let $\mu_m = \varepsilon_m 1_{S_m} / \lambda(S_m)$. For $r \geq 2$ and $m \in \mathbb{N}$ consider the set

$$\Pi_{r,m} = \{\mu_{i_1} * \cdots * \mu_{i_r} \mid \mu_{i_l} \in \{\mu_1, \ldots, \mu_m\}, \exists 1 \leq l \leq r : \mu_{i_l} = \mu_m\},$$

i.e., the collection of products of r elements from $\{\mu_1, \ldots, \mu_m\}$, with at least one of the terms being μ_m. For each such $\pi \in \Pi_{r,m}$ let $j(\pi) < r$ denote the number of terms in π not equal to μ_m and $J(\pi) < r$ be equal to the number of terms in π before the first occurrence of μ_m. We will use j, J respectively when π is understood. In this way, for any $\pi \in \Pi_{r,m}$ we have

$$\pi - \left(\prod_{t=1}^{J} \varepsilon_{i_t} \right) \mu_m * \prod_{t=J+2}^{r} \mu_{i_t} = (\mu_{i_1} * \cdots * \mu_{i_J} * \mu_m - \varepsilon_{i_1} \cdots \varepsilon_{i_J} \mu_m) * \prod_{t=J+2}^{r} \mu_{i_t},$$

so by integration over G we obtain

$$\left\| \pi - \left(\prod_{t=1}^{J} \varepsilon_{i_t} \right) \mu_m * \prod_{t=J+2}^{r} \mu_{i_t} \right\|_1 \leq \|\mu_{i_1} * \cdots * \mu_{i_J} * \mu_m - \varepsilon_{i_1} \cdots \varepsilon_{i_J} \mu_m\|_1. \quad (1)$$

It is now time to pick the sequence of S_m's, so start out by letting S_1 be some symmetric, compact subset of G with $\lambda(S_1) > 0$. The remaining sets will be picked inductively in the following way. Given $\{S_1, \ldots, S_{m-1}\}$ we will pick a compact and symmetric set S_m such that for all $\pi \in \Pi_{r,m}$

$$\left\| \pi - \left(\prod_{t=1}^{J} \varepsilon_{i_t} \right) \mu_m * \prod_{t=J+2}^{r} \mu_{i_t} \right\|_1 < \frac{\gamma_m}{m^{p_m}},$$

where $2 \le r \le p_m$. We will also want this S_m to satisfy

$$\lambda(gS_m \Delta S_m) < \lambda(S_m) \cdot \gamma_m.$$

To do this we will once again apply Følner's condition, but first we should find an appropriate compact set K_m and an upper bound ε_m. For the first requirement on S_m, for each $\pi \in \Pi_{r,m}$ given as $\pi = \mu_{i_1} * \cdots * \mu_{i_n}$, we let

$$K_\pi = K_{i_1} \cdots K_{i_J},$$

i.e., the product of the compact sets appearing on the right hand side of (1). With these K_π's chosen we will define

$$K_m = F_m \cup \bigcup_{r=2}^{p_m} \bigcup_{\pi \in \Pi_{r,m}} K_\pi,$$

and apply Følner's condition to this compact set with $\varepsilon_m = \gamma_m / m^{p_m}$. In this way our set S_m will satisfy $\lambda(gS_m \Delta S_m) < \lambda(S_m) \cdot \gamma_m$, for all $g \in F_m$, but also

$$\left\| \pi - \left(\prod_{t=1}^{J} \varepsilon_{i_t} \right) \mu_m * \prod_{t=J+2}^{r} \mu_{i_t} \right\|_1 < \frac{\varepsilon_{i_1} \cdots \varepsilon_{i_J} \gamma_m}{m^{p_m}} < \frac{\gamma_m}{m^{p_m}}$$

for all $\pi \in \Pi_{r,m}$ with $2 \le r \le p_m$. Notice furthermore that the F_m's are increasing, so we will also have

$$\lambda(gS_m \Delta S_m) < \gamma_m \cdot \lambda(S_m),$$

for all $g \in F_n$, whenever $m \ge n$.

With these sets in mind, we will now define $f \in L^1(G)$ by $f = \sum_{n=1}^{\infty} \mu_n$ and claim that this is the desired function. The function f inherits positivity and symmetry directly from the μ_k's and by our way of choosing (ε_k) we also get $\int f \, \mathrm{d}\lambda_G = 1$. We are now left with showing

$$\|\delta_g * f^{(n)} - f^{(n)}\|_1 = \|(\delta_g - \delta_e) * f^{(n)}\|_1 \to 0$$

for all $g \in G$. For this, let $g \in G$ be given and $m_0 \in \mathbb{N}$ be such that $g \in F_{m_0}$ and thus in F_m for any $m \geq m_0$. Let furthermore $r \geq 2$ be given, such that using our notation from above

$$(\delta_g - \delta_e) * f^{(r)} = (\delta_g - \delta_e) * \mu_1^{(r)} + \sum_{k=2}^{\infty} \sum_{\pi \in \Pi_{r,k}} (\delta_g - \delta_e) * \pi.$$

For a fixed $m \geq m_0$ we will now split up the above sum into two sums

$$\sum_{k=2}^{\infty} \sum_{\pi \in \Pi_{r,k}} (\delta_g - \delta_e) * \pi = \sum_{k=2}^{m-1} \sum_{\pi \in \Pi_{r,k}} (\delta_g - \delta_e) * \pi + \sum_{k=m}^{\infty} \sum_{\pi \in \Pi_{r,k}} (\delta_g - \delta_e) * \pi,$$

which we will denote by Σ_1 and Σ_2, respectively. Let us now try and find some useful upper bounds for these sums. Given $\pi \in \Pi_{r,k}$ either $\pi = \mu_k^{(r)}$, such that $\|\pi\|_1 \leq \|\mu_k\|_1^r$ or else $j > 0$ and then $\|\pi\|_1 \leq \|\mu_{i_{l_1}}\|_1 \cdots \|\mu_{i_{l_j}}\|_1 \|\mu_k\|^{r-j}$. With this in mind a quick count of possible combinations in $\Pi_{r,k}$ reveals that

$$\begin{aligned}
\sum_{\pi \in \Pi_{r,k}} \|\pi\|_1 &\leq \sum_{k=0}^{r-1} \binom{r}{j} \left(\sum_{l=1}^{k-1} \|\mu_l\|_1 \right)^j \|\mu_k\|_1^{r-j} \\
&\leq \left(\sum_{l=1}^{k} \|\mu_l\|_1 \right)^r = \left(\sum_{l=1}^{k} \varepsilon_l \right)^r.
\end{aligned}$$

Next we notice that $\|(\delta_g - \delta_e) * \pi\|_1 \leq \|\delta_g * \pi\|_1 + \|\pi\|_1 = 2\|\pi\|_1$, and hence

$$\|\Sigma_1\|_1 \leq 2 \sum_{k=1}^{m-1} \left(\sum_{l=1}^{k} \varepsilon_l \right)^r.$$

For the bound of $\|\Sigma_2\|_1$ let $k \geq m$ be given and assume furthermore that our fixed r satisfy $2 \leq r \leq P_k$. From our choice of the sequence (S_m) we get

$$\left\| \pi - \prod_{t_0}^{J} \varepsilon_{i_t} \mu_k * \prod_{t=J+2}^{r} \mu_{i_t} \right\|_1 < \gamma_k / k^{P_k},$$

so by combining these results we see that

$$\begin{aligned}
&\left\| \sum_{\pi \in \Pi_{r,k}} (\delta_g - \delta_e) * \pi \right\|_1 \\
&\quad \leq \left\| \sum_{\pi \in \Pi_{r,k}} (\delta_g - \delta_e) * \prod_{t=0} \varepsilon_{i_t} \mu_k * \prod_{t=J+2}^{r} \mu_{i_t} \right\|_1 + 2 \sum_{\pi \in \Pi_{r,k}} \gamma_k / k^{P_k}.
\end{aligned}$$

Since $k \geq m$ we also get $\|(\delta_g - \delta_e) * \mu_k\|_1 < \gamma_k \|\mu_k\|_1$ from our discussion before, and using $\|\mu_i\|_1 = \varepsilon_i$ we obtain

$$\left\| \sum_{\pi \in \Pi_{r,k}} (\delta_g - \delta_e) * \pi \right\|_1 \leq \sum_{\pi \in \Pi_{r,k}} \gamma_k \|\mu_{i_{l_1}}\|_1 \cdots \|\mu_{i_{l_j}}\|_1 \|\mu_k\|_1^{r-j} + 2\gamma_k.$$

By the same combinatorial argument from before we then get

$$\|\Sigma_2\|_1 \leq \sum_{k=m}^{\infty} \left(\gamma_k \left(\sum_{l=1}^{k} \varepsilon_l \right)^r + 2\gamma_k \right) \leq 3 \sum_{k=m}^{\infty} \gamma_k.$$

Combining these arguments we will have for $m \geq m_0$ and $2 \leq r \leq p_m$

$$\|(\delta_g - \delta_e) * f^{(r)}\|_1 \leq 2\varepsilon_1^r + 2 \sum_{k=2}^{m-1} \left(\sum_{l=1}^{k} \varepsilon_l \right)^r + 2 \sum_{k=m}^{\infty} \gamma_k.$$

So if we let $r = p_m$ every time, we get $\lim_{m \to \infty} \|(\delta_g - \delta_e) * f^{(p_m)}\|_1 = 0$. From here we only need $\|f\|_1 \leq 1$ to obtain $\|(\delta_g - \delta_e) * f^{(n)}\|_1 \to 0$ as $n \to \infty$, but $\|f\|_1 = \int |f| \, \mathrm{d}\lambda = \int f \, \mathrm{d}\lambda = 1$ so this is not an issue.

To finish the proof, we will now let μ be the probability measure on G, having f as density with respect to λ. Then μ is clearly absolutely continuous with respect to λ and with symmetric density. By absolute continuity, μ is also spread-out, and since

$$\delta_g * \mu^{(n)} - \mu^{(n)} = \delta_g * f^{(n)} - f^{(n)},$$

for all $g \in G$, we can use Lemma 5.5 and the convergence above to conclude that μ is mixing by convolutions. □

The above theorem will now be the starting point of the series of arguments leading to the proof of the necessity condition in Furstenberg's conjecture. That is, if G is amenable there exists a measure μ on G with full support and only trivial μ-boundaries. The first thing we will do is to introduce, the so-called μ-harmonic functions and then connect this concept to the measure obtained in Theorem 5.4.

Definition 5.6. Let μ be a bounded, regular measure Borel measure on G. We say that $f \in L^\infty(G)$ is μ-harmonic if

$$f(x) = \int f(xt) \, \mathrm{d}\mu(t), \quad \text{for all } x \in G$$

In view of Theorem 5.4 we will in the following focus on measures $\mu \in \mathcal{P}(G)$ that are absolutely continuous with respect to λ. Such a measure μ will have density $\varphi \in \mathrm{Prob}(G)$. We will also focus on measures with a symmetric density φ, i.e., $\varphi = \tilde{\varphi}$.

Proposition 5.7. *Let $\mu \in \mathcal{P}(G)$ be absolutely continuous with respect to λ and with symmetric density $\varphi \in Prob(G)$. Then for $f \in L^\infty$ the following are equivalent*

1. $\mu * \tilde{h} = \tilde{h}$
2. $h * \mu = h$
3. *h is μ-harmonic*

Proof. The equivalence of $h * \mu = h$ and h being μ-harmonic follows since

$$\begin{aligned}(h * \mu)(s) = (h * \varphi)(s) &= (h * \tilde{\varphi})(s) \\ &= \int h(t)\tilde{\varphi}(t^{-1}s)\,\mathrm{d}\lambda(t) \\ &= \int h(t)\varphi(s^{-1}t)\,\mathrm{d}\lambda(t) \\ &= \int h(st)\varphi(t)\,\mathrm{d}\lambda(t) = \int h(st)\,\mathrm{d}\mu(t)\end{aligned}$$

for any $s \in G$. That $\mu * \tilde{h} = \tilde{h}$ and $h * \mu = h$ are equivalent can be obtained through the identities

$$\mu * \tilde{h} = \varphi * \tilde{h} = \tilde{\varphi} * \tilde{h} = \widetilde{h * \varphi} = \widetilde{h * \mu}.$$

which can be easily checked. ☐

Remark 5.8. We note that if $\mu \in \mathcal{P}(G)$ satisfies the hypothesis of Proposition 5.7, then a function $h \in L^\infty(G)$, which is μ-harmonic will automatically belong to $UCB_\ell(G)$ by Lemma 2.4.

From here we will now link the μ-harmonic functions in $UCB_\ell(G)$ to the μ-boundaries. This will be done through a certain (G, μ)-space called the universal μ-boundary, also defined in [4].

Definition 5.9. Let $\mu \in \mathcal{P}(G)$ be given. We say that a (G, μ)-space (X, ν) is a *universal* μ-boundary if any other μ-boundary can be obtained as an equivariant image of (X, ν).

It is proved in [4] (see Thm 10.1) that for each pair (G, μ), there exists a universal μ-boundary and if, moreover, this satisfies a certain minimality condition, then it is unique up to isomorphism. We omit the details of the proof.

It is clear from the definition that if the universal μ-boundary is trivial, then so is any other μ-boundary.

The importance of the concept of universal μ-boundary arises in connection with the Poisson representation of bounded μ-harmonic functions on G. Namely, it is proved in Theorem 12.2 in [4] that for a σ-compact locally compact group G, there is a one-to-one correspondence between μ-harmonic functions in $UCB_\ell(G)$ and continuous functions on the universal μ-boundary. As a consequence is all μ-harmonic functions in $UCB_\ell(G)$ are constant, then the universal μ-boundary is trivial.

Combining these deep results with Proposition 5.7 will now give us the following desired result:

Proposition 5.10. *Let $\mu \in \mathcal{P}(G)$ be absolutely continuous with respect to λ with symmetric density $\varphi \in Prob(G)$. If μ is ergodic by convolutions, then all μ-boundaries are trivial.*

Proof. Assume that μ is ergodic by convolutions. If $f \in L^\infty(G)$ is μ-harmonic which by Remark 5.8 will automatically belong to $UCB_\ell(G)$, then $\mu * \tilde{h} = \tilde{h}$. By the implication *1.* $\implies$ *6.* of Proposition 5.3 we can now conclude that $\tilde{h}$ and hence also h are constant. By the arguments above, this implies that the universal μ-boundary is trivial and hence so is any μ-boundary. □

We have now established that the measure obtained from Theorem 5.4 will admit only trivial μ-boundaries, but it need not have full support. What we will do though is to construct a new measure from μ with full support and no non-trivial μ-boundaries. For this we will introduce some terminology regarding regular Borel probability measures on G.

Definition 5.11. Let $\mu \in \mathcal{P}(G)$. We say that

- μ is balanced if $\mu(A) = \mu(A^{-1})$ for all $A \in \mathcal{B}(G)$.
- μ is adapted if the subgroup generated by $\text{supp}(\mu)$ is dense in G.
- μ is faithful if $\text{supp}(\mu) = G$.

If we let $\tilde{\mu}$ denote the measure $\tilde{\mu}(A) = \mu(A^{-1})$, for $A \in \mathcal{B}(G)$, we could say that μ is balanced if $\mu = \tilde{\mu}$. Our goal is now to find a way of obtaining a faithful measure.

Remark 5.12. It is stated (without proof) in [9] that whenever $\mu \in \mathcal{P}(G)$ is ergodic by convolutions, then μ is adapted. A proof of this result can be found in [6] (see Corollary 4.2.6.), using the theory of a Poisson boundary on μ, which corresponds to the universal μ-boundary in the measure theoretic setting.

Hence, we now know that the ergodic measure we obtained from the amenability of G is adapted, so let us see how this can be used to construct a faithful measure.

Proposition 5.13. *Let μ be a regular Borel probability measure on G and set $\nu = \sum_{n=1}^{\infty} 2^{-n}\mu^{(n)}$.*

1. *If μ is balanced, then $\bigcup_{n=1}^{\infty} \mathrm{supp}\big(\mu^{(n)}\big)$ is a subgroup of G.*
2. *If μ is adapted and balanced, then ν is faithful.*

Proof. 1. We start out by noticing that $\mathrm{supp}(\tilde{\mu}) = \mathrm{supp}(\mu)^{-1}$, so if μ is balanced, then $\mathrm{supp}(\mu) = \mathrm{supp}(\mu) \cup \mathrm{supp}(\mu)^{-1}$ is a symmetric subset of G. Note also that if μ is balanced then $\mu^{(n)}$ is balanced, for all $n \geq 1$. To finish the proof we want to show that $\mathrm{supp}(\mu)^2 \subset \mathrm{supp}(\mu^{(2)})$, and in general $\mathrm{supp}(\mu_1) \cdot \mathrm{supp}(\mu_2) \subset \mathrm{supp}(\mu_1 * \mu_2)$. Let first U, V be Borel sets in G and notice that

$$\begin{aligned}\mu(U) \cdot \nu(V) &= \int_{U \times V} 1_{UV}(xy)\ \mathrm{d}\mu(x)\,\mathrm{d}\nu(y) \\ &\leq \int 1_{UV}(xy)\ \mathrm{d}\mu(x)\,\mathrm{d}\nu(y) = (\mu * \nu)(UV).\end{aligned}$$

Let now $x \in \mathrm{supp}(\mu_1)$ and $y \in \mathrm{supp}(\mu_2)$ be given and choose an open neighbourhood W of $xy \in G$. We can now pick open neighbourhoods $x \in U$ and $y \in V$ such that $UV \subset W$ and hence

$$0 < \mu_1(U)\mu_2(V) \leq (\mu_1 * \mu_2)(UV) \leq (\mu_1 * \mu_2)(W),$$

so $xy \in \mathrm{supp}(\mu_1 * \mu_2)$. Combining these properties of $\mathrm{supp}(\mu)$ will let us conclude that $\bigcup_{n=1} \mathrm{supp}(\mu^{(n)})$ is a subgroup of G.

2. For this part notice first that $\mathrm{supp}(\nu)$ is closed and

$$\bigcup_{n=1}^{\infty} \mathrm{supp}(\mu^n) \subset \mathrm{supp}(\nu).$$

Since μ is adapted and balanced, $\mathrm{supp}(\nu)$ must be equal to G. □

Now this result concerns balanced measures, but the measure μ obtained in Theorem 5.4 is not necessarily balanced. If $\varphi \in \mathrm{Prob}(G)$ is the symmetric density of μ with respect to λ, we see by integration that

$$\mathrm{d}\tilde{\mu}(x) = \varphi(x)\Delta(x^{-1})\,\mathrm{d}\lambda(x) = \Delta(x^{-1})\,\mathrm{d}\mu(x), \text{ for } x \in G.$$

Here we have used that $\mathrm{d}\tilde{\lambda}(x) = \Delta(x^{-1})\,\mathrm{d}\lambda(x)$ for $x \in G$, which can be found in [3] (see Proposition 11.14). Since $\Delta > 0$ we see that $\tilde{\mu}$ and μ agree on null-sets, so we still have $\mathrm{supp}(\mu) = \mathrm{supp}(\tilde{\mu})$. Repeating the proof above will then tell us that the results in Proposition 5.13 hold true for μ, as well.

At this point we have shown that the measure μ we constructed in Theorem 5.4 gives rise to a measure $\nu = \sum_{k=1}^{\infty} 2^{-k}\mu^{(k)}$ with full support. To finish the prove of Furstenberg's conjecture, we should proof that ν only admits trivial ν-boundaries. In accordance to the results proved earlier in this section it suffices to show that ν is absolutely continuous with respect λ, has a symmetric density and is mixing by convolutions.

Proposition 5.14. *Let $\mu \in \mathcal{P}(G)$ be given, such that μ is absolutely continuous with respect to λ, has symmetric density and is mixing by convolutions. Then $\nu \in \mathcal{P}(G)$ defined by $\nu = \sum_{k=1}^{\infty} 2^{-k}\mu^{(k)}$ will inherit all of these properties as well.*

Proof. Let us start out by showing that ν is absolutely continuous with respect to λ. To do this we should notice that if τ, τ' are both absolutely continuous with respect to λ and φ, φ', are the respective densities, then $\tau * \tau'$ is absolutely continuous with respect to λ, and has density $\varphi * \varphi'$. This correlation between convolutions of measures and convolution of densities will let us conclude that ν is absolutely continuous with respect to λ. The density of ν will furthermore be given by

$$g = \sum_{k=1}^{\infty} 2^{-k} f^{(k)},$$

where f is the density of μ. That this density is symmetric follows from the identity $\tilde{\varphi} * \tilde{\psi} = \widetilde{\psi * \varphi}$ and since f commutes with itself

$$\tilde{g} = \sum_{k=1}^{\infty} 2^{-k} \left(\tilde{f}\right)^{(k)} = \sum_{k=1}^{\infty} 2^{-k} f^{(k)} = g.$$

Finally we should show that ν is mixing by convolutions, so let $f \in L_0^1(G)$ be given. As μ was mixing by convolutions we know that $\|f * \mu^{(n)}\| \to 0$, as

$n \to \infty$. The next thing to notice is that

$$\mu * \sum_{k=1}^{\infty} 2^{-k} \mu^{(k-1)} = \nu = \left(\sum_{k=1}^{\infty} 2^{-k} \mu^{(k-1)} \right) * \mu,$$

so when raised to the power n we get

$$\nu^{(n)} = \mu^{(n)} * \left(\sum_{k=1}^{\infty} 2^{-k} \mu^{(k-1)} \right)^n .$$

The measure multiplied onto $\mu^{(n)}$ above is a probability measure, so we can add f into the equation and obtain

$$\|f * \nu^{(n)}\| = \left\| f * \mu^{(n)} * \left(\sum_{k=1}^{\infty} 2^{-k} \mu^{(k-1)} \right)^n \right\| \leq \|f * \mu^{(n)}\|,$$

where $\mu^{(0)} = \delta_e$. Letting $n \to \infty$ will then give us $\|f * \nu^{(n)}\| \to \infty$ and hence ν is mixing by convolutions. □

We have now seen that amenability of G will give us a measure $\nu \in \mathcal{P}(G)$ with full support that only admits trivial ν-boundaries, so we have proved the remaining part of Furstenberg's conjecture.

As a final remark in this project we wish to show that the converse statement of Theorem 5.4 is true, as well. It turns out that if $\mu \in \mathcal{P}(G)$ is absolutely continuous with respect to λ, has symmetric density and is mixing by convolutions then G is both amenable and σ-compact. Showing that the existence of such a probability measure ensures amenability of the group is a matter of combining ergodicity with strong left invariance of a a certain net.

Proposition 5.15. *Assume that there exists a regular Borel probability measure μ on G such that μ is ergodic by convolutions. Then G is amenable.*

Proof. Let $f \in \mathrm{Prob}(G)$ be given and define $\nu_n = f * \left(\frac{1}{n} \sum_{k=1}^{n} \mu^{(k)}\right)$ for all $n \geq 1$. Then $\nu_n \in L^1(G)$ and $\nu_n \geq 0$. Furthermore for any $g \in \mathrm{Prob}(G)$,

$$\begin{aligned}\int g * \mu(s) \, \mathrm{d}\lambda(s) &= \int \int g(t^{-1}s) \, \mathrm{d}\mu(t) \, \mathrm{d}\lambda(s) = \int \int g(t^{-1}s) \, \mathrm{d}\lambda(s) \, \mathrm{d}\mu(t) \\ &= \int \int f(s) \, \mathrm{d}\lambda(s) \, \mathrm{d}\mu(t) = \int 1 \, \mathrm{d}\mu(t) = 1,\end{aligned}$$

and hence $\nu_n \in \text{Prob}(G)$ for all $n \in \mathbb{N}$. For any $g \in G$ the function $\delta_g * f - f$ belongs to $L^1_0(G)$ by left invariance of λ and hence $\|\delta_g * \nu_n - \nu_n\|_1 \to 0$ as n tends to infinity. Then $(\nu_n)_{n\geq 1}$ converges strongly to left invariance and hence G is amenable. □

That the group G will also be σ-compact will follow from the adaptedness of the measure μ, as discussed in Remark 5.12. The arguments are as follows.

Proposition 5.16. *Let μ be a faithful regular Borel probability measure on G. Then every open subgroup H of G has countable index.*

Proof. Assume that H is an open subgroup of G and let $I \subset G$ be a transversal set for H, i.e., G is a disjoint union of the sets tH, for $t \in I$. Since each tH is open and non-empty we have $\mu(tH) > 0$ by faithfulness of μ. Then

$$\begin{aligned}\sum_{t\in I} \mu(tH) &= \sup\left\{\sum_{t\in J} \mu(tH) : J \subset I \text{ finite}\right\} \\ &= \sup\left\{\mu\left(\bigcup_{t\in J} tH\right) : J \subset I \text{ finite}\right\} \\ &\leq \mu(G) = 1\end{aligned}$$

This gives us a convergent sum with strictly positive terms, so there can be at most countably many terms. In conclusion I is countable, and hence H has countable index in G. □

Proposition 5.17. *Let μ be a regular Borel probability measure on G. If μ is adapted then G is σ-compact.*

Proof. Let ρ be the measure defined by $\rho = \frac{1}{2}(\mu + \tilde{\mu})$. Then ρ is adapted and balanced so let ν be the measure defined by $\nu = \sum_{n=1}^{\infty} 2^{-n}\rho^{(n)}$. Then ν is faithful. Let now U be a precompact open neighbourhoood of $e \in G$. Then $H := \bigcup_{n=1}^{\infty} U^n$ is an open subgroup of G and hence

$$G = \bigcup_{t\in I} \bigcup_{n=1}^{\infty} tU^n$$

is a countable union. By the choice of U, $K = \overline{U}$ is compact, so tK^n is compact, for all $t \in I$ and $n \geq 1$. This allows us to write G as a countable union of compact sets namely

$$G = \bigcup_{t\in I} \bigcup_{n\geq 1} tK^n,$$

so G is σ-compact. □

In conclusion this will allows us reformulate Theorem 5.4 as an if and only if statement. More precisely we have shown that

> *If G is a locally compact group, then there exists an absolutely continuous measure $\mu \in \mathcal{P}(B)$ with symmetric density which is ergodic by convolutions if and only if G is σ-compact and amenable.*

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