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Emilie Mai Elkiær

Rigidity of group actions on Banach spaces

Department of Mathematical Sciences, University of Copenhagen

Advisor: Magdalena Musat

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Abstract

Kazhdan's Property (T) is a fundamental notion in the study of groups and has applications in a number of different fields of mathematics. It is initially defined in terms of unitary representations on Hilbert spaces, but was shown to be equivalent with a fixed-point property for affine actions of the group on Hilbert spaces. In this thesis, we study fixed-point and rigidity properties in the broader context of actions of groups on Banach spaces. We study the relations between two such properties for group actions on L^p-spaces – a class of Banach spaces where many of the tools from the study of actions on Hilbert spaces also applies. Furthermore, we study spectral conditions for Property (T) and for the related fixed-point property for actions on L^p-spaces.

Resumé

Kazhdan's egenskab (T) er et fundamentalt begreb i studiet af grupper og har anvendelser i en række forskellige grene af matematikken. Egenskab (T) er oprindeligt defineret i termer af unitære representationer på Hilbertrum, men viste sig at være ækvivalent med en fikspunkt-segenskab for affine gruppevirkninger på Hilbertrum. I dette speciale studerer vi fikspunkts- og rigide egenskaber i den bredere kontekst af gruppevirkninger på Banachrum. Vi studerer relationerne mellem to sådanne egenskaber for gruppervirkninger på L^p-rum – en klasse af Banachrum hvor mange af værktøjerne fra studiet af gruppevirkninger på Hilbertrum også kan benyttes. Ydermere studerer vi spectrale betingelser for egenskab (T) og for dens relaterede fikspunktsegenskab for gruppevirkninger på L^p-rum.

Contents

Introduction

Property (T) was introduced by D. Kazhdan in his paper [26] from 1967 and concerns how a group may act on a Hilbert space. It has proved to be a fundamental notion with applications in a number of different fields such as geometric group theory, ergodic theory, operator algebras and combinatorics among others. Despite its relevance, it took almost 40 years from its introduction until the study of similar rigidity properties for groups in the broader framework of Banach spaces was initiated. The first paper in this direction, by D. Fisher and G. Margulis in 2005 (see [18]) considered actions of Kazhdan groups on L^p-spaces. Property (T_X) and (F_X) were then introduced as generalizations of Property (T) by U. Bader, A. Furman, T. Gelander and N. Monod in 2007 (see [2]). In their paper, relations between these properties for superreflexive Banach spaces, and in particular for L^p-spaces, are studied systematically.

Concrete examples of infinite groups with Property (T) are sparse and often complicated. A celebrated result by A. Żuk gives a sufficient condition for Property (T) in terms of the spectral properties of the *link graph* associated to a generating set (see [55]). In the same paper, A. Żuk gives an application of this result to random groups showing that, though they may be hard to construct, infinite groups with Property (T) are abundant. It is beyond the scope of this thesis to study random groups, but this application does motivate studying Żuk's condition and, further, the question if there exists a similar condition for the Banach space versions of Property (T). Very resently, in [14], T. de Laat and M .de la Salle gave such a sufficient condition for Property (F_{L^p}).

Property (T) has many equivalent reformulations, and in Chapters 1 to 3, we study a number of the different forms it takes. We hereby acquire a range of tools that we shall see in Chapter 4 are applicable when studying Banach space relatives of Property (T) for the class of superreflexive Banach spaces. We include an introduction to superreflexivity of Banach spaces in Appendix A. The main goal of Chapter 4 is to present the main results of the paper [2] by U. Bader, A. Furman, T. Gelander and N. Monod. The main goal of the final chapter of this thesis is to present the mentioned sufficient condition for Property (F_{L^p}) due to T. de Laat and M .de la Salle in [14]. Central in their proofs is the use of techniques from complex interpolations, and we therefore include a short review of this theory in Appendix B.

For the purpose of this thesis, we choose to focus on discrete groups only. This is because we wish to avoid the extra layer of technicalities added when discussing more general locally compact groups. However, many of the results do hold for locally compact groups, as well. We refer the interested reader to [5] for details.

Chapter 1: We introduce Property (T) choosing to define it in terms of invariant and almost invariant vectors for unitary representations. We introduce the notion of a *Kazhdan pair* and show that Property (T) is equivalent with the existence of a finite such. Further, we introduce *Fell's topology* on the unitary dual and proof that Property (T) is equivalent with the existence of a so called *Kazhdan projection* and with the isolation of the trivial representation in the unitary dual. The latter characterization is Kazhdan's original definition.

Chapter 2: We introduce Property (FH), which is the property that all affine actions on a real Hilbert space have a fixed point. We give a thorough introduction to affine actions on general vector spaces and to the 1-cohomology of a group with coefficients in a linear isometric representation on a Banach space. We also give an introduction to the theory of positivity of functions on groups including a famous theorem by Schoenberg. In the end of the chapter, we specialize to Hilbert spaces and show that Property (FH) is equivalent to Property (T). This is a result due to P. Delorme in [15] and A. Guichardet in [20].

Chapter 3: We discuss measure preserving group actions on probability spaces (p.m.p. actions). We introduce *ergodicity* and *strong ergodicity* of such actions and show that Property (T) is equivalent with the property that every ergodic p.m.p. action is strongly ergodic. This is a result due to A. Connes and B. Weiss in [11].

Chapter 4: We introduce two generalizations of Property (T) to the realm of Banach spaces: Property (T_X), which generalizes the definition of Property (T) in terms of invariant and almost invariant vectors, and Property (F_X), which generalizes Property (FH). We shall see that (F_X) is always stronger that (T_X), but that they are not, in general, equivalent. We further discuss their relations when X is an L^p-space.

Chapter 5: We discuss the mentioned spectral condition for Property (T) due to A. Żuk, as well as the spectral condition of the same flavor for Property (F_{L^p}) due to T. de Laat and M .de la Salle. A short introduction to graphs and their spectrum is included. Along the way, we introduce the *regular norm* of operators on (finite) ℓ^p-spaces, we introduce *p-uniform convexity*, and we discuss *Poincaré type inequalities*.

The reader is assumed to be familiar with measure theory and with functional analysis.

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Chapter 1

A rigidity property for group actions on Hilbert spaces

Property (T) was first introduced by D. A. Kazhdan in [26] and concerns actions on Hilbert spaces. We give in Section 1.2 an introduction based mainly on [9]. In Section 1.3, we study the relation between Property (T) and the trivial representation, recovering, in Proposition 1.3.18, the original definition by Kazhdan along with a characterization in terms of the existence of a so called *Kazhdan projection*. We start by reviewing the basics of the representation theory of groups on Hilbert spaces.

1.1 Representations of groups on Hilbert spaces

Definition 1.1.1. A *(linear) representation* of a discrete group Γ is a tuple (π, V) consisting of a vector space V and a map $\pi : \Gamma \to \mathrm{GL}(V)$ which is multiplicative and unital.

Even though we have formally defined a representation to be a tuple (π, V), we shall often refer to the map π as a representation. When V is a normed vector space, we say that a representation (π, V) of Γ is *isometric* if $\pi(g)$ is an isometry, for all $g \in \Gamma$. In that case we shall also refer to (π, V) as a *(linear) isometric representation* or say that π is a *representation by surjective isometries.*

Let H be a Hilbert space. If H is complex, the surjective isometries on H are referred to as *unitary* operators, and the group of all such operators are denoted $\mathcal{U}(H)$. If H is real, we refer to the surjective isometries instead as *orthogonal* operators, and the set of all such operators are denoted by $\mathcal{O}(H)$. Following this terminology, we refer to an isometric representation on a Hilbert space as a *unitary representation*, respectively, an *orthogonal representation* depending on whether the Hilbert space is complex or real

Example 1.1.2. The possibly simplest example of a unitary representation of a group Γ is the *trivial representation* $1_\Gamma : \Gamma \to \mathbb{C}$, given by

$$1_\Gamma(g) = 1, \qquad \text{for all } g \in \Gamma. \qquad \circ$$

Example 1.1.3 (A unitary representation from an action)**.** For a left action $\Gamma \curvearrowright X$ of a discrete group Γ on a set X, we may construct a unitary representation π of Γ on the Hilbert space $\ell^2(X)$ as follows: For each $g \in \Gamma$ and $x \in X$, set $\pi(g)(\delta_x) = \delta_{g.x}$. As $\Gamma \curvearrowright X$ is a left action, we see directly that $\pi : \Gamma \to \mathcal{U}(\ell^2(X))$ is a group homomorphism. $\circ$

Example 1.1.4. Any discrete group Γ has a left action on itself: The *left translation action* given by left multiplication. By Example 1.1.3, this gives rise to a unitary representation

$\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$, given by

$$\lambda_g(\delta_h) = \delta_{gh}, \qquad \text{for all } g, h \in \Gamma.$$

This important special case of Example 1.1.3 is refered to as the *left regular representation.* $\circ$

Let (π, H) be a representation of the discrete group Γ. A subspace $K \subset H$ is said to be Γ-*invariant* if $\pi(g)K \subset K$, for all $g \in \Gamma$. Any representation has the trivial Γ-invariant subspaces H and $\{0\}$. For a Γ-invariant subspace $K \subset H$, we obtain a representation of Γ on K by restriction of π. We refer to this representation as the *subrepresentation* of π defined by K. If π is unitary (orthogonal), then so is every subrepresentation.

Remark 1.1.5. If (π, H) is unitary and $K \subset H$ is Γ-invariant then $K^\perp$ is Γ-invariant, as well. Indeed, for all $\xi \in K^\perp$, $\eta \in K$ and $g \in \Gamma$, we have $\langle \pi(g)\xi\,, \eta\rangle = \langle \xi\,, \pi(g^{-1})\eta\rangle = 0$.

Definition 1.1.6. We say that a representation (π, H) of a discrete group Γ is *irreducible* if H contains no non-trivial Γ-invariant subspaces.

An *intertwining operator* between two representations (π, H) and (ρ, K) of a discrete group Γ is a bounded linear operator $T : H \to K$ satisfying $T\pi(g) = \rho(g)T$, for all $g \in \Gamma$.

Remark 1.1.7. Let (π, H) and (ρ, K) be unitary representations of a discrete group Γ. If $T : H \to K$ is an intertwining operator between π and ρ, then so is $T^* : K \to H$. Indeed, for each $g \in \Gamma$, we have $T^*\rho(g) = (\rho(g^{-1})T)^* = (T\pi(g^{-1}))^* = \pi(g)T^*$. It follows, in particular, that T^*T intertwines π with itself, and that TT^* intertwines ρ with itself.

Definition 1.1.8. The representations (π, H) and (ρ, K) are said to be *equivalent* if there exists an intertwining operator between them which is isometric and onto. In that case we write $\pi \simeq \rho$. We say that π is *contained* in ρ, and write $\pi \leq \rho$, if π is equivalent to a subrepresentation of ρ.

Lemma 1.1.9. *Let (π, H) be a unitary representations of a discrete group Γ, and suppose that $T \in B(H)$ intertwines π with itself. Then so does $f(T)$, for any continuous function on the spectrum of T.*

Proof. By continuity of the functional calculus, it is enough to show the statement on a dense subset of $C(\sigma(T))$. The statement is clear for polynomials, by induction, and the set of polynomials is dense in $C(\sigma(T))$, by the Stone-Weierstrass theorem. $\square$

Proposition 1.1.10. *Let (π, H) and (ρ, K) be two unitary representations of the discrete group Γ, and let $T : H \to K$ be an intertwining operator. Then $(\ker T)^\perp$ and $\overline{\operatorname{Im} T}$ are closed invariant subspaces of H and K, respectively. Moreover, the subrepresentation of π defined by $(\ker T)^\perp$ is equivalent with the subrepresentation of ρ defined by $\overline{\operatorname{Im} T}$.*

Proof. We start by showing that $(\ker T)^\perp$ and $\overline{\operatorname{Im} T}$ are Γ-invariant. For each $\xi \in H$ and $g \in \Gamma$, we have $T\pi(g)\xi = \rho(g)T\xi$. This shows Γ-invariance of $\overline{\operatorname{Im} T}$ directly. Further, if $\xi \in \ker T$, the so is $\pi(g)\xi$. Therefore, $\ker T$ is Γ-invariant, and as (π, H) is unitary, then so is $(\ker T)^\perp$, by Remark 1.1.5.

By Remark 1.1.7, T^*T intertwines π with itself and therefore, $|T| = (T^*T)^{1/2}$ intertwines π with itself, by Lemma 1.1.9. Let $T = U|T|$ be the polar decomposition of T. Then U is a partial isometry on H with initial space $(\ker T)^\perp$ and final space $\overline{\operatorname{Im} T}$. We claim that U, viewed as an isometry $U : (\ker T)^\perp \to \overline{\operatorname{Im} T}$, intertwines the subrepresentations of π and ρ defined by these subspaces. Let $\xi \in (\ker T)^\perp$. Then $U\xi \in \operatorname{Im} T$, so that $U\xi = T\eta = U|T|\eta$, or, equivalently, $\xi = |T|\eta$, for some $\eta \in H$. For each $g \in \Gamma$, we derive that

$$\rho(g)U\xi = \rho(g)T\eta = T\pi(g)\eta = U|T|\pi(g)\eta = U\pi(g)|T|\eta = U\pi(g)\xi.$$

Hence, $\rho(g)U = U\pi(g)$, for all $g \in \Gamma$, where ρ and π are the subrepresentations on the mentioned Γ-invariant subspaces. $\square$

In the following subsections we remind the reader briefly of a few standard constructions for how to construct new representations from existing ones.

1.1.1 Extension and restriction of scalars

We show here how one may construct a unitary representation from an orthogonal one and vice versa using extension, respectively, restriction of scalars.

Given an orthogonal representation (π, H), one can construct a unitary one by taking its *complexification.* The complexification of the real Hilbert space H, denoted by $H_{\mathbb{C}}$, is obtained from H by extension of scalars to the complex numbers – precisely we define $H_{\mathbb{C}}$ as the tensor product $H \otimes_{\mathbb{R}} \mathbb{C}$. The complexification of a representation (π, H) of the group Γ is then the representation $\pi_{\mathbb{C}}$ on $H_{\mathbb{C}}$ defined by

$$\pi_{\mathbb{C}}(g)(\xi \otimes \lambda) = \pi(g)\xi \otimes \lambda, \qquad \text{for } g \in \Gamma,\ \xi \in H, \text{ and } \lambda \in \mathbb{C}.$$

For a unitary representation (ρ, K), we obtain an orthogonal representation as follows: We may view the complex vector space K as a real vector space by restriction of scalar multiplication to $\mathbb{R}$. Viewed as such we equip K with the inner product

$$\langle \xi \,, \eta \rangle_{\mathbb{R}} = \operatorname{Re} \langle \xi \,, \eta \rangle \,, \qquad \xi, \eta \in K.$$

The obtained real Hilbert space is denoted by $K_{\mathbb{R}}$. We denote by $\pi_{\mathbb{R}}$ the representation π considered as acting on $K_{\mathbb{R}}$. It is straight forward to verify that $(\pi_{\mathbb{R}}, K_{\mathbb{R}})$ is an orthogonal representation.

1.1.2 The contragredient representation

Let H be a Hilbert space. The *conjugate Hilbert space*, denoted by $\overline{H}$, is the Hilbert space which is identical to H as an additive group, with scalar multiplication given by

$$(\lambda, \xi) \mapsto \overline{\lambda}\xi, \qquad \text{for } \lambda \in \mathbb{C} \text{ and } \xi \in H,$$

and an inner product given by

$$\langle \xi \,, \eta \rangle_{\overline{H}} = \langle \eta \,, \xi \rangle_H \qquad \text{for } \xi, \eta \in H,$$

Let (π, H) be a representation of Γ. The *contragredient representation*, denoted by $\overline{\pi}$, is the representation of Γ on $\overline{H}$ which is identical to π as a set-theoretic transformation.

We remark that the contragredient representation of a unitary representation is again unitary. Observe also that $\overline{\overline{\pi}} = \pi$.

Lemma 1.1.11. *If a unitary representation (π, H) is the complexification of an orthogonal representation on a real Hilbert space, then it is unitarily equivalent to its contragredient representation.*

Proof. By assumption, $H = H' \otimes \mathbb{C}$, for some real Hilbert space H'. The map $H \to \overline{H}$ defined on elementary tensors by $\xi' \otimes \lambda \mapsto \xi' \otimes \overline{\lambda}$, for $\xi' \in H'$ and $\lambda \in \mathbb{C}$, is a $\mathbb{C}$-linear. It is clear that it is an isometric bijection and that it intertwines π and $\overline{\pi}$. $\square$

1.1.3 Direct sums of representations

Let $(H_i)_{i \in I}$ be a family of Hilbert spaces over either $\mathbb{R}$ or $\mathbb{C}$. For each $i \in I$, denote by $\langle \cdot \,, \cdot \rangle_i$ the inner product on H_i and by $\| \cdot \|_i$ the corresponding norm. The *Hilbert space direct sum* of this family is the Hilbert space

$$\bigoplus_{i \in I} H_i = \left\{ (\xi_i)_{i \in I} \in \prod_{i \in I} H_i \;\middle|\; \sum_{i \in I} \|\xi_i\|_i^2 < \infty \right\},$$

with inner product given by

$$\left\langle (\xi_i)_{i\in I}\,, (\eta_i)_{i\in I}\right\rangle = \sum_{i\in I} \langle \xi_i\,, \eta_i\rangle\,.$$

The *direct sum of a family of representations* $(\pi_i, H_i)_{i\in I}$ of a (discrete) group Γ is the representation $\oplus_i \pi_i$ of Γ on $\bigoplus_{i\in I} H_i$ given by

$$\oplus_i\pi_i(g)\big((\xi_i)_{i\in I}\big) = (\pi_i(g)\xi_i)_{i\in I}\,,$$

for each $g\in\Gamma$ and each $(\xi_i)_{i\in I}\in\bigoplus_{i\in I} H_i$.

Remark 1.1.12. If $(\pi_i, H_i)_{i\in I}$ is a family of unitary representation then their direct sum is a unitary representation, as well.

Example 1.1.13. For a representation (π, H) of a (discrete) group Γ and a Γ-invariant subspace $K\subset H$, one easily verifies that π is equal to the direct sum of the subrepresentations defined by K and its orthogonal complement. ◦

Proposition 1.1.14. *Let $(\pi_i, H_i)_{i\in I}$ be a family of unitary (orthogonal) representations of a discrete group Γ and let (π, K) be an irreducible unitary (orthogonal) representation of Γ. If π is contained in $\oplus_i\pi_i$ then π is contained in π_i, for some $i\in I$.*

Proof. Set $H=\bigoplus_{i\in I} H_i$ and suppose $T: K\to H$ is an intertwining operator between π and $\oplus_i\pi_i$ which is isometric onto a subspace of H. For each $i\in I$, denote by p_i the orthogonal projection onto H_i and let $T_i = p_i\circ T$. Then $T=\oplus_i T_i$, and so, there exists an $i_0\in I$ such that T_{i_0} is non-zero. Moreover, $(\oplus_i\pi_i)\circ T = \oplus_i(\pi_i\circ T_i)$, and so,

$$T_{i_0}\circ\pi = p_{i_0}\circ(T\circ\pi) = p_{i_0}\circ\oplus_i(\pi_i\circ T_i) = \pi_{i_0}\circ T_{i_0}.$$

That is, T_{i_0} is an intertwiner of π and π_{i_0}. In particular, $\ker T_{i_0}$ is an invariant subspace for π, so as π is irreducible and T_{i_0} is non-zero, $\ker T_{i_0}=\{0\}$. By Proposition 1.1.10, it follows that π is equivalent with the subrepresentation of $\oplus_i\pi_i$ defined by $\overline{\operatorname{Im} T_{i_0}}$, which is clearly a suprepresentation of π_{i_0}. □

1.1.4 Tensor products of representations

Let H and K be Hilbert spaces and let $H\odot K$ denote their algebraic tensor product. The Hilbert space tensor product, which we denote by $H\otimes K$, is the completion of $H\odot K$ with respect to the unique inner product satisfying

$$\langle \xi_1\otimes\eta_1\,, \xi_2\otimes\eta_2\rangle = \langle\xi_1\,,\xi_2\rangle\,\langle\eta_1\,,\eta_2\rangle\,,$$

for all $\xi_1,\xi_2\in H$, and all $\eta_1,\eta_2\in K$.

Definition 1.1.15. Let (π, H) and (ρ, K) be unitary representations of the discrete group Γ. Their *tensor product* is the (unique) unitary representation $\pi\otimes\rho$ of Γ on $H\otimes K$ given on elementary tensors by

$$(\pi\otimes\rho)(g)(\xi\otimes\eta) = \pi(g)\xi\otimes\rho(g)\eta,$$

for all $\xi\in H, \eta\in K$ and $g\in\Gamma$.

Let $(\eta_i)_{i\in I}$ be an orthonormal basis for $\overline{K}$. We denote by $HS(\overline{K}, H)$ the set of space of bounded linear operators form $T:\overline{K}\to H$ satisfying

$$\operatorname{Tr}(T^*T) = \sum_{i\in I}\langle T^*T\eta_i\,,\eta_i\rangle_{\overline{K}} < \infty.$$

This is a Hilbert space with inner product

$$(T , S) = \mathrm{Tr}(S^*T), \qquad \text{for } S, T \in HS(\overline{K}, H).$$

The operators in $HS(\overline{K}, H)$ are referred to as *Hilbert-Schmidt operators.* We define a map Φ on the elementary tensors in $H \otimes K$ with values in the space of bounded linear operators $\overline{K} \to H$ by setting, for each pair of vectors $\eta \in K$ and $\xi \in H$,

$$\Phi(\xi \otimes \eta)(\zeta) = \langle \eta , \zeta \rangle_K \xi, \qquad \text{for } \zeta \in \overline{K}.$$

Then Φ is an isometry of set of elementary tensors onto the set of rank 1 operators. The linear extension of Φ to the algebraic tensor product, $H \odot K$, is an isometry onto the set of finite rank operators, and so, Φ extends to a linear Hilbert space isomorphism $H \otimes K \to HS(\overline{K}, H)$ (see [37, Theorem VI.22(f)]). Thus, $HS(\overline{K}, H)$ can be viewed as an alternative realization of the tensor product of H and K. From this point of view, the unitary representation $\pi \otimes \rho$ on $H \otimes K$ corresponds to the unitary representation on $HS(\overline{K}, H)$ given by

$$T \mapsto \pi(g)T\overline{\rho}(g^{-1}), \qquad \text{for } T \in HS(\overline{K}, H). \tag{1.1}$$

To see this, let $T : \overline{K} \to H$ be the rank 1 operator given by $T = \langle \eta , \cdot \rangle_K \xi$, for $\eta \in \mathcal{K}$ and $\xi \in H$. For each $\zeta \in \overline{K}$ and each $g \in \Gamma$,

$$\pi(g)T\overline{\rho}(g^{-1})(\zeta) = \big\langle \eta , \overline{\rho}(g^{-1})\zeta \big\rangle_K \pi(g)\xi = \langle \rho(g)\eta , \zeta \rangle_K \pi(g)\xi = \Phi\big(\pi(g)\xi \otimes \rho(g)\eta\big)(\zeta).$$

The correspondence of equation (1.1) follows.

Lemma 1.1.16. *For any finite dimensional representation π, $\pi \otimes \overline{\pi}$ has an invariant vector.*

Proof. Let (π, H) be a finite dimensional representation. Since H is finite dimensional, $HS(\overline{H}, H)$ contains the identity operator, and the identity operator is clearly invariant under the map given in equation (1.1) with $\rho = \pi$, for any group element, g. □

Proposition 1.1.17. *Let (π, H) and (ρ, K) be unitary representations of the discrete group Γ. Then $\pi \otimes \rho$ contains the trivial representation 1_Γ if and only if there exists a finite dimensional representation of Γ which is contained in both π and $\overline{\rho}$.*

Proof. If σ is a finite dimensional representation of Γ contained in both π and ρ, then $\sigma \otimes \overline{\sigma}$ is contained in $\pi \otimes \overline{\rho}$. It then follows directly from Lemma 1.1.16 that $\pi \otimes \rho$ contains the trivial representation.

Conversely, supper that $\pi \otimes \overline{\rho}$ contains the trivial representation. Then we may find a non-zero $T \in HS(\overline{K}, H)$ such that $\pi(g)T\overline{\rho}(g^{-1}) = T$, for all $g \in \Gamma$. Rearranging, we see that such a T intertwines π and ρ. By [37, Theorem VI.21], T^*T is a compact operator on $\overline{K}$. As T^*T is also non-zero and positive, it has an eigenvalue $\lambda > 0$, and the corresponding eigenspace $E_\lambda \subset \overline{K}$ is finite dimensional and closed. Recall from Remark 1.1.7 that T^*T intertwines π with itself. Hence, E_λ is an invariant subspace. It remains to show that the finite dimensional subrepresentation of π defined by E_λ is contained in $\overline{rho}$, as well. We have, for each $\xi \in E_\lambda$, $\|T\xi\|^2 = \langle T^*T\xi , \xi \rangle = \lambda \|\xi\|^2$. Hence, $\lambda^{-1/2}T$ is an isometry from E_λ onto the closed subspace $T(E_\lambda)$ of H. This finishes the proof. □

1.2 Kazhdan's Property (T)

Definition 1.2.1. Let (π, H) be a unitary representation of the discrete group Γ.

- A non-zero vector $\xi \in H$ is called *invariant* if $\pi(g)\xi = \xi$, for all $g \in \Gamma$. If such a vector exists, we say that the representation (π, H) has a non-zero invariant vector.
- A net $(\xi_i)_{i\in I}$ of unit vectors in H is called *almost invariant* if $\|\pi(g)\xi_i - \xi_i\| \to 0$, for all $g \in \Gamma$. If such a net exists, we say that the representation (π, H) has almost invariant vectors.

Definition 1.2.2. Let Γ be a discrete group. We say that Γ has *property* (T) if any unitary representation of Γ with almost invariant vectors has a non-zero invariant vector.

Example 1.2.3. All finite groups have Property (T) – see Corollary 1.2.15. ◦

In [55, Theorem 3 and 4], A. Żuk showed that there are plenty of infinite groups with property (T). However, not many concrete examples are known. It is beyond the scope of this thesis to proof Property (T) for concrete groups, and we mention here only the most well known example:

Example 1.2.4. The special linear group over the integers, $\mathrm{SL}_n(\mathbb{Z})$, has Property (T), for $n \geq 3$. This was shown by Y. Shalom in [45]. ◦

For a discrete group Γ and a unitary representation (π, H), we denote by $H^{\pi(\Gamma)}$ the closed subspace of H defined by

$$H^{\pi(\Gamma)} := \{\, \xi \in H \mid \pi(g)\xi = \xi, \text{ for all } g \in \Gamma \,\}.$$

That is, $H^{\pi(\Gamma)}$ is the set invariant vectors for the representation of Γ on H. Whenever either the group or the representation is clear from context, we shall write H^{π} or H^{Γ} instead of $H^{\pi(\Gamma)}$. It is clear that this is an invariant subspace for Γ.

Remark 1.2.5. For each $g \in \Gamma$, $\pi(g)$ commutes with the orthogonal projection onto $H^{\pi(\Gamma)}$. Indeed, if P is this projection, invariance of the subspace $H^{\pi(\Gamma)}$ (and its complement) implies that $P\pi(g)\xi = \pi(g)\xi$, for $\xi \in H^{\pi(\Gamma)}$, and $P\pi(g)\xi = 0$, for $\xi \in (H^{\pi(\Gamma)})^{\perp}$. For $\xi \in H$, it follows that $P\pi(g)\xi = P\pi(g)P\xi + P\pi(g)(1-P)\xi = \pi(g)P\xi$. Hence, $P\pi(g) = \pi(g)P$.

Given a unitary representation (π, H) and a closed Γ-invariant subspace $M \subset H$, we get a canonical induced unitary representation of Γ on the quotient Hilbert space H/M.

Lemma 1.2.6. *Let H be a Hilbert space and let M be a closed subspace. The quotient space H/M is isometrically isomorphic to $M^{\perp}$.*

Proof. Let P be the orthogonal projection onto M. Observe that if $\xi - \xi' \in M$, then $(1-P)\xi = (1-P)\xi'$. The map $\Phi : H/M \to M^{\perp}$ given by $\Phi([\xi]) = (1-P)\xi$, for each $\xi \in H$, is therefore well-defined. It is clearly an isomorphism. Further, each $\xi \in H$ can be decomposed uniquely as $\xi = \xi' + \xi''$, for $\xi' \in M$ and $\xi'' \in M^{\perp}$, and we see that $\Phi([\xi]) = \xi''$. It follows from Pythagoras that $\|\Phi([\xi])\| = \|\xi''\| = \|[\xi]\|_{H/M}$. □

Proposition 1.2.7. *A discrete group Γ has property (T) if and only if, for every unitary representation (π, H), the induced representation on the quotient $H/H^{\pi(\Gamma)}$ does not have almost invariant vectors.*

Proof. Let (π, H) be a unitary representation of Γ and let π' denote the induced representation on $H/H^{\pi(\Gamma)}$. Let P be the orthogonal projection onto $H^{\pi(\Gamma)}$. By Lemma 1.2.6, $H/H^{\pi(\Gamma)}$ is isometrically isomorphic to $(H^{\pi(\Gamma)})^{\perp}$ through the map $[\xi] \mapsto (1-P)\xi$, for $\xi \in H$.

Suppose $([\xi_i])_{i\in I}$ is a net of almost invariant vectors in $H/H^{\pi(\Gamma)}$. For each $i \in I$, let $\xi_i' = P\xi_i \in H^{\pi(\Gamma)}$ and $\xi_i'' = (1-P)\xi_i \in (H^{\pi(\Gamma)})^\perp$. For each $i \in I$, we have $\|\xi_i''\| = \|[\xi_i]\|_{H/H^{\pi(\Gamma)}} = 1$. Further, by Remark 1.2.5,

$$\begin{aligned}\left\|\pi(g)\xi_i'' - \xi_i''\right\| &= \|(1-P)(\pi(g)\xi_i - \xi_i)\| = \|[\pi(g)\xi_i - \xi_i]\|_{H/H^{\pi(\Gamma)}} \\ &= \left\|\pi'(g)[\xi_i] - [\xi_i]\right\|_{H/H^{\pi(\Gamma)}} \to 0.\end{aligned}$$

Hence, $(\xi_i'')_{i\in I}$ is a net of almost invariant vectors in $(H^{\pi(\Gamma)})^\perp$. By restricting π to the invariant subspace $(H^{\pi(\Gamma)})^\perp$, we obtain a unitary representation of Γ which has almost invariant vectors but no non-zero invariant vectors. Hence, Γ does not have (T).

Conversely, suppose Γ does not have (T) and let (π, H) be a unitary representation with almost invariant vectors but no non-zero invariant vector. Then $H^{\pi(\Gamma)}$ is trivial, and so $H/H^{\pi(\Gamma)} \cong H$ has a net of almost invariant vectors for Γ. □

1.2.1 Kazhdan pairs

Definition 1.2.8. Let (π, H) be a unitary representation of the discrete group Γ. A non-zero vector $\xi \in H$ is said to be (E,k)-*invariant*, for a subset $E \subset \Gamma$ and a constant $k > 0$, if

$$\sup_{g\in E} \|\pi(g)\xi - \xi\| < k\,\|\xi\|\,.$$

If such a vector exists, we say that (π, H) has a non-zero (E,k)-invariant vector.

Proposition 1.2.9. *A representation (π, H) of a discrete group Γ has almost invariant vectors if and only if, for every finite subset $F \subset \Gamma$ and every $\varepsilon > 0$, there exists a (F, ε)-invariant vector for (π, H).*

Proof. Suppose we may, for every finite subset $F \subset \Gamma$ and every $\varepsilon > 0$, find an (F, ε)-invariant vector $\xi_{(F,\varepsilon)}$ for (π, H). We may take each $\xi_{(F,\varepsilon)}$ to be a unit vector. This defines a net $(\xi_{(F,\varepsilon)})$ indexed by the set of all tuples (F, ε), where $F \subset \Gamma$ is finite and $\varepsilon > 0$. We equip this index set with a direction by setting $(F, \varepsilon) \leq (F', \varepsilon')$ if $F \subset F'$ and $\varepsilon' < \varepsilon$. It is straight forward to verify that $(\xi_{(F,\varepsilon)})$ is a net of almost invariant vectors. The converse statement is trivial. □

Definition 1.2.10. Let Γ be a discrete group. A pair (E,k), where $E \subset \Gamma$ and $k > 0$, is called a *Kazhdan pair* for Γ if every unitary representation which has a non-zero (E,k)-invariant vector, has a non-zero Γ-invariant vector.

Remark 1.2.11. If (E,k) is a Kazhdan pair then so is (E,k'), for all $0 < k' \leq k$.

We shall see in a moment that $(\Gamma, \sqrt{2})$ is a Kazhdan pair for any discrete group Γ. The existence of a Kazhdan pair is therefore nothing special. However, the existence of a Kazhdan pair with a finite Kazhdan set turns out to be equivalent with property (T).

Proposition 1.2.12. *Let Γ be a discrete group. Then Γ has property (T) if and only if there exists a Kazhdan pair (F,k) for Γ with $F \subset \Gamma$ finite.*

Proof. Suppose first that there exists a Kazhdan pair (F,k) for Γ with $F \subset \Gamma$ finite, and let $(\xi_i)_{i\in I}$ be a net of almost invariant vectors for a given unitary representation (π, H) of Γ. Then, for all $g \in \Gamma$, there exists $i_g \in I$ such that $\|\pi(g)\xi_i - \xi_i\| < k$, for all $i \succcurlyeq i_g$. Since I is directed and F is finite, we can take $i_0 \in I$ such that $i_0 \succcurlyeq i_g$, for all $g \in F$. Then $\|\pi(g)\xi_{i_0} - \xi_{i_0}\| < k$, for all $g \in F$, and so ξ_{i_0} is (F,k)-invariant. Since (F,k) is a Kazhdan pair it follows that the representation (π, H) has a non-zero invariant vector, and hence that Γ has property (T).

Suppose instead that Γ has no Kazhdan pair (F, k) with F finite. Then, for all finite subsets $F \subset \Gamma$ and all $k > 0$, there exists a unitary representation with an (F, k)-invariant vector but with with no non-zero invariant vectors. Let

$$I = \{ (F, k) \mid F \subset \Gamma \text{ finite}, \ k > 0 \}.$$

For $(F_1, k_1), (F_2, k_2) \in I$, set $(F_1, k_1) \preccurlyeq (F_2, k_2)$ if $F_1 \subset F_2$ and $k_2 \leq k_1$. Then $(I, \preccurlyeq)$ is a directed set. For each $i = (F_i, k_i) \in I$, let (π_i, H_i) be the aforementioned existing unitary representation of Γ, and let $\xi_i \in H_i$ be an (F_i, k_i)-invariant unit vector, i.e.,

$$\|\pi_i(g)\xi_i - \xi_i\| < k_i, \qquad t \in F_i.$$

Set $H = \bigoplus_{i \in I} H_i$. Then H is a Hilbert space, and each vector ξ_i can be viewed as a unit vector in H through the inclusion $H_i \hookrightarrow H$. Moreover, $\pi = \bigoplus_{i \in I} \pi_i : \Gamma \to B(H)$ is a unitary representation of Γ on H. For all $i \in I$ and for all $g \in \Gamma$, we see that $\|\pi(g)\xi_i - \xi_i\| = \|\pi_i(g)\xi_i - \xi_i\| \to 0$. Hence, $(\xi_i)_{i \in I}$ is a net of almost invariant unit vectors.

Suppose that $\xi \in H$ is a non-zero invariant vector. For each $i \in I$, let $P_i : H \to H_i$ be the orthogonal projection onto H_i. Then, for all $g \in \Gamma$ and all $i \in I$,

$$\pi_i(g)P_i\xi = P_i\pi(g)\xi = P_i\xi.$$

Hence, $P_i\xi$ is invariant with respect to the representation (π_i, H_i). But (π_i, H_i) has no non-zero invariant vectors, and so $P_i\xi = 0$. Since this is true for all $i \in I$, we deduce that $\xi = \sum_{i \in I} P_i\xi = 0$. Hence, (π, H) has no non-zero invariant vectors, and we conclude that Γ does not have property (T). $\square$

We proceed to show in Lemma 1.2.14 the already mentioned result that all discrete groups have a Kazhdan pair. To show this, we shall need the so called circumcenter lemma for Hilbert spaces.

For a Hilbert space H and a non-empty bounded subset $S \subset H$, we define the radius of S to be the number

$$\operatorname{rad}(S) = \inf \left\{ r > 0 \;\middle|\; S \subset \overline{B}(\zeta, r), \text{ for some } \zeta \in H \right\}.$$

An element $\zeta \in H$ is called a *circumcenter* of S if $S \subset \overline{B}\big(\zeta, \operatorname{rad}(S)\big)$

Lemma 1.2.13 (The circumcenter lemma for Hilbert spaces)**.** *Any bounded subset $S \subset H$ of a Hilbert space H has a unique circumcenter. Morover, this circumcenter lies in the closed convex hull of S.*

Proof. For each $n \in \mathbb{N}$, take $\xi_n \in H$ such that $S \subset \overline{B}(\xi_n, \operatorname{rad}(S) + \frac{1}{n})$. We claim that the obtained sequence $(\xi_n)_{n \geq 1}$ is Cauchy.

For each $\eta \in H$ and each $n, m \in \mathbb{N}$, the parallelogram identity yields that

$$\left\| \frac{\xi_n + \xi_m}{2} - \eta \right\|^2 + \left\| \frac{\xi_n - \xi_m}{2} \right\|^2 = 2 \left\| \frac{\xi_n - \eta}{2} \right\|^2 + 2 \left\| \frac{\xi_m - \eta}{2} \right\|^2$$

If $\eta \in S$, we have $\|\xi_n - \eta\| \leq \operatorname{rad}(S) + 1/n$, for each $n \in \mathbb{N}$. Hence, for $\eta \in S$ and $1 \leq n \leq m$,

$$\begin{aligned}\left\| \frac{\xi_n - \xi_m}{2} \right\|^2 &= \frac{1}{2} \|\xi_n - \eta\|^2 + \frac{1}{2} \|\xi_m - \eta\|^2 - \left\| \frac{\xi_n + \xi_m}{2} - \eta \right\|^2 \\ &\leq \left(\operatorname{rad}(S) + \frac{1}{n} \right)^2 - \left\| \frac{\xi_n + \xi_m}{2} - \eta \right\|^2.\end{aligned}$$

Fix $1 \leq n \leq m$. For each $\delta > 0$, we have $S \not\subset \overline{B}\left(\frac{\xi_n+\xi_m}{2}, \operatorname{rad}(S) - \delta\right)$, and so, we may find $\eta \in S$ such that $\|(\xi_n + \xi_m)/2 - \eta\| > \operatorname{rad}(S) - \delta$. Combining this with our preparations above, we see that

$$\left\| \frac{\xi_n - \xi_m}{2} \right\|^2 \leq \left(\operatorname{rad}(S) + \frac{1}{n} \right)^2 - (\operatorname{rad}(S) - \delta)^2$$

It follows that $(\xi_n)_{n\geq 1}$ is, indeed, Cauchy. Let ζ be the point of convergence. For $\eta \in S$ and $\varepsilon > 0$, pick $n \in \mathbb{N}$ such that $n > 2/\varepsilon$ and such that $\|\xi_n - \zeta\| < \varepsilon/2$. Then

$$\|\eta - \zeta\| \leq \|\eta - \xi_n\| + \|\xi_n - \zeta\| < \operatorname{rad}(S) + \varepsilon.$$

We conclude that $S \subset \overline{B}(\zeta, r_0)$. This shows existence of the circumcenter. To see that the circumcenter is unique, suppose $\zeta' \in H$ is also a circumcenter. Let

$$\eta_n = \begin{cases} \zeta & \text{if } n \text{ is odd} \\ \zeta' & \text{if } n \text{ is even} \end{cases}.$$

Clearly, $S \subset \overline{B}(\eta_n, r_0 + \frac{1}{n})$, for all $n \in \mathbb{N}$. By our previous argument, the sequence $(\eta_n)_{n\geq 1}$ is Cauchy, and so, we must have $\zeta = \zeta'$.

It remains to show that ζ lies in $\overline{\operatorname{conv}}(S)$. Suppose for contradiction that this is not the case. Since $\overline{\operatorname{conv}}(S)$ is closed and convex there exists a unique vector $\xi_0 \in \overline{\operatorname{conv}}(S)$ such that $\|\zeta - \xi_0\| = \operatorname{dist}(\zeta, \overline{\operatorname{conv}}(S)) > 0$. Moreover, for each $\xi \in \overline{\operatorname{conv}}(S)$, it holds that $\operatorname{Re}\langle \zeta - \xi_0 \,, \xi - \xi_0 \rangle \leq 0$. Let $M = \{\zeta - \xi_0\}^\perp$. For $\xi \in \overline{\operatorname{conv}}(S)$, we may uniquely write $\xi - \xi_0 = \lambda(\zeta - \xi_0) + \eta$, for some $\lambda \in \mathbb{C}$ (or $\mathbb{R}$, if H is real) and some $\eta \in M$. Observe that

$$\operatorname{Re}(\lambda) \|\zeta - \xi_0\|^2 = \operatorname{Re}\langle \zeta - \xi_0 \,, \xi - \xi_0 \rangle \leq 0.$$

Hence, $\operatorname{Re}(\lambda) \leq 0$. Isolating $\xi - \zeta$ on one side of the decomposition of $\xi - \xi_0$, we see that

$$\begin{aligned} \|\xi - \zeta\|^2 &= |\lambda - 1|^2 \|\zeta - \xi_0\|^2 + \|\eta\|^2 = \left(|\lambda|^2 + 1 - 2\operatorname{Re}(\lambda)\right) \|\zeta - \xi_0\|^2 + \|\eta\|^2 \\ &\geq \left(|\lambda|^2 + 1\right) \|\zeta - \xi_0\|^2 + \|\eta\|^2 = \|\xi - \xi_0\|^2 + \|\zeta - \xi_0\|^2. \end{aligned}$$

The last equality follows from the decomposition of $\xi - \xi_0$. Set $R = \sup\{\, \|\xi - \xi_0\| \mid \xi \in \overline{\operatorname{conv}}(S) \,\}$. This is a finite number because S is bounded, and we have that

$$\|\xi - \zeta\|^2 \geq \|\xi - \xi_0\|^2 \left(1 + \frac{\|\zeta - \xi_0\|^2}{R^2}\right),$$

for each $\xi \in \overline{\operatorname{conv}}(S)$. Set $\delta = (1 + \|\zeta - \xi_0\|^2 / R^2)^{-1/2}$. Then $\|\xi - \xi_0\| \leq \delta \|\xi - \zeta\| \leq \delta \operatorname{rad}(S)$, for all $\xi \in \overline{\operatorname{conv}}(S)$. But then $\overline{\operatorname{conv}}(S) \subset \overline{B}(\xi_0, \delta \operatorname{rad}(S))$, which contradicts the definition of $\operatorname{rad}(S)$ as $\delta < 1$. Thus, we must have $\zeta \in \overline{\operatorname{conv}}(S)$. □

Lemma 1.2.14. *For any discrete group Γ, the pair $(\Gamma, \sqrt{2})$ is a Kazhdan pair.*

Proof. Let (π, H) be a unitary representation of Γ and suppose that ξ is a non-zero $(\Gamma, \sqrt{2})$-invariant vector. We may take ξ to be a unit vector. Let $\mathcal{O}_\xi = \pi(\Gamma)\xi$ denote the orbit of ξ. This is clearly a Γ-invariant subset of H. Moreover, as π is isometric, $\mathcal{O}_\xi$ is bounded, and so, there exists a unique circumcenter of $\mathcal{O}_\xi$, by Lemma 1.2.13. Let ζ be this circumcenter. Then $\pi(g)\zeta$ is a circumcenter for $\pi(g)\mathcal{O}_\xi$, for each $g \in \Gamma$, since π is isometric. But $\pi(g)\mathcal{O}_\xi = \mathcal{O}_\xi$, and so $\pi(g)\zeta = \zeta$, by uniqueness of the circumcenter. Hence, ζ is an invariant vector. It remains to show that $\zeta \neq 0$. For this, recall from Lemma 1.2.13 that $\zeta \in \overline{\operatorname{conv}}\, \mathcal{O}_\xi$. If $\zeta \in \operatorname{conv} \mathcal{O}_\xi$ we may write $\zeta = \sum_{i=1}^n a_i \pi(g_i)\xi$, where each $a_i > 0$ and $\sum_{i=1}^n a_i = 1$. Then

$$\operatorname{Re}\langle \xi \,, \zeta \rangle = \sum_{i=1}^n a_i \operatorname{Re}\langle \xi \,, \pi(g_i)\xi \rangle \geq \inf_{g\in\Gamma} \operatorname{Re}\langle \xi \,, \pi(g)\xi \rangle = 1 - \frac{1}{2} \sup_{g\in\Gamma} \|\xi - \pi(g)\xi\|^2,$$

where we in the last equality use that ξ is a unit vector. This inequality holds for $\zeta \in \overline{\operatorname{conv}}\, \mathcal{O}_\xi$ as well, by continuity. Since ξ is $(\Gamma, \sqrt{2})$-invariant, it follows that $\operatorname{Re} \langle \xi , \zeta \rangle$ is strictly positive. In particular, $\zeta \neq 0$, as we wanted to show. $\square$

Corollary 1.2.15. *Every finite group has property (T).*

Proof. This is immediate from Lemma 1.2.14 and Proposition 1.2.12. $\square$

Proposition 1.2.16. *Let (E, k) be a Kazhdan pair for the discrete group Γ, and suppose that (π, H) is a unitary representation of Γ with the property that there exists a non-zero vector $\xi \in H$ such that $\pi(g)\xi = \xi$, for all $g \in E$. Then ξ is Γ-invariant for π.*

Proof. Let H_0 be the set of all Γ-invariant vectors in H, and let $K = H_0^\perp$. Since $H = H_0 \oplus K$, we can find unique vectors $\xi_0 \in H_0$ and $\eta_0 \in K$ such that $\xi = \xi_0 + \eta_0$. Then, for all $g \in E$,

$$\xi = \pi(g)\xi = \pi(g)\xi_0 + \pi(g)\eta_0 = \xi_0 + \pi(g)\eta_0.$$

Hence, $\pi(g)\eta_0 = \eta_0$, for all $g \in E$, by the uniqueness of the decomposition $\xi = \xi_0 + \eta_0$.

Observe next that both H_0 and K are invariant under π, that is, $\pi(g)H_0 \subset H_0$ and $\pi(g)K \subset K$, for all $g \in \Gamma$. We can then define a unitary representation (π', K) of Γ on K by setting $\pi'(g) = \pi(g)|_K$, for all $g \in \Gamma$. By construction of K, this representation has no non-zero Γ-invariant vectors. As (E, k) is a Kazhdan-pair for Γ it follows that, for all $\eta \in K$, there exists $g \in E$ such that $\|\pi'(g)\eta - \eta\| \geq k \|\eta\|$. In particular, we can find such a $g \in E$ for the vector $\eta_0 \in K$. Hence, $\eta_0 = 0$, and so $\xi = \xi_0 \in H_0$. We conclude that ξ is Γ-invariant for the representation (π, H). $\square$

Corollary 1.2.17. *Let $E \subset \Gamma$ be a subset of the discrete group Γ. If there exists a $k > 0$ such that (E, k) is a Kazhdan pair then E is a generating set for Γ.*

Proof. Let $\Gamma_0 \leq \Gamma$ be the subgroup generated by E, and let π be the unitary representation of Γ on $\ell^2(\Gamma/\Gamma_0)$ induced by the left translation action of Γ on the set Γ/Γ_0 of left cosets (see Example 1.1.3). By construction of Γ_0, the trivial coset is left invariant by all elements of E. It follows from Proposition 1.2.16 that $\delta_{\Gamma_0} \in \ell^2(\Gamma/\Gamma_0)$ is an invariant vector for π. Hence, $\Gamma_0 = \Gamma$. $\square$

Corollary 1.2.18. *Every discrete group with Property (T) is finitely generated.*

Proof. This follows immediately from Proposition 1.2.12 and Corollary 1.2.17. $\square$

Proposition 1.2.19. *Let Γ be a discrete group and let $\Lambda \triangleleft \Gamma$ be a normal subgroup. If Γ has property (T) then so does the quotient Γ/Λ.*

Proof. Let $q : \Gamma \to \Gamma/\Lambda$ be the quotient homomorphism. If $\pi : \Gamma/\Lambda \to \mathcal{U}(H)$ is a unitary representation of the quotient group Γ/Λ then $\pi \circ q : \Gamma \to \mathcal{U}(H)$ is a unitary representation of Γ. If, moreover, π has almost Γ/Λ-invariant vectors then so does $\pi \circ q$. In fact, if $(\xi_i)_{i \in I}$ is a net of almost Γ/Λ-invariant unit vectors in H, then it is almost Γ-invariant as well. As Γ has property (T) we can find a non-zero vector $\xi \in H$ such that $\pi(q(g))\xi = \pi \circ q(g)\xi = \xi$, for all $g \in \Gamma$. We deduce that ξ is Γ/Λ-invariant, and it follows that Γ/Λ has property (T). $\square$

Remark 1.2.20. Let Γ be a discrete group, let $\Lambda \triangleleft \Gamma$ be a normal subgroup and let $q : \Gamma \to \Gamma/\Lambda$ be the quotient homomorphism. If (E, k) is a Kazhdan pair for Γ then $(q(E), k)$ is a Kazhdan pair for the quotient.

1.3 Property (T) and the trivial representation

We point out, in Proposition 1.3.1 below, the easy observation that a unitary representation has invariant vectors if and only if it contains the trivial representation. We shall see hereafter that it is possible to define a weaker notion of containment that captures having almost invariant vectors.

Proposition 1.3.1. *A unitary representation (π, H) of the discrete group Γ contains the trivial representation if and only if there exists a non-zero Γ-invariant vector in H.*

Proof. Suppose $1_\Gamma \leq \pi$, and take $T : \mathbb{C} \to H$ such that $T1_\Gamma = \pi T$. Let $\xi = T(1)$. Then $\pi(g)\xi = \xi$. Conversely, suppose $\pi(g)\xi = \xi$, for some non-zero vector $\xi \in H$. Set $T(z) = z\xi$, for $z \in \mathbb{C}$. Then $\pi(g)T(z) = z\pi(g)\xi = z\xi = T(z) = T1_\Gamma(g)(z)$, for all $g \in \Gamma$. □

Denote by $\mathbb{C}[\Gamma]$ the group ring of the discrete group Γ, i.e., the set of all finite formal linear sums over Γ with coefficients in $\mathbb{C}$ equipped, as usual, with the structure of a *-algebra. Further, we denote by $C^*(\Gamma)$ the universal C^*-algebra of Γ, i.e., the completion of $\mathbb{C}[\Gamma]$ with respect to supremum norm over all norms on $\mathbb{C}[\Gamma]$ obtained from a unitary representation. We refer the reader to [9, Section 2.5] for details on this construction.

A unitary representation (π, H) of Γ induces a *-homomorphism $\mathbb{C}[\Gamma] \to B(H)$ simply by extending π linearly. We shall denote this *-homomorphism by $\hat{\pi}$. By universality, we may extend $\hat{\pi}$ further to a *-homomorphism $C^*(\Gamma) \to B(H)$. With a slight abuse of notation, we shall use $\hat{\pi}$ to denote the extension to all of $C^*(\Gamma)$, as well.

Definition 1.3.2. Let (π, H) and (ρ, K) be unitary representations of the discrete group Γ. We say that π is weakly contained in ρ if, for all $x \in C^*(\Gamma)$,

$$\|\hat{\pi}(x)\| \leq \|\hat{\rho}(x)\| . \tag{1.2}$$

In that case we write $\pi \prec \rho$.

Remark 1.3.3. Since the norm on $C^*(\Gamma)$ majorizes the norm on $\mathbb{C}[\Gamma]$ induced by any representation of Γ, it suffices in the above definition that equation (1.2) holds for all $x \in \mathbb{C}[\Gamma]$.

Lemma 1.3.4. *Let H be a complex Hilbert space. For any $\varepsilon > 0$ and any $n \geq 2$, there exists a $\delta > 0$ such that, if $\xi_1, \ldots, \xi_n \in H$ are unit vectors with $\|\sum_{i=1}^n \xi_i\| \geq n - \delta$, then $\|\xi_i - \xi_j\| < \varepsilon$, for all $1 \leq i, j \leq n$*

Proof. Suppose $\xi_1, \ldots, \xi_n \in H$ is a collection of at least two unit vectors satisfying the inequality $\|\sum_{i=1}^n \xi_i\| \geq n - \delta$, for some $\delta > 0$. Then

$$\sum_{k,l=1}^n \langle \xi_k \,, \xi_l \rangle = \left\| \sum_{i=1}^n \xi_i \right\|^2 \geq n^2 - 2n\delta + \delta^2.$$

On the other hand, as $\xi_1, \ldots, \xi_n$ are unit vectors, we have, for each pair of indices $1 \leq i, j \leq n$,

$$\sum_{k,l=1}^n \langle \xi_k \,, \xi_l \rangle \leq n^2 - 2 + 2 \operatorname{Re} \langle \xi_i \,, \xi_j \rangle .$$

Comparing the two inequalities, it follows that $\operatorname{Re} \langle \xi_i \,, \xi_j \rangle \geq 1 - n\delta + \delta^2/2$. From this, we derive the following inequality, for each pair of indices $1 \leq i, j \leq n$:

$$\|\xi_i - \xi_j\|^2 = 2 - 2 \operatorname{Re} \langle \xi_i \,, \xi_j \rangle \leq 2n\delta - \delta^2.$$

Given $\varepsilon > 0$ and $n \in \mathbb{N}$, we may choose $0 < \delta < n(1 - \sqrt{1 - \varepsilon^2/n^2})$. Rearranging, we see that, for such a δ, the inequality $2n\delta - \delta^2 < \varepsilon^2$ holds. The statement of the lemma then follows from our above derivations. □

Proposition 1.3.5. *Let (π, H) be a unitary representation of the discrete group Γ. Then $1_\Gamma \prec \pi$ if and only if (π, H) has almost invariant vectors.*

Proof. Suppose $(\xi_i)_{i\in I}$ is a net of almost invariant vectors for the representation (π, H), and let $x = \sum_{g\in\Gamma} a_g g \in \mathbb{C}[\Gamma]$ be a finite sum. For a given $\varepsilon > 0$, we may choose $i_0 \in I$ such that $\|\pi(g)\xi_i - \xi_i\| < \varepsilon / \sum_{g\in\Gamma} |a_g|$, for all $i \succcurlyeq i_0$. Then

$$\|\pi(x)\| \geq \|\pi(x)\xi_{i_0}\| = \left\|\sum_{g\in\Gamma} a_g \pi(g)\xi_{i_0}\right\| > \left\|\sum_{g\in\Gamma} a_g \xi_{i_0}\right\| - \varepsilon = \left|\sum_{g\in\Gamma} a_g\right| - \varepsilon = \|1_\Gamma(x)\| - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we deduce that $\|\pi(x)\| \geq \|1_\Gamma(x)\|$. This holds for all $x \in \mathbb{C}[\Gamma]$, and so, $1_\Gamma \prec \pi$, by Remark 1.3.3.

Conversely, suppose $1_\Gamma \prec \pi$. Let $F \subset \Gamma$ be a finite set, put $F_e = F \cup \{e\}$, and let $\varepsilon > 0$. Set $x = \sum_{g\in F_e} g$. By the triangle inequality, and since π is a unitary representation, we have $\|\pi(x)\| \leq \sum_{g\in F_e} \|\pi(g)\| = |F_e|$. On the other hand, since $1_\Gamma \prec \pi$, we see that $\|\pi(x)\| \geq \|1_\Gamma(x)\| = |F_e|$, and so $\|\pi(x)\|$ equals $|F_e|$. Let $\delta > 0$ be as in Lemma 1.3.4 with $n = |F_e|$. By definition of the operator norm, we may take $\xi \in H$ such that

$$\left\|\sum_{g\in F_e} \pi(g)\xi\right\| = \|\pi(x)\xi\| \geq |F_e| - \delta.$$

Lemma 1.3.4 then yields that $\|\pi(g)\xi - \pi(h)\xi\| < \varepsilon$, for all pairs of elements $g, h \in F_e$. Since F_e contains the identity and F is finite, it follows that

$$\sup_{g\in F} \|\pi(g)\xi - \xi\| < \varepsilon.$$

It follows from Proposition 1.2.9 that (π, H) has almost invariant vectors. $\square$

We obtain the following rephrasing of Property (T) as an immediate corollary to Proposition 1.3.5 and Proposition 1.3.1.

Corollary 1.3.6. *A discrete group Γ has property (T) if and only if, whenever a unitary representation weakly contains 1_Γ, it contains 1_Γ.* $\square$

This is less of a characterization of Property (T) than it is a rephrasing of Definition 1.2.2 in fancy terms. Weak containment, however, allows us to define a topology on the unitary dual of a group, namely *Fell's topology*, and we shall see in Proposition 1.3.18 that isolation of the trivial representation this topology is equivalent to having Property (T). We shall take a different route in defining Fell's topology and make the connection to weak containment in Lemma 1.3.9.

Let $\mathcal{A}$ be a unital C^*-algebra. Recall that the kernel of any *-representation of $\mathcal{A}$ is a closed two-sided ideal in $\mathcal{A}$ (see, e.g., [8, Paragraph II.6.1.2]). A two-sided ideal which arises as the kernel of an irreducible *-representation is called a *primitive ideal*. We denote by $\operatorname{Prim}(\mathcal{A})$ the set of all primitive ideals of $\mathcal{A}$, and we endow this set with a topology as follows: For a subset $S \subset \operatorname{Prim}(\mathcal{A})$, its closure is defined as

$$\overline{S} = \left\{ I \in \operatorname{Prim}(\mathcal{A}) \;\middle|\; \bigcap_{J\in S} J \subset I \right\}.$$

One easily verifies that this, indeed, defines a topology on $\operatorname{Prim}(\mathcal{A})$ – see [16, Section 3.1.1]. This topology is known as *Jacobsen's topology*.

Definition 1.3.7. The *unitary dual* of a (discrete) group Γ, denoted by $\widehat{\Gamma}$, is the set of equivalence classes of irreducible unitary representations of Γ.

Let Γ be a discrete group. Define a map $\Phi : \widehat{\Gamma} \to \mathrm{Prim}(C^*(\Gamma))$ by setting $\Phi(\pi) = \ker(\hat{\pi})$. The topology on $\widehat{\Gamma}$ generated by all sets of the form $\Phi^{-1}(U)$, where U is an open set in $\mathrm{Prim}(C^*(\Gamma))$, is known as *Fell's topology*. With this definition of Fell's topology we have the following understanding of closures: If $S \subset \widehat{\Gamma}$ and $\pi \in \widehat{\Gamma}$ then $\pi \in \overline{S}$ if and only if $\bigcap_{\rho \in S} \ker(\hat{\rho}) \subset \ker(\hat{\pi})$. In Corollary 1.3.11 below, we characterize Fell's topology in terms of convergence of nets.

Lemma 1.3.8. *Let $(\pi_i, H_i)_{i\in I}$ be a family of unitary representations of a discrete group Γ. Then the extension, $(\oplus_i \pi_i)\hat{}$, of their direct sum to $C^*(\Gamma)$ equals the direct sum, $\oplus_i \hat{\pi}_i$, of their extensions. Moreover, for each $x \in C^*(\Gamma)$, it holds that*

$$\|\oplus_i \hat{\pi}_i(x)\| = \sup_{i\in I} \|\hat{\pi}_i(x)\|\,.$$

Proof. The extension $(\oplus_i \pi_i)\hat{}$ is the unique *-homomorphism $C^*(\Gamma) \to B(H)$ satisfying $(\oplus_i \pi_i)\hat{}\,(g) = \oplus_i \pi_i(g)$, for all $g \in \Gamma$. On the other hand, for each $i \in I$ and each $g \in \Gamma$, $\hat{\pi}_i(g) = \pi_i(g)$, by definition of $\hat{\pi}_i$. Thus, the first statement of the lemma is direct from uniqueness of $(\oplus_i \pi_i)\hat{}$.

Let $x \in C^*(\Gamma)$. By definition of the norm on the universal C^*-algebra of Γ, it is clear that $\sup_{i\in I} \|\hat{\pi}_i(x)\|$ is finite. For each $\xi = (\xi_i)_{i\in I} \in \bigoplus_{i\in I} H_i$, we have, by definition of the norm on the Hilbert space direct sum, that

$$\big\|\big(\oplus_i \hat{\pi}_i(x)\big)\xi\big\|^2 = \|\oplus_i \hat{\pi}_i(x)\xi_i\|^2 = \sum_{i\in I} \|\hat{\pi}_i(x)\xi_i\|_i^2 \le \left(\sup_{i\in I} \|\hat{\pi}_i(x)\|\right)^2 \|\xi\|^2$$

Hence, $\|\oplus_i \hat{\pi}_i(x)\| \le \sup_{i\in I} \|\hat{\pi}_i(x)\|$. For the opposite inequality, recall that each H_i embeds isometrically into the direct sum $\bigoplus_{i\in I} H_i$ by considering each vector $\xi \in H_i$ as the vector in $\bigoplus_{i\in I} H_i$ with ξ in the i'th coordinate and the zero-vector elsewhere. Thus, for each $i_0 \in I$,

$$\begin{aligned}\|\oplus_i \hat{\pi}_i(x)\| &\ge \sup\{\, \|\oplus_i \hat{\pi}_i(x)\xi\| \mid \xi \in H_{i_0},\ \|\xi_i\|_i = 1 \,\}\\ &= \sup\{\, \|\hat{\pi}_{i_0}(x)\xi\| \mid \xi \in H_{i_0},\ \|\xi_i\|_i = 1 \,\} = \|\hat{\pi}_{i_0}(x)\|\,.\end{aligned}$$

Taking the supremum over $i_0 \in I$, we obtain the inequality $\|\oplus_i \hat{\pi}_i(x)\| \ge \sup_{i\in I} \|\hat{\pi}_i(x)\|$. This finishes the proof of the lemma. □

Lemma 1.3.9. *Let $(\pi_i)_{i\in I}$ be a net in $\widehat{\Gamma}$. The following are equivalent:*

(i) $\bigcap_{i\in I} \ker(\hat{\pi}_i) \subset \ker(\hat{\pi})$,

(ii) $\pi \prec \oplus_{i\in I} \pi_i$,

(iii) $\sup_{i\in I} \|\hat{\pi}_i(x)\| \ge \|\hat{\pi}(x)\|$, *for all* $x \in C^*(\Gamma)$.

Proof. The equivalence of (ii) and (iii) is clear by Lemma 1.3.8 and by definition of weak containment. We proceed to show that (i) and (iii) are equivalent.

Suppose that (iii) holds, for all $x \in C^*(\Gamma)$. Then, if $x \in \bigcap_{i\in I} \ker(\hat{\pi}_i)$, then

$$\|\hat{\pi}(x)\| \le \sup_{i\in I} \|\hat{\pi}_i(x)\| = 0,$$

and so, $x \in \ker(\hat{\pi})$. Hence, (iii) implies (i). Suppose, conversely, that the inclusion in (i) holds. It suffices to show that $\sup_{i\in I} \|\hat{\pi}_i(x)\| \ge \|\hat{\pi}(x)\|$, for all positive elements of $x \in C^*(\Gamma)_+$. Indeed, the inequality then follows for all $x \in C^*(\Gamma)$ by the C^*-identity for the norm. So let $x \in C^*(\Gamma)_+$ be a positive element. For each $t \in \mathbb{R}_+$, let $f_t : \mathbb{R}_+ \to \mathbb{R}_+$ be the continuous function given by $f_t(s) = \max\{s - t, 0\}$. By the properties of the continuous functional calculus, $\hat{\pi}_i(f_t(x)) = f_t(\hat{\pi}_i(x))$, for all $i \in I$ and all $t \in \mathbb{R}_+$. Set $t = \sup_{i\in I} \|\hat{\pi}_i(x)\|$. Then, for all $i \in I$, f_t equals zero on the spectrum of $\hat{\pi}_i(x)$, and so, $\hat{\pi}_i(f_t(x)) = f_t(\hat{\pi}_i(x)) = 0$. By our assumption that the inclusion of (i) holds, it follows that $f_t(\hat{\pi}(x)) = \hat{\pi}(f_t(x)) = 0$. Thus, f_t equals zero on the spectrum of $\hat{\pi}(x)$. But then the spectrum of $\hat{\pi}(x)$ must be contained in the interval $[0, t]$. Because $\hat{\pi}(x)$ is normal, the spectral radius formular implies that

$$\|\hat{\pi}(x)\| \leq t = \sup_{i\in I} \|\hat{\pi}_i(x)\| \, .$$

This shoes that (i) implies (iii). ☐

Proposition 1.3.10. *Let $(\pi_i)_{i\in I}$ be a net in $\widehat{\Gamma}$ and let $\pi \in \widehat{\Gamma}$. Then π is a cluster point for $(\pi_i)_{i\in I}$ if and only if one and hence all of the following equivalent conditions hold:*

(i) For every $i_0 \in I$, we have the inclusion $\bigcap_{i\succcurlyeq i_0} \ker(\hat{\pi}_i) \subset \ker(\hat{\pi})$,

(ii) For every $i_0 \in I$, $\pi \prec \bigoplus_{i\succcurlyeq i_0} \pi_i$,

(iii) We have the inequality $\limsup_{i\in I} \|\hat{\pi}_i(x)\| \geq \|\hat{\pi}(x)\|$, for all $x \in C^*(\Gamma)$.

Proof. We have that π is a cluster point for $(\pi_i)_{i\in I}$ if and only if π lies in the closure of all sets $\{\, \pi_i \mid i \succcurlyeq i_0 \,\}$, for $i_0 \in I$. Hence, π is a cluster point if and only if (i) holds. The equivalence of (i) and (ii) is direct from Lemma 1.3.9.

Suppose that (iii) holds, for all $x \in C^*(\Gamma)$. Fix $i_0 \in I$. If $x \in \bigcap_{i\succcurlyeq i_0} \ker(\hat{\pi}_i)$, then

$$\|\hat{\pi}(x)\| \leq \limsup_{i\in I} \|\hat{\pi}_i(x)\| \leq \sup_{i\succcurlyeq i_0} \|\hat{\pi}_i(x)\| = 0,$$

and so, $x \in \ker(\hat{\pi})$, showing that (iii) implies (i). Suppose, conversely, that the inclusion in (i) holds, for all $i_0 \in I$. By Lemma 1.3.9, this implies that

$$\sup_{i\succcurlyeq i_0} \|\hat{\pi}_i(x)\| \geq \|\hat{\pi}(x)\| \, ,$$

for all $x \in C^*(\Gamma)$ and all $i_0 \in I$. Thus,

$$\limsup_{i\in I} \|\hat{\pi}_i(x)\| = \inf_{i_0\in I} \sup_{i\succcurlyeq i_0} \|\hat{\pi}_i(x)\| \geq \|\hat{\pi}(x)\| \, ,$$

as we wanted to show. ☐

Corollary 1.3.11. *Let $(\pi_i)_{i\in I}$ be a net in $\widehat{\Gamma}$ and let $\pi \in \widehat{\Gamma}$. Then π is a limit point for $(\pi_i)_{i\in I}$ if and only if, for every subnet $(\pi_j)_{j\in J}$ of $(\pi_i)_{i\in I}$, one and hence all of the following equivalent conditions hold:*

(i) $\bigcap_{j\in J} \ker(\hat{\pi}_j) \subset \ker(\hat{\pi})$,

(ii) $\pi \prec \bigoplus_{j\in J} \pi_j$,

(iii) $\sup_{j\in J} \|\hat{\pi}_j(x)\| \geq \|\hat{\pi}(x)\|$, for all $x \in C^*(\Gamma)$.

Proof. The conditions are equivalent by Lemma 1.3.9. We show that π is a limit point if and only if (i) holds. Suppose π is a limit point for $(\pi_i)_{i\in I}$ and let $(\pi_j)_{j\in J}$ be any subnet. Fix $j_0 \in J$. As π is, in particular, a cluster point for $(\pi_j)_{j\in J}$, Proposition 1.3.10 implies that

$$\bigcap_{j\in J} \ker(\hat{\pi}_j) \subset \bigcap_{j_0\preccurlyeq j\in J} \ker(\hat{\pi}_j) \subset \ker(\hat{\pi}).$$

Suppose, conversely, that (i) holds, for all subnets. Let $(\pi_j)_{j\in J}$ be any subnet. For each $j_0 \in J$, $(\pi_j)_{j_0\preccurlyeq j\in J}$ is a subnet of $(\pi_j)_{j\in J}$ and therefore also of $(\pi_i)_{i\in I}$. Our assumption therefore gives us the inclusion $\bigcap_{j_0\preccurlyeq j\in J} \ker(\hat{\pi}_j) \subset \ker(\hat{\pi})$, for all $j_0 \in J$. Thus, π is a cluster point for $(\pi_j)_{j\in J}$, by Proposition 1.3.10. We deduce that π is a cluster point for all subnets of $(\pi_i)_{i\in I}$, and so, π is a limit point, as we wanted to show. ☐

Definition 1.3.12. A non-empty linear subspace I of a unital C^*-algebra $\mathcal{A}$ is called an *algebraic (two-sided) ideal* of $\mathcal{A}$ if $axb \in I$, for all $x \in I$ and all $a, b \in \mathcal{A}$. An algebraic two-sided ideal which is, moreover, closed in norm is called a *(two-sided) ideal* of $\mathcal{A}$.

Let $\mathcal{A}$ be a unital C^*-algebra. For a two-sided ideal I of $\mathcal{A}$, the quotient $\mathcal{A}/I$ is an algebra with addition and multiplication given by addition and multiplication of representatives. Furthermore, the involution given by $(a+I)^* = a^* + I$ is well-defined and the quotient norm on $\mathcal{A}/I$ is a C^*-norm (see [36, Theorem 1.9.1]). We denote by $\pi_I : \mathcal{A} \to \mathcal{A}/I$ the canonical projection. It is clear that π_I is a *-homomorphism.

Lemma 1.3.13. *Let $\mathcal{A}$ be a unital C^*-algebra. Let I and J be two-sided ideals in $\mathcal{A}$ and denote by $\pi_I : \mathcal{A} \to \mathcal{A}/I$ and $\pi_J : \mathcal{A} \to \mathcal{A}/J$ the canonical quotient maps. The map $\Phi : \mathcal{A} \to \mathcal{A}/I \oplus \mathcal{A}/J$ given by*

$$\Phi(a) = (\pi_I(a), \pi_J(a)), \qquad a \in \mathcal{A},$$

is a *-homomorphism. Moreover,

(i) Φ is injective if and only if $I \cap J = \{0\}$,

(ii) Φ is surjective if and only if $I + J = \mathcal{A}$.

Proof. It is clear that Φ is a *-homomorphism because π_I and π_J are. To see that (i) holds, observe that an element a lies in the intersection $I \cap J$ if and only if $\Phi(a) = (0,0)$. It remains to show (ii). Suppose $I + J = \mathcal{A}$ and let $a, b \in \mathcal{A}$ so that $(\pi_I(a), \pi_J(b))$ is a generic element of $\mathcal{A}/I \oplus \mathcal{A}/J$. By assumption, a and b has decompositions $a = a_1 + a_2$ and $b = b_1 + b_2$, for $a_1, b_1 \in I$ and $a_2, b_2 \in J$. Then $\pi_I(a) = \pi_I(a_2 + b_1)$ and $\pi_J(b) = \pi_J(a_2 + b_1)$. Hence, Φ is surjective. Conversely, suppose $I + J \subsetneq \mathcal{A}$ and take $a \in \mathcal{A} \backslash (I + J)$. We claim that $(\pi_I(a), \pi_J(-a))$ is not in the image of Φ. Indeed, if it were, we could find an element $x \in \mathcal{A}$ such that $a - x \in I$ and $a + x \in J$. Adding these two equalities leads to a contradiction. Hence, Φ is not surjective. $\square$

Remark 1.3.14. Since $\hat{1}_\Gamma : C^*(\Gamma) \to \mathbb{C}$ is a surjective *-homomorphism, we see that the quotient C^*-algebra $C^*(\Gamma)/\ker(\hat{1}_\Gamma)$ is *-isomorphic to $\mathbb{C}$. Specifically, the *-isomorphism is the unique map optained from the universal property of the quotient making the following diagram commutative:

$$\begin{tikzcd} C^*(\Gamma) \arrow[r, "\hat{1}_\Gamma"] \arrow[d, "\pi_{\ker \hat{1}_\Gamma}"'] & \mathbb{C} \\ C^*(\Gamma)/\ker \hat{1}_\Gamma \arrow[ur, dashed] & \end{tikzcd}$$

Remark 1.3.15. The kernel of the trivial representation, $\ker(\hat{1}_\Gamma)$, is a maximal ideal in $C^*(\Gamma)$. Indeed, if there were another ideal $\ker(\hat{1}_\Gamma) \subsetneq J \subset C^*(\Gamma)$, we could find $x \in J \backslash \ker(\hat{1}_\Gamma)$. Since such an x cannot be 0, the image of $[x]_{\ker(\hat{1}_\Gamma)}$ under the isomorphism $C^*(\Gamma)/\ker(\hat{1}_\Gamma) \cong \mathbb{C}$ from Remark 1.3.14 is not zero, and so, by normalizing, we find an element $x' \in J$ such that $[x']_{\ker(\hat{1}_\Gamma)}$ maps to 1. But then $x' = 1 + y$, for some $y \in \ker(\hat{1}_\Gamma)$. As J contains $\ker(\hat{1}_\Gamma)$, we deduce that $1 \in J$, and so, $J = C^*(\Gamma)$.

Definition 1.3.16. A two-sided ideal I of a unital C^*-algebra $\mathcal{A}$ is called *complemented* if there exists another two-sided ideal J in $\mathcal{A}$ such that $I \cap J = \{0\}$ and such that $I + J = \mathcal{A}$.

Remark 1.3.17. If I is a complemented ideal in a unital C^*-algebra $\mathcal{A}$, then Lemma 1.3.13 yields that $\mathcal{A}$ is *-isomorphic to $A/I \oplus \mathcal{B}$, for some unital C^*-algebra $\mathcal{B}$.

Let $S \subset \widehat{\Gamma}$ be a subset. A representation $\pi \in S \subset \widehat{\Gamma}$ is called an *isolated point* in S if the singleton $\{\pi\}$ is an open set in the subspace topology, or, equivalently, if $\pi \notin \overline{S \backslash \{\pi\}}$. That is, π is isolated in S if and only if

$$\bigcap_{\rho \in S \backslash \{\pi\}} \ker(\hat{\rho}) \not\subset \ker(\hat{\pi}) \tag{1.3}$$

Proposition 1.3.18. *Let Γ be a discrete group. The following are equivalent:*

(i) *Γ has Property (T),*

(ii) *The trivial representation, 1_Γ, is an isolated point in the unitary dual, $\widehat{\Gamma}$,*

(iii) *The kernel of $\hat{1}_\Gamma$ is a complemented ideal in $C^*(\Gamma)$,*

(iv) *There is a central projection q in $C^*(\Gamma)$ such that $\hat{1}_\Gamma(q) = 1$ and $u_g q = q u_g = q$, for all $g \in \Gamma$.*

Proof. (i)$\Rightarrow$(ii): Suppose 1_Γ is not an isolated point in $\widehat{\Gamma}$. Then 1_Γ lies in the closure of $\widehat{\Gamma}\backslash\{1_\Gamma\}$, and so, we may find a net $(\pi_i)_{i\in I}$ in $\widehat{\Gamma}\backslash\{1_\Gamma\}$ converging to 1_Γ. Let $\pi = \bigoplus_{i\in I} \pi_i$. Then 1_Γ is weakly contained in π, by Corollary 1.3.11. But π does not contain 1_Γ, as otherwise, 1_Γ would be contained in π_{i_0}, for some $i_0 \in I$, by Proposition 1.1.14. By construction, π_{i_0} is an irreducible representation not equivalent to 1_Γ, so this would be a contradiction. Hence, π is a unitary representation weakly containing but not containing 1_Γ, and so, Γ does not have property (T), by Corollary 1.3.6.

(ii)$\Rightarrow$(iii): Let $I_0 = \ker(\hat{1}_\Gamma)$ and let $J_0 = \bigcap_{\pi \in \widehat{\Gamma}\backslash\{1_\Gamma\}} \ker(\hat{\pi})$. Recall that there are enough irreducible representations to separate the points of $C^*(\Gamma)$ (see, e.g., [54, Theorem 13.11(a)] and recall that pure states correspond to irreducible representations via the GNS construction). Therefore, as the irreducible representations of Γ are in 1-1 correspondence with the irreducible representations of $C^*(\Gamma)$, we see that

$$I_0 \cap J_0 = \bigcap_{\pi \in \widehat{\Gamma}} \ker(\hat{\pi}) = \{0\}.$$

If 1_Γ is isolated in $\widehat{\Gamma}$, equation (1.3) yields that $J_0 \not\subset I_0$, and so, $I_0 + J_0$ is an ideal strictly containing I_0. But I_0 is a maximal ideal by Remark 1.3.15. Thus, $I_0 + J_0 = C^*(\Gamma)$. That is, $\ker(\hat{1}_\Gamma)$ is complemented in $C^*(\Gamma)$.

(iii)$\Rightarrow$(iv): By Lemma 1.3.13, Remark 1.3.17 and Remark 1.3.14, we have a *-isomorphism $\Phi : C^*(\Gamma) \to \mathbb{C} \oplus \mathcal{B}$, for some unital C^*-algebra $\mathcal{B}$. Moreover, we have $\hat{1}_\Gamma = \pi_\mathbb{C} \circ \Phi$, where $\pi_\mathbb{C}$ is the canonical projection $\mathbb{C} \oplus \mathcal{B} \to \mathbb{C}$. As Φ is a *-isomophism, there exists a (unique) central projection $q \in C^*(\Gamma)$ such that $\Phi(q) = (1, 0)$. We see immediately that $1_\Gamma(q) = \pi_\mathbb{C}(\Phi(q)) = 1$. Further, for each $g \in \Gamma$, $\pi_\mathbb{C}(\Phi(u_g)) = \hat{1}_\Gamma(u_g) = 1$, and so, we must have $\Phi(u_g) = (1, v_g)$, for some unitary element $v_g \in \mathcal{B}$. It follows that $q u_g = u_g q = q$, for all $g \in \Gamma$.

(iv)$\Rightarrow$(ii): Suppose for contradiction that 1_Γ is not an isolated point and let $(\pi_i)_{i\in I}$ be a net in $\widehat{\Gamma}\backslash\{1_\Gamma\}$ converging to $\hat{1}_\Gamma$. By Corollary 1.3.11, $\sup \|\hat{\pi}_i(q)\| \geq \left|\hat{1}_\Gamma(q)\right| = 1$. Hence, we may find an index $i \in I$ such that $\hat{\pi}_i(q)$ is non-zero. Let H_i be the Hilbert space on which $\hat{\pi}_i$ acts. By the properties of q and since $\hat{\pi}_i$ is a representation, we have, for each $g \in \Gamma$ and each $\xi \in H_i$,

$$\pi_i(g)\big(\hat{\pi}_i(q)\xi\big) = \hat{\pi}_i(u_g)\big(\hat{\pi}_i(q)\xi\big) = \hat{\pi}_i(u_g q)\xi = \hat{\pi}_i(q)\xi.$$

Hence, $\hat{\pi}_i(q)\xi$ is a Γ-invariant vector for π_i, for all $\xi \in H_i$. As $\hat{\pi}_i(q)$ is non-zero, this implies the existence of a non-zero Γ-invariant vector for π_i. But π_i is irreducible, and so, we must have that π_i is equivalent to 1_Γ. But this contradicts that $(\hat{\pi}_i)_{i\in I}$ is a net in $\widehat{\Gamma}\backslash\{1_\Gamma\}$. ☐

Remark 1.3.19. The projection in Proposition 1.3.18 is known as a *Kazhdan projection.*

1.4 Literature

The brief introduction to representations on Hilbert spaces in Section 1.1 is based on [5, Appendix A]. In the introduction to Kazhdan's Property (T) in Section 1.2 we follow mainly [9, Chapter 12.1 and Chapter 6.4]. Another reference for an introduction to Property (T) is [5, Chapter 1]. In the introduction to this book, the reader may also find more references to the literature on the topic. The results of Section 1.3 can be found in [5, Section 1.2 and Appendix F]. The presentation given in Section 1.3 is based mainly on private communication with my supervisor, Magdalena Musat.

Chapter 2

Property (T) as a fixed-point property

We introduce in the last section of this chapter, Section 2.5, a property regarding the existence of fixed points for affine actions on real Hilbert spaces. Our aim is to show that this property, known as Property (FH), is equivalent with Property (T). In the first four sections we shall develop the tools needed to study affine actions and, in particular, to proof the mentioned equivalence. These tools will also be useful to us when studying group actions on Banach spaces in Chapter 4, and we shall therefore not restrict ourselves to the Hilbert space setting before in the final section. In Section 2.1 we introduce what we mean by an affine map and discuss how this set of maps on a normed vector space relates to the set of isometric maps. In Section 2.2 we discuss affine actions of a group on a general vector space, and we define and study the vector space of 1-*cocycles* with respect to a (linear) representation. In Section 2.3, we introduce the *first cohomology* of a group with coefficients in a (linear) representation, and we discuss how one may use analysis to study this object for representations on Banach spaces. In Section 2.4, we give an introduction to the theory of positivity of functions on groups. At first, this section might seem misplaced, but we shall see that the tools developed here will be very useful to us in the final section of this chapter.

2.1 Affine maps and isometries

Definition 2.1.1. A map $f : V \to W$ between vector spaces over the same field $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$ is said to be *affine* if

$$f\left(\lambda x + (1-\lambda)y\right) = \lambda f(x) + (1-\lambda)f(y) \tag{2.1}$$

for all $x, y \in V$ and all $\lambda \in \mathbb{K}$. We denote by $\mathrm{Aff}(V)$ the group of affine bijections on V.

Example 2.1.2. The following are two easy but important examples of affine maps on a vector space V:

- Any linear map is affine. In particular, the set of affine bijections $\mathrm{Aff}(V)$ contains the general linear group of V.
- A *translation* on V is a map T_{x_0} on V of the form $T_{x_0}(x) = x + x_0$, for some fixed vector $x_0 \in V$. All translations are affine. Moreover, all translations are bijections with $T_{x_0}^{-1} = T_{-x_0}$. ○

Remark 2.1.3. We can, equivalently, define a map $f : V \to W$ to be affine if the map $f_0 : V \to W$ given by

$$f_0(x) = f(x) - f(0), \qquad \text{for } x \in V, \tag{2.2}$$

is linear. If f is affine, then f_0 is called the *linear part* of f and the vector $f(0)$ is called the *translation part* of f. Clearly, an affine map is linear, if its translation part is zero.

Remark 2.1.4. An affine map is injective (respectively surjective), if and only if its linear part is injective (respectively surjective).

Definition 2.1.5. A map $f : V \to W$ between normed vector spaces is called an *isometry* if

$$\|f(x) - f(y)\| = \|x - y\| ,$$

for all $x, y \in V$. We denote by $\mathrm{Isom}(V)$ the set of bijective isometries on V.

Remark 2.1.6. Isometries are always continuous and injective.

An affine map is an isometry if and only if its linear part is. Isometries, on the other hand, need not be affine. However, a famous result due to S. Mazur and S. Ulam in [29] show that bijective isometries between real vector spaces are always affine. We present below a proof following [50].

Lemma 2.1.7. *A map $f : V \to W$ between real vector spaces is affine if and only if*

$$f(tx + (1-t)y) = tf(x) + (1-t)f(y) \tag{2.3}$$

for all $x, y \in V$ and all $t \in [0, 1]$.

Proof. We need only proof that, if equation (2.3) hold for all $t \in [0, 1]$, it holds for all $t \in \mathbb{R}$. So let $x, y \in V$ and let $t \in \mathbb{R}\backslash[0, 1]$. We may assume without loss of generality that $t < 0$ (otherwise, exchange the role of x and y). Then $1/(1-t)$ lies in $[0, 1]$, and so,

$$\begin{aligned} f(y) &= f\left(\frac{1}{1-t}(tx + (1-t)y) + \left(1 - \frac{1}{1-t}\right)x\right) \\ &= \frac{1}{1-t}f(tx + (1-t)y) + \left(1 - \frac{1}{1-t}\right)f(x). \end{aligned}$$

equation (2.3) follows by rearrangement. □

Lemma 2.1.8. *Let V and W be normed real vector spaces. If $f : V \to W$ is continuous, then f is affine if and only if*

$$f\left(\tfrac{1}{2}x + \tfrac{1}{2}y\right) = \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y), \tag{2.4}$$

for all $x, y \in V$.

Proof. Suppose the condition of equation (2.4) hold. We obtain equation (2.3), for all dyadic rationals t in $[0, 1]$ and all $x, y \in V$, by induction. Since the dyadic rationals are dense in $[0, 1]$, continuity of f implies that f is affine. □

Let V be a real vector space. For each $z \in V$, we define the map $r_z : V \to V$ by $r_z(x) = 2z - x$. The map r_z is called the *reflection of V in z*. It is an affine map, and it clear that $r_z^2 = \mathrm{id}_V$. Thus, r_z is bijective with $r_z^{-1} = r_z$. Moreover, z is a fixed point for r_z, and it is the only fixed point. If V is normed, the following equalities are direct from the definition:

$$\|r_z(x) - z\| = \|x - z\| , \qquad \|r_z(x) - x\| = 2\|x - z\| , \tag{2.5}$$

for all $x \in V$.

Theorem 2.1.9 (Mazur-Ulam)**.** *Every bijective isometry between real normed vector spaces is affine.*

Proof. Let $f : V \to W$ be a bijective isometry between real normed vector spaces. For $x, y \in V$, set $z = (x+y)/2 \in V$ and $z' = (f(x)+f(y))/2$. We show that $f(z) = z'$. This will imply that f is affine by Lemma 2.1.8.

Let $\mathrm{Isom}(V; x, y)$ denote the set of all bijective isometries on V keeping the points x and y fixed. For each such map g, we have $\|gz - x\| = \|gz - gx\| = \|z - x\|$, and so

$$\|g(z) - z\| \leq \|g(z) - x\| + \|x - z\| = 2\,\|x - z\|\,.$$

Hence,

$$c = \sup\{\,\|g(z) - z\| \mid g \in \mathrm{Isom}(V; x, y)\,\} < \infty.$$

For $g \in \mathrm{Isom}(V; x, y)$, let $\tilde{g} = r_z g^{-1} r_z g$, where r_z is the reflection of V in z. Then $\tilde{g} \in \mathrm{Isom}(V; x, y)$, as well, and so $\|\tilde{g}(z) - z\| \leq c$. Using this together with the equalities in equation (2.5) and the fact that g^{-1} is an isometry, we deduce that

$$2\,\|g(z) - z\| = \left\|r_z\big(g(z)\big) - g(z)\right\| = \left\|g^{-1} r_z g(z) - z\right\| = \|\tilde{g}(z) - z\| \leq c.$$

Taking the supremum of the left hand side over all $g \in \mathrm{Isom}(V; x, y)$, we obtain that $2c \leq c$. As c is non-negative and finite, we must therefore have $c = 0$. Hence, z is a fixed point, for all $g \in \mathrm{Isom}(V; x, y)$.

Let $r_{z'}$ be the reflection of W in z'. Then $g = r_z f^{-1} r_{z'} f$ is in $\mathrm{Isom}(V; x, y)$, and our preparations above then implies that $g(z) = z$. It follows that $r_{z'} f(z) = f(z)$, so as z' is the only fixed point for $r_{z'}$, we conclude that $f(z) = z'$, as we wanted to show. □

2.2 Affine actions and 1-cocycles

Definition 2.2.1. Let Γ be a discrete group and let V a vector space. An *affine action* of Γ on V is a group homomorphism $\Gamma \to \mathrm{Aff}(V)$.

Example 2.2.2. Let V be a vector space. We may reformulate Example 2.1.2 in the language of affine actions:

- The inclusion $\mathrm{GL}(V) \subset \mathrm{Aff}(V)$ is an affine action of $\mathrm{GL}(V)$ on V.
- Any additive subgroup of V acts on V by translation. ∘

Let Γ be a discrete group and $\alpha : \Gamma \to \mathrm{Aff}(V)$ be an affine action. For each $g \in \Gamma$, let $\pi(g) : V \to V$ be the linear part of $\alpha(g)$, and $b(g)$ be the translation part of $\alpha(g)$. Then π is a representation of Γ on V, and $b : \Gamma \to V$ satisfies the relation

$$b(gh) = b(g) + \pi(g)b(h), \tag{2.6}$$

for all $g, h \in \Gamma$, because α is a homomorphism. This relation is called the 1*-cocycle relation* (with respect to π). The representation π is called the *linear part* of α, and the map b is called the *translation part* of α.

We shall change our perspective now and fix a representation π of Γ on a vector space V.

Definition 2.2.3. A map $b : \Gamma \to V$ satisfying the 1-cocycle relation of equation (2.6) is called a *1-cocycle* with respect to π. The set of all 1-cocycles with respect to π is denoted by $Z^1(\Gamma, \pi)$.

Remark 2.2.4. The set $Z^1(\Gamma, \pi)$ of 1-cocycles with respect to a fixed representation π has the structure of a vector space with addition and scalar multiplication given pointwise.

Proposition 2.2.5. *Let Γ be a discrete group and (π, V) a representation of Γ. An action α of Γ on V is an affine action with linear part π if and only if there exists a 1-cocycle $b : \Gamma \to V$ with respect to π such that*

$$\alpha(g)x = \pi(g)x + b(g), \tag{2.7}$$

for all $g \in \Gamma$ and all $x \in V$.

Proof. If α is an affine action with linear part π then the translation part of α has the required properties. Conversely, suppose there exists a map $b : \Gamma \to V$ satisfying the 1-cocycle relation with respect to π and such that equation (2.7) hold, for all $g \in \Gamma$ and all $x \in V$. Then each map $\alpha(g) : V \to V$, for $g \in \Gamma$, is affine, and the 1-cocycle relation ensures that α is a homomorphism. □

Remark 2.2.6. Given a representation π of a discrete group, Proposition 2.2.5 shows that the set of affine actions with linear part π corresponds exactly to the set of 1-cocycles with respect to π. For a 1-cocycle $b \in Z^1(\Gamma, \pi)$, the action α defined by equation (2.7) is referred to as the affine action *associated to* π and b.

Lemma 2.2.7. *Any 1-cocycle $b \in Z^1(\Gamma, \pi)$ has the following properties:*

(i) $b(e) = 0$,

(ii) $\pi(g)b(g^{-1}) = -b(g)$, *for all $g \in \Gamma$.*

Proof. Since, for any representation π, $\pi(e) = \mathrm{id}$, we have

$$b(e) = b(e^2) = 2b(e).$$

We must therefore necessarily have $b(e) = 0$. Then, for all $g \in \Gamma$,

$$0 = b(e) = b(gg^{-1}) = b(g) + \pi(g)b(g^{-1}).$$

This finishes the proof. □

The following is an important example of a 1-cocycle:

Example 2.2.8. Let π be a representation of the discrete group Γ on a vector space V, let $x_0 \in V$, and let $b : \Gamma \to V$ be the map given by

$$b(g) = \pi(g)x_0 - x_0, \qquad \text{for all } g \in \Gamma. \tag{2.8}$$

Then, for each $g, h \in \Gamma$,

$$b(gh) = \pi(gh)x_0 - x_0 = \pi(g)(\pi(h)x_0 - x_0) + \pi(g)x_0 - x_0 = \pi(g)b(h) + b(g).$$

Hence, b is a 1-cocycle with respect to π. Any 1-cocycle which has the form of equation (2.8), for some vector x_0, is called a *1-coboundary* with respect to π. The set of all 1-coboundaries with respect to π is denoted by $B^1(\Gamma, \pi)$. ○

Remark 2.2.9. The set of 1-coboundaries, $B^1(\Gamma, \pi)$, is a subspace of the vector space of all 1-cocycles, $Z^1(\Gamma, \pi)$.

Definition 2.2.10. Let α be an affine action of the discrete group Γ on a vector space V. A *fixed point* for α in V is a vector $x \in V$ satisfying $\alpha(g)x = x$, for all $g \in \Gamma$.

Lemma 2.2.11. *Let Γ be a discrete group, V a vector space, and let α be an affine action of Γ on V with linear part π and translation part b. The following are equivalent:*

(i) α *has a fixed point in* V*,*

(ii) b *is a 1-coboundary,*

(iii) α *is conjugate to* π *via a translation.*

Proof. If $-x_0 \in V$ is a fixed point for α then

$$b(g) = \alpha(g)(-x_0) - \pi(g)(-x_0) = \pi(g)x_0 - x_0.$$

That is, b is the 1-coboundary associated to x_0. Further, if b is the 1-coboundary associated to $x_0 \in V$, then α has the following form:

$$\alpha(g)x = \pi(g)x + b(g) = \pi(g)(x + x_0) - x_0 = T_{x_0}^{-1} \circ \pi(g) \circ T_{x_0}x,$$

for all $g \in \Gamma$ and $x \in V$. That is, α is conjugate to π via the translation T_{x_0}. Finally, if $\alpha = T_{x_0}^{-1} \circ \pi \circ T_{x_0}$, for some $x_0 \in V$, then

$$\alpha(g)(-x_0) = T_{x_0}^{-1} \circ \pi(g) \circ T_{x_0}(-x_0) = T_{x_0}^{-1}(0) = -x_0.$$

Hence, $-x_0$ is a fixed point for α. □

2.3 1-cohomology

For a fixed representation π of a discrete group Γ on a vector space V, we define the *first cohomology* of Γ with coefficients in π as the quotient vector space

$$H^1(\Gamma, \pi) = Z^1(\Gamma, \pi)/B^1(\Gamma, \pi).$$

When all 1-cocycles with respect to π are 1-coboundaries, the first cohomology group vanishes. By Lemma 2.2.11, this vanishing therefore captures that all affine actions of Γ on V with linear part π has a fixed point. We state this observation in Lemma 2.3.1 below.

Lemma 2.3.1. *Let* Γ *be a discrete group and let* π *be a representation on a vector space* V*. Then* $H^1(\Gamma, \pi) = 0$ *if and only if all affine actions of* Γ *on* V *with linear part* π *has a fixed point in* V*.*

Proof. We have that $H^1(\Gamma, \pi) = 0$ if and only if all 1-cocycles on V with respect to π are 1-coboundaries. By Proposition 2.2.5, the affine actions with linear part π correspond exactly to the 1-cocycles on V with respect to π. Hence, by Lemma 2.2.11, $H^1(\Gamma, \pi) = 0$ if and only if all affine actions with linear part π have a fixed point. □

Let X be a topological vector space and let $C(\Gamma, X)$ denote the set of all functions $\Gamma \to X$. We say that a net $(f_i)_{i \in I}$ in $C(\Gamma, X)$ *converges pointwise* to f if $f_i(g) \to f(g)$ as a net in X, for all $g \in \Gamma$. The defines a topology the $C(\Gamma, X)$ known as the *topology of pointwise convergence*. When the topology on X comes from a norm, the induced topology of pointwise convergence on $C(\Gamma, X)$ is generated by the separating family of seminorms $\{p_g\}_{g \in \Gamma}$, with $p_g(f) = \|f(x)\|$, for all $g \in \Gamma$ and $f \in C(\Gamma, X)$. In particular, it is Hausdorff (see [19, Proposition 5.16(a)]). If X is a Banach space, the topology of pointwise convergence on $C(\Gamma, X)$ is complete, as well. If X is normed and Γ is countable, the topology of pointwise convergence on $C(\Gamma, X)$ is metrizable with a translation invariant metric (see [19, Proposition 5.16(b)]).

Let π be a representation of the countable discrete group Γ on a normed vector space X. The vector space of 1-cocycles, $Z^1(\Gamma, \pi)$, inherits the topology of pointwise convergence from the inclusion $Z^1(\Gamma, \pi) \subset C(\Gamma, X)$. We show in the proposition below that, if X is Banach, $Z^1(\Gamma, \pi)$ is complete. Recall that a Hausdorff topological vector space whose topology is generated by a family of seminorms is called a *Fréchet space* if it is complete.

Proposition 2.3.2. *Let (π, X) be a representation of the countable discrete group Γ by linear surjective isometries on a normed vector space X. If X is a Banach space then $Z^1(\Gamma, \pi)$ is a Fréchet space in the topology of pointwise convergence.*

Proof. We need only show that $Z^1(\Gamma, \pi)$ is closed in $C(\Gamma, X)$. To show this, let $(b_n)_{n\geq 1}$ be a sequence in $Z^1(\Gamma, \pi)$ converging to b in $C(\Gamma, X)$. Then, for each pair of group elements $g, h \in \Gamma$, $b_n(gh) \to b(gh)$. But also, $b_n(gh) = b_n(g) + \pi(g)b_n(h) \to b(g) + \pi(g)b(h)$. Since the topology of pointwise convergence on $C(\Gamma, X)$ is Hausdorff we deduce that $b(gh) = b(g) + \pi(g)b(h)$. Hence, b is a 1-cocycle. □

Remark 2.3.3. In general, the set of 1-coboundaries need not be closed in the set of 1-cocycles. If $B^1(\Gamma, \pi)$ is not closed in $Z^1(\Gamma, \pi)$ it follows, in particular, that $H^1(\Gamma, \pi) \neq 0$. Hence, by Lemma 2.3.1, showing that $B^1(\Gamma, \pi)$ is not closed provides a mean for showing the existence of an affine action with linear part π which does not have any fixed points. Loosely speaking, topologizing the vector space of 1-cocycles allows us to use analysis to study fixed-point properties for group actions on Banach spaces.

We show in the proposition below that the closure of $B^1(\Gamma, \pi)$ is connected to the existence of almost invariant vectors for π.

Proposition 2.3.4. *Let (π, X) be a representation of a countable discrete group Γ by linear surjective isometries on a Banach space. Assume that π does not have any non-zero invariant vectors. Then π has almost invariant vectors if and only if $B^1(\Gamma, \pi)$ is not closed in $Z^1(\Gamma, \pi)$ with respect to the topology of pointwise convergence.*

Proof. For each $x \in X$, denote by b_x the 1-coboundary given by $b_x(g) = \pi(g)x - x$, for $g \in \Gamma$. By definition of $B^1(\Gamma, \pi)$, each 1-coboundary has this form. Define a map $\beta : X \to B^1(\Gamma, \pi)$ by setting $\beta(x) = b_x$. Then β is clearly a linear and surjective map. It is also clear that β is continuous as, for each $g \in \Gamma$ and each $x, y \in B$, $p_g(b_x - b_y) \leq 2\|x - y\|$. Further, when π has no non-zero invariant vectors, we see that β is injective.

Suppose that π does not have almost invariant vectors. Then there exists a group element $g_0 \in \Gamma$ and an $\varepsilon > 0$ such that $\|\pi(g_0)x - x\| \geq \varepsilon \|x\|$, for all $x \in X$. Let $(x_i)_{i\in I}$ be a net in X such that the corresponding net $(b_{x_i})_{i\in I}$ in $B^1(\Gamma, \pi)$ converges pointwise to some $b \in Z^1(\Gamma, \pi)$. We may then extract a subnet in the form of a sequence $(x_n)_{n\geq 1}$ such that $p_{g_0}(b_{x_n} - b) \to 0$. For each $n, m \in \mathbb{N}$, we see that

$$\begin{aligned}\|x_n - x_m\| &\leq \varepsilon^{-1} \|\pi(g_0)(x_n - x_m) - (x_n - x_m)\| \\ &\leq \varepsilon^{-1}\big(\|\pi(g_0)x_n - x_n - b(g_0)\| + \|\pi(g_0)x_m - x_m - b(g_0)\|\big) \\ &= \varepsilon^{-1}\big(p_{g_0}(b_{x_n} - b) + p_{g_0}(b_{x_m} - b)\big).\end{aligned}$$

Thus, $(x_n)_{n\geq 1}$ is Cauchy. Let x_0 be the point of convergence. For each $g \in \Gamma$, $p_g(b_{x_n} - b_{x_0}) \leq 2\|x_n - x_0\|$, and so, $(b_{x_n})_{n\geq 1}$ converges pointwise to b_{x_0}. Further, for $g \in \Gamma$ and $\varepsilon > 0$, take $n_0 \in \mathbb{N}$ such that $p_g(b_{x_n} - b_{x_0}) < \varepsilon/3$, for all $n \geq n_0$, and take $i_0 \in I$ such that $p_g(b_{x_i} - b) < \varepsilon/3$, for all $i \succcurlyeq i_0$. Denote by $\phi : \mathbb{N} \to I$ the index map defining $(\xi_n)_{n\geq 1}$ as a subnet and take $n_1 \in \mathbb{N}$ such that $\phi(n) \succcurlyeq i_0$, for every $n \geq n_1$. Let $n_\varepsilon = \max\{n_0, n_1\}$. Then

$$p_g(b_{x_i} - b_{x_0}) \leq p_g(b_{x_i} - b) + p_g\big(b - b_{x_{\phi(n_\varepsilon)}}\big) + p_g(b_{x_{n_\varepsilon}} - b_{x_0}) < \varepsilon,$$

for all $i \succcurlyeq i_0$. Since $g \in \Gamma$ and $\varepsilon > 0$ was arbitrary, this shows that $(b_{x_i})_{i\in I}$ converges to b_{x_0}. We deduce that $b = b_{x_0}$, as the topology of pointwise convergence is Hausdorff. Hence, $B^1(\Gamma, \pi)$ is closed in $Z^1(\Gamma, \pi)$.

Conversely, suppose $B^1(\Gamma, \pi)$ is closed in $Z^1(\Gamma, \pi)$. Then $B^1(\Gamma, \pi)$ is Fréchet by Proposition 2.3.2. Moreover, since Γ is countable, the topology of pointwise convergence is metrizable by a translation invariant metric. It follows by the open mapping theorem (for complete topological vector spaces whose topology is induced by a translation invariant metric) that β is open (see [40, Corollary 2.12(a)]). In particular, the set

$$\beta(B_X(0,1)) = \left\{ b_x \in B^1(\Gamma,\pi) \;\middle|\; \|x\| < 1 \right\},$$

where $B_X(0,1)$ is the open unit ball in X, is open in $B^1(\Gamma,\pi)$. Hence, we can find a finite subset $F \subset \Gamma$ and an $\varepsilon > 0$ such that, whenever $b_x \in B^1(\Gamma,\pi)$ satisfies $\sup_{g\in F} p_g(b_x) < \varepsilon$, then $\|x\| < 1$. For each non-zero $x \in X$, we deduce that

$$\sup_{g\in F} \|\pi(g)x - x\| = \|x\| \sup_{g\in F} \left\|\pi(g)\big(x/\|x\|\big) - \big(x/\|x\|\big)\right\| = \|x\| \sup_{g\in F} p_g(b_{x/\|x\|}) \geq \varepsilon \|x\|.$$

We conclude that (π, B) does not have almost invariant vectors. $\square$

Remark 2.3.5. The assumption that Γ is countable in Proposition 2.3.4 above is used to show one of the implications of the proposition only: if π has almost invariant vectors then $B^1(\Gamma,\pi)$ is not closed. Specifically, it is used in the application of the open mapping theorem. When Γ is not countable, the topology of pointwise convergence on $C(\Gamma, X)$ need not be metrizable, and we do therefore not have the open mapping theorem, as stated in [40, Corollary 2.12(a)].

2.4 Positivity of functions on groups

When studying groups it is often useful to consider the set of complex valued functions on the group of interest instead of looking at the group directly. In particular, studying functions with certain positivity properties will proof to be a useful tool for us. We study in the following two subsections functions of *positive type* and functions *conditionally of negative type*. In Theorem 2.4.27, which is due to I. J. Schoenberg in [43], we show a connection between the two notions. We show immediately hereafter a consequence of this theorem allowing us to construct, from any affine action on a Banach space, a family unitary representations on complex Hilbert spaces with a set of properties that might at first seem technical, but that will later allow us to establish a connection between Kazhdan's Property (T) (or its generalization to the Banach spaces setting) and a property regarding fixed points for affine actions.

2.4.1 Functions of positive type

Definition 2.4.1. Let Γ be a discrete group. A *function of positive type* on Γ is a function $\varphi : \Gamma \to \mathbb{C}$ satisfying

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(g_j^{-1} g_i) \geq 0, \tag{2.9}$$

for all $n \in \mathbb{N}$, $g_1, \dots, g_n \in \Gamma$ and $c_1, \dots, c_n \in \mathbb{C}$. We denote by $\mathcal{P}(\Gamma)$ the set of all functions of positive type on Γ.

Remark 2.4.2. We remark that condition (2.9) is equivalent to requirering that all matrices of the form

$$\begin{pmatrix} \varphi(e) & \varphi(g_2^{-1}g_1) & \cdots & \varphi(g_n^{-1}g_1) \\ \varphi(g_1^{-1}g_2) & \varphi(e) & \cdots & \varphi(g_n^{-1}g_2) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(g_1^{-1}g_n) & \varphi(g_2^{-1}g_n) & \cdots & \varphi(e) \end{pmatrix},$$

for $g_1, \dots, g_n \in \Gamma$, are positive semidefinite.

Remark 2.4.3. We see directly from the definition of positive type that $\mathcal{P}(\Gamma)$ is a convex cone with addition and scalar multiplication defined pointwise. That is, if φ and ψ are of positive type, then so is $\varphi + \psi$ and $t\varphi$, for all $t \geq 0$.

Remark 2.4.4. When $\varphi : \Gamma \to \mathbb{R}$ is real-valued, it suffices for φ to be of positive type that equation (2.9) hold for all $g_1, \ldots, g_n \in \Gamma$ and all collections of *real* numbers $c_1, \ldots, c_n \in \mathbb{R}$.

As a subset of the set of all complex valued functions on Γ, $\mathcal{P}(\Gamma)$ inherits the topology of pointwise convergence. This topology is Hausdorff and, when Γ is countable, it is metrizable with a translation invariant metric (see [19, Proposition 5.16]). Proposition 2.4.5 below shows that $\mathcal{P}(\Gamma)$ is closed in the set of all complex valued functions on Γ.

Proposition 2.4.5. *Let $(\varphi_k)_{k\geq 1}$ be a sequence of functions of positive type on Γ. If the sequence converges pointwise to a function $\varphi : \Gamma \to \mathbb{C}$, then φ is of positive type, as well.*

Proof. For each finite $n \in \mathbb{N}$, and any collections $c_1, \ldots, c_n \in \mathbb{C}$ and $g_1, \ldots, g_n \in \Gamma$, we have

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(g_j^{-1} g_i) = \sum_{i,j=1}^{n} c_i \overline{c_j} \lim_{k\to\infty} \varphi_k(g_j^{-1} g_i) = \lim_{k\to\infty} \sum_{i,j=1}^{n} c_i \overline{c_j} \varphi_k(g_j^{-1} g_i).$$

If $(\varphi_k)_{k\geq 1}$ is a sequence of functions of positive type, the sum on the right-hand side is non-negative, for each $k \in \mathbb{N}$, and so, the limit is non-negative as well. Hence, if φ is the pointwise limit of a sequence of functions of positive type, it is itself of positive type. □

We shall give a few examples of functions of positive type in a moment. Before doing so, we record the following easy observation:

Lemma 2.4.6. *Let Γ be a discrete group and let $\varphi \in \mathcal{P}(\Gamma)$. For all $g \in \Gamma$, it holds that*

(i) $|\varphi(g)| \leq \varphi(e)$,

(ii) $\varphi(g^{-1}) = \overline{\varphi(g)}$.

Proof. Consider the 2×2-matrix

$$\begin{pmatrix} \varphi(e) & \varphi(g) \\ \varphi(g^{-1}) & \varphi(e) \end{pmatrix}.$$

This is a positive matrix by the assumption that φ is of positive type. In particular, it is hermitian and has a positive determinant. □

Example 2.4.7. The function $x \mapsto e^{i\theta x}$ is of positive type on $\mathbb{R}$ as an additive group, for any $\theta \in \mathbb{R}$. Indeed, for $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathbb{R}$, let $v \in M_{1,n}(\mathbb{C})$ be the row-vector $(e^{-i\theta x_1}, \ldots, e^{-i\theta x_n})$. Then $\left(e^{i\theta(x_j - x_k)}\right)_{j,k} = v^* v$. By Remark 2.4.2, this shows the claim. ○

Example 2.4.8. Let (π, H) be a unitary representation of the discrete group Γ and let $\xi \in H$. The function $\langle \pi(\,\cdot\,)\xi\,, \xi\rangle$ is of positive type. Indeed, for each choice of $g_1, \ldots, g_n \in \Gamma$ and $c_1, \ldots, c_n \in \mathbb{C}$, we have

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \langle \pi(g_j^{-1} g_i)\xi\,, \xi\rangle = \left\langle \sum_{i=1}^{n} c_i \pi(g_i)\xi\,, \sum_{j=1}^{n} c_j \pi(g_j)\xi \right\rangle \geq 0.$$

Hence, $\langle \pi(\,\cdot\,)\xi\,, \xi\rangle$ is of positive type. A function of this form is called a *function of positive type associated to* π. We denote by $\mathcal{P}^{\pi}(\Gamma)$ the set of all such functions. ○

Our second example is of particular importance. It turns out that every function of positive type on a discrete group is associated to some unitary representation through a GNS-type construction.

Theorem 2.4.9 (GNS construction for functions of positive type)**.** *Let φ be a function of positive type on a discrete group Γ. There exists a unitary representation (π_φ, H_φ) of Γ and a cyclic unit vector ξ_φ in H_φ such that*

$$\varphi(g) = \langle \pi_\varphi(g)\xi_\varphi \, , \xi_\varphi \rangle \, , \qquad \textit{for all } g \in \Gamma.$$

Moreover, the triplet $(\pi_\varphi, H_\varphi, \xi_\varphi)$ is unique in the sense that, if (π, H, ξ) is any other such triplet, then there exists a Hilbert space isomorphism $T : H_\varphi \to H$ intertwining π_φ and π and which satisfies $T\xi_\varphi = \xi$.

Proof. The case where φ is zero is trivial, so we may assume that this is not the case. For each $g \in \Gamma$, define $\varphi_g : \Gamma \to \mathbb{C}$ by setting $\varphi_g(h) = \varphi(h^{-1}g)$, for all $h \in \Gamma$. Observe that

$$\overline{\varphi_g(h)} = \overline{\varphi(h^{-1}g)} = \varphi(g^{-1}h) = \varphi_h(g), \tag{2.10}$$

for all $g, h \in \Gamma$, by Lemma 2.4.6. Set

$$V = \operatorname{span}_{\mathbb{C}} \{ \varphi_g \mid g \in \Gamma \} .$$

This is a linear subspace of the vector space of all complex valued functions on Γ. For $\varphi = \sum_{i=1}^n a_i \varphi_{g_i}$ and $\psi = \sum_{j=1}^m b_j \varphi_{h_j}$ in V, set

$$\langle \varphi \, , \psi \rangle = \left\langle \sum_{i=1}^n a_i \varphi_{g_i} \, , \sum_{j=1}^m b_j \varphi_{h_j} \right\rangle = \sum_{i=1}^n \sum_{j=1}^m a_i \overline{b_j} \varphi_{g_i}(h_j). \tag{2.11}$$

By equation (2.10), we see that

$$\langle \varphi \, , \psi \rangle = \sum_{j=1}^m \overline{b_j} \varphi(h_j) = \sum_{i=1}^n a_i \overline{\psi(g_i)}.$$

Hence, $\langle \varphi \, , \psi \rangle$ does not depend on the representation of φ and ψ in V, and $\langle \cdot \, , \cdot \rangle$ is therefore well-defined. Moreover, it is clear that $\langle \cdot \, , \cdot \rangle$ is conjugate symmetric and linear in the first entry. Further, since φ is of positive type, it is positive semi-definite. We show next that it is, in fact, positive definite, and so defines an inner product on V.

For any $g \in \Gamma$, we have $\langle \varphi_g \, , \varphi_g \rangle = \varphi(e)$. By Lemma 2.4.6, this is non-zero because φ is non-zero by assumption. Let $\varphi \in V$. For any $g \in \Gamma$, it follows from Cauchy-Schwarz that

$$|\varphi(g)|^2 = | \langle \varphi \, , \varphi_g \rangle |^2 \leq \langle \varphi_g \, , \varphi_g \rangle \langle \varphi \, , \varphi \rangle = \varphi(e) \langle \varphi \, , \varphi \rangle .$$

Hence, if $\langle \varphi \, , \varphi \rangle = 0$, we must have $\varphi = 0$.

Let H_φ be the completion of the pre-Hilbert space V with respect to $\langle \cdot \, , \cdot \rangle$. For each $g \in \Gamma$, set $\pi_\varphi(g)\varphi_h = \varphi_{gh}$, for all $h \in \Gamma$, and extend linearly to V. It is a straight forward computation to check that $\pi_\varphi(g_1)\pi_\varphi(g_2) = \pi_\varphi(g_1 g_2)$, that $\pi_\varphi(e) = \mathrm{id}_V$ and that $\pi_\varphi(g)^* = \pi_\varphi(g^{-1})$, for all $g, g_1, g_2 \in \Gamma$. It follows that $\pi_\varphi(g)$ is well-defined and inner product preserving, for all $g \in \Gamma$. Hence, $\pi_\varphi(g)$ is bounded and may therefore be extended to a unitary operator on H_φ. Let $\xi_\varphi = \varphi_e \in H_\varphi$. It is clear that ξ_φ is cyclic and that

$$\varphi(g) = \langle \varphi_g \, , \varphi_e \rangle = \langle \pi_\varphi(g)\varphi_e \, , \varphi_e \rangle \, , \qquad \text{for all } g \in \Gamma.$$

The triplet $(\pi_\varphi, H_\varphi, \xi_\varphi)$ then has the desired properties.

Suppose (π, H, ξ) is another triplet with the given properties. Define $T' : V \to H$ by setting $T'(\varphi_g) = \pi(g)\xi$ and extending linearly. We have, for $\sum_g a_g \varphi_g \in V$,

$$\begin{aligned} \left\| T'\left(\sum_g a_g \varphi_g \right) \right\|^2 &= \sum_{g,h} a_g \overline{a_h} \langle \pi(h^{-1}g)\xi \, , \xi \rangle = \sum_{g,h} a_g \overline{a_h} \langle \pi_\varphi(h^{-1}g)\xi \, , \xi \rangle \\ &= \left\| \sum_g a_g \pi_\varphi(g)\xi_\varphi \right\|^2 = \left\| \sum_g a_g \varphi_g \right\|^2 . \end{aligned}$$

Hence, T' extends to a well-defined isometry $T : H_\varphi \to H$. Moreover, since $T(V) = \operatorname{span}\{\, \pi(g)\xi \mid g \in \Gamma \,\}$, which is dense in H, we see that T is onto. Hence, T is a Hilbert space isomorphism. Finally, it is clear that $T\xi_\varphi = \xi$, and it is straight forward to check that T intertwines π_φ and π. $\square$

Remark 2.4.10. In the proof of Theorem 2.4.9, we see that, if φ is a *real-valued* function of positive type, then (π_φ, H_φ) is the comlexification of an orthogonal representation.

The next three propositions illustrate how one may construct new functions of positive type from old ones.

Proposition 2.4.11. *Let φ and ψ be functions of positive type on the discrete group Γ. Then their entrywise product, $\varphi \cdot \psi$, is a function of positive type on Γ, as well.*

Proof. Recall that the Schur product of two positive semidefinite matrices is, again, positive semidefinite. Hence, the statement is direct from Remark 2.4.2.

Alternatively, the statement can be proved from Theorem 2.4.9. Let $(\pi_\varphi, H_\varphi, \xi_\varphi)$ and $(\pi_\psi, H_\psi, \eta_\psi)$ be the GNS constructions associated to φ and ψ, respectively. Let π be the unitary representation of Γ on $H_\varphi \otimes H_\psi$ given on elementary tensors by

$$\pi(g)(\xi \otimes \eta) = \pi_\varphi(g)\xi \otimes \pi_\psi(g)\eta, \qquad \text{for all } \xi \in H_\varphi \text{ and } \eta \in H_\psi.$$

Then, for all $g \in \Gamma$,

$$\varphi \cdot \psi(g) = \varphi(g)\psi(g) = \langle \pi_\varphi(g)\xi_\varphi \,, \xi_\varphi \rangle \langle \pi_\psi(g)\eta_\psi \,, \eta_\psi \rangle = \langle \pi(g)(\xi_\varphi \otimes \eta_\psi) \,, \xi_\varphi \otimes \eta_\psi \rangle \,.$$

Hence, $\varphi \cdot \psi$ is of positive type by Example 2.4.8. $\square$

Proposition 2.4.12. *If $\varphi : \Gamma \to \mathbb{C}$ is a function of positive type, then so is $e^{\varphi + \theta}$, for all real numbers $\theta \in \mathbb{R}$.*

Proof. For $\theta \in \mathbb{R}$, e^θ is real. Thus, for any $n \in \mathbb{N}$, $c_1, \ldots, c_n \in \mathbb{C}$ and $g_1, \ldots, g_n \in \Gamma$, we have

$$\sum_{i,j=1}^n c_i \overline{c_j} e^{\varphi(g_j^{-1} g_i) + \theta} = \sum_{i,j=1}^n c_i e^{\theta/2} \overline{c_j e^{\theta/2}} e^{\varphi(g_j^{-1} g_i)}.$$

Hence, it is enough to show the statement for $\theta = 0$.

Observe that e^θ is the pointwise limit of the sequence $(\varphi_n)_{n \geq 1}$ with $\varphi_n = \sum_{k=1}^n \varphi^k / k!$. By inductive use of Proposition 2.4.11, we see that φ^k is a function of positive type, for each $k \in \mathbb{N}$. Since $\mathcal{P}(\Gamma)$ is a convex cone, it follows that φ_n is of positive type, for all $n \in \mathbb{N}$, and so, e^θ is of positive type by Proposition 2.4.5. $\square$

Proposition 2.4.13. *If $\varphi : \Gamma \to \mathbb{C}$ is a function of positive type, then so is $\operatorname{Re} \varphi$.*

Proof. Let A be any positive semidefinite square matrix and denote by $\operatorname{Re} A$ its entrywise real part. For any hermitian square matrix, entrywise complex conjugation amounts to applying the transpose map. Hence, $\operatorname{Re} A = \frac{1}{2}(A + A^T)$. Recall that both the identity map and the transposition map are positive maps. Thus, $\operatorname{Re} A$ is a convex combination of two positive semidefinite matrices, and therefore, it is itself positive semidefinite. The statement of the lemma follows by Remark 2.4.2. $\square$

Example 2.4.14. Applying Proposition 2.4.13 to Example 2.4.7 yields that $x \mapsto \cos(\theta x)$ is a function of positive type on the additive group of $\mathbb{R}$, for any $\theta \in \mathbb{R}$. $\circ$

2.4.2 A theorem by Schoenberg

Definition 2.4.15. Let Γ be a discrete group. A function $\psi : \Gamma \to \mathbb{C}$ is said to be *conditionally of negative type* if it has the following properties:

(i) $\psi(e) = 0$,

(ii) $\psi(g^{-1}) = \overline{\psi(g)}$, for all $g \in \Gamma$,

(iii) For any finite selection of elements $g_1, \ldots, g_n \in \Gamma$ and numbers $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$, it holds that

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \psi(g_j^{-1} g_i) \leq 0. \tag{2.12}$$

Remark 2.4.16. We see directly from the definition that the set of all functions conditionally of negative type on a discrete group Γ is a convex cone.

Remark 2.4.17. As with functions of positive type, when $\psi : \Gamma \to \mathbb{R}$ is real-valued, it suffices that equation (2.12) hold for all collections of *real* numbers $c_1, \ldots, c_n \in \mathbb{R}$ summing to zero.

Example 2.4.18. Let $\varphi : \Gamma \to \mathbb{C}$ be a function of positive type on Γ, and consider the function $\psi : \Gamma \to \mathbb{C}$ given by $\psi(g) = \varphi(e) - \varphi(g)$, for $g \in \Gamma$. It is clear that $\psi(e) = 0$, and further that $\psi(g^{-1}) = \overline{\psi(g)}$, for all $g \in \Gamma$, by Lemma 2.4.6. Let $g_1, \ldots, g_n \in \Gamma$ and $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$. We have

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \psi(g_j^{-1} g_i) = \sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(e) - \sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(g_j^{-1} g_i) = - \sum_{i,j=1}^{n} c_i \overline{c_j} \varphi(g_j^{-1} g_i) \leq 0.$$

Hence, ψ is conditionally of negative type. ○

Example 2.4.19. Let H be a real Hilbert space. Let $\xi_1, \ldots, \xi_n \in H$ and let $c_1, \ldots, c_n \in \mathbb{R}$ with $\sum_{i=1}^n c_i = 0$. We have

$$\sum_{i,j=1}^{n} c_i c_j \left\| \xi_i - \xi_j \right\|^2 = \sum_{i,j=1} c_i c_j \left\| \xi_i \right\|^2 + \sum_{i,j=1} c_i c_j \left\| \xi_j \right\| - 2 \sum_{i,j=1} c_i c_j \left\langle \xi_i \, , \xi_j \right\rangle = -2 \left\| \sum_{i=1}^{n} c_i \xi_i \right\|^2 \leq 0.$$

Hence, $\| \cdot \|^2$ is a function conditionally of negative type on the additive group of H. ○

Proposition 2.4.20. *Let $(\psi_i)_{i \in I}$ be a net of functions conditionally of negative type on the discrete group Γ. If $(\psi_i)_{i \in I}$ converges pointwise to a function $\psi : \Gamma \to \mathbb{C}$, then ψ is a function conditionally of negative type on Γ.*

Proof. Let $g_1, \ldots, g_n \in \Gamma$ and let $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{j=1}^n c_j = 0$. Define,

$$x = \sum_{j,k=1}^{n} c_j \overline{c_k} \psi(g_k^{-1} g_j), \qquad \text{and} \qquad x_i = \sum_{j,k=1}^{n} c_j \overline{c_k} \psi_i(g_k^{-1} g_j), \qquad \text{for each } i \in I$$

Since $\psi_i \to \psi$ and the above sums are finite, $x_i \to x$. We have, by assumption, that $(x_i)_{i \in I}$ is a net in $(-\infty, 0]$, and therefore, $x \leq 0$. Hence, ψ is conditionally of negative type. □

Definition 2.4.21. A function $\Phi : \Gamma \times \Gamma \to \mathbb{C}$ is called a *positive kernel* on the (discrete) group Γ if

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \Phi(g_i, g_j) \geq 0. \tag{2.13}$$

for all $g_1, \ldots, g_n \in \Gamma$ and all $c_1, \ldots, c_n \in \mathbb{C}$.

Remark 2.4.22. For a function $\Phi : \Gamma\times\Gamma \to \mathbb{C}$ and a selection of group elements $g_1, \ldots, g_n \in \Gamma$, let $M_\Phi(g_1, \ldots, g_j)$ denote the matrix whose $(i, j)'th$ entry is $\Phi(g_i, g_j)$. Then Φ is a positive kernel if and only if $M_\Phi(g_1, \ldots, g_n)$ is positive, for all choices of $g_1, \ldots, g_n \in \Gamma$.

Example 2.4.23. For a complex valued function φ on a (discrete) group Γ, define a function $\Phi : \Gamma \times \Gamma \to \mathbb{C}$ by setting $\Phi(g, h) = \varphi(h^{-1}g)$, for all $g, h \in \Gamma$. If φ is of positive type then Φ is a positive kernel. ◦

Proposition 2.4.24. *Let $\Phi, \Psi : \Gamma\times\Gamma \to \mathbb{C}$ be positive kernels. Then their pointwise product $\Phi \cdot \Psi : \Gamma \times \Gamma \to \mathbb{C}$ is a positive kernel, as well.*

Proof. Take $g_1, \ldots, g_n \in \Gamma$ and write $M_\Phi = M_\Phi(g_1, \ldots, g_n)$ and $M_\Psi = M_\Psi(g_1, \ldots, g_n)$. By Remark 2.4.22, M_Φ is a positive matrix, and so, its corresponding Schur multiplier T_{M_Φ} is positive. Since, by assumption, M_Ψ is also positive, we deduce that $M_{\Phi\cdot\Psi} = T_{M_\Phi}(M_\Psi)$ is positive. Then $\Phi \cdot \Psi$ is a positive kernel, by Remark 2.4.22. □

Proposition 2.4.25. *Let $(\Phi_i)_{i\in I}$ be a net of positive kernels on the discrete group Γ. If $(\Phi_i)_{i\in I}$ converges pointwise to a function $\Phi : \Gamma \times \Gamma \to \mathbb{C}$, then Φ is a positive kernel.*

Proof. For $g_1, \ldots, g_n \in \Gamma$ and $c_1, \ldots, c_n \in \mathbb{C}$, define,

$$x = \sum_{j,k=1}^{n} c_j\overline{c_k}\Phi(g_j, g_k), \qquad \text{and} \qquad x_i = \sum_{j,k=1}^{n} c_j\overline{c_k}\Phi_i(g_j, g_k), \qquad \text{for each } i \in I$$

Since $\Phi_i \to \Phi$ and the above sums are finite, $x_i \to x$. We have, by assumption, that $(x_i)_{i\in I}$ is a net in $[0, \infty)$, and therefore, $x \geq 0$. Hence, Φ is a positive kernel. □

Lemma 2.4.26. *Let Γ be a discrete group and $\psi : \Gamma \to \mathbb{C}$ be a function with $\psi(e) = 0$ and $\psi(g^{-1}) = \psi(g)$, for all $g \in \Gamma$. Then ψ is conditionally of negative type if and only if, for each $g_0 \in \Gamma$, the function $\Phi : \Gamma \times \Gamma \to \mathbb{C}$ given by*

$$\Phi(g, h) = -\psi(h^{-1}g) + \psi(g_0^{-1}g) + \psi(h^{-1}g_0), \qquad \textit{for all } g, h \in \Gamma,$$

is a positive kernel.

Proof. Fix $g_0 \in \Gamma$. Suppose first that Φ satisfies equation (2.13). For $g_1, \ldots, g_n \in \Gamma$ and $c_1, \ldots, c_n \in \mathbb{C}$ with $\sum_{i=1}^{n} c_i = 0$, we have

$$0 \leq \sum_{i,j=1}^{n} c_i\overline{c_j}\Phi(g_i, g_j) = -\sum_{i,j=1}^{n} c_i\overline{c_j}\psi(g_j^{-1}g_i).$$

Hence, ψ is conditionally of negative type.

Conversely, suppose ψ is conditionally of negative type. Fix $g_0 \in \Gamma$. For $g_1, \ldots, g_n \in \Gamma$ and $c_1, \ldots, c_n \in \mathbb{C}$, set $c_0 = -\sum_{i=1}^{n} c_i$. Then

$$\begin{aligned}
0 &\geq \sum_{i,j=0}^{n} c_i\overline{c_j}\psi(g_j^{-1}g_i) \\
&= \sum_{i,j=1}^{n} c_i\overline{c_j}\psi(g_j^{-1}g_i) + \overline{c_0}\sum_{i=1}^{n} c_i\psi(g_0^{-1}g_i) + c_0\sum_{j=1}^{n} \overline{c_j}\psi(g_j^{-1}g_0) \\
&= \sum_{i,j=1}^{n} c_i\overline{c_j}\big(\psi(g_j^{-1}g_i) - \psi(g_0^{-1}g_i) - \psi(g_j^{-1}g_0)\big) \\
&= -\sum_{i,j=1}^{n} c_i\overline{c_j}\Phi(g_i, g_j).
\end{aligned}$$

Hence, the function Φ corresponding to g_0 satisfies equation (2.13). Since $g_0 \in \Gamma$ was arbitrary, this finishes the proof. □

Theorem 2.4.27 (Schoenberg). *Let $\psi : \Gamma \to \mathbb{C}$ be a function on the discrete group Γ with $\psi(e) = 0$ and $\psi(g^{-1}) = \overline{\psi(g)}$, for all $g \in \Gamma$. Then ψ is conditionally of negative type if and only if, for every $t \geq 0$, the function $e^{-t\psi} : \Gamma \to \mathbb{C}$ is of positive type.*

Proof. Assume that $e^{-t\psi}$ is of positive type, for all $t \geq 0$. By Example 2.4.18, the function $g \mapsto 1 - e^{-t\psi(g)}$ is conditionally of negative type, and then so is $g \mapsto (1 - e^{-t\psi(g)})/t$, by Remark 2.4.16. This holds for all $t > 0$, and so, the pointwise limit

$$\psi = \lim_{t \to 0^+} \frac{1 - e^{-t\psi}}{t}$$

is conditionally of negative type, as well, by Proposition 2.4.20.

Conversely, assume that ψ is conditionally of negative type. Since the set of functions conditionally of negative type is a convex cone, it suffices to show that $e^{-\psi}$ is of positive type. Consider the function $\Phi : \Gamma \times \Gamma \to \mathbb{R}$ given by

$$\Phi(g,h) = -\psi(h^{-1}g) + \psi(g) + \psi(h^{-1}), \qquad \text{for all } g,h \in \Gamma,$$

This is a positive kernel by Lemma 2.4.26. Then Φ^n is a positive kernel, for all $n \in \mathbb{N}$, by Proposition 2.4.24, and so, e^{Φ} is a positive kernel, by Proposition 2.4.25. For $g_1, \dots, g_n \in \Gamma$ and $c_1, \dots, c_n \in \mathbb{C}$, it follows that

$$\sum_{i,j=1}^{n} c_i \overline{c_j} e^{-\psi(g_j^{-1} g_i)} = \sum_{i,j=1}^{n} c_i \overline{c_j} e^{\Phi(g_i,g_j) - \psi(g_i) - \psi(g_j^{-1})} = \sum_{i,j=1}^{n} c_i e^{-\psi(g_i)} \overline{c_j e^{-\psi(g_j)}} e^{\Phi(g_i,g_j)} \geq 0.$$

This shows that $e^{-\psi}$ is of positive type. □

2.4.3 A consequence of Schoenberg's theorem

Definition 2.4.28. Let V be a vector space. A function $\varphi : V \to \mathbb{C}$ is said to be of *positive type* (respectively, *conditionally of negative type*) if it is a function of positive type (respectively, conditionally of negative type) on the additive group of V.

Lemma 2.4.29. *Let Γ be a discrete group and let ρ be a linear isometric representation of Γ on a Banach space X. If $\|\cdot\|^p$ is conditionally of negative type on X, for some $p \geq 1$, and if $b : \Gamma \to X$ is a 1-cocycle, then the map $g \mapsto \|b(g)\|_p^p$ is conditionally of negative type.*

Proof. Since ρ is an isometric representation, we have, for $g, h \in \Gamma$,

$$\left\|b(h^{-1}g)\right\| = \left\|b(h^{-1}) + \rho(h^{-1})b(g)\right\| = \left\|\rho(h)b(h^{-1}) + b(g)\right\|_p = \|b(g) - b(h)\|\,.$$

Here, the last equality follows from Lemma 2.2.7. It follows directly that $g \mapsto \|b(g)\|^p$ is conditionally of negative type. □

Proposition 2.4.30. *Let Γ be a discrete group. Let X be a Banach space and assume that $\|\cdot\|^p$ is conditionally of negative type on X, for some fixed $p \geq 1$. Let (ρ, X) be a representation by linear surjective isometries of Γ on X, and let α be an affine isometric action of Γ on X with linear part ρ. For each $t > 0$, there exists a unitary representation (π_t, H_t) of Γ on a complex Hilbert space, and a continuous map Φ_t from X to the unit sphere of H_t such that the following properties hold:*

(i) $\langle \Phi_t(x)\,, \Phi_t(y) \rangle = e^{-t\|x-y\|^p}$, *for all $x, y \in X$,*

(ii) $\pi_t(g)\Phi_t(x) = \Phi_t(\alpha(g)x)$, *for all $g \in \Gamma$ and $x \in X$,*

(iii) the linear span of $\Phi_t(X)$ is dense in H_t.

Moreover, the pair (π_t, H_t) is unique up to isomorphism.

Proof. Consider the semidirect product $X \rtimes \Gamma$ of the additive group of X with Γ. Recall that the group operation is given by

$$(x, g)(y, h) = (\rho(g)y + x, gh), \qquad \text{for } x, y \in X,\ g, h \in \Gamma.$$

Let $\tilde{\rho}$ be the representation of $X \rtimes \Gamma$ by linear surjective isometries given by

$$\tilde{\rho}(x, g) = \rho(g), \qquad \text{for } (x, h) \in X \rtimes \Gamma.$$

Let b be the translation part of α and define a map $\tilde{b} : X \rtimes \Gamma \to X$ by setting

$$\tilde{b}(x, g) = b(g) + x, \qquad \text{for } (x, h) \in X \rtimes \Gamma.$$

For $x, y \in X$ and $g, h \in \Gamma$, we have

$$\begin{aligned} \tilde{b}\big((x, g)(y, h)\big) &= \tilde{b}(\rho(g)y + x, gh) = b(gh) + \rho(g)y + x = b(g) + x + \rho(g)(b(h) + y) \\ &= \tilde{b}(x, g) + \tilde{\rho}(x, g)\tilde{b}(y, h). \end{aligned}$$

Hence, $\tilde{b}$ is a 1-cocycle on $X \rtimes \Gamma$ with respect to $\tilde{\rho}$. Define $\psi : X \rtimes \Gamma \to \mathbb{R}$ by

$$\psi(x, g) = \left\|\tilde{b}(x, g)\right\|^p = \|b(g) + x\|^p, \qquad \text{for } (x, g) \in X \rtimes \Gamma.$$

Then ψ is conditionally of negative type, by Lemma 2.4.29. By Schoenberg's theorem, Theorem 2.4.27, the function $e^{-t\psi}$ on $X \rtimes \Gamma$ is of positive type, for each $t > 0$. Fix $t > 0$ and let (H_t, Π_t, ξ_t) be the GNS triple associated to $e^{-t\psi}$ (see Theorem 2.4.9). Since Γ sits inside $X \rtimes \Gamma$ via the embedding $g \mapsto (0, g)$, we obtain a unitary representation π_t of Γ on H_t by restriction of Π_t to Γ. Define $\Phi_t : X \to H_t$ by

$$\Phi_t(x) = \Pi_t(x, e)\xi_t, \qquad \text{for } x \in X.$$

We claim that the triple (π_t, H_t, Φ_t) has the desired properties.

Property (i) is direct from our construction. Indeed, for $x, y \in X$, we have

$$\begin{aligned} \langle \Phi_t(x)\,, \Phi_t(y) \rangle &= \langle \Pi_t(x, e)\xi_t\,, \Pi_t(y, e)\xi_t \rangle \\ &= \left\langle \Pi_t(y, e)^{-1}\Pi_t(x, e)\xi_t\,, \xi_t \right\rangle \\ &= \langle \Pi_t(x - y, e)\xi_t\,, \xi_t \rangle \\ &= e^{-t\psi(x-y,e)} \\ &= e^{-t\|x-y\|^p}. \end{aligned}$$

For the last equality, we use that $b(e) = 0$, as shown in Lemma 2.2.7.

For $g \in \Gamma$, observe that

$$\begin{aligned} \langle \Phi_t(b(g))\,, \pi_t(g)\xi_t \rangle &= \langle \Pi_t(b(g), e)\xi_t\,, \Pi_t(0, g)\xi_t \rangle = \left\langle \Pi_t(\rho(g^{-1})b(g), g^{-1})\xi_t\,, \xi_t \right\rangle \\ &= \left\langle \Pi_t(-b(g^{-1}), g^{-1})\xi_t\,, \xi_t \right\rangle = e^{-t\psi(-b(g^{-1}),g^{-1})} = 1. \end{aligned}$$

Since $\Phi_t(b(g))$ and $\pi_t(g)\xi_t$ are unit vectors, we obtain that $\|\Phi_t(b(g)) - \pi_t(g)\xi_t\| = 0$. Hence,

$$\Pi_t(b(g), e)\xi_t = \Phi_t(b(g)) = \pi_t(g)\xi_t, \tag{2.14}$$

for all $g \in \Gamma$. Using this, we deduce, for $g \in \Gamma$ and $x \in X$, that

$$\begin{aligned}\pi_t(g)\Phi_t(x) &= \pi_t(g)\Pi_t(x, e)\xi_t = \Pi_t(\rho(g)x, e)\pi_t(g)\xi_t \\ &= \Pi_t(\rho(g)x, e)\Pi_t(b(g), e)\xi_t = \Pi_t(\rho(g)x + b(g), e)\xi_t \\ &= \Phi(\alpha(g)\xi).\end{aligned}$$

This shows property (ii).

Using equation (2.14) again, we derive, for each $g \in \Gamma$ and each $x \in X$, that

$$\Pi_t(x, g)\xi_t = \Pi_t(x, e)\pi_t(g)\xi_t = \Pi_t(x, e)\Pi_t(b(g), e)\xi_t = \Pi_t(b(g) + x, e)\xi_t = \Phi_t(b(g) + x)$$

Property (iii) follows directly, as ξ_t is cyclic for Π_t.

Suppose (ρ_t, K_t, Ψ_t) is another triplet satisfying the properties (i)–(iii). Going backwards through our calculations, we see that $(\rho_t, K_t, \Psi_t(0))$ is a GNS triplet for $e^{-t\psi}$. Uniqueness of the representation (π_t, H_t) then follows from uniqueness of the GNS construction. □

Remark 2.4.31. Since the unitary representation (π_t, H_t) in Proposition 2.4.30 is (a restriction) of the GNS representation associated to a real-valued function of positive type, it follows from Remark 2.4.10 that it is the complexification of an orthogonal representation.

Proposition 2.4.32. *Let (ρ, X) be representation of the discrete group Γ by linear surjective isometries on a Banach space X and suppose that $p \geq 1$ is such that $\|\cdot\|^p$ is conditionally of negative type. Let α be an affine isometric action of Γ with linear part ρ, let $t > 0$ and let (π_t, H_t, Φ_t) be the associated triple from Proposition 2.4.30. If $(x_n)_{n\geq 1}$ is a sequence in X diverging in norm to infinity then $(\Phi_t(\xi_n))_{n\geq 1}$ tends weakly to zero in H_t.*

Proof. Let $(x_n)_{n\geq 1}$ be a sequence in X such that $\|x_n\| \to \infty$ as n tends to infinity. By Proposition 2.4.30(i), we have, for every $x \in X$ and every $n \in \mathbb{N}$,

$$\langle \Phi_t(x_n) , \Phi_t(x)\rangle = e^{-t\|x_n - x\|^p}.$$

Hence, $\langle \Phi_t(x_n) , \Phi_t(x)\rangle \to 0$ as n tends to infinity, because $t > 0$. Let $\zeta \in H_t$ and let $\varepsilon > 0$. Since $\operatorname{span}\{\, \Phi_t(x) \mid x \in X \,\}$ is dense in H_t, by Proposition 2.4.30(iii), we may find $x \in X$ such that $\|\zeta - \Phi_t(x)\| < \varepsilon/2$. Take $n_\varepsilon \in \mathbb{N}$ such that $\langle \Phi_t(x_n) , \Phi_t(x)\rangle < \varepsilon/2$, for all $n \geq n_\varepsilon$. Then, as $\|\Phi_t(x_n)\| = 1$, for all $n \in \mathbb{N}$, we have

$$|\langle \Phi_t(x_n) , \zeta\rangle| \leq \langle \Phi_t(x_n) , \Phi_t(x)\rangle + \|\zeta - \Phi_t(x)\| < \varepsilon,$$

by Cauchy-Schwarz. Hence, $\langle \Phi_t(\xi_n) , \zeta\rangle \to 0$ as n tends to infinity, for all $\zeta \in H_t$. □

Proposition 2.4.33. *Let α be an affine isometric action of the discrete group Γ on a Banach space X and suppose that $p \geq 1$ is such that $\|\cdot\|^p$ is conditionally of negative type. For each $t > 0$, let (π_t, H_t, Φ_t) be the triple from Proposition 2.4.30 associated to α. If $(t_n)_{n\geq 1}$ is a sequence of strictly positive real numbers tending to zero, then $\bigoplus_n \pi_{t_n}$ weakly contains the trivial representation.*

Proof. For each $n \in \mathbb{N}$, let $\xi_n = \Phi_{t_n}(0) \in H_{t_n}$. Then $\|\xi_n\| = 1$, by Proposition 2.4.30(i). Further, by Proposition 2.4.30(ii),

$$\|\pi_{t_n}(g)\xi_n - \xi_n\|^2_{H_{t_n}} = \|\Phi_{t_n}(\alpha(g)0) - \Phi_{t_n}(0)\|^2_{H_{t_n}} = 2 - 2e^{-t_n\|\alpha(g)0\|^p}.$$

Hence, for each $g \in \Gamma$, $\|\pi_{t_n}(g)\xi_n - \xi_n\|^2 \to 0$ as n tends to infinity. Let $H = \bigoplus_n H_{t_n}$ be the Hilbert space direct sum and let $\pi = \bigoplus_n \pi_{t_n}$. Each H_{t_n} embeds canonically into H, and so, we may view each ξ_n as a vector in H. Note that $\pi(g)\xi_n = \pi_{t_n}(g)\xi_n$, for all $g \in \Gamma$ and all $n \in \mathbb{N}$. Hence, $(\xi_n)_{n\geq 1}$ is a sequence of almost Γ-invariant vectors for π. By Proposition 1.3.5, this implies that π weakly contains 1_Γ. □

2.5 Affine isometric actions on Hilbert spaces

Let Γ be a discrete group. An *isometric action*, respectively, an *affine isometric action* of Γ on a vector space V is a group homomorphism of Γ into $\mathrm{Isom}(V)$, respectively $\mathrm{Aff\,Isom}(V)$. We remark that, if V is a real vector space, any isometric action on V is automatically an affine action, by the Mazur-Ulam theorem, Theorem 2.1.9.

Definition 2.5.1. A discrete group Γ has property (FH) if every affine isometric action of Γ on a real Hilbert space has a fixed point.

We have seen already in Lemma 2.3.1 that the existence of a fixed point, for all affine actions of a discrete group Γ with the same orthogonal (or unitary) representation π as linear part, is captured by the vanishing of the first cohomology group with coefficients in π. It will therefore be no surprise to the reader that Property (FH), as defined above, can be reformulated in terms of the vanishing of all the first cohomology groups of Γ with coefficients in some orthogonal representation of Γ on a real Hilbert space.

Proposition 2.5.2. *A discrete group Γ has Property (FH) if and only if $H^1(\Gamma,\pi)=0$, for every orthogonal representation π of Γ on a real Hilbert space.*

Proof. By Lemma 2.3.1, Γ has Property (FH) if and only if $H^1(\Gamma,\pi)=0$, for all orthogonal representations π which is the linear part of any affine isometric action of Γ on a real Hilbert space. Since an affine action is isometric if and only if its linear part is an isometric representation, the statement of the proposition follows. $\square$

Our aim is to show that Property (FH) is equivalent to Property (T). First, we show in the next two propositions two characterizations of the affine isometric actions on real Hilbert spaces that have fixed points.

Proposition 2.5.3. *Let α be an affine isometric action of the discrete group Γ on a Hilbert space H and let b be its translation part. The following are equivalent:*

(i) The action α has a fixed point in H,

(ii) b is a 1-coboundary.

(iii) b is bounded,

(iv) All orbits of α are bounded,

(v) Some orbit of α is bounded,

Proof. The equivalence of (i) and (ii) is proven in Lemma 2.2.11. Suppose b is a 1-coboundary and take ξ such that $b(g)=\pi(g)\xi-\xi$, for all $g\in\Gamma$. Then $\|b(g)\|\le 2\|\xi\|$, and so, b is bounded. This proves $(ii)\Rightarrow(iii)$. Further, for $\xi\in H$, we have that

$$\|\alpha(g)\xi\| = \|\pi(g)\xi+b(g)\| \le \|\xi\|+\|b(g)\|\,.$$

Thus, if b is bounded then all orbits of α are bounded. This shows $(iii)\Rightarrow(iv)$. The implication $(iv)\Rightarrow(v)$ is trivial. Finally, assume that $\xi\in H$ is such that the orbit

$$\mathcal{O}_\xi=\{\,\alpha(g)\xi \mid g\in\Gamma\,\}$$

is bounded. Let $\zeta\in H$ be the unique circumcenter of $\mathcal{O}_\xi$ from Lemma 1.2.13. For each $g\in\Gamma$, the center of $\alpha(g)\mathcal{O}_\xi$ is $\alpha(g)\zeta$, because α is an isometry. But clearly $\alpha(g)\mathcal{O}_\xi=\mathcal{O}_\xi$, and so, uniqueness of the center yields that $\alpha(g)\zeta=\zeta$. That is, ζ is a fixed point for α. This establishes the implication $(v)\Rightarrow(i)$. $\square$

Recall from Example 2.4.19 that, when H is a real Hilbert space, $\|\cdot\|^2$ is a function conditionally of negative type on the additive group of H. Hence, given an affine isometric action on a real Hilbert space, we may apply Proposition 2.4.30.

Proposition 2.5.4. *Let α be an affine isometric action of the discrete group Γ on a real Hilbert space H. For each $t > 0$, let (π_t, H_t) be the unitary representation from Proposition 2.4.30 associated to α. The following are equivalent:*

(i) The action α has a fixed point in H,

(ii) For all $t > 0$, (π_t, H_t) has a non-zero invariant vector,

(iii) For some $t > 0$, (π_t, H_t) has a non-zero invariant vector,

Proof. Suppose α has a fixed point $\xi \in H$. It follows directly from Proposition 2.4.30(ii) that, for each $t > 0$, $\Phi_t(\xi)$ is an invariant vector for π_t. Suppose instead that α does not have any fixed points in H. Fix $t > 0$ and suppose that $\zeta \in H_t$ is an invariant vector for π_t. By Proposition 2.5.3, all orbits of α are unbounded. Hence, for $\xi \in H$, we may find a sequence $(g_n)_{n\geq 1}$ in Γ such that $\|\alpha(g_n)\xi\|$ diverges to infinity. Then $(\Phi_t(\alpha(g_n)\xi))_{n\geq 1}$ converges weakly to 0 , by Proposition 2.4.32. Since ζ is invariant, we have, for all $n \in \mathbb{N}$,

$$\langle \Phi_t(\xi)\,, \zeta\rangle = \left\langle \Phi_t(\xi)\,, \pi(g_n^{-1})\zeta\right\rangle = \langle \pi(g_n)\Phi_t(\xi)\,, \zeta\rangle = \langle \Phi_t(\alpha(g_n)\xi)\,, \zeta\rangle\,.$$

The last equality is direct from Proposition 2.4.30(ii). We have that $\langle \Phi_t(\alpha(g_n)\xi)\,, \zeta\rangle \to 0$, and so, we deduce that $\langle \Phi_t(\xi)\,, \zeta\rangle = 0$. As the vector $\xi \in H$ was arbitrary, we see that ζ is orthogonal to the set $\Phi_t(H)$. By Proposition 2.4.30(iii), the linear span of $\Phi_t(H)$ is dense in H_t, and so, we may conclude that $\zeta = 0$. This finishes the proof. □

With these preparations, we are ready to proof the equivalence between Property (T) and Property (FH) for countable groups. The implication $(T) \Rightarrow (FH)$ was proven by P. Delorme in [15, Théorème V.1]. The other implication $(FH) \Rightarrow (T)$ is due to A. Guichardet in [20, Théorème 1].

Theorem 2.5.5 (Delorme-Guichardet)**.** *A countable discrete group Γ has property (T) if and only if it has property (FH).*

Proof. Assume that Γ does not have Property (FH) and let α be an affine isometric action on a Hilbert space H without fixed points. For each $t > 0$, let (π_t, H_t) be the unitary representation from Proposition 2.4.30 associated to α and t. By Proposition 2.5.4, it holds for all $t > 0$, that (π_t, H_t) has no non-zero invariant vectors. Set

$$\pi = \bigoplus_{n=1}^{\infty} \pi_{1/n}$$

By Proposition 1.1.14, π has no non-zero invariant vectors. However, by Proposition 2.4.33, π does have almost invariant vectors, and so, Γ does not have Property (T).

Assume instead that Γ does not have Property (T). Then we may find a unitary representation (π, H) with almost invariant vectors but no non-zero Γ-invariant vectors. The orthogonal representation $(\pi_{\mathbb{R}}, H_{\mathbb{R}})$ defined in Subsection 1.1.1 is then also a representation with almost invariant vectors but no non-zero invariant vector. Proposition 2.3.4 implies that $B^1(\Gamma, \pi_{\mathbb{R}})$ is not closed. As discussed in Remark 2.3.3, it follows that $H^1(\Gamma, \pi_{\mathbb{R}}) \neq 0$, and so, Γ does not have Property (FH), by Proposition 2.5.2. □

Remark 2.5.6. The assumption that Γ is countable in Theorem 2.5.5 is needed for the implication $(FH) \Rightarrow (T)$ only. The above proof of this relies on the implication from Proposition 2.3.4 that the existence of almost invariant vectors implies non-closure of the set of

1-coboundaries – for the use of countability in the proof of this implication, see Remark 2.3.5. When Γ is not countable, Property (FH) need not imply Property (T). Examples of uncountable discrete groups with Property (FH) but without Property (T) where provided by Y. de Cornulier in [12].

We end this chapter with yet another characterization of Property (T), which the equivalence with Property (FH) allows us to show. Recall from Corollary 1.3.6 that Property (T) can be rephrased as the property that any representation weakly containing the trivial representation contains the trivial representation. A seemingly weaker property is, that weak containment of the trivial representation implies containment of some finite dimensional representation. For countable discrete groups, this is equivalent with Property (T).

Lemma 2.5.7. *Let α be an affine action of the discrete group Γ on a real Hilbert space H. Let $\Delta(\alpha)$ be the associated diagonal affine action of Γ on $H \oplus H$. For each $t > 0$, let (π_t, H_t) and (ρ_t, K_t) be the unitary representations from Proposition 2.4.30 associated to α and $\Delta(\alpha)$, respectively. Then ρ_t is unitarily equivalent to $\pi_t \otimes \pi_t$.*

Proof. Fix $t > 0$, let $\Phi_t : H \to H_t$ be as in Proposition 2.4.30 and let $\Psi_t : H \oplus H \to H_t \otimes H_t$ be given by

$$\Psi_t(\xi, \xi') = \Phi_t(\xi) \otimes \Phi_t(\xi'), \qquad \text{for } \xi, \xi' \in H.$$

We show that the triplet $(\pi_t \otimes \pi_t, H_t \otimes H_t, \Psi_t)$ has the properties of Proposition 2.4.30 with respect to $\Delta(\alpha)$. First, since the linear span of $\Phi_t(H)$ is dense in H_t, it is clear that the linear span of $\Psi_t(H \oplus H)$ is dense in $H_t \otimes H_t$. Property (i) is easily derived using the definitions of norms on tensor products and direct sums. Indeed, for all $g \in \Gamma$ and all $\xi, \xi', \eta, \eta' \in H$, we have

$$\left\langle \Psi_t(\xi, \xi') \, , \Psi_t(\eta, \eta') \right\rangle = e^{-t\|\xi - \eta\|^2} e^{-t\|\xi' - \eta'\|^2} = e^{-t\|(\xi,\xi') - (\eta,\eta')\|^2}.$$

Finally, for all $g \in \Gamma$ and all $\xi, \xi', \eta, \eta' \in H$,

$$\begin{aligned}\Psi_t(\Delta(\alpha)(g)(\xi, \xi')) = \Psi_t(\alpha(g)\xi, \alpha(g\xi')) &= \Phi_t(\alpha(g)\xi) \otimes \Phi_t(\alpha(g)\xi') \\ &= \pi_t(g)\Phi_t(\xi) \otimes \pi_t(g)\Phi_t(\xi') = (\pi_t \otimes \pi_t)(g)\Psi_t(\xi, \xi').\end{aligned}$$

This shows that property (ii) holds for the triplet $(\pi_t \otimes \pi_t, H_t \otimes H_t, \Psi_t)$. The uniqueness statement of Proposition 2.4.30 then implies that ρ_t and $\pi_t \otimes \pi_t$ are unitarily equivalent. □

Theorem 2.5.8. *Let Γ be a countable discrete group. Then Γ has Property (T) if and only if any unitary representation of Γ weakly containing the trivial representation contains a non-zero finite dimensional subrepresentation.*

Proof. The 'only if' statement is direct from Corollary 1.3.6. Conversely, suppose that any unitary representation of Γ weakly containing the trivial representation contains a non-zero finite dimensional subrepresentation. We show that Γ has Property (FH), which, by the Delorme-Guichardet theorem, Theorem 2.5.5, implies that Γ has Property (T).

Let α be an affine isometric action of Γ on a real Hilbert space H. For each $t > 0$, let (π_t, H_t, Φ_t) be the triplet from Proposition 2.4.30 associated to α. Let $(t_n)_{n \geq 1}$ be a sequence in $(0, \infty)$ converging to 0. To ease notation we will write π_n instead of π_{t_n}, for $n \in \mathbb{N}$. The representation $\pi = \bigoplus_n \pi_n$ weakly contains 1_Γ, by Proposition 2.4.33. By assumption, this implies that π contains a non-zero finite dimensional representation, and so, $\pi \otimes \overline{\pi}$ contains 1_Γ, by Proposition 1.1.17. We have $\pi \otimes \overline{\pi} = \bigoplus_{n_1, n_2} \pi_{n_1} \otimes \overline{\pi}_{n_2}$, and so, 1_Γ is contained in $\pi_{n_1} \otimes \overline{\pi}_{n_2}$, for some $n_1, n_2 \in \mathbb{N}$, by Proposition 1.1.14. Applying Proposition 1.1.17 again, we deduce that π_{n_1} contains a finite dimensional representation. Since π_{n_1} is the complexification of an orthogonal representation (see Remark 2.4.31), Lemma 1.1.11 implies that the same finite dimensional representation is contained in $\overline{\pi}_{n_1}$. Applying Proposition 1.1.17 once more, we deduce that 1_Γ is contained in $\pi_{n_1} \otimes \pi_{n_1}$. By Lemma 2.5.7, $\pi_{n_1} \otimes \pi_{n_1}$ is unitarily equivalent to the representation (ρ_{t_n}, K_{t_n}) from Proposition 2.4.30 associated to the diagonal action $\Delta(\alpha)$ on $H \oplus H$. Hence, ρ_{t_n} has a Γ-invariant vector. By Proposition 2.5.3, we conclude that $\Delta(\alpha)$ has a fixed point in $H \oplus H$, and so, α has a fixed point in H. Thus, Γ has property (FH), which was what we aimed to show. $\square$

2.6 Literature

Section 2.1, Section 2.2, Section 2.3 and Section 2.5 are all based on [5, Chapter 2] and Section 2.4 is based on [5, Appendix C]. In Proposition 2.4.11 and Proposition 2.4.24 we use that the Schur product of two positive semidefinite matrices is, again, positive semidefinite. This is a well-known result attributed to I. Schur in [44, Theorem VII]. For a different proof, see [21, Proposition 2.6]. The consequence of Schoenberg's theorem stated in Proposition 2.4.30 and its proof can be found in [5, Proposition 2.11.1].

Chapter 3

Property (T) and ergodic actions

We present in Theorem 3.3.1 a characterization of Property (T) in terms of how a group acts on probability spaces. Central to this characterization is understanding the connection between orthogonal representations and actions on probability spaces. We investigate this connection in Section 3.2. First, we present two properties of actions on probability spaces: *ergodicity* and *strong ergodicity*. We show in Proposition 3.1.8 that ergodicity is equivalent with that a certain representation does not contain the trivial representation, and in Proposition 3.1.15 that strong ergodicity is equivalent with this representation not weakly containing the trivial representation.

3.1 Actions on measure spaces and ergodicity

Let Γ be a discrete group and let Ω be a measurable space. An action of Γ on Ω as a set is a group homomorphism from Γ to the group of bijections of Ω. We say that an action of Γ on Ω is *measurable* if the map

$$\begin{aligned} \Gamma \times \Omega &\longrightarrow \Omega \\ (g, x) &\longmapsto gx \end{aligned}$$

is measurable, when Γ is equipped with the power set σ-algebra. We denote by $\Gamma \curvearrowright \Omega$ that Γ acts measurably on Ω. When Ω is equipped with a measure μ, we shall also use the notation $\Gamma \curvearrowright (\Omega, \mu)$ for an action of Γ on Ω.

Let Ω and Ω' be measurable spaces. For a measurable map $\theta : \Omega \to \Omega'$ and a meaure μ on Ω, we define the *image measure* $\theta_* \mu$ on Ω' as follows:

$$\theta_*\mu(B) = \mu(\theta^{-1}(B)), \qquad \text{for all measurable } B \subset \Omega'.$$

Let (Ω, μ) and (Ω', μ') be measure spaces. A measurable map $\theta : \Omega \to \Omega'$ is said to be (μ, μ')-*measure preserving* if $\theta_*\mu = \mu'$. Whenever the measures are understood from context, we shall simple say that θ is measure preserving. If θ is a measure preserving transformation on (Ω, μ), we say that μ is *invariant* under θ.

Lemma 3.1.1. *Let (Ω, μ) and (Ω', μ') be a measure spaces, and let $\theta : \Omega \to \Omega'$ be a measure preserving map. If $f : \Omega' \to \mathbb{R}$ is integrable then so is $f \circ \theta : \Omega \to \mathbb{R}$, and*

$$\int_\Omega f \circ \theta \,\mathrm{d}\mu = \int_{\Omega'} f \,\mathrm{d}\mu'$$

Proof. It is enough to show the lemma for simple functions. But this is clear as $1_{B'} \circ \theta = 1_{\theta^{-1}(B')}$, for any measurable subset $B' \subset \Omega'$. $\square$

Definition 3.1.2. A *measure preserving transformation* of the measure space (Ω, μ) is a measure preserving bijective map $\theta : \Omega \to \Omega$ with measurable inverse.

Remark 3.1.3. If θ is a measure preserving transformation on some measure space then its inverse is also measure preserving.

Suppose Γ acts measurably on a measurable space Ω and let μ be a measure on Ω. For each $g \in \Gamma$, we denote by $g.\mu$ the image measure of μ on Ω under the transformation $\omega \mapsto g.\omega$. That is, $g.\mu(B) = \mu(g^{-1}B)$, for each measurable subset $B \subset \Omega$. This defines an action on the set of all measures on Ω. Indeed, $e.\mu = \mu$ trivially, and, for each $g, h \in \Gamma$ and each measurable set $B \subset \Omega$,

$$(gh).\mu(B) = \mu((gh)^{-1}B) = \mu(h^{-1}g^{-1}B) = h.\mu(g^{-1}B) = g.(h.\mu)(B).$$

We say that the measure μ is Γ-*invariant* if μ is a fixed point for this action. In that case, the action $\Gamma \curvearrowright (\Omega, \mu)$ is said to be *measure preserving*. If μ is a probability measure we say that the action is *probability measure preserving*, or, *p.m.p.* in short. Put differently, a measure preserving action of Γ on a measure space (Ω, μ) is a group homomorphism from Γ to the group of measure preserving transformations of Ω.

3.1.1 Ergodicity and invariant vectors

A measurable subset $B \subset \Omega$ is said to be Γ-*invariant* for the action $\Gamma \curvearrowright \Omega$ if $gB \subset B$, for all $g \in \Gamma$.

Definition 3.1.4. A measurable action of a discrete group Γ on a probability space (Ω, μ) is said to be *ergodic* if all Γ-invariant subsets $B \subset \Omega$ are trivial in the sense that $\mu(B) = 0$ or $\mu(B) = 1$.

Let (Ω, μ) be a probability space and denote by $L^2(\Omega, \mu)$ the Hilbert space of square integrable complex valued functions on Ω. For a p.m.p. action of a discrete group Γ on Ω, define a unitary representation π_μ of Γ on $L^2(\Omega, \mu)$ by setting

$$\pi_\mu(g)f(x) = f(g^{-1}x), \qquad \text{for } f \in L^2(\Omega, \mu),\ g \in \Gamma, \text{ and } x \in \Omega. \tag{3.1}$$

It follows from Lemma 3.1.1 that $\pi(g)$, for each $g \in \Gamma$, is an isometry, and it is then clear that $(\pi_\mu, L^2(\Omega, \mu))$ is a unitary representation.

Remark 3.1.5. Let $L^2(\Omega, \mu; \mathbb{R})$ denote the Hilbert space of all square integrable real valued functions on Ω. The restriction of π_μ to this subspace is an orthogonal representation. Moreover, π_μ is the complexification of this restriction.

We say that a function $f \in L^2(\Omega, \mu)$ is Γ-*invariant* if $\pi_\mu(g)f = f$ a.e., for all $g \in \Gamma$. Equivalently, f is Γ-invariant if, for each $g \in \Gamma$, there exists a null-set $N_g \subset \Omega$ such that $f(gx) = f(x)$, for all $x \in \Omega \backslash N_g$. The set of all such functions is denoted by $L^2(\Omega, \mu)^{\pi_\mu}$

Lemma 3.1.6. *For each Γ-invariant function $f \in L^2(\Omega, \mu)^{\pi_\mu}$ there exists a measurable function $\tilde{f}$ with $\tilde{f} = f$ a.e., and such that $\tilde{f}(gx) = \tilde{f}(x)$, for all $g \in \Gamma$ and all $x \in \Omega$.*

Proof. For each $g \in \Gamma$, set $N_g = \{\, x \in \Omega \mid f(gx) \neq f(x) \,\}$. This is a measurable set with $\mu(N_g) = 0$, by assumption. Set $N = \bigcup_{g \in \Gamma} N_g$. Suppose $x \in N$ but that $gx \notin N$. In particular, $gx \notin N_{g^{-1}}$, and so $f(x) = f\big(g^{-1}(gx)\big) = f(gx)$. Take $h \in \Gamma$ such that $x \in N_h$. Then $f((hg^{-1})gx) = f(hx) \neq f(x) = f(gx)$. But this shows that $gx \in N_{hg^{-1}}$, which is a contradiction. Therefore, if $x \in N$ then so is gx, for all $g \in \Gamma$. The function $\tilde{f} = f 1_{\Omega \backslash N}$ satisfies the required properties. $\square$

Remark 3.1.7. In the proof of Lemma 3.1.6 above, it is only important that f is measurable and not that it is square-integrable.

Let $L^2_0(\Omega,\mu)$ be the subspace of $L^2(\Omega,\mu)$ consisting of all functions with zero mean, i.e.,

$$L^2_0(\Omega,\mu) = \left\{ f \in L^2(\Omega,\mu) \;\middle|\; \int_\Omega f \,\mathrm{d}\mu = 0 \right\}.$$

This is a closed Γ-invariant subspace. Since μ is a finite measure, all constant functions belong to $L^2(\Omega,\mu)$, and we see that $L^2_0(\Omega,\mu)$ is the orthogonal complement to 1_Ω. Denote by π^0_μ the restriction of π_μ to $L^2_0(\Omega,\mu)$.

Proposition 3.1.8. *Let Γ be a discrete group and (Ω,μ) a probability space. An action of Γ on Ω is ergodic if and only if the unitary representation π^0_μ has no non-zero invariant vectors.*

Proof. Suppose $f \in L^2_0(\Omega,\mu)$ is Γ-invariant. By Lemma 3.1.6, we may assume that $f(gx) = f(x)$, for all $g \in \Gamma$ and all $x \in \Omega$. Let $\mathcal{O}_x$ denote the orbit of x. For $\varepsilon > 0$, $n \in \mathbb{N}_0$ and $m \in \{0,1,\dots,n-1\}$, let

$$A_{n,m,\varepsilon} = \left\{ re^{i2\pi\theta} \;\middle|\; r \in [n\varepsilon,(n+1)\varepsilon),\, \theta \in \left[\tfrac{m}{n},\tfrac{m+1}{n}\right) \right\}.$$

Since f is constant on the orbits, each preimage $f^{-1}(A_{n,m,\varepsilon})$ is the union of a collection of orbits. Hence, $f^{-1}(A_{n,m,\varepsilon})$ is Γ-invariant. Further, note that, for a fixed $\varepsilon > 0$, the family $(A_{n,m,\varepsilon})_{n,m}$ is a partition of $\mathbb{C}$, and so $\big(f^{-1}(A_{n,m,\varepsilon})\big)_{n,m}$ is a partition of Ω.

If the action of Γ on Ω is ergodic, we must, for each $\varepsilon > 0$, have $\mu\big(f^{-1}(A_{n_0,m_0,\varepsilon})\big) = 1$, for some $n_0 \in \mathbb{N}_0$ and $m_0 \in \{0,1,\dots,n_0-1\}$, and $\mu\big(f^{-1}(A_{n,m,\varepsilon})\big) = 0$, for all $(n,m) \neq (n_0,m_0)$. For $k \in \mathbb{N}$, let $\varepsilon_k = \varepsilon/2^k$ and take $n_k \in \mathbb{N}_0$ and $m_k \in \{0,1,\dots,n_k-1\}$ such that $\mu\big(f^{-1}(A_{n_k,m_k,\varepsilon_k})\big) = 1$. Then $\overline{(A_{n_k,m_k,\varepsilon_k})_{k\geq 1}}$ is a decreasing sequence of sets of Lebesgue measure tending to zero. Hence $\bigcap_{k\geq 1} \overline{A_{n_k,m_k,\varepsilon_k}} = \{z\}$, for some $z \in \mathbb{C}$. By continuity of measures from above,

$$\mu\big(f^{-1}(\{z\})\big) = \inf_{k\geq 1} \mu\big(f^{-1}\big(\overline{A_{n_k,m_k,\varepsilon_k}}\big)\big) = 1.$$

Hence, f is constant almost everywhere. As f has mean zero, f is equivalent to 0.

Conversely, if f not equivalent to 0, it must take at least two distinct values on non-null-sets, because it has mean zero. Hence, we can find an $\varepsilon > 0$ and two pairs of non-negative integers $(n_1,m_1) \neq (n_2,m_2)$ with $m_1 < n_1$ and $m_2 < n_2$ such that $f^{-1}(A_{n_1,m_1,\varepsilon}) > 0$ and $f^{-1}(A_{n_2,m_2,\varepsilon}) > 0$. But then the action cannot be ergodic. $\square$

3.1.2 Diagonal actions and ergodicity

If Γ acts on Ω, we define the associated *diagonal action* on $\Omega \times \Omega$ by

$$\begin{aligned} \Gamma \times (\Omega \times \Omega) &\longrightarrow \Omega \times \Omega \\ \big(g,(x,y)\big) &\longmapsto (gx,gy) \end{aligned}$$

If α is the action of Γ on Ω, its associated diagonal action is denoted by $\Delta(\alpha)$. Observe that if α is measurable, then so is $\Delta(\alpha)$.

Remark 3.1.9. A diagonal action, $\Delta(\alpha)$, induces a unitary representation $\pi_{\mu\otimes\mu}$ on $L^2(\Omega \times \Omega, \mu\otimes\mu)$, as defined in equation (3.7). This representation is given by

$$\pi_{\mu\otimes\mu}(g)f(x,y) = f(g^{-1}x, g^{-1}y),$$

for $f \in L^2(\Omega\times\Omega,\mu\otimes\mu)$, $g \in \Gamma$ and $x,y \in \Omega$. We remark that $L^2(\Omega\times\Omega,\mu\otimes\mu)$ is isometrically isomorphic to $L^2(\Omega,\mu)^{\otimes 2}$ and that $\pi_{\mu\otimes\mu}$ is unitarily equivalent to $\pi_\mu^{\otimes 2}$ with this isomorphism as an intertwiner.

Definition 3.1.10. We say that a measurable action of a discrete group Γ on a probability space (Ω, μ) is *weakly mixing* if the diagonal action of Γ on the product measure space $(\Omega \times \Omega, \mu \otimes \mu)$ is ergodic.

Proposition 3.1.11. *Let Γ be a discrete group and (Ω, μ) a probability space. An action of Γ on Ω is weakly mixing if and only if the unitary representation $(\pi_\mu^0)^{\otimes 2}$ on $L_0^2(\Omega, \mu)^{\otimes 2}$ has no non-zero invariant vectors.*

Proof. This follows directly from Proposition 3.1.8 and Remark 3.1.9. □

Corollary 3.1.12. *Every weakly mixing action is ergodic.*

Proof. Let $\Gamma \curvearrowright \Omega$ be a non-ergodic action. Then π_μ^0 contains 1_Γ, by Proposition 3.1.8. Since π_μ^0 is the complexification of an orthogonal representation (see Remark 3.1.5), its contragredient representation $\overline{\pi}_\mu^0$ contains 1_Γ, as well. Hence, $(\pi_\mu^0)^{\otimes 2}$ also contains 1_Γ, by Proposition 1.1.17. It follows from Proposition 3.1.11 that $\Gamma \curvearrowright \Omega$ is not weakly mixing. □

3.1.3 Strong ergodicity and almost invariant vectors

Let $\Gamma \curvearrowright (\Omega, \mu)$ be a p.m.p. action. A sequence $(B_n)_{n\geq 1}$ of subsets of Ω is said to be *asymptotically invariant* if

$$\lim_{n\to\infty} \mu(gB_n \triangle B_n) = 0.$$

Note that $\mu(gB\triangle B) \leq \mu(gB)+\mu(B) = 2\mu(B)$ and $\mu(gB\triangle B) \leq \mu\big(B^{\complement}\big)+\mu\big((gB)^{\complement}\big) = 2\mu\big(B^{\complement}\big)$, for all measurable subsets $B \subset \Omega$ and all $g \in \Gamma$. Hence, if either $\mu(B_n) \to 0$ or $\mu(B_n) \to 1$, as n tends to infinity, then $(B_n)_{n\geq 1}$ is trivially asymptotically invariant. If $(B_n)_{n\geq 1}$ is an asymptotically invariant sequence, we say that $(B_n)_{n\geq 1}$ is *non-trivial* if

$$\liminf_{n\in\mathbb{N}} \mu(B_n)\mu\big(B_n^{\complement}\big) > 0.$$

Definition 3.1.13. A p.m.p. action is said to be *strongly ergodic* if there are no non-trivial asymptotically invariant sequences.

Remark 3.1.14. Suppose $\Gamma \curvearrowright (\Omega, \mu)$ is a p.m.p. action on the probability space (Ω, μ). If $B \subset \Omega$ is Γ-invariant, then the constant sequence $(B)_{n\geq 1}$ is asymptotically invariant. Hence, if $\Gamma \curvearrowright (\Omega, \mu)$ is strongly ergodic then $(B)_{n\geq 1}$ cannot be non-trivial, and so either $\mu(B) = 0$ or $\mu(B) = 1$. That is, any strongly ergodic action is ergodic.

Proposition 3.1.15. *Let Γ be a discrete group, (Ω, μ) a probability space and $\Gamma \curvearrowright (\Omega, \mu)$ a p.m.p. action. If $\Gamma \curvearrowright (\Omega, \mu)$ is not strongly ergodic then $1_\Gamma \prec \pi_\mu^0$.*

Proof. Suppose $(B_n)_{n\geq 1}$ is an asymptotically invariant sequence in (Ω, μ). We may assume that $\mu(B_n)\mu\big(B_n^{\complement}\big) > 0$, for all $n \in \mathbb{N}$. Otherwise, remove the finitely many sets from the sequence which do not satisfy this. For each $n \in \mathbb{N}$, define the function $f_n : \Omega \to \mathbb{R}$ by

$$f_n = 1_{B_n} - \mu(B_n)1_\Omega.$$

It is straight forward to compute that $f_n \in L_0^2(\Omega, \mu)$, and that

$$\|f_n\|^2 = \int_{B_n} \big(1 - \mu(B_n)\big)^2 \,\mathrm{d}\mu + \int_{B_n^{\complement}} \mu(B_n)^2 \,\mathrm{d}\mu = \mu(B_n)\mu\big(B_n^{\complement}\big).$$

Further,

$$\begin{aligned}\left\langle \pi_\mu^0(g) f_n \,, f_n \right\rangle &= \int_\Omega f_n(g^{-1}x) f(x)\,\mathrm{d}\mu(x) \\ &= \mu(gB_n \triangle B_n)\big(1-\mu(B_n)\big)\big(-\mu(B_n)\big) + \mu(gB_n \cap B_n)\big(1-\mu(B_n)\big)^2 \\ &\quad + \big(1-\mu(gB_n \cup B_n)\big)\mu(B_n)^2\end{aligned}$$

Since μ is Γ-invariant, we have the equalities:

$$\mu(gB_n \triangle B_n) + \mu(gB_n \cap B_n) = \mu(gB_n \cup B_n) = 2\mu(B_n) - \mu(gB_n \cap B_n).$$

Using these, we derive that

$$\begin{aligned}\left\langle \pi_\mu^0(g) f_n \,, f_n \right\rangle &= -\mu(gB_n \triangle B_n)\big(1-\mu(B_n)\big)\mu(B_n) + \big(\mu(B_n) - \tfrac{1}{2}\mu(gB_n \triangle B_n)\big)\big(1-\mu(B_n)\big)^2 \\ &\quad + \big(1-\mu(B_n) + \tfrac{1}{2}\mu(gB_n \triangle B_n) - \mu(gB_n \triangle B_n)\big)\mu(B_n)^2 \\ &= -\mu(gB_n \triangle B_n)\mu(B_n) - \tfrac{1}{2}\mu(gB_n \triangle B_n)\big(1-2\mu(B_n)\big) + \big(1-\mu(B_n)\big)\mu(B_n) \\ &= -\tfrac{1}{2}\mu(gB_n \triangle B_n) + \mu(B_n)\mu(B_n^{\complement}).\end{aligned}$$

From this, we deduce that that

$$\left\| \pi_\mu^0(g) f_n - f_n \right\|^2 = 2\left\| f_n \right\|^2 - 2\left\langle \pi_\mu^0(g) f_n \,, f_n \right\rangle = \mu(gB_n \triangle B_n).$$

For each $n \in \mathbb{N}$, let $\tilde{f}_n = f_n / \left\| f_n \right\|$. Then $\big(\tilde{f}_n\big)_{n\geq 1}$ is a sequence of almost invariant vectors for π_μ^0. It follows by Proposition 1.3.5 that 1_Γ is weakly contained in π_μ^0. □

Remark 3.1.16. The converse of the above statement is not true. An example of a strongly ergodic p.m.p. action of the free $\mathbb{F}_3$ where the associated representation π_μ^0 has almost invariant vectors were provided by K. Schmidt in [42, Example 2.7].

3.2 Orthogonal representations and actions on measure spaces

We present here a way to construct, from any orthogonal representation π of a discrete group, a p.m.p. action with the property that the associated unitary representation, as defined in equation (3.7), is equivalent with the extension of π to the *symmetric Fock space*, which we define in Subsection 3.2.2 below. Our first step towards establishing this connection between arbitrary orthogonal representations and actions on probability spaces is to show that any real Hilbert space is isometrically isomorphic to a subspace of an L^2-space. This is the content of Lemma 3.2.9.

3.2.1 Gaussian Hilbert spaces

We recall very briefly a few definitions and basic facts from probability theory that we shall need. Let (Ω, μ) be a probability space. A *(real-valued) random variable* on Ω is a measurable function on Ω with values in $\mathbb{R}$. The *distribution* of a random variable $X : \Omega \to \mathbb{R}$ is the *image measure* μ_X on $\mathbb{R}$ given by

$$\mu_X(B) = \mu(X^{-1}(B)), \qquad \text{for all Borel subsets } B \subset \mathbb{R}.$$

If there exists a measurable function $f_X : \mathbb{R} \to \mathbb{R}$ such that, for all Borel subsets $B \subset \mathbb{R}$,

$$\mu_X(B) = \int_B f_X \,\mathrm{d}\lambda,$$

where λ is the Lebesgue measure, we say that X has *density* and refer to f_X as the *density function* for X. If X is integrable or positive valued, we define its *expectation value* as

$$\mathbb{E}[X] = \int_\Omega X \,\mathrm{d}\mu = \int_\mathbb{R} x \,\mathrm{d}\mu_X(x).$$

If $\mathbb{E}[X] = 0$, we say that X is *centered*. If $X \in L^2(\Omega, \mu; \mathbb{R})$, the *variance* of X is the number

$$\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

We define its *characteristic function*, $\varphi_X : \mathbb{R} \to \mathbb{C}$, of a random variable X by

$$\varphi_X(t) = \mathbb{E}[e^{-itX}], \qquad \text{for } t \in \mathbb{R}.$$

Remark 3.2.1. A random variable is uniquely characterized by its characteristic function (see [25, Theorem 14.1]).

Two random variables X and Y are said to be *independent* if the probability of the event that $X \in A$ and $Y \in B$ equals the product of the probability of the individual events, for all Borel sets $A, B \subset \mathbb{R}$.

A real-valued random variable X is said to be *Gaussian* if it is either constant or has a density function of the form

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}, \tag{3.2}$$

for some real numbers $\sigma > 0$ and $m \in \mathbb{R}$. If X is Gaussian with density as in equation (3.2) then $\mathbb{E}[X] = m$ and $\mathbb{V}[X] = \sigma^2$. The *standard Gaussian distribution* is the Gaussian distribution with $m = 0$ and $\sigma = 1$.

Remark 3.2.2. If X and Y are independent Gaussian random variables on the same probability space with mean and variance (m_X, σ_X^2) and (m_Y, σ_Y^2), respectively, then their sum is again a Gaussian random variable with mean and variance $(m_X + m_Y, \sigma_X^2 + \sigma_Y^2)$. In particular, the sum of two independent centered Gaussian random variables is also a centered Gaussian random variable.

Lemma 3.2.3. *The L^2-limit of a sequence of centered Gaussian random variables is a centered Gaussian random variable.*

Proof. Let $(X_n)_{n\geq 1}$ be a sequence of centered Gaussian random variables on a probability space (Ω, μ), and let, for each $n \in \mathbb{N}$, ν_n be the distribution of X_n and let $\sigma_n^2 = \mathbb{V}[X_n]$. Suppose that $(X_n)_{n\geq 1}$ converges in $L^2(\Omega, \mu; \mathbb{R})$ to a random variable X. Then clearly $(\sigma_n^2)_{n\geq 1}$ is a converging sequence with limit $\sigma^2 = \mathbb{V}[X]$. As (Ω, μ) is a finite measure space, $\|\cdot\|_1 \leq \|\cdot\|_2$, and so, $\mathbb{E}[X] = \lim_n \mathbb{E}[X_n] = 0$. Then $\mathbb{V}[X] = \|X\|_2^2$, and so, if $\mathbb{V}[X] = 0$ then $X = 0$. Otherwise, the Lebesgue Dominated Convergence theorem implies that, for every $f \in C_c(\mathbb{R})$,

$$\int_\mathbb{R} f \,\mathrm{d}\nu = \lim_{n\to\infty} \int_\mathbb{R} f \,\mathrm{d}\nu_n = \lim_{n\to\infty} \frac{1}{\sigma_n\sqrt{2\pi}} \int_\mathbb{R} e^{-x^2/2\sigma_n^2} f(x) \,\mathrm{d}x = \frac{1}{\sigma\sqrt{2\pi}} \int_\mathbb{R} e^{-x^2/2\sigma^2} f(x) \,\mathrm{d}x,$$

where ν is the distribution of X. Hence, ν has density $x \mapsto \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$. ☐

Definition 3.2.4. A *Gaussian Hilbert space* is a closed subspace of $L^2(\Omega, \mu; \mathbb{R})$, for some probability space (Ω, μ), consisting of centered Gaussian random variables.

Remark 3.2.5. Let $(X_i)_{i\in I}$ be a family of pairwise independent Gaussian random variables on some probability space (Ω, μ). The closed linear span $\overline{\operatorname{span}}_\mathbb{R} \{ X_i \mid i \in I \}$ is a Hilbert subspace of $L^2(\Omega, \mu; \mathbb{R})$. By Remark 3.2.2 and Lemma 3.2.3, this is a Gaussian Hilbert space. We refer to this space as the Gaussian Hilbert space *generated by the family* $(X_i)_{i\in I}$.

Definition 3.2.6. A measurable space $(\Omega, \mathcal{B})$ is said to be *standard Borel* if there exists a metric d on Ω such that (Ω, d) is a complete separable metric space and such that $\mathcal{B}$ is the Borel σ-algebra associated to d.

Definition 3.2.7. Let Ω be a measurable space. A measure μ on Ω is said to be *non-atomic* if $\mu(\{\omega\}) = 0$, for all $\omega \in \Omega$.

Definition 3.2.8. Let Ω be a topological space equipped with a σ-algebra. A measure μ on Ω is said to be *outer regular* if

$$\mu(B) = \inf \{ \mu(U) \mid B \subset U,\, U \text{ open and measurable} \},$$

for all measurable sets $B \subset \Omega$.

Lemma 3.2.9. *Any real Hilbert space is isometrically isomorphic to a Gaussian Hilbert subspace of $L^2(\Omega, \mu; \mathbb{R})$, for a standard Borel space Ω and a non-atomic outer regular probability measure μ.*

Proof. Let H be a real Hilbert space and let $(e_i)_{i\in I}$ be an orthonormal basis. Denote by ν the standard Gaussian distribution and by $\mathcal{B}_1$ the Borel σ-algebra on $\mathbb{R}$. Let $\Omega = \prod_{i\in I} \mathbb{R}$. We equip Ω with the product σ-algebra $\mathcal{B} = \bigotimes_{i\in I} \mathcal{B}_1$ and the product measure $\mu = \otimes_{i\in I}\nu$. For each $i \in I$, let $X_i : \Omega \to \mathbb{R}$ be the canonical i'th projection, i.e., $X_i((\omega_j)_{j\in I}) = \omega_i$. Then each X_i is a standard Gaussian random variable and the family $(X_i)_{i\in I}$ is pairwise independent and form an orthonormal set in $L^2(\Omega, \mu; \mathbb{R})$. Let $K = \overline{\operatorname{span}} \{ X_i \mid i \in I \}$ be the Gaussian Hilbert space generated by the family $(X_i)_{i\in I}$ (see Remark 3.2.5). Then $(X_i)_{i\in I}$ is an orthonormal basis of K and the map $H \to K$ given by $e_i \mapsto X_i$ is an isometric isomorphism. □

Let Ω be a set, Ω' a measurable set and let $(X_i)_{i\in I}$ be a family of maps $\Omega \to \Omega'$. The *σ-algebra generated by* $(X_i)_{i\in I}$ is the smallest σ-algebra on Ω containing all sets of the form $T_i^{-1}(B)$, where $i \in I$ and $B \subset \Omega'$ is measurable. We denote this σ-algebra by $\sigma(X_i : i \in I)$. If $(\Omega, \mathcal{B})$ is a measurable space, we say that the family $(X_i)_{i\in I}$ generates the σ-algebra on Ω if $\sigma(X_i : i \in I) = \mathcal{B}$.

Remark 3.2.10. Let K be a Gaussian Hilbert space in $L^2(\Omega, \mathcal{B}, \mu)$. If $\mathcal{B}_K$ denotes the σ-algebra generated be the members of K then $K \subset L^2(\Omega, \mathcal{B}_K, \mu; \mathbb{R}) \subset L^2(\Omega, \mathcal{B}, \mu; \mathbb{R})$. If K is generated by the family $(X_i)_{i\in I}$ then $\mathcal{B}_K = \sigma(X_i : i \in I)$.

Lemma 3.2.11. *Let $X \in L^2(\Omega, \mu; \mathbb{R})$ be a centered Gaussian random variable with variance $\sigma^2 = \mathbb{E}[X^2] > 0$. Then $e^X \in L^2(\Omega, \mu; \mathbb{R})$ and*

$$\mathbb{E}\big[e^X\big] = e^{\frac{1}{2}\mathbb{E}[X^2]}. \tag{3.3}$$

Proof. Note first, that as $x \mapsto e^x$ is continuous, e^X is a measurable map on Ω. For $a, b \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} e^{bx} e^{-ax^2} \,\mathrm{d}x = \int_{-\infty}^{\infty} e^{-a(x-b/2a)^2} e^{-b^2/4a} \,\mathrm{d}x = \sqrt{\frac{\pi}{a}} e^{-b^2/4a},$$

where the last equality follows from the change of variables $y = x - b/2a$. We deduce that

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^x e^{-x^2/2\sigma^2} \,\mathrm{d}x = e^{\sigma^2/2}$$

This is exactly equation (3.3). □

Lemma 3.2.12. *Let K be a Gaussian Hilbert space in $L^2(\Omega, \mu; \mathbb{R})$. If K generates the σ-algebra on Ω then the linear span of the set $\big\{ e^X \bigm| X \in K \big\}$ is dense in $L^2(\Omega, \mu; \mathbb{R})$.*

Proof. Let $Y \in L^2(\Omega, \mu; \mathbb{R})$ be orthogonal to $\left\{ e^X \,\middle|\, X \in K \right\}$. We show that then Y is necessarily zero. The set map $\tilde{\nu}$ given by

$$\tilde{\nu}(B) = \int_B Y \, \mathrm{d}\mu, \qquad \text{for all measurable } B \subset \Omega,$$

is a bounded signed measure on Ω. Let $(X_i)_{i \in I}$ be an orthonormal basis of K. For a finite subset $\{i_1, \ldots, i_n\} \subset I$, let ν be the image measure of $\tilde{\nu}$ under the map $\Omega \to \mathbb{R}^n$ given by

$$\omega \mapsto (X_{i_1}(\omega), \ldots, X_{i_n}(\omega)).$$

For each $a = (a_1, \ldots, a_n)^T \in \mathbb{R}^n$, the random variable $a_1 X_{i_1} + \cdots + a_n X_{i_n}$ is in K, and so,

$$\begin{aligned} \int_{\mathbb{R}^n} e^{\langle x, a \rangle} \, \mathrm{d}\nu(x) &= \int_\Omega e^{a_1 X_{i_1}(\omega) + \cdots + a_n X_{i_n}(\omega)} \, \mathrm{d}\tilde{\nu}(\omega) \\ &= \int_\Omega e^{a_1 X_{i_1}(\omega) + \cdots + a_n X_{i_n}(\omega)} Y(\omega) \, \mathrm{d}\mu(\omega) = 0. \end{aligned}$$

Further, for each $a = (a_1, \ldots, a_n)^T \in \mathbb{C}^n$, $\left| \int_{\mathbb{R}^n} e^{\langle x, a \rangle} \, \mathrm{d}\nu(x) \right| \leq \int_{\mathbb{R}^n} e^{\langle x, \operatorname{Re} a \rangle} \, \mathrm{d}\nu(x) = 0$. Hence, ν is the distribution of a random variable with characteristic function equal to zero. Since random variables are characterized by their characteristic functions, we deduce $\nu = 0$. By construction of ν, this implies that $\tilde{\nu}$ vanishes on the σ-algebra generated by $\{X_{i_1}, \ldots, X_{i_n}\}$.

Let $\mathcal{D}$ be the union of all $\sigma(X_{i_1}, \ldots, X_{i_n})$ over all finite subsets $\{i_1, \ldots, i_n\} \subset I$ of indices. Then $\mathcal{D}$ is stable under finite intersections and

$$\sigma(\mathcal{D}) = \sigma(X_i : i \in I) = \sigma(X : X \in K).$$

What we have showed is that $\tilde{\nu}$ vanishes on $\mathcal{D}$, or, equivalently, that the non-negative measures $\tilde{\nu}^+$ and $\tilde{\nu}^-$ associated to the positive, respectively, negative parts of Y agree on all sets of $\mathcal{D}$. Therefore, by uniqueness of measures (see [41, Theorem 5.7]), $\tilde{\nu}^+$ and $\tilde{\nu}^-$ agree on the σ-algebra generated by K. By the assumption that K generates the σ-algebra on Ω, it follows that $\tilde{\nu} = 0$ on all measurable subsets of Ω. Hence, $Y = 0$, as we wanted to show. $\square$

3.2.2 The symmetric Fock space

Let H be a Hilbert space. For each $n \in \mathbb{N}$, we denote by S_n the symmetric group on $\{1, \ldots, n\}$. Each $\sigma \in S_n$ induces an operator T_σ on $H^{\otimes n}$ defined on elementary tensors by

$$T_\sigma(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}, \qquad \text{for } \xi_1, \ldots, \xi_n \in H.$$

Note that T_σ is a surjective isometry and that $T_\sigma \circ T_\tau = T_{\sigma \circ \tau}$, for all $\sigma, \tau \in S_n$.

The *n´th symmetric tensor power* of H is the closed subspace of $H^{\otimes n}$ defined by

$$S^n(H) = \left\{ \xi \in H^{\otimes n} \,\middle|\, T_\sigma \xi = \xi \text{ for all } \sigma \in S_n \right\}.$$

The orthogonal projection $P_n : H^{\otimes n} \to S^n(H)$ is given by

$$P_n = \frac{1}{n!} \sum_{\sigma \in S_n} T_\sigma.$$

For $\xi_1, \ldots, \xi_n \in H$, we denote by $\xi_1 \odot \cdots \odot \xi_n$ the image of $\xi_1 \otimes \cdots \otimes \xi_n$ under P_n.

Set $S^0(H) = \mathbb{R}$ if H is real and $\mathbb{C}$ if H is complex.

Definition 3.2.13. The *symmetric Fock space* of H is the Hilbert space direct sum:

$$S(H) = \bigoplus_{n \geq 0} S^n(H).$$

Let $T : H \to H$ be a linear map. We define the n'th tensor power of T as the map $T^{\otimes n} : H^{\otimes n} \to H^{\otimes n}$ given by $T^{\otimes n}(\xi_1 \otimes \cdots \otimes \xi_n) = T\xi_1 \otimes \cdots \otimes T\xi_n$. We get a canonical extension of T to the symmetric Fock space by taking the direct sum of (the restrictions of) all tensor powers of T. We denote this extension by $S(T)$. Note that if T is injective, surjective or an isometry, then so is $T^{\otimes n}$, for all n, and therefore so is $S(T)$.

Define the map $\operatorname{Exp} : H \to S(H)$ by

$$\operatorname{Exp}(\xi) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \xi^{\odot n}, \qquad \text{for all } \xi \in H.$$

Remark 3.2.14. For all $\xi, \eta \in H$, it holds that

$$\langle \operatorname{Exp}(\xi) \, , \operatorname{Exp}(\eta) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \xi \, , \eta \rangle^n = e^{\langle \xi \, , \eta \rangle} \tag{3.4}$$

Lemma 3.2.15. *The set* $\{ \operatorname{Exp}(\xi) \mid \xi \in H \}$ *is linearly independent and its linear span is dense in the symmetric Fock space,* $S(H)$.

Proof. We start by proving linear independence. We denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$ depending on whether H is real or complex. Suppose there exists a finite collection of distinct vectors $\xi_1, \dots \xi_n \in H$, and, for these vectors, a collection of scalars $\lambda_1, \dots, \lambda_n \in \mathbb{K}$ such that $\sum_{i=1}^n \lambda_i \operatorname{Exp}(\xi_i) = 0$. Consider the sets

$$E_{i,j} = \{ \eta \in H \mid \langle \xi_i - \xi_j \, , \eta \rangle \neq 0 \},$$

where $1 \leq i, j \leq n$ and $i \neq j$. These are open and dense sets in H. Hence, $\bigcap_{i \neq j} E_{i,j}$ is dense in H, by the Baire Category Theorem. In particular, this intersection is non-empty, and so, we may find $\eta \in H$ such that $\langle \xi_i \, , \eta \rangle \neq \langle \xi_j \, , \eta \rangle$, whenever $i \neq j$. Then, for all $z \in \mathbb{C}$,

$$0 = \left\langle \sum_{i=1}^n \lambda_i \operatorname{Exp}(\xi_i) \, , \operatorname{Exp}(z\eta) \right\rangle = \sum_{i=1}^n \lambda_i e^{z \langle \xi_i \, , \eta \rangle}.$$

Linear independence of the set $\{ \operatorname{Exp}(\xi) \mid \xi \in H \}$ then follows from linear independence of the set of functions of the form $z \mapsto e^{\theta z}$ on $\mathbb{C}$.

We proceed to show that $\operatorname{span}\{ \operatorname{Exp}(\xi) \mid \xi \in H \}$ is dense in $S(H)$. For each $\xi \in H$, define a function $f_\xi : \mathbb{R} \to S(H)$ by $f_\xi(t) = \operatorname{Exp}(t\xi)$. For each $n \in \mathbb{N}$, it is straight forward to check that the n'th derivative of f in zero is

$$f_\xi^{(n)}(0) = \sqrt{n!} \, \xi^{\odot n}.$$

Since $f_\xi(t) \in \operatorname{span}\{ \operatorname{Exp}(\xi) \mid \xi \in H \}$, for all $t \in \mathbb{R}$ and any vector $\xi \in H$, it follows that $\xi^{\odot n} \in \operatorname{span}\{ \operatorname{Exp}(\xi) \mid \xi \in H \}$, for all $\xi \in H$ and all $n \geq 0$. We conclude that $\operatorname{span}\{ \operatorname{Exp}(\xi) \mid \xi \in H \}$ equals the direct sum of all the n'th symmetric tensor powers of H as vector spaces. Hence, $\operatorname{span}\{ \operatorname{Exp}(\xi) \mid \xi \in H \}$ is dense in $S(H)$. $\square$

3.2.3 Measure preserving transformations and isometries of L^2-spaces

For a measure preserving transformation θ of the measure space (Ω, μ), we define a linear map $\theta^* : L^2(\Omega, \mu; \mathbb{R}) \to L^2(\Omega, \mu; \mathbb{R})$ by setting

$$\theta^*(f) = f \circ \theta^{-1}, \qquad \text{for } f \in L^2(\Omega, \mu; \mathbb{R}).$$

By Lemma 3.1.1, this is a surjective isometry, and we see that $(\theta^*)^{-1} = (\theta^{-1})^*$.

The induced surjective isometry θ^* has the following property: For every pair of measurable subsets $A, B \subset \Omega$,

$$\theta^*(1_A 1_B) = \theta^*(1_{A\cap B}) = 1_{A\cap B} \circ \theta^{-1} = (1_A \circ \theta^{-1})(1_B \circ \theta^{-1}) = \theta^*(1_A)\theta^*(1_B).$$

Moreover, since $(\theta^*)^{-1} = (\theta^{-1})^*$, the similar equality follows for $(\theta^*)^{-1}$. We shall see in Lemma 3.2.21 that the converse is true when Ω is a standard Borel space and μ is an outer regular non-atomic probability measure.

Lemma 3.2.16. *If a measure preserving transformation θ on a finite measure space (Ω, μ) is equal to the identity transformation of Ω almost everywhere then $\theta^*(1_B) = 1_B$, for all measurable $B \subset \Omega$. If Ω is standard Borel, the converse is also true.*

Proof. We have, for any measurable $B \subset \Omega$, the following equality in $L^2(\Omega, \mu)$:

$$\theta^*(1_B) = 1_B \circ \theta^{-1} = 1_{\theta(B)}.$$

Hence, $\theta^*(1_B) = 1_B$ if and only if the measurable set

$$\widetilde{B} = \big\{ \omega \in \Omega \bigm| 1_B(\omega) \neq 1_{\theta(B)}(\omega) \big\} = B \triangle \theta(B)$$

has measure zero. Further, θ is equal almost everywhere to the identity transformation if and only if the measurable set $A = \{\theta(\omega) \neq \omega\}$ has measure zero.

For a subset $B \subset \Omega$, if $\omega \in B\triangle\theta(B)$, we have, in particular, that $\omega \neq \theta^{-1}(\omega)$. Hence, for any subset $B \subset \Omega$, the symmetric difference $B\triangle\theta(B)$ is contained in A. Therefore, if $\mu(B\triangle\theta(B)) > 0$, for some measurable subset $B \subset \Omega$, then A has strictly positive measure, as well. This shows the first statement of the lemma.

Assume that Ω is a standard Borel space, let $(\omega_i)_{i\in I}$ be a countable dense set and denote by d the metric on Ω inducing the Borel σ-algebra on Ω. Suppose θ is not equal almost everywhere to the identity transformation so that A has strictly positive measure. Then we can find an $a \in A$ with the property that, for each $\varepsilon > 0$ there exists an $i \in I$ with $a \in B(\omega_i, \varepsilon)$ such that $\mu(B(\omega_i, \varepsilon)) > 0$. Since Ω is Hausdorff and $\theta(a) \neq a$, we may find an open neighborhood $a \in V \subset \Omega$ such that $V \cap \theta(V) = \emptyset$. Take $\varepsilon > 0$ such that $B(a, \varepsilon) \subset V$. Take $i_0 \in I$ such that $a \in B(\omega_i, \varepsilon/2)$ and $\mu(B(\omega_i, \varepsilon/2)) > 0$. If $\omega \in B(\omega_i, \varepsilon/2)$, then $d(\omega, a) \leq d(\omega, \omega_i) + d(\omega_i, a) < \varepsilon$. Hence, $B(\omega_i, \varepsilon/2) \subset V$. We conclude that V is a set of strictly positive measure. Since V also have the property that $V \cap \theta(V) = \emptyset$, it follows that $1_V \neq 1_{\theta(V)}$. This shows the second statement of the lemma. $\square$

Remark 3.2.17. It follows directly from the above lemma that two measure preserving transformations θ_1 and θ_2 of the same finite standard Borel measure space are equal almost everywhere if and only if $\theta_1^*(1_B) = \theta_2^*(1_B)$, for all measurable $B \subset \Omega$. Indeed, this is clear using the easy observation that $(\theta_1 \circ \theta_2)^* = \theta_1^* \circ \theta_2^*$.

Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. We define an equivalence relation on $\mathcal{B}$ by setting $A \sim B$ whenever $\mu(A\triangle B) = 0$.

Remark 3.2.18. For any pair of measurable sets $A, B \in \mathcal{B}$, their indicator functions 1_A and 1_B are equal in $L^2(\Omega, \mu; \mathbb{R})$ if and only if $A \sim B$.

Denote by $\widetilde{\mathcal{B}}$ the quotient of $\mathcal{B}$ with this relation, and by $\widetilde{B}$ the equivalence class of a set $B \subset \Omega$. For $A, B \in \mathcal{B}$ with $A \sim B$, we have $0 \leq \mu(A \backslash (A \cap B)) \leq \mu(A \triangle B) = 0$, and so, $\mu(A) = \mu(A \cap B)$. By symmetry, it follows that $\mu(A) = \mu(B)$. We may therefore define an induced measure $\tilde{\mu}$ on $\widetilde{\mathcal{B}}$ by setting $\tilde{\mu}(\widetilde{B}) = \mu(B)$, for each set $B \subset \Omega$. The pair $(\widetilde{\mathcal{B}}, \tilde{\mu})$ is called the *measure algebra* associated to the measure space $(\Omega, \mathcal{B}, \mu)$.The operations of complementation, intersections and countable unions of the elements of $\widetilde{B}$ are defined in the natural way:

$$\begin{aligned}
\widetilde{B}^{\complement} &= \widetilde{B^{\complement}}, && \text{for } B \in \mathcal{B}, \\
\widetilde{A} \cap \widetilde{B} &= \widetilde{A \cap B}, && \text{for } A, B \in \mathcal{B}, \\
\bigcup_{i \in I} \widetilde{B_i} &= \widetilde{\bigcup_{i \in I} B_i}, && \text{for } I \text{ countable and } (B_i)_{i \in i} \subset \mathcal{B}.
\end{aligned}$$

Definition 3.2.19. A *measure algebra automorphism* of $(\widetilde{\mathcal{B}}, \tilde{\mu})$ is a measure preserving bijection $\widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ which preserves complements and countable unions.

Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. A measure preserving transformation $\theta : \Omega \to \Omega$ induces in a natural way a measure algebra automorphism as follows: We may define a set map $\Theta : \mathcal{B} \to \mathcal{B}$ by setting $\Theta(B) = \theta(B)$, for $B \in \mathcal{B}$. Note that this is well-defined by the requirement that θ is a bijection with measurable inverse. For any map $\eta : \Omega \to \Omega$, we have the set inclusion $\eta(A) \triangle \eta(B) \subset \eta(A \triangle B)$, and, if η is a bijection, the inclusion is an equality. Hence, if $A \sim B$, we see that $\mu(\theta(A) \triangle \theta(B)) = \mu(\theta(A \triangle B)) = \mu(A \triangle B) = 0$, so that $\theta(A) \sim \theta(B)$. Therefore, Θ descends to a well-defined map $\widetilde{\Theta} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ given by $\widetilde{\Theta}(\widetilde{B}) = \widetilde{\theta(B)}$. By the assumption that θ is bijective, it is clear that $\widetilde{\Theta}$ is a measure algebra automorphism of $(\widetilde{\mathcal{B}}, \tilde{\mu})$.

For standard Borel spaces with σ-finite outer regular non-atomic measures, the converse is also true. That is, any measure algebra automorphism is induced by a measure preserving transformation as described above. This result is due to J. von Neumann in [51], and we state it below without proof.

Proposition 3.2.20. *Let $(\Omega, \mathcal{B}, \mu)$ be a standard Borel space equipped with a σ-finite outer regular non-atomic measure, and let $\Theta : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ be a measure algebra automorphism. There exists a measure preserving transformation $\theta : \Omega \to \Omega$ such that $\Theta(\widetilde{B}) = \widetilde{\theta(B)}$, for all $B \in \mathcal{B}$.*

We are now ready to state and proof the announced result characterizing the surjective isometries on a given L^2-space coming from a measure preserving transformation.

Lemma 3.2.21. *Let $(\Omega, \mathcal{B}, \mu)$ be a standard Borel space equipped with a non-atomic outer regular probability measure. If $T : L^2(\Omega, \mu; \mathbb{R}) \to L^2(\Omega, \mu; \mathbb{R})$ is a surjective isometry satisfying the equalities*

$$T(1_A 1_B) = T(1_A)T(1_B) \qquad \textit{and} \qquad T^{-1}(1_A 1_B) = T^{-1}(1_A)T^{-1}(1_B),$$

for all pairs of measurable subsets $A, B \subset \Omega$, then there exists a measure preserving transformation θ of Ω such that $T = \theta^*$.

Proof. For $B \in \mathcal{B}$, we have $T(1_B)^2 = T(1_B^2) = T(1_B)$, by assumption. Hence, $T(1_B)$ takes values in $\{0, 1\}$, and so, there exists $B' \in \mathcal{B}$ such that $T(1_B) = 1_{B'}$. For each $B \in \mathcal{B}$, we make such a choice, and wefine a map $\Theta : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}$ by setting

$$\Theta(\widetilde{B}) = \widetilde{B'}, \qquad \text{for } B \in \mathcal{B}.$$

By Remark 3.2.18, this map is well-defined. We claim that it is a measure algebra automorphism. First, it is clear that Θ is measure preserving because T is an isometry. Let $B \in \mathcal{B}$. Using linearity of T, we see that

$$T(1_\Omega) = T(1_B) + T(1_{B^\complement}) = 1_{B'} + 1_{(B^\complement)'} = 1_{B' \cup (B^\complement)'} + 1_{B' \cap (B^\complement)'},$$

Since T is an isometry $T(1_\Omega) = 1_{\Omega'}$, for some set Ω' of measure 1. Hence, $\mu(B' \cap (B^\complement)') = 0$ and $\mu(B' \cup (B^\complement)') = 1$. Using the easy fact that $A\Delta(C^\complement) = (A\Delta C)^\complement$, it follows that

$$0 \leq \mu((B')^\complement \cap (B^\complement)') = 1 - \mu(B' \cap (B^\complement)') \leq 1 - \mu(B' \cup (B^\complement)') = 0.$$

This shows that $(B')^\complement \sim (B^\complement)'$, and so, Θ preserves complements. Finally, let $(B_i)_{i\in I}$ be a countable family of pairwise disjoint sets in $\mathcal{B}$. By linearity and continuity of T, we have

$$\sum_{i\in I} 1_{B_i'} = \sum_{i\in I} T(1_{B_i}) = T\left(\sum_{i\in I} 1_{B_i}\right) = T\big(1_{\bigcup_{i\in I} B_i}\big) = 1_{\left(\bigcup_{i\in I} B_i\right)'}.$$

Since the right-hand side is an indicator function, we can find a set $A \in \mathcal{B}$ of measure 1 such that the sets $(B_i \cap A)_{i\in I}$ are pairwise disjoint, and we have

$$1_{\bigcup_{i\in I} B_i'} = 1_{\bigcup_{i\in I}(B_i'\cap A)} = 1_{\left(\bigcup_{i\in I} B_i\right)'},$$

in $L^2(\Omega,\mu;\mathbb{R})$. By Remark 3.2.18, this shows that Θ preserves countable unions. Thus, Θ is a measure algebra automorphism. By Proposition 3.2.20, there exists a measure preserving transformation θ of Ω such that $\Theta(\widetilde{B}) = \widetilde{\theta(B)}$, for all $B \in \mathcal{B}$. Hence, for any $B \in \mathcal{B}$, $\theta(B) \sim B'$. By Remark 3.2.18, this is equivalent to

$$\theta^* 1_B = 1_{\theta(B)} = 1_{B'} = T(1_B).$$

Hence, θ^* and T agree on all simple functions, and therefore on all measurable functions. $\square$

3.2.4 Construction of a p.m.p. action from an orthogonal representation

Theorem 3.2.22. *Let H be a separable real Hilbert space and let (Ω,μ) be a standard Borel probability space such that μ is non-atomic. Let $K \subset L^2(\Omega,\mu;\mathbb{R})$ be a Gaussian Hilbert space generating the σ-algebra on Ω. Let $\Phi : H \to K$ be an isometric isomorphism. Then Φ extends to an isometric isomorphism $\widetilde{\Phi} : S(H) \to L^2(\Omega,\mu;\mathbb{R})$ with the property that, for every surjective isometry $T : H \to H$, there exists a measure preserving transformation θ_T of (Ω,μ) such that $\widetilde{\Phi} \circ S(T) = \theta_T^* \circ \widetilde{\Phi}$. Moreover, θ_T is unique up to null-sets.*

Proof. Let $\xi, \eta \in H$ and set $X = \Phi(\xi)$ and $Y = \Phi(\eta)$. By Lemma 3.2.11, the random variables e^X, e^Y and e^{X+Y} are in $L^2(\Omega,\mu;\mathbb{R})$, and we have

$$\begin{aligned}
\left\langle e^{X-\frac{1}{2}\mathbb{E}[X^2]}, e^{Y-\frac{1}{2}\mathbb{E}[Y^2]} \right\rangle &= \int_\Omega e^{X-\frac{1}{2}\mathbb{E}[X^2]} e^{Y-\frac{1}{2}\mathbb{E}[Y^2]}\,\mathrm{d}\mu \\
&= e^{-\frac{1}{2}(\mathbb{E}[X^2]+\mathbb{E}[Y^2])}\,\mathbb{E}\big[e^{X+Y}\big] \\
&= e^{-\frac{1}{2}(\mathbb{E}[X^2]+\mathbb{E}[Y^2])} e^{\frac{1}{2}\mathbb{E}[(X+Y)^2]} \\
&= e^{\mathbb{E}[XY]} = e^{\langle X, Y\rangle} = e^{\langle \Phi(\xi), \Phi(\eta)\rangle}.
\end{aligned}$$

Further, using that Φ is an isometry together with Remark 3.2.14 we deduce that

$$\left\langle e^{X-\frac{1}{2}\mathbb{E}[X^2]}, e^{Y-\frac{1}{2}\mathbb{E}[Y^2]} \right\rangle = e^{\langle \xi, \eta\rangle} = \langle \mathrm{Exp}(\xi), \mathrm{Exp}(\eta)\rangle. \tag{3.5}$$

Define a map $\widetilde{\Phi} : \{\, \mathrm{Exp}(\xi) \mid \xi \in H \,\} \to L^2(\Omega, \mu; \mathbb{R})$ by

$$\widetilde{\Phi}(\mathrm{Exp}(\xi)) = e^{\Phi(\xi) - \frac{1}{2}\mathbb{E}[\Phi(\xi)^2]}, \qquad \text{for } \xi \in H.$$

By equation (3.5), $\widetilde{\Phi}$ is an isometry on $\{\, \mathrm{Exp}(\xi) \mid \xi \in H \,\}$. Hence, $\widetilde{\Phi}$ extends to an isometric linear map $S(H) \to L^2(\Omega, \mu; \mathbb{R})$, by Lemma 3.2.15, which we also denote by $\widetilde{\Phi}$. By Lemma 3.2.12, this extension is surjective.

Let $T : H \to H$ be a surjective isometry. Then its canonical extension, $S(T)$, to the symmetric Fock space of H is a surjective isometry, as well. Therefore, $R = \widetilde{\Phi} \circ S(T) \circ \widetilde{\Phi}^{-1}$ is a surjective isometry on $L^2(\Omega, \mu; \mathbb{R})$. For any $\zeta \in H$, we have,

$$R\big(\widetilde{\Phi}(\mathrm{Exp}(\zeta))\big) = \widetilde{\Phi}\big(S(T)\big(\mathrm{Exp}(\zeta)\big)\big) = \widetilde{\Phi}(\mathrm{Exp}(T\zeta)) = e^{\Phi(T\zeta) - \frac{1}{2}\mathbb{E}[\Phi(T\zeta)^2]}. \tag{3.6}$$

Let $\xi, \eta \in H$ and set $X = \widetilde{\Phi}(\mathrm{Exp}(\xi))$ and $Y = \widetilde{\Phi}(\mathrm{Exp}(\eta))$. We have, by definition of $\widetilde{\Phi}$,

$$\begin{aligned} XY &= e^{\Phi(\xi) - \frac{1}{2}\mathbb{E}[\Phi(\xi)^2]} e^{\Phi(\eta) - \frac{1}{2}\mathbb{E}[\Phi(\eta)^2]} \\ &= e^{\Phi(\xi+\eta) - \frac{1}{2}\mathbb{E}[\Phi(\xi+\eta)^2]} e^{\mathbb{E}[\Phi(\xi)\Phi(\eta)]} \\ &= e^{\mathbb{E}[\Phi(\xi)\Phi(\eta)]} \widetilde{\Phi}(\mathrm{Exp}(\xi + \eta)). \end{aligned}$$

Hence, XY lies in $L^2(\Omega, \mu; \mathbb{R})$. By equation (3.6), we see that

$$\begin{aligned} R(X) &= e^{\Phi(T\xi) - \frac{1}{2}\mathbb{E}[\Phi(T\xi)^2]}, \\ R(Y) &= e^{\Phi(T\eta) - \frac{1}{2}\mathbb{E}[\Phi(T\eta)^2]}, \end{aligned}$$

and that

$$\begin{aligned} R(XY) &= e^{\mathbb{E}[\Phi(\xi)\Phi(\eta)]} e^{\Phi(T(\xi+\eta)) - \frac{1}{2}\mathbb{E}[\Phi(T(\xi+\eta))^2]} \\ &= e^{\Phi(T(\xi+\eta)) - \frac{1}{2}\left(\mathbb{E}[\Phi(T\xi)^2] + \mathbb{E}[\Phi(T\eta)^2]\right)} e^{\mathbb{E}[\Phi(\xi)\Phi(\eta)] - \mathbb{E}[\Phi(T\xi)\Phi(T\eta)]}. \end{aligned}$$

Since Φ and T are isometries, we have

$$\mathbb{E}[\Phi(\xi)\Phi(\eta)] = \langle \xi \,, \eta \rangle = \langle T\xi \,, T\eta \rangle = \mathbb{E}[\Phi(T\xi)\Phi(T\eta)].$$

Together with our above derivations, we deduce that

$$R(XY) = R(X)R(Y).$$

By continuity, this equality holds for all X and Y in $L^2(\Omega, \mu; \mathbb{R})$. In particular, for any pair of indicator functions 1_A and 1_B with A and B measurable. Further, as T was any surjective isometry and since $R^{-1} = \widetilde{\Phi} \circ S(T^{-1}) \circ \widetilde{\Phi}^{-1}$, the same equality holds for R^{-1}. Hence, by Lemma 3.2.21, there exists a measure preserving transformation θ_T of (Ω, μ) such that $R = \theta_T^*$. This finishes the proof of the theorem. $\square$

Remark 3.2.23. Let H, $K \subset L^2(\Omega, \mu; \mathbb{R})$ and $\Phi : H \to K$ be as in Theorem 3.2.22 above. Consider the 1-dimensional subspace $S^0(H)$ of the symmetric Fock space of H. It is clear from Lemma 3.2.15 that this is the subspace spanned by the vector $\mathrm{Exp}(0)$, where 0 is the zero-vector in H. By definition of $\widetilde{\Phi}$ in the proof of Theorem 3.2.22, we see that $\widetilde{\Phi}(\mathrm{Exp}(0)) = 1_\Omega$. Hence, $\widetilde{\Phi}(S^0(H)) = \mathbb{R}1_\Omega$. Since $\widetilde{\Phi}$ is an isometric isomorphism, it follows that the restriction of $\widetilde{\Phi}$ to the orthogonal complement of $S^0(H)$ is an isometry onto $L^2_0(\Omega, \mu; \mathbb{R}) = 1_\Omega^\perp$.

Let (π, H) be an orthogonal representation of the discrete group Γ on a real Hilbert space. For each $n \in \mathbb{N}$, let $\pi^{\otimes n}$ be the representation of Γ on $H^{\otimes n}$ given on elementary tensors by $\pi^{\otimes n}(g)(\xi_1 \otimes \cdots \otimes \xi_n) = \pi(g)\xi_1 \otimes \cdots \otimes \pi(g)\xi_n$. Clearly, the n'th symmetric tensor power of H, $S^n(H)$, is a Γ-invariant subspace of $H^{\otimes n}$. The restriction of $\pi^{\otimes n}$ to $S^n(H)$ is called the *n'th symmetric tensor power* of π and is denoted by $S^n(\pi)$. For $n = 0$, let $S^0(\pi)$ be the trivial representation on $\mathbb{R}$. The representation (π, H) then induces in a canonical way a representation on the symmetric Fock space of H by taking the direct sum $S(\pi) = \bigoplus_{n=0}^{\infty} S^n(\pi)$ of all n'th symmetric tensor powers of π. Precisely, $S(\pi)$ is the representation on $S(H)$ given by $S(\pi)(g) = S(\pi(g))$, for each $g \in \Gamma$.

Remark 3.2.24. For any pair of non-negative integers $k, l \in \mathbb{N}_0$, and any real Hilbert space H,, we have the inclusion $S^{k+l}(H) \subset S^k(H) \otimes S^l(H)$. In particular, for a representation π of a discrete group Γ on H, $S^{k+l}(\pi)$ is a subrepresentation of $S^k(\pi) \otimes S^l(\pi)$.

Corollary 3.2.25. *Let (π, H) be an orthogonal representation of the countable discrete group Γ on a real and separable Hilbert space. There exists a probability space (Ω, μ) and a measure preserving action of Γ on Ω such that the associated representation of Γ on $L^2(\Omega, \mu; \mathbb{R})$ is equivalent with the representation of Γ on the symmetric Fock space of H induced by π.*

Proof. By Lemma 3.2.9, we may find a Gaussian Hilbert space $K \subset L^2(\Omega, \mu; \mathbb{R})$, where Ω is a standard Borel space and μ is a non-atomic probability measure, and an isometric isomorphism $\Phi : H \to K$. Further, by Remark 3.2.10, we may assume that K induces the σ-algebra on Ω. Then, Theorem 3.2.22 implies that Φ extends to an isometric isomorphism $\widetilde{\Phi} : S(H) \to L^2(\Omega, \mu; \mathbb{R})$. Moreover, for each $g \in \Gamma$, $\pi(g)$ is a surjective isometry of H, and so induces a measure preserving transformation θ_g of (Ω, μ) with the property that $\widetilde{\Phi} \circ S(\pi(g)) = \theta_g^* \circ \widetilde{\Phi}$. In particular, for $g = e$, we see that $\theta_e^* = \mathrm{id}_{L^2(\Omega,\mu;\mathbb{R})}$. Further, for every $g, h \in \Gamma$, we have

$$\theta_{gh}^* = \widetilde{\Phi} \circ S(\pi(gh)) \circ \widetilde{\Phi}^{-1} = \widetilde{\Phi} \circ S(\pi(g)) \circ \widetilde{\Phi}^{-1} \circ \widetilde{\Phi} \circ S(\pi(h)) \circ \widetilde{\Phi}^{-1} = \theta_g^* \circ \theta_h^*.$$

Hence, by Lemma 3.2.16 and Remark 3.2.17, $\theta_e = \mathrm{id}_\Omega$ a.e., and $\theta_g \circ \theta_h = \theta_{gh}$ a.e., for all $g, h \in \Gamma$. Define

$$\Omega_0 = \bigcap_{g \in \Gamma} \{ \, \omega \in \Omega \mid \theta_e(\theta_g(\omega)) = \theta_g(\omega) \, \}.$$

Then Ω_0 is a Γ-invariant set of measure 1. We equip Ω_0 with the trace σ-algebra. Define an action α of Γ on Ω_0 by setting $\alpha(g, \omega) = \theta_g(\omega)$, for all $g \in \Gamma$ and $\omega \in \Omega_0$. For each measurable $B \subset \Omega_0$, we see that

$$\alpha^{-1}(B) = \{ \, (g, \omega) \in \Gamma \times \Omega_0 \mid \theta_g(\omega) \in B \, \} = \bigcup_{g \in \Gamma} \{g\} \times \theta_g^{-1}(B).$$

Hence, α is a measurable action. The unitary representation of Γ on $L^2(\Omega_0, \mu; \mathbb{R}) \cong L^2(\Omega, \mu; \mathbb{R})$ associated to this action is given by

$$\pi_\mu(g) f = f \circ \theta_{g^{-1}} = \theta_g^*(f), \tag{3.7}$$

for $f \in L^2(\Omega_0, \mu; \mathbb{R})$ and $g \in \Gamma$. Thus, it is clear that $\widetilde{\Phi}$ intertwines $\bigoplus_{n=0}^{\infty} S^n(\pi)$ and π_μ. □

3.3 The Connes-Weiss characterisation

With the preparations of the two previous sections, we are ready to state and proof a characterization of Property (T) in terms of ergodicity of actions on measure spaces. This characterization is due to A. Connes and B. Weiss in [11]. We follow the proof presented in [5, Theorem 6.3.4].

Theorem 3.3.1 (Connes-Weiss)**.** *Let Γ be a discrete group. The following are equivalent:*

(i) *Γ has property (T),*

(ii) *every p.m.p. ergodic action of Γ is strongly ergodic,*

(iii) *every p.m.p. weakly mixing action of Γ is strongly ergodic.*

Proof. If $\Gamma \curvearrowright (\Omega, \mu)$ is a p.m.p. ergodic action then π_μ^0 does not contain 1_Γ, by Proposition 3.1.8. Therefore, if Γ has (T), then π_μ^0 does not weakly contain 1_Γ, and it follows from Proposition 3.1.15 that $\Gamma \curvearrowright (\Omega, \mu)$ is strongly ergodic. This shows $(i) \Rightarrow (ii)$. The implication $(ii) \Rightarrow (iii)$ is direct from Corollary 3.1.12. We proceed to show the implication $(iii) \Rightarrow (i)$.

Assume that Γ does not have (T) and let (π, H) be a unitary representation with almost invariant vectors but no non-zero invariant vector. Note that H must then necessarily be infinite dimensional, as otherwise its unit ball would be compact. By Theorem 2.5.8, we may assume that (π, H) contains no non-zero finite dimensional representations. Moreover, it is clear from the proof of Theorem 2.5.8 and Remark 2.4.31 that we can also assume this representation to be the complexification of an orthogonal representation. That is, $H = H' \otimes \mathbb{C}$, for some real Hilbert space H', and $\pi = \pi'_{\mathbb{C}}$, for some orthogonal representation π' of Γ on H'.

By Corollary 3.2.25, we may find a probability space (Ω, μ) and a p.m.p. action on Ω such that the associated representation π'_μ on $L^2(\Omega, \mu; \mathbb{R})$ is equivalent to the representation $\bigoplus_{n=0}^\infty S^n(\pi')$ of Γ on the symmetric Fock space of H' induced by π'. This equivalence naturally lifts to the complexifications π_μ on $L^2(\Omega, \mu)$ and $\bigoplus_{n=0}^\infty S^n(\pi)$ on $S(H)$. Using Remark 3.2.23, we obtain the following isometric isomorphisms intertwining the corresponding subrepresentations:

$$\bigoplus_{n=1}^{\infty} S^n(H) \cong L_0^2(\Omega, \mu).$$

Suppose $\bigoplus_{n=1}^\infty S^n(\pi)$ contains a finite dimensional subrepresentation, ρ. We may assume that ρ is irreducible, as any finite dimensional representation contains an irreducible one. By Proposition 1.1.14, we may find an $n \in \mathbb{N}$ such that $S^n(\pi)$ contains ρ. Since π is the complexification of an orthogonal representation, then so is $S^n(\pi)$. Therefore, Proposition 1.1.17 implies that $S^n(\pi) \otimes S^n(\pi)$ contains the trivial representation. Therefore, $S^1(\pi) \otimes \big(S^{n-1}(\pi) \otimes S^n(\pi)\big)$ contains the trivial representation, by Remark 3.2.24. Observing that $S^1(\pi) = \pi$ and applying Proposition 1.1.17 to $\pi \otimes \big(S^{n-1}(\pi) \otimes S^n(\pi)\big)$ then yields that (π, H) contains a finite dimensional subrepresentation. But this is a contradiction. Hence, $\bigoplus_{n=1}^\infty S^n(\pi)$ contains no finite dimensional subrepresentation, and so neither does π_μ^0. Recalling that π_μ^0 is the complexification of an orthogonal representation and applying Proposition 1.1.17 and Proposition 3.1.11 to π_μ^0, we deduce that the action of Γ on (Ω, μ) is weakly mixing.

Let $(\xi_n)_{n \geq 1}$ be a sequence of almost invariant (unit) vectors for (π, H). We may assume that each ξ_n lies in the real Hilbert space H'. Then, for each $g \in \Gamma$ and each $n \in \mathbb{N}$, $\|\pi(g)\xi_n - \xi_n\|^2 = 2(1 - \langle \pi(g)\xi_n \,, \xi_n \rangle)$. As $\|\pi(g)\xi_n - \xi_n\| \to 0$, we see that

$$\lim_{n\to\infty} \langle \pi(g)\xi_n \,, \xi_n \rangle = 1,$$

for all $g \in \Gamma$. By Cauchy-Schwarz, we have $\langle \pi(g)\xi_n \,, \xi_n \rangle \in [-1, 1]$. Define $\alpha_n(g) \in [0, 1]$ by

$$\langle \pi(g)\xi_n \,, \xi_n \rangle = \cos(\alpha_n(g)\pi).$$

Note that as $\langle \pi(g)\xi_n \,, \xi_n \rangle \to 1$, we have

$$\lim_{n\to\infty} \alpha_n(g) = 0.$$

In particular, $\alpha_n(g) \in [0, 1/2]$, for n large enough.

Let $\widetilde{\Phi} : S(H') \to L^2(\Omega, \mu; \mathbb{R})$ be the isometric isomorphism from Theorem 3.2.22 and Corollary 3.2.25 intertwining $\bigoplus_{n=0}^{\infty} S^n(\pi')$ and π'_μ. For each $n \in \mathbb{N}$ and each $g \in \Gamma$, set

$$X_n = \widetilde{\Phi}(\xi_n) \qquad \text{and} \qquad X_n^g = \pi'_\mu(g) X_n.$$

Since $\widetilde{\Phi}$ extends an isometric isomorphism of H' onto a Gaussian Hilbert space in $L^2(\Omega, \mu; \mathbb{R})$, we see directly that each X_n is a centered Gaussian random variable. Further, for each $n \in \mathbb{N}$, each $g \in \Gamma$ and each Borel set $B \subset \mathbb{R}$, we see that $\omega \in (X_n^g)^{-1}(B)$ if and only if $g^{-1}\omega \in X^{-1}(B)$. Thus, $(X_n^g)^{-1}(B) = gX^{-1}(B)$. Since the action of Γ on (Ω, μ) is measure preserving, this implies that the distribution of X_n^g is the same as the distribution of X_n, for each $g \in \Gamma$. Hence, each X_n^g is a centered Gaussian random variable on Ω. The variance of each of these random variables is given by $\|X_n^g\|_2 = \left\|\pi'_\mu(g)\widetilde{\Phi}(\xi_n)\right\|_2 = \|\xi_n\| = 1$.

Fix $n \in \mathbb{N}$ and $g \in \Gamma$, and write X_n^g in its orthogonal decomposition with respect to the subspace spanned by X_n:

$$X_n^g = \langle X_n^g , X_n \rangle \, X_n + Z_n = \cos(\alpha_n(g)\pi) X_n + Z_n,$$

with $Z_n \in X_n^\perp$. If $\alpha_n(g) = 0$, then $X_n^g = X_n$ and $Z_n = 0$. Otherwise, X_n and Z_n are independent Gaussian random variables (see [25, Corollary 16.1]). In the latter case, set

$$Y_n = \frac{1}{\sin(\alpha_n(g)\pi)} Z_n.$$

Then Y_n is a centered Gaussian random variable with variance 1. Moreover, X_n and Y_n are independent and $X_n^g = \cos(\alpha_n(g)\pi)X_n + \sin(\alpha_n(g)\pi)Y_n$. We see that the joint distribution m of X_n and Y_n on $\mathbb{R}^2$ is the standard Gaussian measure.

Set

$$A_n = \{\, \omega \in \Omega \mid X_n(\omega) \geq 0 \,\}.$$

Observe that $gA_n = \{\, \omega \in \Omega \mid X_n^g(\omega) \geq 0 \,\}$, for all $g \in \Gamma$. Since the distribution of each X_n^g is centered and symmetric, we see that $\mu(gA_n) = 1/2$, for each $n \in \mathbb{N}$ and each $g \in \Gamma$. Further, we see immediately that

$$gA_n \triangle A_n = \{X_n^g \geq 0 \text{ and } X_n < 0\} \cup \{X_n^g < 0 \text{ and } X_n \geq 0\}.$$

For each $n \in \mathbb{N}$ and $g \in \Gamma$, let $B_{n,g}$ be the subset of $\mathbb{R}^2$ given by

$$\begin{aligned} B_{n,g} = &\left\{\, (x,y) \in \mathbb{R}^2 \;\middle|\; \cos(\alpha_n(g)\pi)x + \sin(\alpha_n(g)\pi)y \geq 0 \text{ and } x < 0 \,\right\} \\ &\cup \left\{\, (x,y) \in \mathbb{R}^2 \;\middle|\; \cos(\alpha_n(g)\pi)x + \sin(\alpha_n(g)\pi)y < 0 \text{ and } x \geq 0 \,\right\}. \end{aligned}$$

By the abstract change of variables formula, we see immediately that $\mu(gA_n \triangle A_n) = m(B_{n,g})$. Rearranging, obtain the following expression for the set $B_{n,g}$:

$$\begin{aligned} B_{n,g} = &\left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; y \geq -\frac{1}{\tan(\alpha(g)\pi)} x \text{ and } x < 0 \right\} \\ &\cup \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; y < -\frac{1}{\tan(\alpha(g)\pi)} x \text{ and } x \geq 0 \right\}. \end{aligned}$$

Since m is rotation invariant, a simple geometric consideration yields that $m(B_{n,g}) = \alpha_n(g)$. But $\alpha_n(g) \to 0$, for each $g \in \Gamma$, as n tends to infinity. Hence, $(A_n)_{n \geq 1}$ is a non-trivial asymptotically invariant sequence in Ω. $\square$

3.4 Literature

This chapter is based on [5, Section 6.3 and A.7].

Chapter 4

Kazhdan type rigidity properties for actions on Banach spaces

After our survey on Kazhdan's Property (T) in its different forms, we proceed to study this type of rigidity property in the more general setting of actions on Banach spaces. We shall study two such properties: Property (T_X), which generalizes Property (T) as stated in Proposition 1.2.7, and Property (F_X) with generalizes Serre's Property (FH). Both generalizations are due to U. Bader, A. Furman, T. Gelander and N. Monod in [2]. We define these properties in Section 4.1 where we also discuss their relations. In Section 4.2 we discuss Property (T_X) in the setting where X is a superreflexive Banach space. Superreflexive Banach spaces, which includes the L^p-spaces, is a class of Banach spaces with very nice properties. For example, we shall see that for isometric representations on superreflexive Banach spaces, the subspace of invariant vectors is complemented. We provide a brief introduction to superreflexivity in Appendix A. In Section 4.3, we study affine actions on superreflexive Banach spaces. We show that, for this class of Banach spaces, all bounded subsets have a unique circumcenter, and this allows us to show that the existence of a fixed point is equivalent with all orbits being bounded. We remark that these two properties of superreflexive Banach spaces – complementation of the subspace of invariant vectors and the existence of a unique circumcenter – will allow us to use some of the tools we developed in the previous chapters when studying Property (T) in the Hilbert space setting. We end the chapter by discussing, in Section 4.4, the relations between Property (T), (T_X) and (F_X) when X is an L^p-space. We remark that, for actions on L^p-spaces, the consequence of Schoenberg's theorem discussed in Subsection 2.4.3 applies.

4.1 Property (T) in the Banach space setting

4.1.1 Property (T_X)

For a Banach space X, we denote by $\mathcal{O}(X)$ the set of linear surjective isometries on X. Note that for a real Hilbert space, H, $\mathcal{O}(H)$ is the set of orthogonal operators (justifying the notation), and for a complex Hilbert space, it is the set of unitary operators. A *linear isometric representation* of a discrete group Γ is then a tuble (ρ, X) consisting of a Banach space X and a group homomorphism $\rho : \Gamma \to \mathcal{O}(X)$.

The notions of Γ-invariant and almost Γ-invariant from Definition 1.2.1 carries over directly to the setting of linear isometric representations on Banach spaces. For a linear isometric representation (ρ, X) of a group Γ, let $X^{\rho(\Gamma)}$ denote the set of Γ-invariant vectors:

$$X^{\rho(\Gamma)} = \{\, x \in X \mid \rho(g)x = x, \text{ for all } g \in \Gamma \,\}.$$

This is a closed linear subspace of X invariant under the action of Γ through ρ.

Definition 4.1.1. Let Γ be a discrete group and let $\mathscr{B}$ be a class of Banach spaces. We say that Γ has property $(T_{\mathscr{B}})$ if, for all linear isometric representations (ρ, X) with $X \in \mathscr{B}$, the induced representation of Γ on $X/X^{\rho(\Gamma)}$ does not have almost invariant vectors. When $\mathscr{B}$ consists of only one Banach space X, we write property (T_X) instead of $(T_{\mathscr{B}})$.

Remark 4.1.2. For the class $\mathscr{H}$ of all Hilbert spaces, Proposition 1.2.7 states that Property $(T_{\mathscr{H}})$, as defined above, is equivalent to property (T).

4.1.2 Property (F_X)

Let X be a Banach space and let Γ be a discrete group. As in Chapter 2, $\operatorname{Aff}\operatorname{Isom}(X)$ denotes the set of affine isometries on X, and an affine isometric action of Γ on X is a group homomorphism $\Gamma \to \operatorname{Aff}\operatorname{Isom}(X)$.

Definition 4.1.3. Let Γ be a discrete group and let $\mathscr{B}$ be a class of Banach spaces. We say that Γ has property $(F_{\mathscr{B}})$ if all affine isometric actions of Γ on any $X \in \mathscr{B}$ has a fixed point. When $\mathscr{B}$ consists of only one Banach space, X, we write (F_X) instead of $(F_{\mathscr{B}})$.

Remark 4.1.4. When $\mathscr{B}$ is the class of all real Hilbert spaces, we recover Serre's Property (FH) (see Definition 2.5.1).

We saw in Proposition 2.5.2 that property (FH) is equivalent to the vanishing of the first cohomology group with coefficients in an orthogonal representation. The proof relies on our analysis in Section 2.2 of affine actions on general vector spaces, and so, generalizes easily to the Banach space setting.

Proposition 4.1.5. *Let X be a Banach space. A discrete group Γ has property (F_X) if and only if $H^1(\Gamma, \rho) = \{0\}$, for all linear isometric representations ρ of Γ on X.*

Proof. For a linear isometric representation ρ of Γ on X, we have, as in the proof of Proposition 2.5.2, that $H^1(\Gamma, \rho) = 0$ if and only if all affine actions with linear part ρ have a fixed point. Moreover, we remark that the affine isometric actions on X are in 1-to-1 correspondence to the 1-cocycles with respect to the linear isometric representations on B. Therefore, $H^1(\Gamma, \rho) = 0$, for all linear isometric representations ρ of Γ on X if and only if all affine isometric actions of Γ on X has a fixed point. $\square$

4.1.3 Relations between Kazhdan type rigidity properties

In the Hilbert space setting, we have seen that Property (T) and Property (FH) are equivalent (for discrete groups) – this is the content of the Theorem 2.5.5 by P. Delorme and A. Guichardet. In the general setting, where we consider actions on the members of any specified class of Banach spaces $\mathscr{B}$, only one implication remains true: Property $(F_{\mathscr{B}})$ is, in general, stronger than Property $(T_{\mathscr{B}})$. The proof of this is a straight forward generalization of the proof for the case where $\mathscr{B} = \mathscr{H}$ is the class of all Hilbert spaces.

Theorem 4.1.6. *For any Banach space X, Property (F_X) implies Property (T_X).*

Proof. Let Γ be a discrete group, let X be a Banach space, and assume that Γ does not have property (T_X). We may then find an isometric representation (ρ, X) of Γ on X such that the induced representation on $X/X^{\rho(\Gamma)}$ has almost invariant vectors. By Proposition 2.3.4, $B^1(\Gamma, \rho)$ is not closed. By Remark 2.3.3 and Proposition 4.1.5, this implies that Γ does not have Property (F_X). $\square$

We illustrate the failure of the opposite implication in the example below, which is taken from [2, Example 2.22]. We remark that we don't need to look outside the class of Hilbert spaces for an example.

Example 4.1.7 ((T_X) $\not\Rightarrow$ (F_X))**.** Consider the Banach space $\mathbb{R}$ with the usual Euclidean norm. On this space, we have only two linear isometries: the identity map and the antipodal map $x \mapsto -x$. Composing the antipodal map with itself yields the identity map, and so, we see that $\mathcal{O}(\mathbb{R})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ as a group. If Γ is any discrete group acting on $\mathbb{R}$ by linear surjective isometries then either the action is trivial, in which case all vectors in $\mathbb{R}$ are fixed, or there exists a $g \in \Gamma$ for which $g.x = -x$, for all $x \in \mathbb{R}$. In the latter case, the action has no non-zero fixed points, but neither does it have almost invariant vectors. Hence, all discrete groups have property ($T_{\mathbb{R}}$). Let $\Gamma \subset \mathbb{R}$ be a non-zero additive subgroup of $\mathbb{R}$, e.g., $\Gamma = \mathbb{Z}$ or $\mathbb{R}$, and consider the affine isometric action of Γ on $\mathbb{R}$ by translation (see Example 2.2.2). Clearly, such an action has no fixed-points, and so, Γ constitutes an example of a group with Property ($T_{\mathbb{R}}$) but without Property ($F_{\mathbb{R}}$). ◦

We proceed to discuss the relations of the classical Property (T) with Property (T_X), respectively Property (F_X), for a given Banach space X. In general, the existence of fixed points for any affine action on a given fixed Banach space X, or the property that, for any linear isometric representation ρ on X, the induced representation on X/X^ρ does not have almost invariant vectors, does not say anything about how a group may act on other Banach spaces. In particular, Property (T_X) and (F_X) do not, in general, imply Property (T). The example below, which can be found in [2, Example 2.25] and in [13, Examples 2.5], show that (T) is also not stronger that (T_X).

Example 4.1.8 ((T) $\not\Rightarrow$ (T_X))**.** Let Γ be any finitely generated infinite discrete group and let $c_0(\Gamma)$ be the subspace of $\ell^\infty(\Gamma)$ consisting of complex valued functions on Γ tending to zero. Consider the left regular representation of Γ on $\ell^\infty(\Gamma)$, i.e.,

$$g.a(h) = a(g^{-1}h), \qquad a \in \ell^\infty(\Gamma), g, h \in \Gamma.$$

It is straight forward to verify that this is an action by linear surjective isometries. If $a \in \ell^\infty(\Gamma)$ is invariant under this action, then $a(g) = g^{-1}.a(e) = a(e)$, for all $g \in \Gamma$. That is, any invariant function is a constant function. Hence, the left regular representation on $c_0(\Gamma)$ has no non-zero invariant vectors. It does, however, have almost invariant vectors. To see this, let F_1 be a finite generating set containing the identity, and set, for each $k \in \mathbb{N}$, $F_k = F_1^k$. This gives us an increasing sequence $F_1 \subset F_2 \subset \cdots$ of finite sets in Γ such that $\Gamma = \bigcup_{k\in\mathbb{N}} F_k$. For notational convenience we set $F_0 = \emptyset$. Define a sequence of functions $(a_n)_{n\geq 1}$ in $c_0(\Gamma)$ by setting, for each $h \in \Gamma$,

$$a_n(h) = \frac{1}{k^{1/n}}, \qquad \text{if } h \in F_k \backslash F_{k-1},\ k \geq 1.$$

We claim that $(a_n)_{n\geq 1}$ is a sequence of almost invariant vectors. For $g \in \Gamma$, let $\ell(g)$ be the word length of g with respect to F_1. We have

$$\begin{aligned}\|g.a_n - a_n\|_\infty &= \sup_{h\in\Gamma} |g.a_n(h) - a_n(h)| \\ &\leq \sup_{h\in F_{\ell(g)}} |g.a_n(h) - a_n(h)| + \sup_{h\notin F_{\ell(g)}} |g.a_n(h) - a_n(h)|\end{aligned}$$

Given $\varepsilon > 0$, pick $n_\varepsilon \in \mathbb{N}$ such that $1 - 1/k^{1/n_\varepsilon} < \varepsilon/3$, for all $k \leq 2\ell(g)$. If $h \in F_{\ell(g)}$ then $g^{-1}h \in F_{2\ell(g)}$, and so, we see that the first term in the above inequality is bounded by $2\varepsilon/3$. Turning our attention to the latter term, suppose $h \in F_k \backslash F_{k-1}$, for some $k > \ell(g)$. Then

$$|g.a_n(h) - a_n(h)| \leq \max\left\{\left|\frac{1}{k^{1/n}} - \frac{1}{\big(k - \ell(g)\big)^{1/n}}\right|, \left|\frac{1}{k^{1/n}} - \frac{1}{\big(k + \ell(g)\big)^{1/n}}\right|\right\}.$$

Consider the function $1/x^{1/n} - 1/(x+r)^{1/n}$ on $[1, \infty)$ for a fixed $r > 0$. The differential quotient is negative on the entire interval $(1, \infty)$. Hence, the maximum of this function is attained at $x = 1$. From this we deduce that

$$|g.a_n(h) - a_n(h)| \leq 1 - \frac{1}{\big(1 + \ell(g)\big)^{1/n}},$$

for all $h \notin F_{\ell(g)}$. Hence, $\|g.a_n - a_n\|_\infty < 2\varepsilon/3 + \varepsilon/3 = \varepsilon$, for all $n \geq n_\varepsilon$. As $g \in \Gamma$ and $\varepsilon > 0$ were arbitrary, we deduce that $(a_n)_{n\geq 1}$ is, indeed, a sequence of almost invariant vectors.

We have now shown that, for any finitely generated infinite discrete group Γ, the left-regular representation on $c_0(\Gamma)$ has almost invariant vectors but no non-zero invariant vectors. Hence, no such group can have Property $(T_{c_0(\Gamma)})$. However, finitely generated infinite discrete groups with Property (T) do exist: $\mathrm{SL}_3(\mathbb{Z})$ is a well-known such example. ◦

Since $(F_X) \Rightarrow (T_X)$, this also show that (T) is not stronger that (F_X). A more direct example is given in [53] – see also [13, Example 2.7].

When X is an L^p-space, there is more to say about the relation between Property (T) and its Banach space relatives. We shall use the notation (T_{L^p}) and (F_{L^p}) for a fixed $1 \leq p \leq \infty$, to mean that a group has Property (T_X), respectively (F_X), for any L^p-space X. The aim of the rest of this chapter is to establish the following implications for a discrete group Γ:

- If Γ has Property (T) then Γ has Property (F_{L^p}), for all $1 \leq p \leq 2$.
- Γ has Property (T) if and only if it has Property (T_{L^p}), for all $1 \leq p < \infty$.

4.2 Property (T_X) for superreflexive spaces

4.2.1 The dual representation

Let Γ be a discrete group. We associate to each linear isometric representation (ρ, X) of Γ a *dual representation* ρ^* of Γ on X^* by setting

$$(\rho^*(g)\lambda)(x) = \lambda(\rho(g^{-1})x), \qquad g \in \Gamma, \lambda \in X^*, x \in X. \tag{4.1}$$

Proposition 4.2.1. *Let (ρ, X) be a linear isometric representation of the discrete group Γ. The dual representation ρ^*, as defined in equation (4.1), is a linear isometric representation of Γ on X^*.*

Proof. For each $g \in \Gamma$ and each $\lambda \in X^*$, it is clear that $\rho^*(g)\lambda$ is a linear functional on X. Furthermore, it is bounded, as verified by the following straight forward computation:

$$\begin{aligned} \|\rho^*(g)\lambda\| &= \sup\{\, |(\rho^*(g)\lambda)(x)| \mid x \in X,\ \|x\| = 1\,\} \\ &= \sup\left\{\, \left|\lambda(\rho(g^{-1})x)\right| \;\middle|\; x \in X,\ \|x\| = 1\,\right\} \\ &\leq \sup\left\{\, \|\lambda\| \left\|\rho(g^{-1})x)\right\| \;\middle|\; x \in X,\ \|x\| = 1\,\right\} \\ &= \|\lambda\|\,. \end{aligned} \tag{4.2}$$

Hence, $\rho^*(g)$ is a well-defined map on X^*. Moreover, it is clear that $\rho^*(g)$ is linear.

To see that $\rho^*(g)$ is an isometry, let $\varepsilon > 0$ and take $x \in X$ with $\|x\| = 1$ such that $|\lambda(x)| > \|\lambda\| - \varepsilon$. Then

$$\|\rho^*(g)\lambda\| \geq |(\rho^*(g)\lambda)(\rho(g)x)| = |\lambda(x)| > \|\lambda\| - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we deduce that $\|\rho^*(g)\lambda\| \geq \|\lambda\|$. This together with the inequality of (4.2) implies that $\rho^*(g)$ is an isometry.

Finally, we show that ρ^* is a group homomorphism $\Gamma \to \mathcal{O}(X^*)$. Unitality follows directly from unitality of ρ, and so, it remains to show that ρ^* is multiplicative. For $g, h \in \Gamma$, $\lambda \in X^*$ and $x \in X$, we have

$$(\rho^*(gh)\lambda)(x) = \lambda\big(\rho(h^{-1}g^{-1})x\big) = \lambda\big((\rho(h^{-1}) \circ \rho(g^{-1}))x\big) = \big((\rho^*(g) \circ \rho^*(h))\lambda\big)(x)$$

Hence, $\rho^*(gh) = \rho^*(g) \circ \rho^*(h)$, as we wanted to show. □

Let (ρ, X) be a linear isometric representation. Write ρ^{**} for the double dual representation $(\rho^*)^*$ on X^{**}. For each $x \in X$, denote by $\hat{x}$ the corresponding element of X^{**}. Then, for $g \in \Gamma$, $\lambda \in X^*$ and $x \in X$, we have

$$(\rho^{**}(g)\hat{x})(\lambda) = \hat{x}(\rho^*(g^{-1})\lambda) = (\rho^*(g^{-1})\lambda)(x) = \lambda(\rho(g)x) = (\rho(g)x)\hat{\ }(\lambda).$$

Hence, when restricted to X (viewed as a subspace of X^{**}), $\rho^{**}(g)$ agrees with $\rho(g)$. In particular, when X is reflexive, $\rho^{**} = \rho$.

Remark 4.2.2. When X is a reflexive Banach space, it follows from the above discussion that there is a 1-1 correspondence between the set of linear isometric representations on X and the set of linear isometric representations on its dual X^*.

4.2.2 Splitting off of the invariant vectors

Definition 4.2.3. A closed subspace M of a Banach space X is said to be *complemented* if there exists another closed subspace N of X such that $X = M + N$ and $M \cap N = \{0\}$.

A very convenient property of Hilbert spaces is that all closed subspaces are complemented. In more general Banach spaces, this need not be true. In fact, J. Lindenstrauss and L. Tsafriri showed in [27] that there exists a non-complemented closed subspace in any infinite dimensional Banach space not isomorphic to a Hilbert space. This lack of guarantee poses a challenge when moving from the Hilbert space setting to the general setting of Banach spaces.

Let X be a Banach space, let Γ be a discrete group and let (ρ, X) be a linear isometric representation of Γ on X. We aim to show that, when X is superreflexive, the closed subspace of Γ-invariant vectors is complemented.

Proposition 4.2.4. *Let (ρ, X) be a linear isometric representation of the discrete group Γ on a superreflexive Banach space X. For each $x \in X$, respectively $\lambda \in X^*$, be non-zero, and denote by x^*, respectively λ^*, the unique functional of Proposition A.2.5.*

(i) If $x \in X^{\rho(\Gamma)}$ then $x^* \in (X^*)^{\rho^*(\Gamma)}$.

(ii) If $\lambda \in (X^*)^{\rho^*(\Gamma)}$ then $\lambda^* \in X^{\rho(\Gamma)}$.

Proof. Suppose $x \in X^{\rho(\Gamma)}$ is Γ-invariant. For each $g \in \Gamma$, $\rho^*(g)x^*$ is a linear functional on X with $\|\rho^*(g)x^*\| = \|x^*\| = 1$. Moreover, Γ-invariance of x implies that

$$(\rho^*(g)x^*)(x) = x^*(\rho(g^{-1})x) = x^*(x) = \|x\|\,.$$

By uniqueness of the linear functional associated to x in Proposition A.2.5, we deduce that $\rho^*(g)x^* = x^*$. As $g \in \Gamma$ was arbitrary, this shows (i). The proof of (ii) is analogous. □

For a subset M of X, the *annihilator* of M is the subset of X^* given by

$$M^\perp = \{\, f \in X^* \mid M \subset \ker f \,\}\,.$$

For a subset $N \subset X^*$, the *pre-annihilator* of N is the subset of X given by

$$N_\perp = \{\, x \in X \mid N \subset \ker \hat{x} \,\} .$$

We remark that, if X is reflexive, the pre-annihilator of any subset $N \subset X^*$ coincides with the annihilator under the identification of X with X^{**}.

We denote by $X'(\rho)$ the pre-annihilator of $(X^*)^{\rho^*(\Gamma)}$, i.e.,

$$X'(\rho) = (X^*)_\perp^{\rho^*(\Gamma)} = \left\{ x \in X \,\middle|\, \lambda(x) = 0, \text{ for all } \lambda \in (X^*)^{\rho^*(\Gamma)} \right\}.$$

This is a closed linear subspace of X (see [30, Proposition 1.10.15(a)]). Further, it is clear that $X'(\rho)$ is invariant under the action of Γ through ρ.

Lemma 4.2.5. *Let (ρ, X) be a linear isometric representation of the group Γ on a super-reflexive Banach space. For all $x \in X^{\rho(\Gamma)}$, $\operatorname{dist}(x, X'(\rho)) \geq \|x\|$.*

Proof. Let $x \in X^{\rho(\Gamma)}$ be non-zero and let x^* be the unique linear functional of Proposition A.2.5. By Proposition 4.2.4, $x^* \in (X^*)^{\rho^*(\Gamma)}$, and so, for every $y \in X'(\rho)$, $x^*(y) = 0$. Therefore,

$$\|x\| = x^*(x) = x^*(x - y) \leq \|z - y\| .$$

The statement of the lemma follows by taking the infimum over all $y \in X'(\rho)$. □

Lemma 4.2.6. *If X is reflexive and if $\lambda \in X^*$ vanishes on $X'(\rho)$ then λ is invariant for ρ^*.*

Proof. Let $\lambda_0 \in X^*$ and suppose that λ_0 is not invariant for ρ^*. As $(X^*)^{\rho^*(\Gamma)}$ is a closed subspace of X^*, the Hahn-Banach Theorem (see [19, Theorem 5.8(a)]) yields the existence of an element $x \in X^{**}$ such that $x(\lambda_0) \neq 0$ and $x(\lambda) = 0$, for all $\lambda \in (X^*)^{\rho^*(\Gamma)}$. As X is reflexive, we may view x as an element of X. Then $x \in X'(\rho)$ and we see that $\lambda_0|_{X'(\rho)}$ is not equal to zero. □

Proposition 4.2.7. *Let (ρ, X) be a linear isometric representation of the discrete group Γ. If X is superreflexive then*

$$X = X^{\rho(\Gamma)} \oplus X'(\rho).$$

In particular, the subspace of Γ-invariant vectors is complemented.

Proof. We start by showing that the intersection of the two subspaces $X^{\rho(\Gamma)}$ and $X'(\rho)$ is zero. So let $x \in X^{\rho(\Gamma)} \cap X'(\rho)$. Since $x \in X^{\rho(\Gamma)}$, Lemma 4.2.5 yields that $\|x\| \leq \operatorname{dist}(x, X'(\rho))$. But as $x \in X'(\rho)$, as well, the distance on the right-hand side of this inequality is zero. Thus, $x = 0$, and we conclude that $X^{\rho(\Gamma)} \cap X'(\rho) = \{0\}$.

We proceed to show that $X^{\rho(\Gamma)} + X'(\rho)$ is a closed subspace. It is clear that it is a subspace, so we need only show that it is closed. Let $(x_n)_{n\geq 1}$ be a sequence in $X^{\rho(\Gamma)}$ and $(y_n)_{n\geq 1}$ a sequence in $X'(\rho)$ such that $x_n + y_n \to z$, as n tends to infinity, for some $z \in X$. We claim that the sequence $(x_n)_{n\geq 1}$ is Cauchy. Indeed, for every $n, m \in \mathbb{N}$, Lemma 4.2.5 implies that

$$\begin{aligned} \|x_n - x_m\| &\leq \operatorname{dist}(x_n - x_m, X'(\rho)) \leq \|(x_n - x_m) - (y_n - y_m)\| \\ &\leq \|(x_n - y_n) - z\| + \|(x_m - y_m) - z\| . \end{aligned}$$

It follows that $(x_n)_{n\geq 1}$ is Cauchy using convergence of $(x_n + y_n)_{n\geq 1}$ to z. Let x be point of convergence for $(x_n)_{n\geq 1}$. Since $X^{\rho(\Gamma)}$ is closed, $x \in X^{\rho(\Gamma)}$. The sequence $(y_n)_{n\geq 1}$ then converges to $z-x$, and so, $z-x \in X'(\rho)$, as this subspace is also closed. We conclude that $X^{\rho(\Gamma)} + X'(\rho)$ is closed, as well.

Finally, we show that $X^{\rho(\Gamma)} + X'(\rho)$ equals all of X. Assume for contradiction that this is not the case. By the Hahn-Banach Theorem (see [19, Theorem 5.8(a)]), there exists a linear functional $\lambda \in X^*$ with $\|\lambda\| = 1$ and such that $\lambda|_{X^{\rho(\Gamma)}+X'(\rho)} = 0$. Let $\lambda^* \in X^{**} = X$ be the unique linear functional of Proposition A.2.5 with $\|\lambda^*\| = 1$ and $\lambda^*(\lambda) = \|\lambda\| = 1$. We have, in particular, $\lambda|_{X'(\rho)} = 0$, and so, $\lambda \in (X^*)^{\rho^*(\Gamma)}$, by Lemma 4.2.6. Proposition 4.2.4 then implies that $\lambda^* \in X^{\rho(\Gamma)}$. But this contradicts that $\lambda|_{X^{\rho(\Gamma)}} = 0$. We conclude that $X^{\rho(\Gamma)} + X'(\rho) = X$. This finishes the proof. ☐

Remark 4.2.8. As a direct consequence of Proposition 4.2.7 above, we see that, when X is a superreflexive Banach space, a discrete group Γ has property (T_X) if and only if, for every linear isometric representation ρ on X, the restriction of ρ to $X'(\rho)$ does not have almost invariant vectors.

4.2.3 Property (T_B) and the dual space

Lemma 4.2.9. *Let (ρ, X) be a linear isometric representation of the group Γ. If X is superreflexive then*

$$\rho^*(g)x^* = (\rho(g)x)^*,$$

for all $x \in X$ and all $g \in \Gamma$.

Proof. For each $g \in \Gamma$, we see that $\|\rho^*(g)x^*\| = \|x^*\| = 1$ and, further, that

$$\rho^*(g)x^*(\rho(g)x) = x^*\big(\rho(g^{-1})(\rho(g)x)\big) = x^*(x) = \|x\| = \|\rho(g)x\|.$$

The claim follows by the uniqueness part of Proposition A.2.5. ☐

Proposition 4.2.10. *Let (ρ, X) be a linear isometric representation on a superreflexive Banach space. The dual representation ρ^* has almost invariant vectors if and only if ρ does.*

Proof. It suffices to show one direction as ρ^{**} may be identified with ρ, for any linear isometric representation on a reflexive Banach space. Recall that, since X is superreflexive, it has an equivalent uniformly smooth norm. The norm on X^* with respect to this equivalent norm is then uniformly convex. Let $(x_n)_{n\geq 1}$ be a sequence of almost invariant vectors for ρ. Uniform continuity of $(\,\cdot\,)^* : X \to X^*$ on the unit sphere of X (see Proposition A.2.7) together with Lemma 4.2.9 implies that $(x_n^*)_{n\geq 1}$ is a sequence of almost invariant vectors for ρ^*. ☐

Theorem 4.2.11. *Let X be a Banach space and let $M \subset X$ be a closed subspace.*

(i) $(X/M)^*$ is isometrically isomorphic to $M^\perp$.

(ii) M^* is isometrically isomorphic to $X^*/M^\perp$.

Proof. (i): Let $\pi : X \to X/M$ denote the quotient map. For each $f \in (X/M)^*$, $f \circ \pi \in X^*$ and we have, for each $x \in M$, $f \circ \pi(x) = f([0]) = 0$. Hence, $f \circ \pi \in M^\perp$. Therefore, $\pi^* : (X/M)^* \to M^\perp$ given by $\pi^*(f) = f \circ \pi$, for $f \in (X/M)^*$, is a well-defined map. If $g \in X^*$ is such that $M \subset \ker g$, then g factors through X/M, by the universal property of the universal property of the quotient. We see directly from this that π^* is surjective. Since the quotient map is a contraction, we have $\|\pi^*(f)\| \leq \|f\|$, for all $f \in (X/M)^*$. For the opposite inequality, let $(x_n)_{n\geq 1}$ be a sequence in X with $\|[x_n]\| < 1$, for each $n \in \mathbb{N}$, and such that $|f([x_n])| \to \|f\|$. By definition of the norm on the quotient X/M, we may, for each $n \in \mathbb{N}$, find $y_n \in M$ such that $\|x_n - y_n\| < 1$. It follows that

$$\|\pi^*(f)\| \geq \sup_{n\in\mathbb{N}} |\pi^*(f)(x_n - y_n)| = \sup_{n\in\mathbb{N}} |f([x_n])| = \|f\| .$$

Hence, π^* is a linear isometry of $(X/M)^*$ onto $M^\perp$.

(ii): For each $f \in M^*$, the Hahn-Banach extension theorem ensures the existence of a linear functional $F \in X^*$ such that $F|_M = f$ – we may even choose F with $\|F\| = \|f\|$. We remark that if F and F' are both extensions of f, then $(F - F')(x) = 0$, for all $x \in M$, and so, $F - F' \in M^\perp$. Hence, the map $\Phi : M^* \to X^*/M^\perp$ given by

$$\Phi(f) = F + M^\perp, \qquad \text{for } f \in M^*,$$

does not depend on the choice of extension $F \in B^*$ of f. Linearity of Φ is immediate from this independence. Since, for every $F \in X^*$, F extends $F|_M \in M^*$ to all of X, it is clear that Φ is surjective. Finally, for $f \in M^*$, let $F \in X^*$ be an extension with $\|F\| = \|f\|$. Then

$$\left\|F + M^\perp\right\| = \inf\left\{ \left\|F'\right\| \;\middle|\; F - F' \in M^\perp \right\} \leq \|F\| = \|f\| .$$

Further, if F' is any extension of f, then $F' = F + G$, for some $G \in M^\perp$, and we have that

$$\begin{aligned} \left\|F'\right\| &= \|F + G\| = \sup\{ \|(F+G)(x)\| \mid x \in X, \|x\| = 1 \} \\ &\geq \sup\{ \|(F+G)(x)\| \mid x \in M, \|x\| = 1 \} \\ &= \sup\{ \|f(x)\| \mid x \in M, \|x\| = 1 \} = \|f\| . \end{aligned}$$

It follows that Φ is an isometry. This finishes the proof. □

Proposition 4.2.12. *Let (ρ, X) be a linear isometric representation of the discrete group Γ on a superreflexive Banach space X. Then there exists an isometric isomorphism between $(X'(\rho))^*$ and $(X^*)'(\rho^*)$ intertwining the associated subrepresentations of ρ and ρ^*.*

Proof. Recall that $X'(\rho)$ is a closed subspaces of X. Hence, by Theorem 4.2.11, $(X'(\rho))^*$ is isometrically isomorphic to $X^*/X'(\rho)^\perp$. Since $(X^*)^{\rho^*(\Gamma)}$ is a closed subspace of X^* in the weak*-topology, [40, Theorem 4.7(b)] implies that

$$X'(\rho)^\perp = ((X^*)^{\rho^*(\Gamma)}_\perp)^\perp = (X^*)^{\rho^*(\Gamma)}.$$

Hence, $(X'(\rho))^*$ is isometrically isomorphic to $X^*/(X^*)^{\rho^*(\Gamma)}$. As X^* is superreflexive, it decomposes into the direct sum $(X^*)^{\rho^*(\Gamma)} \oplus (X^*)'(\rho^*)$, by Proposition 4.2.7. Hence, the quotient $X^*/(X^*)^{\rho^*(\Gamma)}$ is isometrically isomorphic to $(X^*)'(\rho^*)$.

It remains to show that the isometric isomorphism described above intertwines the actions of Γ on $(X'(\rho))^*$ and $(X^*)'(\rho^*)$, respectively. Let $\Phi : (X'(\rho))^* \to (X^*)'(\rho^*)$ denote this isometric isomorphism. Let $f \in (X'(\rho))^*$. If $F_0 \in X^*$ is any extension of f, then $\Phi(f)$ is the part of F_0 which lies in $(X^*)'(\rho^*)$. Precisely, F_0 decomposes (uniquely) as $F_0 = F + F'$, for some $F \in (X^*)'(\rho^*)$ and $F' \in (X^*)^{\rho^*(\Gamma)}$, and we have $\Phi(f) = F$. For each $g \in \Gamma$, it is clear that $\rho^*(g)F_0$ is an extension of $(\rho|_{X'(\rho)})^*(g)f$, because $X'(\rho)$ is Γ-invariant. Further $\rho^*(g)F_0$ decomposes as $\rho^*(g)F + F'$, where $\rho^*(g)F \in (X^*)'(\rho^*)$. Hence

$$\Phi\big((\rho|_{X'(\rho)})^*(g)f\big) = \rho^*(g)F = \rho^*|_{(X^*)'(\rho^*)}(g)\Phi(f)$$

This holds for all $f \in (X'(\rho))^*$, and so we may conclude that $\Phi \circ \left(\rho|_{X'(\rho)}\right)^* = \rho^*|_{(X^*)'(\rho^*)} \circ \Phi$. This concludes the proof. □

Corollary 4.2.13. *Let X be a superreflexive Banach space and let Γ be a discrete group. Then Γ has Property (T_X) if and only if it has Property (T_{X^*}).*

Proof. Recall from Remark 4.2.8 that Γ has Property (T_X) if and only if, for any linear isometric representation ρ on X, the restriction of ρ to $X'(\rho)$ does not have almost invariant vectors. By Proposition 4.2.10 and Proposition 4.2.12, we see that $\rho|_{X'(\rho)}$ has almost invariant vectors if and only if $\rho^*|_{(X^*)'(\rho^*)}$ does. So, as the dual map is a 1-1-correspondence between representations on a given reflexive Banach space and its dual (see Remark 4.2.2), we may deduce that Γ has Property (T_X) if and only it has Property (T_{X^*}). □

4.3 Affine isometric actions on superreflexive spaces

For affine actions in the setting of Hilbert spaces, we saw in Proposition 2.5.3 that having a fixed point is equivalent to all orbits being bounded. We show in Proposition 4.3.2 below that this result holds more generally for all superreflexive Banach spaces.

For a Banach space X and a non-empty bounded subset $S \subset X$, we define the *radius* of S to be the number

$$\operatorname{rad}(S) = \inf \left\{ r > 0 \,\middle|\, S \subset \overline{B}(x, r), \text{ for some } x \in X \right\}.$$

An element $x \in X$ is called a *circumcenter* of S if $S \subset \overline{B}\big(x, \operatorname{rad}(S)\big)$.

Lemma 4.3.1. *If X is a reflexive Banach space then any non-empty bounded subset has a circumcenter. If, moreover, X is uniformly convex, then the circumcenter is unique.*

Proof. Assume that X is reflexive so that each closed ball of X is weakly compact (see [17, Theorem 3.31]). For each $n \in \mathbb{N}$, take $x_n \in X$ such that $S \subset \overline{B}\big(x_n, \operatorname{rad}(S) + \frac{1}{n}\big)$. Fix $y_0 \in S$. Then $(x_n)_{n\geq 1}$ is a sequence in the weakly compact set $\overline{B}\big(y_0, \operatorname{rad}(S)+1\big)$. Hence, there exists a point $x \in X$ and a subsequence $(x_{n_k})_{k\geq 1}$ converging weakly to x. We show that this x is a circumcenter of S. Let $y \in S$. If $y = x$, y trivially lies in $\overline{B}\big(x, \operatorname{rad}(S)\big)$. Therefore, suppose $y \neq x$. By Hahn-Banach's Theorem (see, e.g., [19, Theorem 5.8]), there exists $f \in X^*$ with $\|f\| = 1$ such that $f(y - x) = \|y - x\|$. For $\varepsilon > 0$, take $n_\varepsilon \in \mathbb{N}$ such that $|f(x_n - x)| < \varepsilon/2$, for all $n \geq n_\varepsilon$, and such that $1/n_\varepsilon < \varepsilon/2$. Then, for all $n \geq n_\varepsilon$,

$$\|y - x\| = f(y - x) \leq |f(y - x_n)| + |f(x_n - x)| < \|y - x_n\| + \varepsilon/2 < \operatorname{rad}(S) + \varepsilon.$$

Since, $\varepsilon > 0$ was arbitrary, we deduce that $\|y - x\| \leq \operatorname{rad}(S)$. Hence, $S \subset \overline{B}\big(x, \operatorname{rad}(S)\big)$, which is exactly what it means for x to be a circumcenter.

Assume now that X is uniformly convex and suppose x and x' are two different circumcenters of S. Set $\varepsilon = \|x - x'\| > 0$. By uniform convexity (see Lemma A.2.3), we may find a $\delta > 0$ such that, whenever $y, y' \in X$ are such that $\|y\|, \|y'\| \leq \operatorname{rad}(S)$ and such that $\|y - y'\| \geq \varepsilon$, then $\|(y + y')/2\| \leq \operatorname{rad}(S) - \delta$. For each $y \in S$, $\|x - y\|$ and $\|x' - y\|$ are both bounded by $\operatorname{rad}(S)$ while $\|(x - y) - (x' - y)\| = \|x - x'\| = \varepsilon$. Thus,

$$\left\| \frac{x + x'}{2} - y \right\| = \left\| \frac{(x - y) + (x' - y)}{2} \right\| \leq \operatorname{rad}(S) - \delta.$$

We deduce that S is contained in $B\big((x + x')/2, \operatorname{rad}(S) - \delta\big)$. But this contradicts the minimality of $\operatorname{rad}(S)$, and so, we must have $x = x'$. □

Lemma 4.3.2. *Let α be an affine isometric action of a discrete group Γ on a superreflexive Banach space X and let b be its translation part. The following are equivalent:*

(i) The action α has a fixed point in X,

(ii) b is a 1-coboundary.

(iii) b is bounded,

(iv) All orbits of α are bounded,

(v) Some orbit of α is bounded,

Proof. The proof of Lemma 2.5.3 generalizes directly to this setting replacing Lemma 1.2.13 with Lemma 4.3.1 and using that X has an equivalent uniformly convex norm. $\square$

4.4 Kazhdan type rigidity properties for L^p-spaces

We are now ready to show the announced relations between Property (T), (T_{L^p}) and (F_{L^p}). We show in Theorem 4.4.4 that Property (T) implies Property (F_{L^p}), for $1 \leq p \leq 2$. The proof of this implication follows the same lines as the proof of the implication $(T) \Rightarrow (FH)$ in Theorem 2.5.5. Recall that this proof is based on the consequence of Schoenberg's theorem presented in Proposition 2.4.30. To apply this proposition in the same way, we need to show that, for any L^p-space with $1 \leq p \leq 2$, $\| \cdot \|_p^p$ is a function conditionally of negative type.

Theorem 4.4.1. *The function $e^{-t|\cdot|^\alpha}$ is positive definite on $\mathbb{R}$, for $0 \leq \alpha \leq 2$ and $t > 0$.*

Proof. For $0 < \alpha < 2$ and for any fixed $x \in \mathbb{R}$, the integral

$$\int_{-\infty}^{\infty} \frac{1 - \cos xs}{|s|^{1+\alpha}} \, \mathrm{d}s$$

is convergent. Indeed, $\alpha > 0$ ensures convergence over the domain away from zero while $\alpha < 2$ ensures convergence around zero. The substitution $h(s) = xs$ together with a symmetry observation yields

$$\int_0^1 \frac{1 - \cos xs}{|s|^{1+\alpha}} \, \mathrm{d}s = 2 |x|^\alpha \int_0^\infty \frac{1 - \cos s}{s^{1+\alpha}} \, \mathrm{d}s.$$

The integral on the right-hand side depends only on α. We let

$$C_\alpha = \frac{1}{2} \left(\int_0^\infty \frac{1 - \cos s}{s^{1+\alpha}} \, \mathrm{d}s \right)^{-1},$$

so that

$$- |x|^\alpha = C_\alpha \int_{-\infty}^{\infty} \frac{\cos xs - 1}{|s|^{1+\alpha}} \, \mathrm{d}s \tag{4.3}$$

Fix $0 < \alpha < 2$ for a moment, let $t > 0$ and define, for each $n \in \mathbb{N}$, a function φ_n on $\mathbb{R}$ by

$$\varphi_n(x) = tC_\alpha \int_{|s| \geq 1/n} \frac{\cos xs - 1}{|s|^{1+\alpha}} \, \mathrm{d}s = tC_\alpha \int_{|s| \geq 1/n} \frac{\cos xs}{|s|^{1+\alpha}} \, \mathrm{d}s - tC_\alpha \int_{|s| \geq 1/n} \frac{1}{|s|^{1+\alpha}} \, \mathrm{d}s$$

Observe that the removal of a neighborhood around zero ensures that the two integrals on the right-hand side of the last equality converge. The latter of these is a number independent of x. Recall from Example 2.4.14 that cos is positive definite on $\mathbb{R}$. Linearity and monotonicity of the integral then implies that $x \mapsto \int_{|s| \geq 1/n} \cos xs / |s|^{1+\alpha} \, \mathrm{d}s$ is positive definite. We deduce by Proposition 2.4.12 that e^{φ_n} is positive definite, as well. By the Lebesgue Dominated Convergence Theorem and by equation (4.3), $\varphi_n(x)$ converges to $-t |x|^\alpha$, for each $x \in \mathbb{R}$. Hence, e^{φ_n} converges pointwise to $e^{-t|\cdot|^\alpha}$, by continuity of the exponential function. Hence, $e^{-t|\cdot|^\alpha}$ is positive definite, by Proposition 2.4.5. Using Proposition 2.4.5 and continuity once more gives us the statement for $\alpha = 0$ and $\alpha = 2$, as well. $\square$

Remark 4.4.2. For $\alpha > 2$, the function $\varphi = e^{-|\cdot|^\alpha}$ is *not* positive definite on $\mathbb{R}$ – see [52, Corollary 4.11].

Corollary 4.4.3. *The function $e^{-t\|\cdot\|_p^p}$ on ℓ_n^p, $\ell^p(\mathbb{N})$ respectively on $L^p(\Omega,\mu)$, is positive definite, for $1 \le p \le 2$ and for all $t > 0$.*

Proof. Fix $1 \le p \le 2$ and $t > 0$. Consider first $e^{-t\|\cdot\|_p^p}$ a function on ℓ_n^p, for some $n \in \mathbb{N}$. For each n-tuble $x = (x_1, \ldots, x_n) \in \ell_n^p$, we have

$$e^{-t\|x\|_p^p} = \prod_{i=1}^n e^{-t|x_i|^p}.$$

Hence, positive definiteness of $e^{-\|\cdot\|_p^p}$ as a function on ℓ_n^p follows directly from Theorem 4.4.1 and Proposition 2.4.11.

Next, consider $e^{-t\|\cdot\|_p^p}$ a function on $\ell^p(\mathbb{N})$. For each $n \in \mathbb{N}$, define $\varphi_n : \ell^p(\mathbb{N}) \to \mathbb{R}$ by

$$\varphi_n(x) = \prod_{i=1}^n e^{-t|x_i|^p}, \qquad x = (x_i)_{i\ge 1} \in \ell^p(\mathbb{N}).$$

Then φ_n is positive definite by the same argument as above. Clearly, $(\varphi_n)_{n\ge1}$ converges pointwise to $e^{-t\|\cdot\|_p^p}$, which is therefore positive definite, by Proposition 2.4.5.

Finally, consider $e^{-t\|\cdot\|_p^p}$ a function on $L^p(\Omega,\mu)$. For a finite collection of simple functions $f_1, \ldots, f_n \in L^p(\Omega,\mu)$, we may pick a finite collection of pairwise disjoint subsets $A_1, \ldots, A_m$ of Ω such that each of the simple functions can be written as $f_i = \sum_{k=1}^m a_{i,k} A_k$, for numbers $a_{i,1}, \ldots, a_{i,m} \in \mathbb{C}$. For each $1 \le i \le n$, set $x_i = (\mu(A_1)^{1/p} a_{i,1}, \ldots, \mu(A_m)^{1/p} a_{i,m})$. Then each x_i lies in ℓ_m^p, and

$$\begin{aligned}\|f_i - f_j\|_p^p &= \int_\Omega |f_i(\omega) - f_j(\omega)|^p \, \mathrm{d}\mu = \sum_{k=1}^m |a_{i,k} - a_{j,k}|^p \, \mu(A_k) \\ &= \sum_{k=1}^m \left|\mu(A_k)^{1/p} a_{i,k} - a_{j,k}\mu(A_k)^{1/p}\right|^p = \|x_i - x_j\|_{\ell_m^p}^p .\end{aligned}$$

Hence, for any collection $c_1, \ldots, c_n \in \mathbb{R}$,

$$\sum_{i,j=1}^n c_i c_j e^{-t\|f_i - f_j\|_p^p} = \sum_{i,j=1}^n c_i c_j e^{-t\|x_i - x_j\|_{\ell_m^p}^p} \ge 0,$$

because $e^{-t\|\cdot\|_p^p}$ is a positive definite function on ℓ_m^p. Since the simple functions are dense in any L^p-space, and since the norm and the exponential function are continuous functions, it follows that $e^{-t\|\cdot\|_p^p}$ is positive definite as a function on $L^p(\Omega,\mu)$. ☐

Theorem 4.4.4. *Property (T) implies Property (F_{L^p}), for all $1 \le p \le 2$.*

Proof. Assume that Γ does not have Property $(F_{L^p(\mu)})$, for some $1 \le p \le 2$, and let α be an affine isometric action of Γ on $L^p(\mu)$ without fixed points. By Corollary 4.4.3 and Theorem 2.4.27, $\|\cdot\|_p^p$ is conditionally of negative type on $L^p(\mu)$, and so, we may apply Theorem 2.4.30. For each $t > 0$, let (π_t, H_t) be the unitary representation from Proposition 2.4.30 associated to α and t. By Proposition 2.5.4, it holds for all $t > 0$, that (π_t, H_t) has no non-zero invariant vectors. Set

$$\pi = \bigoplus_{n=1}^\infty \pi_{1/n}$$

By Proposition 1.1.14, π has no non-zero invariant vectors. But by Proposition 2.4.33, π does have almost invariant vectors, and so, Γ does not have Property (T). ☐

We end the chapter by showing that Property (T) and Property (T_{L^p}) are equivalent. The proof of the implication $(T) \Rightarrow (T_{L^p})$ is based on Theorem 4.4.4 together with a duality argument. The implication $(T_{L^p}) \Rightarrow (T)$ is based on the Connes-Weiss characterization presented in Theorem 3.3.1.

Let Γ be a discrete group. Given a p.m.p. action of Γ on a probability space (Ω, μ) we get in a canonical way an induced isometric representation of Γ on the Banach space $L^p(\Omega, \mu)$ of complex-valued p-integrable functions, for each $1 \leq p \leq \infty$, by setting

$$\rho_\mu(g)f(x) = f(g^{-1}x), \qquad \text{for } f \in L^p(\Omega, \mu),\ g \in \Gamma, \text{ and } x \in \Omega.$$

Lemma 3.1.1 ensures that $\rho_\mu(g)$ is an isometry, for each $g \in \Gamma$, and it is then clear that $(\rho_\mu, L^p(\Omega, \mu))$ is an isometric representation. Notice that for the case $p = 2$, we recover the definition of equation (3.7). The closed subspace of fixed points for the representation ρ_μ is denoted by $L^p(\Omega, \mu)^{\rho_\mu}$. Further, let $L^p_0(\Omega, \mu)$ be the subspace of $L^p(\Omega, \mu)$ consisting of all functions with zero mean, i.e.,

$$L^p_0(\Omega, \mu) = \left\{ f \in L^p(\Omega, \mu) \;\middle|\; \int_\Omega f \,\mathrm{d}\mu = 0 \right\}.$$

This is a closed Γ-invariant subspace. Since μ is a finite measure, all constant functions belong to $L^p(\Omega, \mu)$. Denote by ρ^0_μ the restriction of ρ_μ to $L^p_0(\Omega, \mu)$.

Lemma 4.4.5. *Let Γ be a discrete group acting on a probability space (Ω, μ), let $1 \leq p \leq \infty$ and let ρ_μ denote the induced representation on $L^p(\Omega, \mu)$. If the action is ergodic then the set of fixed points for ρ_μ equals the set of constant functions.*

Proof. It is clear that all constant functions are fixed points for ρ_μ. Conversely, if $\Gamma \curvearrowright (\Omega, \mu)$ is ergodic we may find $x_0 \in \Omega$ such that the measure of its orbit $\mathcal{O}_{x_0}$ is 1. Let $f \in L^p(\Omega, \mu)^{\rho_\mu}$ be a fixed point. By Lemma 3.1.6 and Remark 3.1.7, f is a.e. equal to a function which is constant on the orbits, and, in particular, which is constant on $\mathcal{O}_{x_0}$. Hence, f is a.e. equal to a constant function. $\square$

Theorem 4.4.6. *Let Γ be a discrete group. Then Γ has Property (T) if and only if it has Property (T_{L^p}), for some and hence for all $1 \leq p < \infty$.*

Proof. If Γ has Property (T) then Γ has Property (F_{L^p}), for all $1 \leq p \leq 2$, by Theorem 4.4.4. Since any L^p-space with $2 < p < \infty$ is the dual of an L^q-space with $1 < q < 2$, it follows from Corollary 4.2.13 that Γ has Property (T_{L^p}), for $2 < p < \infty$, as well.

Suppose Γ does not have property (T). By Theorem 3.3.1, there exists a probability space (Ω, μ) and a p.m.p. action of Γ on (Ω, μ) which is ergodic but not strongly ergodic. Let $(B_n)_{n \geq 1}$ be a non-trivial asymptotically invariant sequence of measurable subsets of Ω. By the proof of Theorem 3.3.1, we may take this sequence such that $\mu(B_n) = 1/2$. For each $n \in \mathbb{N}$, define $f_n : \Omega \to \mathbb{C}$ by

$$f_n = 2 1_{B_n} - 1.$$

Observe that $|f_n| = 1_\Omega$, and so, f_n lies in $L^p(\Omega, \mu)$, for all $1 \leq p < \infty$ with $\|f_n\|_p = 1$. Moreover, each f_n satisfies that $\int_\Omega f_n \,\mathrm{d}\mu = 0$.

Fix $1 \leq p < \infty$ and consider the canonical representation ρ_μ of Γ on $L^p(\Omega, \mu)$ induced by the action $\Gamma \curvearrowright (\Omega, \mu)$. Since this action is ergodic, $L^p(\Omega, \mu)^{\rho_\mu} = \mathbb{C}1_\Omega$, by Lemma 4.4.5. The set of constant functions is complemented in $L^p(\Omega, \mu)$, and its canonical complement is

$$L^p_0(\Omega, \mu) = \left\{ f \in L^p(\Omega, \mu) \;\middle|\; \int_\Omega f \,\mathrm{d}\mu = 0 \right\}.$$

Hence, $L^p(\Omega,\mu)/L^p(\Omega,\mu)^{\rho_\mu} \cong L_0^p(\Omega,\mu)$. As already mentioned, $(f_n)_{n\geq 1}$ is a sequence in $L_0^p(\Omega,\mu)$. For each $g \in \Gamma$, $\rho_\mu(g)f_n - f_n = 2(\rho_\mu(g)1_{B_n} - 1_{B_n}) = 2(1_{gB_n} - 1_{B_n})$, and so

$$\|\rho_\mu(g)f_n - f_n\|_p^p = 2^p \int_\Omega |1_{gB_n} - 1_{B_n}| \,\mathrm{d}\mu = 2^p \mu(gB_n \triangle B_n).$$

As $(B_n)_{n\geq 1}$ is asymptotically invariant, we see directly that $\|\rho_\mu(g)f_n - f_n\|_p \to 0$. Hence, $(f_n)_{n\geq 1}$ is a sequence of almost invariant vectors, and so, Γ does not have (T_{L^p}). $\square$

4.5 Literature

This chapter is based on the paper *Property (T) and rigidity for actions on Banach spaces* by U. Bader, A. Furman, T. Gelander and N. Monod [2], and all results regarding the relations between Property (T), (T_X) and (F_X) that we present are first published in this paper. We remark that the implication $(T) \Rightarrow (F_{L^p})$ can also be proved without considering Property (F_{L^p}) (see [2, Section 4.a])- A reference for circumcenters in Banach spaces can be found in [6, p. 26-27]. The proof that the p-norm is conditionally of negative type (see Corollary 4.4.3) can be found in [52, Theorem 4.10].

Chapter 5

Spectral conditions

We present in Section 5.2 a sufficient condition for Property (T) due to A. Żuk in terms of the spectral properties of a certain graph associated to a generating set. In Section 5.1 we give a brief introduction to graphs and their spectrum. In Section 5.3, we present a sufficient condition for Property (F_{L^p}) of the same flavor as Żuk's condition. This result is due to T. de Laat and M. de la Salle in [14], and we follow the proof of their paper. A central tool in their proof is complex interpolation. A brief overview of the needed results from this theory is given in Appendix B.

5.1 Graphs

We give here a short introduction to graphs and their spectrum, the main purpose of which is to fix notation and terminology.

5.1.1 Basic definitions

Definition 5.1.1. A *graph* $\mathcal{G}$ is an ordered pair (V, E) consisting of a set V of vertices and a set E of edges equipped with two maps $\mathfrak{s}, \mathfrak{r} : E \to V$ referred to as the *source* and the *range* map, respectively.

Unless we have explicitly named the vertex and the edge set of a graph $\mathcal{G}$, we shall use the notation $V(\mathcal{G})$, respectively, $E(\mathcal{G})$ to refer to these. A graph is said to be *finite* if the size of its vertex and edge sets are finite.

Let $\mathcal{G} = (V, E)$ be a graph. An edge $e \in E$ with $\mathfrak{s}(e) = s$ and $\mathfrak{r}(e) = t$ is interpreted as a (directed) edge from vertex s to vertex t, and s and t are called the *source* respectively the *range* of e. We shall often write (s, t) for an edge in E from s to t. The edge (s, t) is a *multiple edge* if it occurs more than once in E. The graph $\mathcal{G}$ has no multiple edges if the map $E \to V \times V$ given by $e \mapsto (\mathfrak{s}(e), \mathfrak{r}(e))$ is injective. We remark that if a graph has multiple edges, writing (s, t) is ambiguous. For the purpose of this thesis, we are only interested in graphs with no multiple edges. If $\mathfrak{s}(e) = \mathfrak{r}(e)$ we say that e is a *loop*.

Definition 5.1.2. A graph with no multiple edges and no loops is said to be *simple*.

Definition 5.1.3. A graph $\mathcal{G} = (V, E)$ is said to be *undirected* if whenever $(s, t) \in E$ then $(t, s) \in E$ as well, and if, in that case, (s, t) and (t, s) occurs equally many times in E. A graph is *directed* if it is not undirected. A directed graph is said to be *oriented* if it has no symmetric edges, i.e., if whenever $(s, t) \in E$ it follows that $(t, s) \notin E$.

Remark 5.1.4. A graph $\mathcal{G}$ may be graphically represented by drawing a point for every vertex and an arrow from vertex s to vertex t if (s, t) is an edge of $\mathcal{G}$. An undirected graph is then, by our definition, a graph where the arrows come in pairs. Alternatively, one may define an undirected graph by letting the edges be unordered pairs instead of ordered tuples. In this picture an undirected graph is represented by connecting two vertices with a line whenever there is an edge between them. The two pictures are, of course, equivalent.

Definition 5.1.5. Let $\mathcal{G} = (V, E)$ be a simple undirected graph. An *orientation* of $\mathcal{G}$ is a simple oriented graph $\vec{\mathcal{G}} = (V, \vec{E})$ obtained from $\mathcal{G}$ by assigning an orientation to each edge.

Let s and t be two vertices in a graph $\mathcal{G} = (V, E)$. If $(s, t) \in E$ we say that t is a *direct successor* of s, and that s is a *direct predecessor* of t. If t is a direct successor of s, we write $t \sim s$. Two vertices are said to be *adjacent* if one is the direct successor of the other.

Definition 5.1.6. Let $\mathcal{G} = (V, E)$ be a graph. For a vertex $s \in V$ we define the *neighborhood* of s to be the set N_s of all its direct successors, i.e., $N_s = \{t \in V \mid (s, t) \in E\}$.

Definition 5.1.7. Let $\mathcal{G} = (V, E)$ be a graph. The *degree* of a vertex $s \in V$ is the number $\deg_{\mathcal{G}}(s) = |N_s|$. The map $\deg_{\mathcal{G}} : V \to \mathbb{N}_0$ is referred to as the *degree function*. If the graph is clear from context, we shall omit it from the notation and write deg instead of $\deg_{\mathcal{G}}$. If all vertices in a graph have the same degree, we say that the graph is *regular*, or, more explicitly, *d-regular* when the degree is d.

Definition 5.1.8. A *path* in a graph is a finite or infinite list of edges $e_1 e_2 \cdots$ such that $\mathfrak{r}(e_i) = \mathfrak{s}(e_{i+1})$, for all i. A graph $\mathcal{G}$ is said to be *connected* if there is a path between any two distinct vertices of $\mathcal{G}$. A maximal connected subgraph of $\mathcal{G}$ is called a *component* of $\mathcal{G}$.

5.1.2 The spectrum of a finite graph

For a set S, we denote by $\mathbb{C}^S$ the set of complex valued functions on S.

Definition 5.1.9. Let $\mathcal{G}$ be a finite simple graph. The *(discrete) Laplace operator* on $\mathcal{G}$ is the operator Δ on $\mathbb{C}^{V(\mathcal{G})}$ given by

$$\Delta f(s) = f(s) - \frac{1}{\deg(s)} \sum_{t \sim s} f(t), \qquad s \in V,$$

for all $f \in \mathbb{C}^{V(\mathcal{G})}$.

Remark 5.1.10. The canonical representation of Δ as a matrix is referred to as the *discrete Laplacian matrix*. Its entries are given by

$$\Delta_{s,t} = \begin{cases} -\frac{1}{\deg(s)}, & \text{if } (s,t) \in E, \\ 1, & \text{if } s = t, \\ 0, & \text{otherwise} \end{cases} .$$

Definition 5.1.11. The spectrum of a finite simple graph $\mathcal{G}$ is the spectrum of the discrete Laplace operator. We denote this spectrum by $\sigma(\mathcal{G})$.

For a finite graph $\mathcal{G}$ on n vertices, the *adjacency matrix* $A_{\mathcal{G}} = (a_{st})$ is the $n \times n$-matrix with the (s, t)'th entry equal to the number of edges from vertex s to vertex t. We remark that the sum of the entries of the s'th row of $A_{\mathcal{G}}$ equals the degree of vertex s. If $\mathcal{G}$ is an undirected graph, then $A_{\mathcal{G}}$ is a symmetric matrix. If $\mathcal{G}$ is simple, $A_{\mathcal{G}}$ consists of zeros and ones, and its diagonal entries are zero.

The *degree matrix* $D_{\mathcal{G}} = \operatorname{diag}(\deg(1), \ldots, \deg(n))$ is the diagonal matrix containing information on the degree of each vertex. The *normalized adjacency matrix* is the matrix

$$M_{\mathcal{G}} = D_{\mathcal{G}}^{-1} A_{\mathcal{G}}.$$

All rows of $M_\mathcal{G}$ sum up to 1. If $\mathcal{G}$ is an undirected graph, then $M_\mathcal{G}$ is self-adjoint with respect to the inner product on $\mathbb{C}^{V(\mathcal{G})}$ weighted by the degree matrix. Considered an operator on $\mathbb{C}^{V(\mathcal{G})}$, we refer to $M_\mathcal{G}$ as the *Markov operator*. We denote by 1_n the $n \times n$ identity matrix. By Remark 5.1.10, we see directly that the discrete Laplacian matrix is given by

$$\Delta = 1_n - M_\mathcal{G}.$$

Lemma 5.1.12 (Gershgorin's circle theorem). *Let $A = (a_{ij})$ be a complex $n \times n$ matrix. For each $i \in \{1, \dots, n\}$, let $R_i = \sum_{j \neq i} |a_{ij}|$. If λ is an eigenvalue of A then there is an $i \in \{1, \dots, n\}$ such that $\lambda \in \overline{B}(a_{ii}, R_i)$. The closed ball $\overline{B}(a_{ii}, R_i)$ is called a* Gershgorin disc.

Proof. Suppose λ is an eigenvalue of A and let v be a corresponding eigenvector. We may choose v such that one of its entries equal 1, say $v_i = 1$, and such that $|v_j| \leq 1$, for $1 \leq j \leq n$. Otherwise, divide v by the entry which is numerically largest. We have that

$$\lambda = \lambda v_i = (Av)_i = \sum_{j=1}^{n} a_{ij} v_j = a_{ii} + \sum_{j \neq i} a_{ij} v_j.$$

Rearringing, taking the absolute value and applying the triangle inequality, it follows immediately that $|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| = R_i$. Hence, $\lambda \in \overline{B}(a_{ii}, R_i)$, as we wanted to show. $\square$

Remark 5.1.13. Let $\mathcal{G}$ be a finite graph. If $\mathcal{G}$ is simple, the s'th Gershgorin disc associated to the adjacency matrix $A_\mathcal{G}$ is centered with radius equal to the degree at vertex s. Hence, all Gershgorin discs of $M_\mathcal{G}$ are equal to the closed unit ball in $\mathbb{C}$.

Corollary 5.1.14. *Let $\mathcal{G} = (V, E)$ be a simple finite graph. The spectrum of $\mathcal{G}$ is contained in $\overline{B}(1, 1)$. If, moreover, $\mathcal{G}$ is undirected, its spectrum is contained in $[0, 2]$.*

Proof. Since 1_n and $M_\mathcal{G}$ trivially commutes, we have $\sigma(\Delta) \subset 1 - \sigma(M_\mathcal{G})$. It is then direct from Lemma 5.1.12 together with Remark 5.1.13 that $\sigma(\mathcal{G}) = \sigma(\Delta) \subset \overline{B}(1, 1)$.

For the second statement, observe that

$$D_\mathcal{G}^{1/2} \Delta D_\mathcal{G}^{-1/2} = 1_n - D_\mathcal{G}^{-1/2} A_\mathcal{G} D_\mathcal{G}^{-1/2}.$$

If $\mathcal{G}$ is undirected, the matrix on the right-hand side is symmetric. Being similar to a real symmetric matrix, we deduce that the discrete Laplacian matrix has real eigenvalues. $\square$

Definition 5.1.15. The *gradient* on $\mathcal{G}$ is the operator $\nabla : \mathbb{C}^{V(\mathcal{G})} \to \mathbb{C}^{E(\mathcal{G})}$ given by

$$\nabla f(s, t) = f(s) - f(t), \qquad (s, t) \in E(\mathcal{G}),$$

for all $f \in \mathbb{C}^{V(\mathcal{G})}$.

Lemma 5.1.16. *Let $\mathcal{G}$ be a finite simple undirected graph. For each $f \in \mathbb{C}^{V(\mathcal{G})}$, it holds that*

$$2 \sum_{s \in V(\mathcal{G})} \overline{f(s)} \Delta f(s) \deg(s) = \sum_{(s,t) \in E(\mathcal{G})} |\nabla f(s, t)|^2.$$

Proof. Let $f \in \mathbb{C}^{V(\mathcal{G})}$. The assumption that $\mathcal{G}$ is undirected ensures the following equality:

$$\sum_{(s,t) \in E(\mathcal{G})} \overline{f(t)} f(s) = \sum_{(s,t) \in E(\mathcal{G})} \overline{f(s)} f(t).$$

In particular, the above sum is real. Further, since there is a bijective set correspondence between $E(\mathcal{G})$ and $\bigcup_{s\in S}\{s\} \times N_s$, we see that

$$\begin{aligned}\sum_{(s,t)\in E(\mathcal{G})} |\nabla f(s,t)|^2 &= \sum_{(s,t)\in E(\mathcal{G})} |f(s)-f(t)|^2 = 2\sum_{(s,t)\in E(\mathcal{G})} |f(s)|^2 - 2\sum_{(s,t)\in E(\mathcal{G})} \overline{f(s)}f(t) \\ &= 2\sum_{s\in S} |f(s)|^2 \deg(s) - 2\sum_{s\in S}\sum_{t\sim s} \overline{f(s)}f(t) \\ &= 2\sum_{s\in S}\left(|f(s)|^2 - \frac{1}{\deg(s)}\sum_{t\sim s}\overline{f(s)}f(t)\right)\deg(s) \\ &= 2\sum_{s\in V(\mathcal{G})} \overline{f(s)}\Delta f(s)\deg(s).\end{aligned}$$

This proves the claimed equality. □

Corollary 5.1.17. *Let $\mathcal{G}$ be a finite simple undirected graph. Then the kernel of Δ consists of all functions that are constant on the components of $\mathcal{G}$. In particular, if $\mathcal{G}$ is connected, then* $\ker(\Delta) = \mathbb{C}1_{V(\mathcal{G})}$.

Proof. It is straight forward to verify using the definition of Δ that $\ker(\Delta)$ contains all functions which are constant on the components of $\mathcal{G}$. The other inclusion follows directly from Lemma 5.1.16. □

Definition 5.1.18. The smallest non-zero eigenvalue of a finite simple undirected graph $\mathcal{G}$ is referred to as the *spectral gap* of $\mathcal{G}$ and is denoted by $\lambda_1(\mathcal{G})$.

5.2 A spectral condition for Property (T)

5.2.1 A spectral characterization

Let Γ be a discrete group and let $S \subset \Gamma$ be a finite subset. For a unitary representation (π, H), we define an operator $h_\pi \in B(H)$ by

$$h_\pi = \frac{1}{|d|}\sum_{s\in S} d(s)\pi(s), \tag{5.1}$$

where $d : S \to \mathbb{R}$ is a strictly positive function and $|d| = \sum_{s\in S} d(s)$. One can view h_π as the average of the representation over the subset S with respect to the weight given by the function d. We proof in Proposition 5.2.6 a characterization of Property (T) for finitely generated groups in terms of the spectrum of an operator of this form.

Remark 5.2.1. Let H be a Hilbert space, let $\xi_1, \dots, \xi_n \in H$ be a finite collection of unit vectors and let $\xi = \sum_{i=1}^n a_i\xi_i$ be a convex combination. For each unit vector $\eta \in H$, the Cauchy-Schwarz inequality gives us the estimate

$$\sum_{i=1}^n \|\eta - \xi_i\|^2 = 2 - 2\operatorname{Re}\langle \eta, \xi\rangle = 2\operatorname{Re}\langle \eta, \eta - \xi\rangle \le \|\eta - \xi\|.$$

Hence, if $\|\eta - \xi\| < \varepsilon$, for some unit vector $\eta \in H$ and some $\varepsilon > 0$, then $\|\eta - \xi_i\| < \sqrt{2\varepsilon/a_i}$, for each $1 \le i \le n$. Loosely speaking, if ξ is close to a unit vector η, then all ξ_i are close to η, as well.

Lemma 5.2.2. *Let Γ be a finitely generated discrete group and let S be a finite symmetric generating set. Let $d : S \to \mathbb{R}$ be a strictly positive symmetric function and set $|d| = \sum_{s\in S} d(s)$. Then h_π, as defined in equation (5.1), is a self-adjoint operator with spectrum $\sigma(h_\pi) \subset [-1,1]$. Moreover, a non-zero vector $\xi \in H$ is invariant for (π, H) if and only if ξ is a an eigenvector for h_π with eigenvalue 1, and a net $(\xi_i)_{i\in I}$ in H is almost invariant for (π, H) if and only if $\langle h_\pi \xi_i , \xi_i\rangle$ converges to 1 in $\mathbb{C}$.*

Proof. Since (π, H) is unitary, S is a symmetric set and d is a symmetric function, we see directly that h_π is self-adjoint. Further, being a convex combination of unitaries, h_π has norm bounded by 1, and since its spectrum is real, it follows that $\sigma(h_\pi) \subset [-1,1]$.

If $\xi \in H$ is a non-zero invariant vector, we see directly that ξ must be an eigenvector for h_π with eigenvalue 1. Conversely, if $\xi \in H$ is any unit vector, Remark 5.2.1 yields that

$$\frac{1}{|d|}\sum_{s\in S} d(s)\, \|\pi(s)\xi - \xi\|^2 = 2\big(1 - \mathrm{Re}\,\langle \xi , h\xi\rangle\big) \leq \|h\xi - \xi\| \,. \tag{5.2}$$

Hence, if ξ is a unit eigenvector for h_π with eigenvalue 1, then $\|\pi(s)\xi - \xi\| = 0$, for all $s \in \Gamma$. Since S is a generating set for Γ, it follows that ξ is invariant. The general case for non-unit eigenvectors of h_π with eigenvalue 1 follows by rescaling.

The statement regarding nets of almost invariant vectors follows directly from the first equality in equation (5.2) using that S is a finite set and that S generates Γ. □

Definition 5.2.3. Let Γ be a discrete group. A unitary representation (π, H) of Γ is said to be *universal* if it weakly contains all other unitary representations.

Remark 5.2.4. Universal representations always exist. Take, for instance, the direct sum of all GNS-representations (see Theorem 2.4.9) corresponding to functions of positive type on the group.

Remark 5.2.5. Suppose (π, H) is a universal representation of the discrete groups Γ, and let (ρ, K) be any other unitary representation of Γ. Since $\rho \prec \pi$, we have a well-defined *-homomorphism $\Phi : \hat{\pi}(C^*(\Gamma)) \to \hat{\rho}(C^*(\Gamma))$ given by $\Phi(\hat{\pi}(x)) = \hat{\rho}(x)$, for all $x \in C^*(\Gamma)$. If Γ has a finite symmetric generating set S and if $d : S \to \mathbb{R}$ is a fixed strictly positive symmetric function, let h_π and h_ρ denote the operators from Lemma 5.2.2 belonging to (π, H) and (ρ, K), respectively. Then $h_\rho = \Phi(h_\pi)$. In particular, we have the inclusion $\sigma(h_\rho) \subset \sigma(h_\pi)$ (see, e.g., [54, Proposition 9.1]).

Proposition 5.2.6. *Let Γ be a finitely generated discrete group and let S be a finite symmetric generating set. Let (π, H) be a universal representation and let $h = h_\pi \in B(H)$ be as in equation (5.1), for some strictly positive symmetric function $d : S \to \mathbb{R}$. Then Γ has Property (T) if and only if 1 is an isolated point in $\sigma(h)$. Moreover, if $\sigma(h) \subset [-1, 1-\varepsilon] \cup \{1\}$, for some $0 < \varepsilon < 1$, then $(S, \sqrt{2\varepsilon})$ is a Kazhdan pair for Γ.*

Proof. Suppose Γ has property (T). Since (π, H) is universal, $1_\Gamma \prec \pi$, so as Γ has Property (T), we deduce that $1_\Gamma \leq \pi$ (see Corollary 1.3.6). Then π has a non-zero invariant vector, by Proposition 1.3.1, and so, Lemma 5.2.2 yields that 1 is in the spectrum of h. We must show that 1 is isolated. Because h is a finite linear combination of operators of the form $\pi(s)$, for $s \in \Gamma$, any Γ-invariant subspace of H is reducing for h (see Remark 1.1.5). In particular, H^π is a reducing subspace for h, and we may therefore consider h a direct sum of the its restrictions to H^π and its compliment. Thus,

$$\sigma(h) = \sigma\big(h|_{H^\pi}\big) \cup \sigma\big(h|_{(H^\pi)^\perp}\big).$$

By Lemma 5.2.2, $h|_{H^\pi}$ is the identity operator on H^π, and so, $\sigma\big(h|_{H^\pi}\big) = \{1\}$. Since $\sigma(h)$ is a compact subset of $[-1,1]$, it therefore suffices to show that $1 \notin \sigma\big(h|_{(H^\pi)^\perp}\big)$. Suppose for contradiction that 1 is contained in the spectrum of $h|_{(H^\pi)^\perp}$ so that $h|_{(H^\pi)^\perp} - 1$ is not invertible. Since this is a normal operator, it follows that there exists a sequence of unit vectors $(\xi_n)_{n\geq 1}$ in $(H^\pi)^\perp$ such that $\|(h-1)\xi_n\| \to 0$ (see [40, Theorem 12.12(c)]). By Cauchy-Schwarz, we see that $\langle h\xi_n\,, \xi_n\rangle \to 1$. But then $(\xi_n)_{n\geq 1}$ is a sequence of almost invariant vectors for (π, H), by Lemma 5.2.2. Since $(\xi_n)_{n\geq 1}$ lies in $(H^\pi)^\perp$, this contradicts Proposition 1.2.7.

Suppose now that the spectrum of h is contained in $[-1, 1-\varepsilon] \cup \{1\}$, for some $0 < \varepsilon < 1$. Let (ρ, K) be any unitary representation of Γ and let h_ρ be as in Lemma 5.2.2. Then $\sigma(h_\rho) \subset [-1, 1-\varepsilon] \cup \{1\}$, by Remark 5.2.5. Let P be the spectral projection of h_ρ onto the spectral subset $\{1\}$. Notice that P coincides with the projection onto K^ρ, by Lemma 5.2.2. By the spectral theorem, h_ρ has a spectral decomposition $h_\rho = P + \sum_i \lambda_i P_i$, where each $\lambda_i \in [-1, 1-\varepsilon]$. Clearly, $\sum_i P_i \leq 1 - P$, and so,

$$(1-h_\rho) - \varepsilon(1-P) = (1-\varepsilon)(1-P) - \sum_i \lambda_i P_i \geq (1-\varepsilon)\Big((1-P) - \sum_i P_i\Big) \geq 0.$$

Hence, $\varepsilon(1-P) \leq 1 - h_\rho$. It follows, for each $\xi \in K$, that

$$\begin{aligned}2\varepsilon \left\|(1-P)\xi\right\|^2 &= 2\left\langle \varepsilon(1-P)\xi\,, \xi\right\rangle \leq 2\left\langle (1-h_\rho)\xi\,, \xi\right\rangle = \frac{1}{|d|}\sum_{s\in S} d(s)\big(\|\xi\|^2 - \operatorname{Re}\left\langle \rho(s)\xi\,, \xi\right\rangle\big) \\ &= \frac{1}{|d|}\sum_{s\in S} d(s)\left\|\rho(s)\xi - \xi\right\|^2 \leq \max_{s\in S}\left\|\rho(s)\xi - \xi\right\|^2.\end{aligned}$$

Thus, if ξ is a $(S, \sqrt{2\varepsilon})$-invariant vector for (ρ, K), then $\|(1-P)\xi\| < \|\xi\|$. This implies that P is a non-zero projection so that K^ρ is non-trivial. As (ρ, K) was an arbitrary unitary representation, we may conclude that $(S, \sqrt{2\varepsilon})$ is a Kazhdan pair for Γ. $\square$

5.2.2 ℓ^2-spaces on finite graphs

Let $\mathcal{G}$ be a finite simple undirected graph. The degree function induces a finite discrete measure on the vertex set of $\mathcal{G}$ by assigning a subset of vertices, $S \subset V(\mathcal{G})$, the measure $\sum_{s\in S}\deg(s)$. Observe that

$$\sum_{s\in V(\mathcal{G})} \deg(s) = |E(\mathcal{G})|\,.$$

We shall in the following consider any finite graph $\mathcal{G}$ a probability space equipped with the normalized measure induced by the degree function. We denote by $\ell^2(\mathcal{G})$ the ℓ^2-space of complex valued functions on the vertices of $\mathcal{G}$ equipped with inner product and norm obtained using the described probability measure. Notice that $\ell^2(\mathcal{G})$ equals $\mathbb{C}^{V(\mathcal{G})}$ as a vector space because $\mathcal{G}$ is finite – the notation therefore merely indicates the additional inner product structure. For a complex Hilbert space H, we denote by $\ell^2(\mathcal{G}; H)$ the tensor product $\ell^2(\mathcal{G}) \otimes H$. As a set, $\ell^2(\mathcal{G}; H)$ consists of all functions on the vertices of $\mathcal{G}$ with values in H. We see that $\ell^2(\mathcal{G}; H)$ is, again, a Hilbert space with inner product given by

$$\frac{1}{|E(\mathcal{G})|}\sum_{s\in V(\mathcal{G})} \left\langle f(s)\,, g(s)\right\rangle \deg(s),$$

for all $f, g \in \ell^2(\mathcal{G}; H)$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product in H.

We have an isometric inclusion of H into $\ell^2(\mathcal{G}; H)$ given by $\xi \mapsto 1_{V(\mathcal{G})} \otimes \xi$, where $1_{V(\mathcal{G})} \otimes \xi$ is the function that takes the value ξ everywhere. To ease notation we shall in the following identify the subspace of constant functions with H and simply write ξ instead of $1_{V(\mathcal{G})} \otimes \xi$. For each $f \in \ell^2(\mathcal{G}; H)$, we denote by

$$\mathbb{E}(f) = \frac{1}{|E(\mathcal{G})|} \sum_{s \in V(\mathcal{G})} f(s) \deg(s)$$

the *mean* of f. Then $\mathbb{E}$ is a map $\ell^2(\mathcal{G}; H) \to H$. With the identification of the subspace of constant functions and H, we may also consider $\mathbb{E}$ an operator on $\ell^2(\mathcal{G}; H)$.

Lemma 5.2.7. *Let $\mathcal{G}$ be a finite simple undirected and connected graph, and let H be a complex Hilbert space. Considered an operator on $\ell^2(\mathcal{G}; H)$, the mean, $\mathbb{E}$, is the orthogonal projection onto the subspace of constant functions.*

Proof. Let $f \in \ell^2(\mathcal{G}; H)$. For each $\xi \in H$, we have

$$\begin{aligned}
\langle f - \mathbb{E}(f)\,, \xi \rangle_{\ell^2(\mathcal{G}) \otimes H} &= \frac{1}{|E(\mathcal{G})|} \sum_{s \in V(\mathcal{G})} \left\langle f(s) - \frac{1}{|E(\mathcal{G})|} \sum_{t \in V} f(t) \deg(t)\,, \xi \right\rangle \deg(s) \\
&= \frac{1}{|E(\mathcal{G})|} \sum_{s \in V(\mathcal{G})} \langle f(s)\,, \xi \rangle \deg(s) - \frac{1}{|E(\mathcal{G})|^2} \sum_{s \in V(\mathcal{G})} \left\langle \sum_{t \in V} f(t) \deg(t)\,, \xi \right\rangle \deg(s) \\
&= 0.
\end{aligned}$$

Hence, $\mathbb{E}(f)$ is, indeed, the orthogonal projection of f onto the subspace of constant functions on $\mathcal{G}$ with values in H (see, e.g., [41, Theorem 21.5]). ☐

Remark 5.2.8. By the Pythagoras theorem, the equality

$$\|f - \mathbb{E}(f)\|^2_{\ell^2(\mathcal{G};H)} = \|f\|^2_{\ell^2(\mathcal{G};H)} - \|\mathbb{E}(f)\|^2$$

holds, for all $f \in \ell^2(\mathcal{G}; H)$.

It is sometimes interesting to consider spaces of functions on the edges of a graph instead of the vertices. For a finite graph $\mathcal{G}$, we make $E(\mathcal{G})$ into a probability space by equipping it with the normalized uniform discrete measure. We denote by $\ell^2(E(G))$ the ℓ^2-space of complex valued functions on the edges of $\mathcal{G}$ equipped with the 2-norm associated to this probability measure. For a Hilbert space H, we denote by $\ell^2(E(\mathcal{G}); H)$ the tensor product $\ell^2(E(\mathcal{G})) \otimes H$. This is, again, a Hilbert space with inner product given by

$$\frac{1}{|E(\mathcal{G})|} \sum_{(s,t) \in V(\mathcal{G})} \langle f(s,t)\,, g(s,t) \rangle\,.$$

for all $f, g \in \ell^2(E(\mathcal{G}); H)$, where $\langle\, \cdot\,, \cdot\, \rangle$ denotes the inner product in H.

For a Hilbert space H with identity operator I_H, we define the discrete Laplace operator on $\ell^2(\mathcal{G}; H)$ as the tensor product $\Delta \otimes I_H$, and the gradient operator $\ell^2(\mathcal{G}; H) \to \ell^2(E(\mathcal{G}); H)$ as the tensor product $\nabla \otimes I_H$. We remark that, for any probability spaces Ω and Ω', and any bounded linear operator $T : L^2(\Omega) \to L^2(\Omega')$, the tensor product operator $T \otimes I_H$ is a bounded linear operator $L^2(\Omega; H) \to L^2(\Omega'; H)$ with norm $\|T \otimes I_H\| = \|T\|$ and spectrum $\sigma(T \otimes I_H) = \sigma(T)$. These observations hold, in particular, for the discrete Laplace operator and the gradient operator. Whenever the context admits it, we shall omit $\otimes I_H$ from the notation. Lemma 5.1.16 generalizes to this setting as follows:

Lemma 5.2.9. *Let $\mathcal{G}$ be a finite simple undirected graph and let H be a complex Hilbert space. For each $f \in \ell^2(\mathcal{G}; H)$, it holds that*

$$2 \langle \Delta f \, , f \rangle_{\ell^2(\mathcal{G};H)} = \| \nabla f(s,t) \|^2_{\ell^2(E(\mathcal{G});H)} \, .$$

Proof. Let $f \in \ell^2(\mathcal{G}; H)$. Since $\mathcal{G}$ is undirected, we have

$$\sum_{(s,t) \in E(S)} \langle f(t) \, , f(s) \rangle = \sum_{(s,t) \in E(S)} \langle f(s) \, , f(t) \rangle \, .$$

In particular, this implies that the above sum is real. Since there is a bijective set correspondence between $E(\mathcal{G})$ and $\bigcup_{s \in V(\mathcal{G})} \{s\} \times N_s$, we see that

$$\begin{aligned}
\sum_{(s,t) \in E(\mathcal{G})} \| \nabla f(s,t) \|^2 &= \sum_{(s,t) \in E(\mathcal{G})} \langle f(s) - f(t) \, , f(s) - f(t) \rangle \\
&= 2 \sum_{(s,t) \in E(\mathcal{G})} \langle f(s) \, , f(s) \rangle - 2 \sum_{(s,t) \in E(\mathcal{G})} \langle f(t) \, , f(s) \rangle \\
&= 2 \sum_{s \in V(\mathcal{G})} \langle f(s) \, , f(s) \rangle \deg(s) - 2 \sum_{s \in V(\mathcal{G})} \sum_{t \sim s} \langle f(t) \, , f(s) \rangle \\
&= 2 \sum_{s \in V(\mathcal{G})} \left(\langle f(s) \, , f(s) \rangle - \frac{1}{\deg(s)} \sum_{t \sim s} \langle f(t) \, , f(s) \rangle \right) \deg(s) \\
&= 2 \sum_{s \in V(\mathcal{G})} \langle \Delta f(s) \, , f(s) \rangle \deg(s) \\
&= 2 \, |E(\mathcal{G})| \, \langle \Delta f \, , f \rangle_{\ell^2(S, \pi, \deg)}
\end{aligned}$$

This proves the claimed equality. ◻

Proposition 5.2.10. *Let $\mathcal{G}$ be a finite simple undirected and connected graph, and let H be a complex Hilbert space. For each $f : V(\mathcal{G}) \to H$, it holds that*

$$\lambda_1(\mathcal{G}) \, \| f - \mathbb{E}(f) \|^2_{\ell^2(\mathcal{G};H)} \leq \frac{1}{2 \, |E(\mathcal{G})|} \sum_{(s,t) \in E(\mathcal{G})} \| \nabla f(s,t) \|^2 \, . \tag{5.3}$$

Proof. Denote by I_H the identity operator on H and by $I_{\mathcal{G}}$ the identity operator on $\ell^2(\mathcal{G})$. Let P be the orthogonal projection in $B(\ell^2(\mathcal{G}))$ onto the subspace $\mathbb{C}1_{V(\mathcal{G})}$ of constant functions on $\mathcal{G}$. Then $\mathbb{E} = P \otimes I_H$. By Lemma 5.1.17 and by the assumption that $\mathcal{G}$ is connected, P is the spectral projection of Δ onto the simple eigenvalue 0. Hence, the spectral decomposition of Δ has the form $\Delta = \sum_{i=1}^n \lambda_i P_i$, for some $1 \leq n < |V(\mathcal{G})|$, where $P + \sum_{i=1}^n P_i = I_{\mathcal{G}}$ and where the eigenvalues $0 < \lambda_1(\mathcal{G}) = \lambda_1 \leq \cdots \leq \lambda_n \leq 2$ are ordered non-decreasingly. It follows that

$$\Delta \otimes I_H = \sum_{i=1}^n \lambda_i P_i \otimes I_H \geq \lambda_1(\mathcal{G}) \sum_{i=1}^n P_i \otimes I_H = \lambda_1(\mathcal{G})(I_{\mathcal{G}} - P) \otimes I_H.$$

From this, we deduce that

$$\begin{aligned}
\lambda_1(\mathcal{G}) \, \| f - \mathbb{E}(f) \|^2_{\ell^2(\mathcal{G};H)} &= \lambda_1(\mathcal{G}) \, \| f - (P \otimes I_H) f \|^2_{\ell^2(\mathcal{G};H)} \\
&= \lambda_1(\mathcal{G}) \, \langle (I_{\mathcal{G}} \otimes I_H - P \otimes I_H) f \, , f \rangle_{\ell^2(\mathcal{G};H)} \\
&\leq \langle (\Delta \otimes I_H) f \, , f \rangle_{\ell^2(\mathcal{G};H)} \, .
\end{aligned}$$

Equation (5.3) then follows from Lemma 5.2.9. ◻

5.2.3 Żuk's condition for Property (T)

For a group Γ and a subset $S \subset \Gamma$, the *link graph* associated to S, denoted by $L(S)$, is the graph defined as follows:

- The vertices of $L(S)$ are the elements of S.
- The edges of $L(S)$ are the ordered pairs $(s,t) \in S \times S$ satisfying $s^{-1}t \in S$.

We denote by $V(S)$ and $E(S)$ the set of vertices, respectively, edges of $L(S)$. By definition, the graph $L(S)$ is finite if and only if S is. In this case, observe that

$$\sum_{s\in S} \deg(s) = |E(S)|.$$

If S contains the identity element, e, then all vertices have self-loops. Otherwise, no vertices do. If S is symmetric then $L(S)$ is undirected. Moreover, in this case the degree function is symmetric, i.e., $\deg(s) = \deg(s^{-1})$, for each $s \in V(S)$. If $L(S)$ is connected the degree function is strictly positive.

Assume that Γ is finitely generated, and let S be a finite symmetric generating set not containing the identity, so that $L(S)$ is a finite simple undirected graph. We may assume that $L(S)$ is connected. Otherwise, $(S \cup S^2)\backslash\{e\}$ is another generating set with the stated properties and such that the associated link graph is connected. In [55], A. Żuk proved a sufficient condition for Property (T) in terms of the spectrum of the link graph. We state and proof this condition in Theorem 5.2.11 below following [9, Theorem 12.1.15].

Theorem 5.2.11 (Żuk's condition)**.** *Let Γ be a discrete finitely generated group, and let S be a finite symmetric generating set not containing the identity and such that $L(S)$ is connected. Let $\lambda_1 = \lambda_1(L(S))$. If $\lambda_1 > 1/2$ then*

$$\left(S, \sqrt{2\left(2-\lambda_1^{-1}\right)}\,\right)$$

is a Kazhdan pair for Γ.

Proof. Let (π, H) be a universal representation of Γ and let $h = h_\pi$ be as in Lemma 5.2.2 using the degree function as the strictly positive symmetric function on S. Set $d = \sum_{s\in S}\deg(s)$. By Proposition 5.2.6, it suffices to show that

$$\sigma(h) \subset [-1, \lambda_1^{-1} - 1] \cup \{1\}.$$

Fix $\xi \in H$ and define $f : S \to H$ by $f(s) = \pi(s)\xi$. As an element of $\ell^2(S, \deg)$, f has mean is equal to $h\xi$ and norm $\|f\|_{\ell^2(S,\deg)} = \|\xi\|$. Proposition 5.2.10 yields the inequality

$$\begin{aligned}
\lambda_1\left(\|\xi\|^2 - \|h\xi\|^2\right) &= \lambda_1\left(\|f\|^2_{\ell^2(S,\deg)} - \|\mathbb{E}(f)\|^2\right) \le \frac{1}{2d}\sum_{(s,t)\in E(S)} \|f(t)-f(s)\|^2 \\
&= \frac{1}{2d}\sum_{(s,t)\in E(S)} \left\|\xi - \pi(s^{-1}t)\xi\right\|^2 = \frac{1}{2d}\sum_{s\in S}\|\xi - \pi(s)\xi\|^2\deg(s) \\
&= \|\xi\|^2 - \frac{1}{d}\sum_{s\in S}\langle \pi(s)\xi\,,\xi\rangle \deg(s) = \|\xi\|^2 - \langle h\xi\,,\xi\rangle\,.
\end{aligned}$$

Or, using that h is self-adjoint, $\lambda_1\left\langle (I_H - h^2)\xi\,,\xi\right\rangle \le \langle (I_H - h)\xi\,,\xi\rangle$, for any vector $\xi \in H$. If λ is any eigenvalue of h, we see directly that $\lambda_1(1-\lambda^2) \le 1-\lambda$. Further, since h is normal, all points in its spectrum are approximate eigenvalues (see, e.g., [40, Theorem 12.12(c)]), and so, this inequality holds for all $\lambda \in \sigma(h)$. The roots of the polynomial $x \mapsto -\lambda_1 x^2 + x + \lambda_1 - 1$ are 1 and $1 - \lambda_1^{-1}$. Thus, $\sigma(h) \cap (\lambda_1^{-1} - 1, 1) = \emptyset$, which was what we aimed to show. $\square$

5.3 A spectral condition for Property (F_{L^p})

Żuk's condition states that a sufficiently large spectral gap of a link graph associated to a (nice) generating set ensures property (T). We discuss in this section a condition of this flavor which ensures Property (F_{L^p}). This condition is due to T. de Laat and M. de la Salle in [14].

5.3.1 ℓ^p-spaces on finite graphs

For finite set S equipped with a probability measure ν on its power set, a $1 \leq p \leq \infty$ and a normed space X, we denote by $\ell^p(S,\nu;X)$ the tensor product $\ell^p(S,\nu) \otimes X$ equipped with the p-norm

$$\|f\|^p_{(S,\nu;X)} = \sum_{s\in S} \|f\|^p \, \nu(\{s\}),$$

for $f \in \ell^p(S,\nu;X)$, if $p < \infty$. For $p = \infty$, the norm is given by $\|f\|_{\ell^\infty(S;X)} = \max_{s\in S} \|f(s)\|$. Since S is finite, it is straight forward to verify that, whenever X is a Banach space, $\ell^p(S,\nu;X)$ is, again, a Banach space. Whenever the measure is clear from context, we shall omit it from the notation.

Let $\mathcal{G}$ be a finite graph. As before, we consider $\mathcal{G}$ (or rather, the vertex set of $\mathcal{G}$) a probability space equipped with the normalized measure induced by the degree function. For each $1 \leq p < \infty$ and each Banach space X, $\ell^p(\mathcal{G};X)$, as defined above, is the Banach space of X-valued functions on the vertices of $\mathcal{G}$ with norm given by

$$\|f\|^p_{\ell^p(\mathcal{G};X)} = \frac{1}{|E(\mathcal{G})|} \sum_{s\in V(\mathcal{G})} \|f(s)\|^p \deg(s),$$

for all $f \in \ell^p(\mathcal{G};X)$. As we have equipped $\mathcal{G}$ with a probability measure, the map $x \mapsto 1_{\mathcal{G}} \otimes x$ is an isometric embedding of X into $\ell^p(\mathcal{G};X)$, and we shall, as in the Hilbert space setting, identify X with the subspace $1_{\mathcal{G}} \otimes X$ of constant X-valued functions on $V(\mathcal{G})$. With this identification in mind, we write x instead of $1_{\mathcal{G}} \otimes x$ for the function which is equal to x everywhere.

We shall keep the notation from the Hilbert space setting and denote by $\mathbb{E}$ the operator on $\ell^p(\mathcal{G};X)$ given by

$$\mathbb{E}(f) = \frac{1}{|E(\mathcal{G})|} \sum_{s\in V(\mathcal{G})} f(s) \deg(s),$$

for all $f \in \ell^p(\mathcal{G};X)$. We denote by $\ell^p_0(\mathcal{G};X)$ the kernel of $\mathbb{E}$, i.e.,

$$\ell^p_0(\mathcal{G};X) = \left\{ f \in \ell^p(\mathcal{G};X) \;\middle|\; \sum_{s\in V(\mathcal{G})} f(s)\deg(s) = 0 \right\}.$$

It is easy to see that $\mathbb{E}$ is continuous, and so, $\ell^p_0(\mathcal{G};X)$ is a closed subspace of $\ell^p(\mathcal{G};X)$. Moreover, $\ell^p_0(\mathcal{G};X)$ is complemented. Specifically, the complement is the closed subspace of constant functions, $1_{\mathcal{G}} \otimes X$, and we have a vector space isomorphism

$$\begin{aligned} \ell^p(\mathcal{G};X) &\longrightarrow \ell^p_0(\mathcal{G};X) \oplus (1_{\mathcal{G}} \otimes X) \\ f &\longmapsto (f - \mathbb{E}(f), \mathbb{E}(f)) \end{aligned}$$

Hence, as a vector space, $\ell_0^p(\mathcal{G};X)$ is isomorphic to the quotient of $\ell^p(\mathcal{G};X)$ by the constant functions. There are therefore two norms which are natural to put on $\ell_0^p(\mathcal{G};X)$: The subspace norm and the quotient norm. In general, they are related by

$$\inf_{x\in X} \|f-x\|_{\ell^p(\mathcal{G};X)} \leq \|f-\mathbb{E}(f)\|_{\ell^p(\mathcal{G};X)} \,. \tag{5.4}$$

Recall that when $p=2$ and X is a Hilbert space, $\mathbb{E}$ is the orthogonal projection onto the subspace of constant functions. In particular, equation (5.4) is an equality.

Another class of function spaces associated to a finite graph $\mathcal{G}$ that is of interest to us are the ℓ^p-spaces of Banach space valued functions on the edges of the graph. As in the Hilbert space setting, we equip $E(\mathcal{G})$ with the uniform probability measure. For a Banach space X, $\ell^p(E(\mathcal{G});X)$ is the ℓ^p-space of X-valued functions on $E(\mathcal{G})$ with norm given by

$$\|f\|^p_{\ell^p(E(\mathcal{G});X)} = \frac{1}{|E(\mathcal{G})|}\sum_{(s,t)\in E(\mathcal{G})} \|f(s,t)\|^p \,.$$

For Banach spaces X and Y, we denote by $L(X,Y)$ the space of bounded linear operators from X to Y. When $X=Y$, we write $L(X)=L(X,X)$. Unlike in the Hilbert space setting, for Banach spaces X,Y,X' and Y', the operator norm on $L(X,Y)\otimes L(X',Y')$ inherited from $L(X\otimes X', Y\otimes Y')$ need not be a cross norm. That is, for $T: X\to Y$ and $S: X'\to Y'$, $\|T\otimes S\|$ need not be equal to $\|T\|\,\|S\|$.

Definition 5.3.1. Let S and S' be finite sets equipped with probability measures on their power sets. An operator $T:\ell^p(S)\to\ell^p(S')$ is called *regular* if

$$\|T\|_{reg} = \sup\left\{ \|T\otimes \mathrm{id}_X\|_{\ell^p(S;X)\to\ell^p(S';X)} \;\middle|\; X \text{ is a Banach space}\right\} < \infty.$$

In that case, the number $\|T\|_{reg}$ is referred to as the *regular norm* of T.

We show in Lemma 5.3.2 below that, for operators on an ℓ^2-space of a finite set, the supremum in the above definition is attained at an ℓ^∞-space. For each $1\leq p\leq\infty$, denote by $\ell_n^p=\ell^p(\{1,\dots,n\})$ the ℓ^p-space on n points. For a bounded linear operator $A\in B(\ell_n^2)$, we denote by $(A_{i,j})_{i,j=1}^n\in M_n(\mathbb{C})$ its canonical matrix representation.

Lemma 5.3.2. *For any $n\in\mathbb{N}$ and bounded linear operator A on ℓ_n^2, it holds that*

$$\|A\|_{reg} = \left\|A\otimes \mathrm{id}_{\ell_n^\infty}\right\|_{L(\ell_n^2\otimes\ell_n^\infty)} = \left\|\left(|A_{i,j}|\right)_{i,j=1}^n\right\|.$$

Proof. Let X be a Banach space and let $x_1,\dots,x_n\in X$ be such that $\sum_{j=1}^n \|x_j\|^2\leq 1$. By the triangle inequality,

$$\begin{aligned}\|A\otimes \mathrm{id}_X\, x\|^2_{\ell_n^2\otimes X} &= \sum_{i=1}^n\left\|\sum_{j=1}^n A_{i,j}x_j\right\|^2 \leq \sum_{i=1}^n\left|\sum_{j=1}^n |A_{i,j}|\,\|x_j\|\right|^2\\ &\leq \sup_{v\in\ell_n^2}\sum_{i=1}^n\left|\sum_{j=1}^n|A_{i,j}|\,v_j\right|^2 = \left\|\left(|A_{i,j}|\right)_{i,j=1}^n\right\|.\end{aligned}$$

Taking first the supremum over all $x\in\ell_n^2\otimes X$ with norm 1 and then over all Banach spaces X, it follows that

$$\|A\|_{reg} = \sup\left\{\|A\otimes\mathrm{id}_X\|_{L(\ell_n^2\otimes X)} \;\middle|\; X\text{ is a Banach space}\right\} \leq \left\|\left(|A_{i,j}|\right)_{i,j=1}^n\right\|. \tag{5.5}$$

For $x = (x_1, \dots, x_n) \in \ell_n^2 \otimes \ell_n^\infty$, write $x_j = (x_j^1, \dots, x_j^n) \in \ell_n^\infty$, for $1 \le j \le n$. We have

$$\left\| A \otimes \mathrm{id}_{\ell_n^\infty} x \right\|^2 = \sum_{i=1}^n \left\| \sum_{j=1}^n A_{i,j} x_j \right\|_{\ell_n^\infty}^2 = \sum_{i=1}^n \max_{k=1,\dots,n} \left| \sum_{j=1}^n A_{i,j} x_j^k \right|^2 . \tag{5.6}$$

Take $v = (v_1, \dots, v_n) \in \ell_n^2$ with $\|v\|_{\ell_n^2} = 1$ such that

$$\left\| \left(|A_{i,j}| \right)_{i,j=1}^n \right\|^2 = \left\| \left(|A_{i,j}| \right)_{i,j=1}^n v \right\|_{\ell_n^2}^2 = \sum_{i=1}^n \left| \sum_{j=1}^n |A_{i,j}| \, v_j \right|^2 . \tag{5.7}$$

For each pair of indices $1 \le i, j \le n$, choose $\theta_{i,j} \in \mathbb{R}$ such that $A_{i,j} = e^{i\theta_{i,j}} |A_{i,j}|$. Then, for each pair of indices $1 \le j, k \le n$, set $x_j^k = e^{-i\theta_{k,j}} v_j$. Then $x = (x_1, \dots, x_n)$ with $x_j = (x_j^1, \dots, x_j^n)$, for $1 \le j \le n$, defines a vector in $\ell_n^2 \otimes \ell_n^\infty$ with

$$\|x\|_{\ell_n^2 \otimes \ell_n^\infty}^2 = \sum_{j=1}^n \max_{k=1,\dots,n} |x_j^k|^2 = \sum_{j=1}^n |v_j|^2 = \|v\|_{\ell_n^2}^2 = 1.$$

For each fixed $1 \le i \le n$,

$$\max_{k=1,\dots,n} \left| \sum_{j=1}^n A_{i,j} x_j^k \right| \ge \left| \sum_{j=1}^n A_{i,j} x_j^i \right| = \left| \sum_{j=1}^n |A_{i,j}| \, v_j \right|^2 .$$

Summing over i, inserting equations (5.6) and (5.7) and taking the square root yields the inequality $\left\| A \otimes \mathrm{id}_{\ell_n^\infty} x \right\| \ge \left\| \left(|A_{i,j}| \right)_{i,j=1}^n \right\|$. Because $x \in \ell_n^2 \otimes \ell_n^\infty$ has norm 1, it follows that

$$\left\| \left(|A_{i,j}| \right)_{i,j=1}^n \right\| \le \left\| A \otimes \mathrm{id}_{\ell_n^\infty} \right\|_{L(\ell_n^2 \otimes \ell_n^\infty)} . \tag{5.8}$$

Putting together the inequalities (5.5) and (5.8), we have showed that

$$\|A\|_{reg} \le \left\| \left(|A_{i,j}| \right)_{i,j=1}^n \right\| \le \left\| A \otimes \mathrm{id}_{\ell_n^\infty} \right\|_{L(\ell_n^2 \otimes \ell_n^\infty)} \le \|A\|_{reg} ,$$

where the last inequality is trivial. Hence, all inequalities are equalities, which is what we aimed to show. □

For a fixed Banach space X, we define, for each $\varepsilon \in [0, 1]$, a number

$$\Delta_X^{fin}(\varepsilon) = \sup \left\{ \|T \otimes \mathrm{id}_X\|_{L(\ell^2(S;X))} \;\middle|\; S \text{ finite}, \|T\|_{L(\ell^2(S))} \le \varepsilon, \|T\|_{reg} \le 1 \right\} . \tag{5.9}$$

Then Δ_X^{fin} is a non-decreasing function on $[0, 1]$.

Lemma 5.3.3. *Let X be an L^p-space with $p \ge 2$. Then*

$$\Delta_X^{fin}(\varepsilon) \le \varepsilon^{2/p}, \tag{5.10}$$

for all $\varepsilon \in [0, 1]$.

Proof. Let $X = L^p(\mu)$ be an L^p-space with $p \ge 2$. Then X is $2/p$-interpolation space between $L^2(\mu)$ and $L^\infty(\mu)$. Let S be any finite set equipped with a probability measure on its power set and let T be a bounded linear operator on $\ell^2(S)$ with $\|T\| \le \varepsilon$, for some $\varepsilon > 0$ and $\|T\|_{reg} \le 1$. We have $\left\| T \otimes \mathrm{id}_{L^2(\mu)} \right\| = \|T\| \le \varepsilon$, since $\ell^2(S; L^2(\mu))$ is a Hilbert space, and $\|T \otimes \mathrm{id}_{L^\infty(\mu)}\| \le \|T\|_{reg} \le 1$, by definition of the regular norm. equation (5.10) then follows directly from the Riesz-Thorin theorem for Bochner spaces, Theorem B.3.2. □

5.3.2 Poincaré type inequalities and fixed points

For a Banach space X and for $1 \leq p \leq \infty$, we define the discrete Laplace operator on $\ell^p(\mathcal{G}; X)$ as the tensor product $\Delta \otimes \mathrm{id}_X$, and the gradient operator $\ell^p(\mathcal{G}; X) \to \ell^p(E(\mathcal{G}); X)$ as the tensor product $\nabla \otimes \mathrm{id}_X$. A central part of the proof of Theorem 5.2.11 is the validity of the inequality (5.3) of Proposition 5.2.10. This is a so called *Poincaré type inequality.* More generally, for $1 \leq p \leq \infty$, a *p-Poincaré inequality* associated to a finite graph $\mathcal{G}$ and a Banach space X, is an inequality of the form

$$\inf_{x \in X} \|f - x\|_{\ell^p(\mathcal{G}; X)} \leq C \|\nabla f\|_{\ell^p(E(\mathcal{G}); X)}, \tag{5.11}$$

for some constant $C > 0$, which holds for all $f \in \ell^p(\mathcal{G}; X)$. Rearranging equation (5.3), we see that this is, indeed, a 2-poincaré inequality with constant $C = 1/\sqrt{2\lambda_1(\mathcal{G})}$. We remark here that this constant is strictly less than one if and only if $\lambda_1(\mathcal{G}) > 1/2$. Hence, the criterion in Żuk's theorem that a group Γ admits a (nice) link graph with spectral gap strictly greater than $1/2$ exactly ensures the validity of a 2-poincaré inequality with constant strictly less than 1. With this in mind, we show in Theorem 5.3.4 below how one can construct a fixed point of an action on a Banach space from a p-Poincaré inequality on the ℓ^p-space of the link graph with a constant strictly less than one.

Theorem 5.3.4. *Let Γ be a discrete finitely generated group, and let S be a finite symmetric generating set not containing the identity and such that $L(S)$ is connected. Let X be a Banach space. If, for some $1 < p < \infty$, there exists a constant $0 < C < 1$ such that*

$$\inf_{x \in X} \|f - x\|_{\ell^p(L(S); X)} \leq C \|\nabla f\|_{\ell^p(E(S); X)}, \tag{5.12}$$

for all $f : S \to X$, then Γ has Property (F_X).

Proof. Suppose $1 < p < \infty$ and $0 < C < 1$ are such that equation (5.12) hold, for all $f : S \to X$, and let $\Gamma \curvearrowright X$ be an action by affine isometries. Let $\mathcal{E}$ be the set of all functions $\Gamma \to X$ satisfying $t.f(s) = f(ts)$, for all $s, t \in \Gamma$. Observe that $\mathcal{E}$ is a convex set. For each $x \in X$, denote by $f_x : \Gamma \to X$ the function $f_x(s) = s.x$, for all $s \in \Gamma$. It is clear that $\mathcal{E}$ contains all functions of this form. In particular, $\mathcal{E}$ is non-trivial.

We may regard any $f \in \mathcal{E}$ as an element of $\ell^p(L(S); X)$ by restriction of its domain. Since X is a Banach space and S is finite, $\ell^p(L(S); X)$ is a Banach space, as well. Denote be d the metric on $\ell^p(L(S); X)$ induced by the norm, i.e., for any $f, f' \in \mathcal{E}$,

$$d(f, f') = \|f - f'\|_{\ell^p(L(S); X)} = \left(\frac{1}{d} \sum_{s \in S} \|f(s) - f'(s)\|^p \deg(s) \right)^{1/p}.$$

For any two functions $f, f' : S \to X$, we define

$$E(f, f') = \left(\frac{1}{d} \sum_{(s,t) \in E(S)} \|f(s) - f'(t)\|^p \right)^{1/p}.$$

To simplify notation, we shall write $E(f) = E(f, f)$. Notice that $E(f) = \|\nabla f\|_{\ell^p(E(S); X)}$. In particular, the map $\ell^p(L(S); X) \to \mathbb{R}$ given by $f \mapsto E(f)$ is continuous. Furthermore, we remark that E is symmetric as a map $X^S \times X^S \to \mathbb{R}$. This is because S is symmetric, by assumption.

Fix a constant $c \in (C, 1)$. For any non-zero function $f \in \mathcal{E}$, we may pick $x \in X$ such that

$$\|f - x\|_{\ell^p(L(S); X)} \leq c \|\nabla f\|_{\ell^p(E(S); X)}. \tag{5.13}$$

Then

$$E(f, f_x)^p = \frac{1}{d} \sum_{(s,t)\in E(S)} \|f(s) - t.x\|^p = \frac{1}{d} \sum_{(s,t)\in E(S)} \left\|f(t^{-1}s) - x\right\|^p$$
$$= \frac{1}{d} \sum_{s\in S} \|f(s) - x\|^p \deg(s) = \|f - x\|^p_{\ell^p(L(S);X)}.$$

Hence, $E(f, f_x) \leq cE(f)$. Further, by the Minkowsky inequality on $\ell^p(E(S); X)$,

$$E\left(\frac{f + f_x}{2}\right) = \frac{1}{2}\left(\frac{1}{d} \sum_{(s,t)\in E(S)} \|f(s) - t.x + s.x - f(t)\|^p\right)^{1/p}$$
$$\leq \frac{1}{2}\left(\frac{1}{d} \sum_{(s,t)\in E(S)} \|f(s) - t.x\|^p\right)^{1/p} + \frac{1}{2}\left(\frac{1}{d} \sum_{(s,t)\in E(S)} \|s.x - f(t)\|^p\right)^{1/p}$$
$$= \frac{1}{2}E(f, f_x) + \frac{1}{2}E(f_x, f) = E(f, f_x).$$

Summerizing, we have established the following inequalities:

$$E\left(\frac{f + f_x}{2}\right) \leq E(f, f_x) \leq cE(f), \tag{5.14}$$

for any $f \in \mathcal{E}$ and with x chosen such that equation (5.13) is satisfied. Furthermore,

$$d\left(f, \frac{f + f_x}{2}\right) = \frac{1}{2}\|f - f_x\|_{\ell^p(L(S);X)} = \frac{1}{2}\left(\frac{1}{d} \sum_{t\in S} \|f(t) - t.x\|^p \deg(t)\right)^{1/p}$$
$$= \frac{1}{2}\left(\frac{1}{d} \sum_{(s,t)\in E(S)} \|f(t) - f(s) + f(s) - t.x\|^p\right)^{1/p}$$
$$\leq \frac{1}{2}\left(\frac{1}{d} \sum_{(s,t)\in E(S)} \|f(t) - f(s)\|^p\right)^{1/p} + \frac{1}{2}\left(\frac{1}{d} \sum_{(s,t)\in E(S)} \|f(s) - t.x\|^p\right)^{1/p}$$
$$= \frac{1}{2}\|\nabla f\|_{\ell^p(E(S);X)} + \frac{1}{2}E(f, f_x) < E(f). \tag{5.15}$$

Here, the inequality in the third line is Minkowsky's inequality on $\ell^p(E(S); X)$, and the last inequality follows from equation (5.14) using that $c < 1$.

Pick any non-zero function $f_0 \in \mathcal{E}$. As $\mathcal{E}$ is a convex set, we obtain a sequence $(f_n)_{n\geq 1}$ in $\mathcal{E}$ by inductively defining

$$f_{n+1} = \frac{f_n + f_{x_n}}{2},$$

for each $n \in \mathbb{N}_0$, where x_n is chosen such that equation (5.13) is satisfied with f_n and x_n in place of f and x. By equation (5.14), this sequence satisfies, for each $n \in \mathbb{N}$,

$$E(f_n) \leq cE(f_{n-1}) \leq \cdots \leq c^n E(f_0). \tag{5.16}$$

Together with equation (5.15), it follows that $d(f_n, f_{n+1}) \leq E(f_n) \leq c^n E(f_0)$. As $0 < c < 1$, this implies that $(f_n)_{n\geq 1}$ is Cauchy when considered a sequence in $\ell^p(L(S); X)$. Let f_∞ denote its limit in $\ell^p(L(S); X)$. Moreover, as $0 < c < 1$, we see from equation (5.16) that $(E(f_n))_{n\geq 0}$ is a strictly decreasing sequence in $[0, \infty)$, so as $E : \ell^p(L(S); X) \to \mathbb{R}$ is continuous, we have $E(f_\infty) = \lim_{n\to\infty} E(f_n) = 0$. Hence, $\|f_\infty(s) - f_\infty(t)\| = 0$, for all pairs $(s,t) \in E(S)$. As $L(S)$ is connected, we deduce that f_∞ is constant. Let $x \in X$ be the constant value of f_∞. Because no vertex of $L(S)$ is isolated, and because S is symmetric, we may, for each $s \in S$, find a $t \in S$ such that $st \in S$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}\|s.x - x\| &= \|s.f_\infty(t) - f_\infty(st)\| \\ &\leq \|s.f_\infty(t) - s.f_n(t)\| + \|f_n(st) - f_\infty(st)\| \\ &\leq 2\,\|f_\infty - f_n\|_{\ell^p(L(S);X)}\,,\end{aligned}$$

and it follows that x is fixed by S. Since S is a generating set, we conclude that x is a fixed point for the action by Γ. Hence, Γ has Property (F_X). □

5.3.3 From small Markov operators to p-Poincaré inequalities for L^p-spaces

To obtain a spectral condition for Property (F_X) from Theorem 5.3.4 with the same flavor as that of Żuk's criterion, what we need is a condition on the spectrum of the link graph which ensures the validity of inequality (5.12) with a constant strictly less than one. Recall that the spectrum of a finite graph $\mathcal{G}$ is closely related to that of its Markov operator – we saw in the proof of Corollary 5.1.14 that $\sigma(\mathcal{G}) \subset 1 - \sigma(M_\mathcal{G})$. With the assumption that $\mathcal{G}$ is undirected, the spectrum of $M_\mathcal{G}$ is real and contained in $[-1,1]$. If, moreover, $\mathcal{G}$ is connected then 1 is a simple eigenvalue and the eigenspace in $\ell^p(\mathcal{G};X)$ is the subspace of constant functions. It is clear that a spectral gap of $\mathcal{G}$ strictly greater than δ is ensured by the condition that $\sigma(M_\mathcal{G}) \subset [-1, 1-\delta) \cup \{1\}$. A stronger condition is to require that the restriction of the Markov operator to $\ell^p_0(\mathcal{G};X)$ has norm bounded by some $0 < \varepsilon < 1$, or, equivalently, that its spectrum away from the simple eigenvalue 1 is contained in $[-\varepsilon,\varepsilon]$. Such a condition leads to a 2-sided spectral gap of the Laplace operator. When X is *p-uniformly convex*, which we define in a moment, we shall see in Theorem 5.3.14 that a (large enough) 2-sided spectral gap leads to the validity of a p-Poincaré inequality with constant strictly less than one. With this discussion in mind, consider the following lemma:

Lemma 5.3.5. *Let $\mathcal{G}$ be a finite graph and let X be a Banach space. If the restriction of the Markov operator to $\ell^p_0(\mathcal{G};X)$ has norm bounded by $0 < \varepsilon < 1$ then*

$$\|f\|_{\ell^p(\mathcal{G};X)} \leq \frac{1}{1-\varepsilon}\,\|\Delta f\|_{\ell^p(\mathcal{G};X)}\,, \tag{5.17}$$

for all $f \in \ell^p_0(\mathcal{G};X)$.

Proof. Let $f \in \ell^p_0(\mathcal{G};X)$. Recall that $\Delta = I_\mathcal{G} - M_\mathcal{G}$. By the triangle inequality,

$$\|f\|_{\ell^p(\mathcal{G};X)} \leq \|\Delta f\|_{\ell^p(\mathcal{G};X)} + \|M_\mathcal{G} f\|_{\ell^p(\mathcal{G};X)} \leq \|\Delta f\|_{\ell^p(\mathcal{G};X)} + \varepsilon\,\|f\|_{\ell^p(\mathcal{G};X)}\,.$$

The claimed inequality follows by rearrangement. □

Recalling equations (5.4) and (5.11), we see that a bound on the norm of Δf in terms of the norm of ∇f will turn equation (5.17) into a Poincaré type inequality. In fact, we always have such a bound: Let $f \in \ell^p(\mathcal{G};X)$. For any fixed $s \in V(\mathcal{G})$, the triangle inequality yields

$$\left\| f(s) - \frac{1}{\deg(s)}\sum_{t\sim s} f(t)\right\| \leq \frac{1}{\deg(s)}\sum_{t\sim s}\|f(s) - f(t)\|.$$

This together with Hölder's inequality on $\ell^1(E(\mathcal{G});X)$ yields

$$
\begin{aligned}
|E(\mathcal{G})|^{1/p}\,\|\Delta f\|_{\ell^p(\mathcal{G};X)} &\leq \sum_{s\in V(\mathcal{G})} \left\| f(s) - \frac{1}{\deg(s)}\sum_{t\sim s} f(t)\right\| \deg(s)^{1/p} \\
&\leq \sum_{s\in V(\mathcal{G})}\sum_{t\sim s} \|f(s)-f(t)\| \deg(s)^{1/p-1} \\
&\leq |E(\mathcal{G})|\,\|\nabla f\|_{\ell^1(E(\mathcal{G});X)} \leq |E(\mathcal{G})|\,\|\nabla f\|_{\ell^1(E(\mathcal{G});X)}\,.
\end{aligned}
$$

Since $|E(\mathcal{G})| \geq 1$, it follows that $\|\Delta f\|_{\ell^p(\mathcal{G};X)} \leq \|\nabla f\|_{\ell^p(\mathcal{G};X)}$. This, however, is not strong enough, because we need a p-Poincaré inequality with constant strictly less than one. We shall see in Proposition 5.3.12 that an improvement of this inequality is possible when X is a p-uniformly convex Banach space. For a Banach space X, denote by $\delta_X : (0,2] \to [0,1]$ the modulus of convexity as defined in equation (A.2).

Definition 5.3.6. A uniformly convex Banach space X is said to be *p-uniformly convex*, for a fixed $2 \leq p < \infty$, if there exists a constant $c > 0$ such that $\delta_X(\varepsilon) \geq c\varepsilon^p$.

Remark 5.3.7. There is nothing preventing us from defining p-uniform convexity for p between 1 and 2. However, this definition would be empty as no p-uniformly convex spaces exist, for $1 < p < 2$.

Example 5.3.8. The L^p-spaces are p-uniformly convex, for $p \geq 2$. ◦

The following useful reformulation of p-uniform convexity is due to [3, Proposition 7]. We state it here without proof.

Lemma 5.3.9. *A uniformly convex Banach space X is p-uniformly convex, for a fixed $2 \leq p < \infty$, if and only if there exists a constant $K \geq 1$ such that*

$$
\|x\|^p + \frac{1}{K^p}\|y\|^p \leq \frac{\|x+y\|^p + \|x-y\|^p}{2}, \tag{5.18}
$$

for all $x, y \in X$. The infimum over all $K \geq 1$ such that equation (5.18) holds, for all $x, y \in X$, is denoted by $K_p(X)$.

The improvement of the inequality $\|\Delta f\|_{\ell^p(\mathcal{G};X)} \leq \|\nabla f\|_{\ell^p(\mathcal{G};X)}$ for p-uniformly convex Banach spaces is based on an improvement of the inequality $\|\mathbb{E}(f)\| \leq \|f\|_{\ell^p(\mathcal{G};X)}$. We state and proof the improvement of this basic inequality of the Bochner integral in the setting of Banach space valued functions on a general probability space (Ω,ν). Here, the mean is given by the integral $\mathbb{E}(f) = \int_\Omega f\,\mathrm{d}\nu$. The proof is due to G. Pisier in [35, Proposition 2.4]. We follow the proof in [31, Lemma 6.5].

Proposition 5.3.10. *Let X be a p-uniformly convex Banach space, for $p \geq 2$, and let (Ω,ν) be a probability space. There exists a $C > 0$ such that, for every $f \in L^p(\Omega,\nu;X)$,*

$$
\|\mathbb{E}(f)\|^p + C\,\|f - \mathbb{E}(f)\|_p^p \leq \|f\|_p^p\,. \tag{5.19}
$$

Proof. Define a constant C depending only on p and X by

$$
C = \inf\left\{ \frac{\|f\|_p^p - \|\mathbb{E}(f)\|^p}{\|f-\mathbb{E}(f)\|_p^p} \;\middle|\; f \in L^p(\Omega,\nu;X),\ \|f-\mathbb{E}(f)\|_p^p > 0 \right\}.
$$

Then C is non-negative by Jensen's inequality. Fix $C' > C$ and take $f \in L^p(\Omega,\nu;X)$ with $\|f - \mathbb{E}(f)\|_p^p > 0$ such that

$$
C'\,\|f-\mathbb{E}(f)\|_p^p > \|f\|_p^p - \|\mathbb{E}\,f\|^p\,.
$$

Fix $K > K_p(X)$. Inserting the vectors $\frac{1}{2}(f + \mathbb{E}(f))$ and $\frac{1}{2}(f - \mathbb{E}(f))$ into equation (5.18) yields the inequality

$$2^{1-p} \|f + \mathbb{E}(f)\|^p + \frac{2^{1-p}}{K^p} \|f - \mathbb{E}(f)\|^p \leq \|f\|^p + \|E(f)\|^p .$$

Putting these inequalities together, we derive that

$$\begin{aligned} C' \|f - \mathbb{E}(f)\|_p^p &> \int \Big(2^{1-p} \|f + \mathbb{E}(f)\|^p + \frac{2^{1-p}}{K^p} \|f - \mathbb{E}(f)\|^p - 2 \|\mathbb{E}(f)\|^p \Big) \, \mathrm{d}\nu \\ &= 2^{1-p} \left(\|f + \mathbb{E}(f)\|_p^p - \|2\,\mathbb{E}(f)\|^p \right) + \frac{2^{1-p}}{K^p} \|f - \mathbb{E}(f)\|_p^p . \end{aligned}$$

Further, by definition of C, we have

$$\|f + \mathbb{E}(f)\|_p^p - \|2\,\mathbb{E}(f)\|^p = \|f + \mathbb{E}(f)\|_p^p - \left\| \int (f + \mathbb{E}(f)) \, \mathrm{d}\nu \right\|^p \geq C \|f - \mathbb{E}(f)\|_p^p .$$

Inserting this into the inequality above, we get

$$C' \|f - \mathbb{E}(f)\|_p^p > 2^{1-p} \left(C + \frac{1}{K^p} \right) \|f - \mathbb{E}(f)\|_p^p .$$

By assumption on f, it follows that $C' > 2^{1-p}(C + 1/K^p)$. As $K > K_p(X)$ and $C' > C$ were arbitrary, we may replace K with $K_p(X)$ and C' with C. Rearranging, it follows that

$$C > \frac{1}{(2^{p-1} - 1) K_p(X)^p} .$$

In particular, $C > 0$, and it is clear by construction that equation (5.19) is satisfied with this constant, for all $f \in L^p(\Omega, \nu; X)$. □

Proposition 5.3.10 holds, in particular, when X is an L^p-space, for $p \geq 2$. In this case, one can proof inequality (5.19) using interpolation theory. This yields a more explicit constant than what we obtained for general p-uniformly convex spaces.

Proposition 5.3.11. *Let $p \geq 2$, let $L^p(\mu)$ be an L^p-space, and let (Ω, ν) be a probability space. Then equation (5.19) holds, for all $f \in L^p(\Omega, \nu; L^p(\mu))$, with $C = 2^{2-p}$.*

Proof. Consider the sum space $L^\infty(\mu) + L^2(\mu)$, and recall from Proposition B.1.1 that this is a Banach space. Define, for each $z \in S$, an operator

$$T_z : L^1(\Omega, \nu; L^\infty(\mu) + L^2(\mu)) \longrightarrow \big(L^\infty(\mu) + L^2(\mu) \big) \oplus L^1\big(\Omega, \nu; L^\infty(\mu) + L^2(\mu)\big)$$

by setting

$$T_z(f) = \big(\mathbb{E}(f) \, , \, 2^{z-1} \big(f - \mathbb{E}(f) \big) \big) .$$

The direct sum in the codomain of T_z is, a priori, algebraic. We may equip this direct sum with any p-norm to consider T_z an operator between normed spaces; we shall decorate the symbol $\oplus$ accordingly. Regardless of the chosen norm, it is clear that the resulting operators $(T_z)_{z \in S}$ will constitute an admissible family of bounded linear operators.

For any $2 \leq q \leq \infty$, Proposition B.1.6 and Theorem B.2.1 implies that $L^q(\mu)$ is embedded continuously into $L^\infty(\mu) + L^2(\mu)$. Since (Ω, ν) is a probability space, it follows that

$$L^q(\Omega, \nu; L^q(\mu)) \hookrightarrow L^1\big(\Omega, \nu; L^\infty(\mu) + L^2(\mu)\big),$$

where the embedding is continuous. We may therefore consider T_z as an operator on any of the Bochner spaces on the left-hand side above. Observe that, for $2 \leq q \leq \infty$ and $f \in L^q\big(\Omega, \nu; L^q(\mu)\big)$, Hölder's inequality yields the following bound on the norm of the Bochner integral of f:

$$\|\mathbb{E}(f)\|_{L^q(\Omega,\nu;L^q(\mu))} = \|\mathbb{E}(f)\|_{L^q(\mu)} \leq \int \|f(\omega)\|_{L^q(\mu)} \, \mathrm{d}\nu(\omega) \leq \|f\|_{L^q(\Omega,\nu;L^q(\mu))} . \tag{5.20}$$

It follows that the codomain of T_z, as an operator on $L^q\big(\Omega, \nu; L^q(\mu)\big)$ may be restricted to $L^q(\mu) \oplus_q L^q\big(\Omega, \nu; L^q(\mu)\big)$.

By Theorem B.2.1, we have the following isometric isomorphisms:

$$\begin{gathered}[L^\infty\big(\Omega, \nu; L^\infty(\mu)\big), L^2\big(\Omega, \nu; L^2(\mu)\big)]_{2/p} \cong L^p\big(\Omega, \nu; L^p(\mu)\big), \\ [L^\infty(\mu) \oplus_\infty L^\infty\big(\Omega, \nu; L^\infty(\mu)\big), L^2(\mu) \oplus_2 L^2\big(\Omega, \nu; L^2(\mu)\big)]_{2/p} \cong L^p(\mu) \oplus_p L^p\big(\Omega, \nu; L^p(\mu)\big).\end{gathered}$$

For $\mathrm{Re}(z) = 0$, consider T_z an operator on $L^\infty\big(\Omega, \nu; L^\infty(\mu)\big)$. For each $t \in \mathbb{R}$ and each $f \in L^\infty\big(\Omega, \nu; L^\infty(\mu)\big)$, we derive from equation (5.20) that

$$\begin{aligned}\|T_{it} f\|_{L^\infty(\mu)\oplus L^\infty(\Omega,\nu;L^\infty(\mu))} &= \max\left\{\|\mathbb{E}(f)\|_{L^\infty(\mu)} \, , \frac{1}{2} \operatorname*{ess\,sup}_{\omega\in\Omega} \|f(\omega) - \mathbb{E}(f)\|_{L^\infty(\mu)}\right\} \\ &\leq \max\left\{\|\mathbb{E}(f)\|_{L^\infty(\mu)} \, , \frac{1}{2} \|f\|_{L^\infty(\Omega,\nu;L^\infty(\mu))} + \frac{1}{2} \|\mathbb{E}(f)\|_{L^\infty(\mu)}\right\} \\ &\leq \|f\|_{L^\infty(\Omega,\nu;L^\infty(\mu))} .\end{aligned}$$

Hence, for all $t \in \mathbb{R}$,

$$\|T_{it}\|_{L^\infty(\Omega,\nu;L^\infty(\mu)) \to L^\infty(\mu)\oplus L^\infty(\Omega,\nu;L^\infty(\mu))} \leq 1.$$

For $\mathrm{Re}(z) = 1$, consider T_z an operator on the Hilbert space $L^2\big(\Omega, \nu; L^2(\mu)\big)$. For each $f \in L^2\big(\Omega, \nu; L^2(\mu)\big)$, it is straight forward to verify that f and $f - \mathbb{E}(f)$ are orthogonal. An application of Pythagoras then yields that

$$\begin{aligned}\|T_{it+1} f\|^2_{L^2(\mu)\oplus L^2(\Omega,\nu;L^2(\mu))} &= \|\mathbb{E}(f)\|^2_{L^2(\mu)} + \|f - \mathbb{E}(f)\|^2_{L^2(\Omega,\nu;L^2(\mu))} \\ &= \|\mathbb{E}(f)\|^2_{L^2(\Omega,\nu;L^2(\mu))} + \|f - \mathbb{E}(f)\|^2_{L^2(\Omega,\nu;L^2(\mu))} \\ &= \|f\|^2_{L^2(\Omega,\nu;L^2(\mu))} .\end{aligned}$$

Hence, for all $t \in \mathbb{R}$,

$$\|T_{it+1}\|_{L^2(\Omega,\nu;L^2(\mu)) \to L^2(\mu)\oplus L^2(\Omega,\nu;L^2(\mu))} = 1.$$

All requirements for Stein's interpolation theorem, Theorem B.2.4, are then fulfilled, and we deduce that $T_{2/p}$ is bounded as an operator on $L^p\big(\Omega, \nu; L^p(\mu)\big) \to L^p(\mu) \oplus L^p\big(\Omega, \nu; L^p(\mu)\big)$. Going through the proof of Theorem B.2.4, we see that

$$\big\|T_{2/p}\big\|_{L^p(\Omega,\nu;L^p(\mu)) \to L^p(\mu)\oplus L^p(\Omega,\nu;L^p(\mu))} \leq 1. \tag{5.21}$$

For each $f \in L^p\big(\Omega, \nu; L^p(\mu)\big)$, we have

$$\begin{aligned}\big\|T_{2/p} f\big\|^p_{L^p(\mu)\oplus L^p(\Omega,\nu;L^p(\mu))} &= \|\mathbb{E}(f)\|^p_{L^p(\mu)} + \Big\|2^{2/p-1}\big(f - \mathbb{E}(f)\big)\Big\|^p_{L^p(\Omega,\nu;L^p(\mu))} \\ &= \|\mathbb{E}(f)\|^p_{L^p(\mu)} + 2^{2-p} \int \|f - \mathbb{E}(f)\|^p_{L^p(\mu)} \, \mathrm{d}\nu.\end{aligned}$$

Hence, inequality (5.21) is exactly the inequality of equation (5.19) with $C = 2^{2-p}$. $\square$

Proposition 5.3.12. *Let $p \geq 2$ and let X be a p-uniformly convex Banach space. Let $\mathcal{G}$ be a finite simple unoriented graph. There exists a $C > 0$ such that, for every $f \in \ell_0^p(\mathcal{G}; X)$,*

$$\|\Delta f\|_{\ell^p(\mathcal{G};X)}^p \leq \|\nabla f\|_{\ell^p(E(\mathcal{G});X)}^p - \frac{C}{|E(\mathcal{G})|} \sum_{(s,t)\in E(\mathcal{G})} \|f(s) - M_{\mathcal{G}} f(t)\|^p. \tag{5.22}$$

Proof. Let $f \in \ell_0^p(\mathcal{G}; X)$. For each fixed $t \in V(\mathcal{G})$, we may consider the function $\nabla f(\,\cdot\,, t)$ an element of $\ell^p(N_t; X)$ – the ℓ^p-space of X-valued functions on N_t with N_t equipped with the uniform probability measure. Denote by $\mathbb{E}$ the mean operator on $\ell^p(N_t; X)$. Then

$$\mathbb{E}(\nabla f(\,\cdot\,, t)) = \frac{1}{\deg(t)} \sum_{s\sim t} \nabla f(s,t) = f(t) - \frac{1}{\deg(t)} \sum_{s\sim t} f(s) = f(t) - M_{\mathcal{G}} f(t) = \Delta f(t),$$

and

$$\begin{aligned} \|\nabla f(\,\cdot\,, t) - \mathbb{E}(\nabla f(\,\cdot\,, t))\|_{\ell^p(N_t;X)}^p &= \frac{1}{\deg(t)} \sum_{s\sim t} \left\|\nabla f(s,t) - \big(f(t) - M_{\mathcal{G}} f(t)\big)\right\|^p \\ &= \frac{1}{\deg(t)} \sum_{s\sim t} \|M_{\mathcal{G}} f(t) - f(s)\|^p . \end{aligned}$$

Since $V(\mathcal{G})$ is finite, we can take $C > 0$ such that equation (5.19) of Proposition 5.3.10 holds on $\ell^p(N_t; X)$, for all $t \in V(\mathcal{G})$. Then

$$\|\mathbb{E}(\nabla f(\,\cdot\,, t))\|^p + C \,\|\nabla f(\,\cdot\,, t) - \mathbb{E}(\nabla f(\,\cdot\,, t))\|_{\ell^p(N_t;X)}^p \leq \|\nabla f(\,\cdot\,, t)\|_{\ell^p(N_t;X)}^p .$$

Inserting our above preparations, this inequality reads,

$$\|\Delta f(t)\|^p + \frac{C}{\deg(t)} \sum_{s\sim t} \|f(s) - M_{\mathcal{G}} f(t)\|^p \leq \frac{1}{\deg(t)} \sum_{s\sim t} \|\nabla f(s,t)\|^p . \tag{5.23}$$

Each term in this inequality is a function of t, and may, as such, be viewed as elements of $\ell^p(\mathcal{G})$. We integrate each term over t separately:

$$\frac{1}{|E(\mathcal{G})|} \sum_{t\in V(\mathcal{G})} \|\Delta f(t)\|^p \deg(t) = \|\Delta f\|_{\ell^p(\mathcal{G};X)}^p ,$$

$$\frac{1}{|E(\mathcal{G})|} \sum_{t\in V(\mathcal{G})} \left(\frac{1}{\deg(t)} \sum_{s\sim t} \|f(s) - M_{\mathcal{G}} f(t)\|^p \right) \deg(t) = \frac{1}{|E(\mathcal{G})|} \sum_{(s,t)\in E(\mathcal{G})} \|f(s) - M_{\mathcal{G}} f(t)\|^p ,$$

$$\frac{1}{|E(\mathcal{G})|} \sum_{t\in V(\mathcal{G})} \left(\frac{1}{\deg(t)} \sum_{s\sim t} \|\nabla f(s,t)\|^p \right) \deg(t) = \|\nabla f\|_{\ell^p(E(\mathcal{G};X))}^p .$$

Inserting this into equation (5.23) and rearranging yields equation (5.22). □

Remark 5.3.13. Let $\mathcal{G}$ and X be as in Proposition 5.3.12 above. Denote by $\pi_k : E(\mathcal{G}) \to V(\mathcal{G})$ the k'th projection, for $k = 1, 2$. For each function $f \in \ell^p(\mathcal{G}; X)$, the composition $f \circ \pi_k$ lies in $\ell^p(E(\mathcal{G}); X)$ and has norm equal to the norm of f. An application of the triangle inequality on $\ell^p(E(\mathcal{G}); X)$ yields that

$$\begin{aligned} \left(\frac{1}{|E(\mathcal{G})|} \sum_{(s,t)\in E(\mathcal{G})} \|f(s) - M_{\mathcal{G}} f(t)\|^p \right)^{1/p} &= \|f \circ \pi_1 - M_{\mathcal{G}} f \circ \pi_2\|_{\ell^p(E(\mathcal{G});X)} \\ &\geq \|f\|_{\ell^p(\mathcal{G};X)} - \|M_{\mathcal{G}} f\|_{\ell^p(\mathcal{G};X)} . \end{aligned}$$

Inserting this into equation (5.22) gives us the inequality

$$\|\Delta f\|_p^p \leq \|\nabla f\|_p^p - C\big(\|f\|_{\ell^p(\mathcal{G};X)} - \|M_{\mathcal{G}} f\|_{\ell^p(\mathcal{G};X)}\big)^p. \tag{5.24}$$

Theorem 5.3.14. *Let X be a p-uniformly convex Banach space, for $p \geq 2$. There exist $\varepsilon, \delta > 0$ (depending on X) such that the following holds: If $\mathcal{G}$ is a finite simple undirected and connected graph with $\|M_{\mathcal{G}}\|_{B(\ell_0^p(\mathcal{G};X))} \leq \varepsilon$ then*

$$\|f\|_{\ell^p(\mathcal{G};X)} \leq (1-\delta) \|\nabla f\|_{\ell^p(E(\mathcal{G});X)}, \tag{5.25}$$

for all $f \in \ell_0^p(\mathcal{G};X)$.

Proof. Suppose the norm of $M_{\mathcal{G}}$ restricted to $\ell_0^p(\mathcal{G};X)$ is bounded by some $0 < \varepsilon < 1$. Let $f \in \ell_0^p(\mathcal{G};X)$. By Lemma 5.3.5, Proposition 5.3.12, and Remark 5.3.13, we have that

$$\|f\|^p_{\ell^p(\mathcal{G};X)} \leq \frac{1}{(1-\varepsilon)^p} \|\Delta f\|^p_{\ell^p(\mathcal{G};X)} \leq \frac{1}{(1-\varepsilon)^p} \|\nabla f\|^p_{\ell^p(E(\mathcal{G});X)} - C \|f\|^p_{\ell^p(\mathcal{G};X)},$$

where $C > 0$ is a constant depending only on p and X. Rearranging and taking the p'th root, we obtain the inequality

$$\|f\|_{\ell^p(\mathcal{G};X)} \leq \frac{1}{(1-\varepsilon)(1+C)^{1/p}} \|\nabla f\|_{\ell^p(E(\mathcal{G});X)}. \tag{5.26}$$

If $\varepsilon < 1 - \frac{1}{(1+C)^{1/p}}$, the constant in the above inequality is strictly smaller than 1. The theorem then follows with $\delta = 1 - \frac{1}{(1-\varepsilon)(1+C)^{1/p}}$. $\square$

Remark 5.3.15. Recall from Proposition 5.3.11 that, when X is an L^p-space, we may choose $C = 2^{2-p}$. Hence, any $\varepsilon < 1 - \frac{1}{(1+2^{2-p})^{1/p}}$ will suffice in Theorem 5.3.14. We remark further that the strict inequality $\frac{2}{p2^p} < 1 - \frac{1}{(1+2^{2-p})^{1/p}}$ holds, for all $p \geq 2$. Hence, we may use any $\varepsilon \leq 2/(p2^p)$ in Theorem 5.3.14 when X is an L^p-space with $p \geq 2$.

Lemma 5.3.16. *Let Γ be a discrete finitely generated group, and let S be a finite symmetric generating set not containing the identity and such that the associated link graph, $L(S)$, is connected. Let $2 \leq p < \infty$. If, for any L^p-space X,*

$$\|M_S \otimes \mathrm{id}_X\|_{B(\ell_0^p(L(S);X))} \leq 2/(p2^p)$$

then Γ has Property (F_{L^p}).

Proof. Fix $p \geq 2$ and suppose Γ admits a finite generating set with the stated properties and such that $\|M_S\|_{B(\ell_0^p(L(S);X))} \leq 2/(p2^p)$, for any L^p-space X. By Theorem 5.3.14 and Remark 5.3.15, we may find a $\delta > 0$ such that equation (5.25) holds. We obtain from this the p-poincaré inequality

$$\inf_{x \in X} \|f - x\|_{\ell^p(L(S);X)} \leq \|f - \mathbb{E}(f)\|_{\ell^p(L(S);X)} \leq (1-\delta) \|\nabla f\|_{\ell^p(E(S);X)},$$

which holds for all $f \in \ell^p(L(S);X)$. Since X is any L^p-space, it follows from Theorem 5.3.4 that Γ has Property (F_{L^p}). $\square$

Lemma 5.3.17. *Let $\mathcal{G}$ be a finite simple undirected graph and let X be a Banach space. If $\|M_{\mathcal{G}}\|_{B(\ell_0^2(\mathcal{G}))} \leq \varepsilon$, for some $\varepsilon > 0$, then*

$$\|M_{\mathcal{G}} \otimes \mathrm{id}_X\|_{B(\ell_0^2(\mathcal{G};X))} \leq 2\Delta_X(\varepsilon/2).$$

Proof. Denote by P the orthogonal projection onto $\ell_0^2(\mathcal{G})$ and consider the operator $M_{\mathcal{G}}P$ on $\ell^2(\mathcal{G})$. By assumption,

$$\|M_{\mathcal{G}}P\|_{B(\ell^2(\mathcal{G}))} = \|M_{\mathcal{G}}\|_{B(\ell_0^2(\mathcal{G}))} \leq \varepsilon.$$

Further, recall that P is given by $Pf = f - \mathbb{E}(f)$, for all $f \in \ell^2(\mathcal{G})$, so as the constant functions are invariant for $M_{\mathcal{G}}$, we see that $M_{\mathcal{G}}P = M_{\mathcal{G}} - \mathbb{E}$. Both $M_{\mathcal{G}}$ and $\mathbb{E}$ are operators on $\ell^2(\mathcal{G})$ with operator norm equal to 1, and their canonical matrix representations have all non-negative entries. Thus, by the triangle inequality and by Lemma 5.3.2,

$$\|M_{\mathcal{G}}P\|_{reg} \leq \|M_{\mathcal{G}}\|_{reg} + \|\mathbb{E}\|_{reg} = \|M_{\mathcal{G}}\|_{B(\ell^2(\mathcal{G}))} + \|\mathbb{E}\|_{B(\ell^2(\mathcal{G}))} = 2.$$

It follows directly from the definition of Δ_X that

$$\|M_{\mathcal{G}} \otimes \mathrm{id}_X\|_{B(\ell_0^2(\mathcal{G};X))} = \|M_{\mathcal{G}}P \otimes \mathrm{id}_X\|_{B(\ell^2(\mathcal{G};X))} \leq 2\Delta_X\left(\varepsilon/2\right),$$

as we wanted to show. □

Theorem 5.3.18. *Let Γ be a discrete finitely generated group, and let S be a finite symmetric generating set not containing the identity and such that the associated link graph, $L(S)$, is connected. Let $2 \leq p < \infty$. If*

$$\|M_S\|_{B(\ell_0^2(L(S)))} \leq \frac{2}{p^{p^2/4}2^{p^3/4}}$$

then Γ has Property (F_{L^p}).

Proof. For each $q \geq 2$, we denote by P the projection onto $\ell_0^q(L(S))$ along the subspace of constant functions, i.e., $Pf = f - \mathbb{E}(f)$, for all $f \in \ell^q(L(S))$. Let X be an L^p-space. If $\|M_S\|_{B(\ell_0^2(L(S)))} \leq \varepsilon$, for some $\varepsilon > 0$, Lemma 5.3.17 and Lemma 5.3.3 gives us that

$$\|M_SP \otimes \mathrm{id}_X\|_{B(\ell^2(L(S);X))} = \|M_S \otimes \mathrm{id}_X\|_{B(\ell_0^2(L(S);X))} \leq 2\left(\varepsilon/2\right)^{2/p} = \varepsilon^{2/p}2^{1-2/p}.$$

As the constant functions are invariant for M_S, we see that $M_SP = M_S - \mathbb{E}$. One easily verifies that $M_S \otimes \mathrm{id}_X$ and $\mathbb{E} \otimes \mathrm{id}_X$ both have norm 1 as operators on $\ell^\infty(L(S);X)$ (without any assumptions on X). Hence,

$$\|M_SP \otimes \mathrm{id}_X\|_{B(\ell^\infty(L(S);X))} \leq \|M_S \otimes \mathrm{id}_X\|_{B(\ell^\infty(L(S);X))} + \|\mathbb{E} \otimes \mathrm{id}_X\|_{B(\ell^\infty(L(S);X))} = 2.$$

By complex interpolation of Bochner space, Theorem B.3.1 and by the Riesz-Thorin theorem for Bochner spaces, Theorem B.3.2, we deduce that $M_SP \otimes \mathrm{id}_X$ is bounded as an operator on $\ell^p(L(S);X)$, and we have the following bound on the norm:

$$\begin{aligned}\|M_SP \otimes \mathrm{id}_X\|_{B(\ell^p(L(S);X))} &\leq \|M_SP \otimes \mathrm{id}_X\|^{2/p}_{B(\ell^2(L(S);X))}\|M_SP \otimes \mathrm{id}_X\|^{1-2/p}_{B(\ell^\infty(L(S);X))} \\ &\leq \varepsilon^{4/p^2}2^{2/p-4/p^2}2^{1-2/p} = \varepsilon^{4/p^2}2^{1-4/p^2}\end{aligned}$$

For $\varepsilon = 2/(p^{p^2/4}2^{p^3/4})$, we obtain the bound

$$\|M_S \otimes \mathrm{id}_X\|_{B(\ell_0^p(L(S);X))} = \|M_SP \otimes \mathrm{id}_X\|_{B(\ell^p(L(S);X))} \leq \frac{2}{p2^p}.$$

The theorem therefore follows from Lemma 5.3.16. □

5.4 Literature

The proof of Żuk's condition for Property (T) presented in Theorem 5.2.11 can be found in [9, Theorem 12.1.15]. Section 5.3 follows the paper by T. de Laat and M. de la Salle [14].

Appendix A

Geometry of Banach spaces

We assume the reader is familiar with ultrapowers. We shall use $\mathcal{U}$ to denote a free ultrafilter on some (infinite) index set.

A.1 Superreflexivity

Definition A.1.1. A Banach space is said to be *superreflexive* if every ultrapower is reflexive.

Remark A.1.2. Superreflexive Banach spaces are, in particular, reflexive. To see this, note that the principal filter on the one-point set is an ultrafilter, and that the ultrapower of any Banach space with respect to this filter equals itself.

We state and proof below some permanence properties of superreflexive Banach spaces. Later, in Theorem A.4.4, we shall present a different characterization of superreflexive Banach spaces in terms of certain properties of the norm.

Proposition A.1.3. *Let X be a superreflexive Banach space. Then any Banach space isomorphic to a closed subspace of X is superreflexive, as well.*

Proof. Let M be a closed subspace of B and let $M_{\mathcal{U}}$ be an ultrapower of M. Then $M_{\mathcal{U}}$ is a closed subspace of $X_{\mathcal{U}}$. Since X is superreflexive, $M_{\mathcal{U}}$ is reflexive, and we deduce that M is superreflexive. The statement follows for any Banach space isomorphic to M, as Banach space isomorphisms lift to ultrapowers and as reflexivity is an isomorphism invariant. □

Theorem A.1.4. *Let X be a Banach space over $\mathbb{K}(= \mathbb{R}$ or $\mathbb{C})$, I an index set and $\mathcal{U}$ a free ultrafilter on I. The map $j : (X^*)_{\mathcal{U}} \to (X_{\mathcal{U}})^*$ given by*

$$j([(f_i)_{i\in I}])([(x_i)_{i\in I}]) = \lim_{\mathcal{U}} f_i(x_i), \qquad \text{for } [(f_i)_{i\in I}] \in (X^*)_{\mathcal{U}} \text{ and } [(x_i)_{i\in I}] \in X_{\mathcal{U}},$$

is a well-defined isometric embedding of $(X^*)_{\mathcal{U}}$ into $(X_{\mathcal{U}})^*$. If, moreover, the ultrapower $X_{\mathcal{U}}$ is reflexive, then j is surjective.

Proof. Let $[(x_i)_{i\in I}] \in B_{\mathcal{U}}$ and $[(f_i)_{i\in I}] \in (X^*)_{\mathcal{U}}$. For every $i \in I$,

$$|f_i(x_i)| \le \|f_i\| \, \|x_i\| \le \left(\sup_{i\in I} \|f_i\|\right)\left(\sup_{i\in I} \|x_i\|\right) < \infty.$$

Hence, the net $(f_i(x_i))_{i\in I}$ is contained in $\overline{B}_{\mathbb{K}}(0, M)$, for $M = \|(f_i)_{i\in I}\|_\infty \cdot \|(x_i)_{i\in I}\|_\infty$. As $\overline{B}_{\mathbb{K}}(0, M)$ is compact, the limit $\lim_{\mathcal{U}} f_i(x_i)$ exists, and a $\mathbb{K}$ is Hausdorff it is unique.

Define a map $j : (X^*)_{\mathcal{U}} \to (X_{\mathcal{U}})^*$ by setting

$$j([(f_i)_{i\in I}])([(x_i)_{i\in I}]) = \lim_{\mathcal{U}} f_i(x_i),$$

for every $[(f_i)_{i\in I}] \in (X^*)_{\mathcal{U}}$ and $[(x_i)_{i\in I}] \in X_{\mathcal{U}}$. It is direct from the linearity of the ultralimit that $j([(f_i)_{i\in I}])$ is a linear map. To see that $j([(f_i)_{i\in I}])$ is bounded, note that $|\cdot| : \mathbb{K} \to \mathbb{R}$ is continuous, and so, $|\lim_{\mathcal{U}} f_i(x_i)| = \lim_{\mathcal{U}} |f_i(x_i)|$. Further, multiplicativity and monotonicity of the ultralimit ensures that

$$\begin{aligned}\left|j([(f_i)_{i\in I}])([(x_i)_{i\in I}])\right| &= \left|\lim_{\mathcal{U}} f_i(x_i)\right| = \lim_{\mathcal{U}} |f_i(x_i)| \leq \lim_{\mathcal{U}} \|f_i\| \, \|x_i\| \\ &= \lim_{\mathcal{U}} \|f_i\| \lim_{\mathcal{U}} \|x_i\| = \left\|[(f_i)_{i\in I}]\right\|_{\mathcal{U}} \left\|[(x_i)_{i\in I}]\right\|_{\mathcal{U}}.\end{aligned}$$

Hence, $\left\|j([(f_i)_{i\in I}])\right\| \leq \left\|[(f_i)_{i\in I}]\right\|_{\mathcal{U}}$.

We proceed to show that j is an isometry. Let $\varepsilon > 0$. For every $i \in I$, take $x_i \in X$ with $\|x_i\| = 1$ such that $|f_i(x_i)| \geq \|f_i\| - \varepsilon$. Then $(x_i)_{i\in I} \in \ell_\infty(I, X)$ and $\left\|[(x_i)_{i\in I}]\right\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|x_i\| = 1$. We may assume that $f_i(x_i)$ is real and positive, for each $i \in I$, (otherwise, multiply x_i with the appropriate phase). Then

$$\begin{aligned}\left\|j([(f_i)_{i\in I}])\right\| &\geq \left|j([(f_i)_{i\in I}])([(x_i)_{i\in I}])\right| = \left|\lim_{\mathcal{U}} f_i(x_i)\right| = \lim_{\mathcal{U}} f_i(x_i) \\ &\geq \lim_{\mathcal{U}}(\|f_i\| - \varepsilon) = \left\|[(f_i)_{i\in I}]\right\|_{\mathcal{U}} - \varepsilon.\end{aligned}$$

It follows that $\left\|j([(f_i)_{i\in I}])\right\| \geq \left\|[(f_i)_{i\in I}]\right\|_{\mathcal{U}}$. Hence, j is an isometry.

Assume now that $X_{\mathcal{U}}$ is reflexive, and suppose for contradiction that $j((X^*)_{\mathcal{U}})$ is a proper closed subspace of $(X_{\mathcal{U}})^*$. By the Hahn-Banach Theorem (see [19, Theorem 5.8(a)]), we may find $\varphi \in (X_{\mathcal{U}})^{**}$ with $\|\varphi\| = 1$ such that the restriction of φ to $j((X^*)_{\mathcal{U}})$ equals zero. If $X_{\mathcal{U}}$ is reflexive, then $\varphi = \hat{x}$, for some $x \in X_{\mathcal{U}}$ with $\|x\|_{\mathcal{U}} = \|\varphi\| = 1$. Let $(x_i)_{i\in I} \in \ell^\infty(I, X)$ be a representative of x. Invoking the Hahn-Banach Theorem once more (see [19, Theorem 5.8(b)]), we find a net $(f_i)_{i\in I}$ in $(X^*)_{\mathcal{U}}$ with $\|f_i\| = 1$ and $f_i(x_i) = \|x_i\|$, for each $i \in I$. Then $f = [(f_i)_{i\in I}]$ is an element of $(X^*)_{\mathcal{U}}$ with $\|f\|_{\mathcal{U}} = \lim_{\mathcal{U}} \|f_i\| = 1$, and we arrive at the contradiction

$$0 = \hat{x}(j(f)) = j(f)(x) = \lim_{\mathcal{U}} f_i(x_i) = \lim_{\mathcal{U}} \|x_i\| = \|x\|_{\mathcal{U}} = 1.$$

Hence, we must have that j is surjective. □

Corollary A.1.5. *A Banach space is superreflexive if and only if its dual is, as well.*

Proof. Suppose X is a superreflexive Banach space. Then $(X^*)_{\mathcal{U}}$ is (isometrically) isomorphic to $(X_{\mathcal{U}})^*$, for any ultrafilter $\mathcal{U}$, by Theorem A.1.4. Since reflexivity is an isomorphism invariant, we deduce that X^* is superreflexive, as well. Conversely, if X^* is superreflexive, then X^{**} is superreflexive, by the above argument. Since superreflexive Banach spaces are, in particular, reflexive (see Remark A.1.2), we may deduce by Proposition A.1.3 that X is superreflexive. □

A.2 Uniform convexity

Definition A.2.1. A Banach space X is said to be *uniformly convex* if, for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that, for all elements $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, it holds that

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta(\varepsilon). \tag{A.1}$$

Observe that if $\delta \geq 0$ is such that inequality (A.1) is satisfied for two given elements $x, y \in X$ of norm 1, then any smaller $\delta' \geq 0$ also satisfies this inequality for these elements, and the biggest $\delta \geq 0$ can be is $1 - \|(x+y)/2\|$. This leads us to define the *modulus of* *convexity* as the biggest $\delta(\varepsilon)$ satisfying inequality (A.1) for all norm 1 elements of X at least ε apart:

$$\delta_X(\varepsilon) = \inf\left\{ 1 - \left\| \frac{x+y}{2} \right\| \;\middle|\; \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon \right\}. \tag{A.2}$$

Remark A.2.2. The modulus of convexity is a monotonically increasing function in $\varepsilon > 0$. Indeed, as ε increases, the infimum is taken over a smaller set. Further, we remark that X is uniformly convex if and only if the modulus of convexity is strictly positive, for all $\varepsilon > 0$.

Lemma A.2.3. *Let X be a Banach space. Then X is uniformly convex if and only if, for all $r > 0$ and for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that, whenever $x, y \in X$ with $\|x\|, \|y\| \leq r$ are such that $\|x - y\| \geq \varepsilon$, it holds that*

$$\left\| \frac{x+y}{2} \right\| \leq r - \delta(\varepsilon).$$

Proof. It is clear that the stated condition implies that X is uniformly convex. Suppose, conversely, that X is uniformly convex. Set $r = 1$ and fix $\varepsilon > 0$. Put $\varepsilon_0 = \min\{\varepsilon/4, \frac{1}{2}\delta_X(\varepsilon/2), 1\}$. Then $\varepsilon - 2\varepsilon_0 \geq \varepsilon/2$ so that $\delta_X(\varepsilon - 2\varepsilon_0) \geq \delta_X(\varepsilon/2) > \varepsilon_0$. Set

$$\delta(\varepsilon) = \min\{\varepsilon_0/2, \delta_X(\varepsilon - 2\varepsilon_0) - \varepsilon_0\}.$$

Then $0 < \delta(\varepsilon) < 1$. Let $x, y \in X$ elements with $\|x\|, \|y\| \leq 1$ and such that $\|x - y\| \leq \varepsilon$. If either $\|x\| \leq 1 - \varepsilon_0$ or $\|y\| \leq 1 - \varepsilon_0$ (or both) then

$$\left\| \frac{x+y}{2} \right\| \leq 1 - \frac{\varepsilon_0}{2} \leq 1 - \delta(\varepsilon).$$

Suppose therefore that $x, y \in X$ are such that $1 - \varepsilon_0 \leq \|x\|, \|y\| \leq 1$. Set $x' = x / \|x\|$ and $y' = y / \|y\|$. We have $\|x' - x\| = 1 - \|x\| \leq \varepsilon_0$, and similarly for y. Hence,

$$\left\|x' - y'\right\| \geq \|x - y\| - \left\|x' - x\right\| - \left\|y' - y\right\| \geq \varepsilon - 2\varepsilon_0.$$

By construction, $\varepsilon - 2\varepsilon_0 > 0$, and so, uniform convexity implies that $\|(x' + y')/2\| \leq 1 - \delta_X(\varepsilon - 2\varepsilon_0)$. It follows that

$$\left\| \frac{x+y}{2} \right\| \leq \frac{1}{2}\left(\left\|x - x'\right\| + \left\|x' - y'\right\| + \left\|y' + y\right\|\right) \leq 1 - \left(\delta_X(\varepsilon - 2\varepsilon_0) - \varepsilon_0\right) \leq 1 - \delta(\varepsilon).$$

We conclude that the stated condition holds for $r = 1$. Finally, let $r > 0$ be arbitrary. For $\varepsilon > 0$, let $x, y \in B$ be such that $\|x\|, \|y\| \leq r$ and such that $\|x - y\| \geq \varepsilon$. Set $x' = x/r$ and $y' = y/r$, then $\|x'\|, \|y'\| \leq 1$ and $\|x' - y'\| \geq \varepsilon/r$. By our above derivations, this implies that

$$\left\| \frac{x+y}{2} \right\| = r \left\| \frac{x'+y'}{2} \right\| \leq r - r\delta(\varepsilon/r),$$

where $\delta(\varepsilon)$ is as constructed above. Hence, the stated condition holds for all $r > 0$. □

A.2.1 The duality map

Recall from the Hahn-Banach Theorem (see [19, Theorem 5.8(b)]) that, for each element x of a Banach space X, there exists a linear functional $x^* \in X^*$ with $\|x^*\| = 1$ and $x^*(x) = \|x\|$. A useful fact in uniformly convex Banach spaces, which we show in Proposition A.2.5 below, is that this functional is unique with these properties. In fact, this result is true in the larger class of *strictly convex* Banach spaces defined below.

Definition A.2.4. A Banach space X is said to be *strictly convex* if, whenever $x, y \in X$ with $\|x\| = \|y\| = 1$ are distinct elements, then $\|(x+y)/2\| < 1$.

Proposition A.2.5. *Let X be a Banach space with a strictly convex dual. For every non-zero $x \in X$ there exists a unique linear functional $x^* \in X^*$ with $\|x^*\| = 1$ and $x^*(x) = \|x\|$.*

Proof. The existence of such a functional is ensured by the Hahn-Banach Theorem (see [19, Theorem 5.8(b)]). To show uniqueness, assume that $\|x\| = 1$ and suppose $\lambda, \lambda' \in X^*$ are both functionals with $\|\lambda\| = \|\lambda'\| = 1$ and $\lambda(x) = \lambda'(x) = \|x\| = 1$. Then

$$\left\| \frac{\lambda + \lambda'}{2} \right\| \geq \frac{\lambda + \lambda'}{2}(x) = \frac{1}{2}\lambda(x) + \frac{1}{2}\lambda'(x) = 1.$$

Thus, if X^* is strictly convex, we must have $\lambda = \lambda'$. Finally, if $x \in X$ is any non-zero element and λ is a linear functional with $\|\lambda\| = 1$ and $\lambda(x) = \|x\|$, then $\lambda(x/\|x\|) = 1$. Hence, uniqueness of the functional associated to x follows from uniqueness of the functionals associated to the unit elements. □

For a strictly convex Banach space X, we shall refer to the map $^*(\,\cdot\,) : X \to X^*$ which maps each element $x \in X$ to the unique functional $x^* \in X^*$ with $\|x^*\| = 1$ and $x^*(x) = \|x\|$ as the *duality map*. We remark that this map may be defined for all Banach spaces; only in general it is a set-valued map. Returning to uniformly convex Banach spaces, we shall end our short discussion on this map by showing a useful continuity property in Proposition A.2.7 below.

Lemma A.2.6. *Let X be a Banach space, let $x, y \in X$ and let $x^*, y^* \in X^*$ denote functionals with $\|x^*\| = \|y^*\| = 1$, $x^*(x) = \|x\|$ and $y^*(y) = \|y\|$. If $\|x\| = \|y\| = 1$ then*

$$\|x^* + y^*\| + \|x - y\| \geq 2. \tag{A.3}$$

Proof. For any $x, y \in X$, we have

$$(x^* + y^*)(x + y) + (x^* - y^*)(x - y) = 2(\|x\| + \|y\|).$$

Therefore, if $\|x\| = \|y\| = 1$, we see that

$$\begin{aligned} 4 &= (x^* + y^*)(x + y) + (x^* - y^*)(x - y) \\ &\leq \|x^* + y^*\| \, \|x + y\| + \|x^* - y^*\| \, \|x - y\| \\ &\leq 2\,\|x^* + y^*\| + 2\,\|x - y\| \,. \end{aligned}$$

Inequality (A.3) follows by rearrangement. □

Proposition A.2.7. *Let X be a Banach space. If X^* is uniformly convex, the duality map $^*(\,\cdot\,) : X \to X^*$ is uniformly continuous on the unit sphere of X.*

Proof. Let $\varepsilon > 0$. Since X^* is uniformly convex, we may take $\delta > 0$ such that, whenever $f, f' \in X^*$ with $\|f\| = \|f'\| = 1$ are such that $\|f + f'\| > 2 - \delta$ then $\|f - f'\| < \varepsilon$. For each pair of elements $x, y \in X$ with $\|x\|, \|y\| = 1$ such that $\|x - y\| < \delta$, Lemma A.2.6 yields that

$$\|x^* + y^*\| \geq 2 - \|x - y\| > 2 - \delta. \tag{A.4}$$

By the choice of $\delta > 0$, this implies that $\|x^* - y^*\| < \varepsilon$. Hence, the duality map is uniformly continuous on the unit sphere of X. □

A.3 Uniform smoothness

Uniform convexity is not preserved under taking duals, but it does have a dual property: *uniform smoothness.* We define this property below and proof that uniform convexity and uniform smoothness are, indeed, duals of each other.

Definition A.3.1. Let X be a Banach space. The *modulus of smoothness* of X is the function $\rho_B : (0, \infty) \to [0, \infty)$ defined, for each $t > 0$, by

$$\rho_B(t) = \tfrac{1}{2} \sup \{ \|x + ty\| + \|x - ty\| - 2 \mid \|x\| = \|y\| = 1 \} .$$

The Banach space X is said to be *uniformly smooth* if $\rho_X(t)/t \to 0$, as $t \to 0$.

Remark A.3.2. The definition of the modulus of smoothness is equivalent to the following:

$$\rho_X(t) = \tfrac{1}{2} \sup \{ \|x + ty\| + \|x - ty\| - 2 \mid \|x\| = 1, \|y\| \leq 1 \} .$$

In particular, it follows that ρ_X is a monotonously increasing function. Further, ρ_X is bounded above by the identity function on $(0, \infty)$, and so, or every $t > 0$, $\rho_X(t)/t$ lies in the interval $[0, 1]$.

The following lemma, which can be found in [4, Lemma 3, p. 208], establishes a connection between the modulus of smoothness of a Banach space and the modulus of convexity of its dual.

Lemma A.3.3. *Let X be a Banach space. For every $t > 0$, the following equalities hold:*

$$\rho_{X^*}(t) = \sup_{0<\varepsilon\leq 2} \left(\frac{t\varepsilon}{2} - \delta_X(\varepsilon) \right), \tag{A.5}$$

$$\rho_X(t) = \sup_{0<\varepsilon\leq 2} \left(\frac{t\varepsilon}{2} - \delta_{X^*}(\varepsilon) \right). \tag{A.6}$$

Proof. Fix $t > 0$. Let $\varepsilon > 0$ and let $x, y \in X$ with $\|x\| = \|y\| = 1$ be such that $\|x - y\| \geq \varepsilon$ and such that $x \neq -y$. By the Hahn-Banach Theorem [19, Theorem 5.8(b)], we may find $f, g \in X^*$ with $\|f\| = \|g\| = 1$ such that $f(x + y) = \|x + y\|$ and $g(x - y) = \|x - y\|$. Then

$$\begin{aligned} 2\rho_{X^*}(t) &\geq \|f + tg\| + \|f - tg\| - 2 \geq |(f + tg)(x)| + |(f - tg)(y)| - 2 \\ &\geq |f(x + y) + tg(x - y)| - 2 \geq \|x + y\| + t\varepsilon - 2. \end{aligned}$$

Rearranging, it follows that $\rho_{X^*}(t) \geq t\varepsilon/2 + \|x + y\|/2 - 1$. Since $x, y \in X$ were arbitrary unit elements at least ε apart, we deduce that $\rho_{X^*}(t) \geq t\varepsilon/2 - \delta_X(\varepsilon)$. Taking the supremum over $0 < \varepsilon \leq 2$ then yields that

$$\rho_{X^*}(t) \geq \sup_{0<\varepsilon\leq 2} \left(\frac{t\varepsilon}{2} - \delta_X(\varepsilon) \right).$$

To show the converse inequality, take $f, g \in X^*$ with $\|f\| = \|g\| = 1$. Let $\eta > 0$. We may find $x, y \in X$ with $\|x\| = \|y\| = 1$ such that $(f+tg)(x) \geq \|f + tg\| - \eta$ and $(f-tg)(y) \geq \|f - tg\| - \eta$. Then we see that

$$\|f + tg\| + \|f - tg\| \leq (f + tg)(x) + (f - tg)(y) + 2\eta \leq \|x + y\| + t\,\|x - y\| + 2\eta.$$

Assume that $x \neq y$ and set $\varepsilon = \|x - y\|$. Then $2 - \|x + y\| \geq 2\delta_X(\varepsilon)$. Inserting this into the above inequality, we get

$$\|f + tg\| + \|f - tg\| \leq 2 - 2\delta_B(\varepsilon) + t\varepsilon + 2\eta \leq 2 + 2 \sup_{0<\varepsilon\leq 2} \left(\frac{t\varepsilon}{2} - \delta_B(\varepsilon) \right) + 2\eta.$$

By definition of the modulus of smoothness, and since $\eta > 0$ was arbitrary, we deduce that

$$\rho_{X^*}(t) \leq \sup_{0<\varepsilon\leq 2} \left(\frac{t\varepsilon}{2} - \delta_X(\varepsilon) \right).$$

The desired equality (A.5) follows.

We proceed to show equation (A.6). Observe first that, since X embeds isometrically into X^{**}, $\rho_X(t) \leq \rho_{X^{**}}(t)$, for all $t > 0$. Hence, we need only show that the left-hand side of (A.6) is greater than or equal to the right-hand side. For this, let $\varepsilon > 0$ and take $f, g \in X^*$ with $\|f\| = \|g\| = 1$ and $\|f - g\| \geq \varepsilon$. Let $\eta > 0$ and take $x, y \in X$ with $\|x\| = \|y\| = 1$ and such that $(f + g)(x) \geq \|f + g\| - \eta$ and $(f - g)(y) \geq \|f - g\| - \eta$. Then

$$\begin{aligned} 2\rho_X(t) &\geq \|x + ty\| + \|x - ty\| - 2 \geq |f(x + ty)| + |g(x - ty)| - 2 \\ &\geq |(f + g)(x) + t(f - g)(y)| - 2 \geq \|f + g\| + t\varepsilon - 2\eta - 2. \end{aligned}$$

Rearranging and recalling that $\eta > 0$ was arbitrary, it follows that $\rho_X(t) \geq t\varepsilon/2 + \|f + g\|/2 - 1$. As $f, g \in X^*$ were arbitrary unit elements at least ε apart, we deduce that $\rho_X(t) \geq t\varepsilon/2 - \delta_{X^*}(\varepsilon)$. Taking the supremum over $0 < \varepsilon \leq 2$ then yields that

$$\rho_X(t) \geq \sup_{0<\varepsilon\leq 2} \left(\frac{t\varepsilon}{2} - \delta_{X^*}(\varepsilon) \right).$$

Equation (A.6) follows. □

Theorem A.3.4. *A Banach space is uniformly convex (uniformly smooth) if and only if its dual is uniformly smooth (uniformly convex).*

Proof. Let X be a uniformly convex Banach space and suppose, for contradiction, that X^* is not uniformly smooth. Then we may find a decreasing sequence of positive numbers $(t_n)_{n\geq 1}$ converging to zero and such that the sequence $(\rho_{X^*}(t_n)/t_n)_{n\geq 1}$ converges to a number $a > 0$. By Lemma A.3.3, it follows that

$$\sup_{0<\varepsilon\leq 2} \left(\frac{\varepsilon}{2} - \frac{\delta_X(\varepsilon)}{t_n} \right) \to a \qquad \text{as } n \to \infty.$$

Take $N \in \mathbb{N}$ such that $\sup_{0<\varepsilon\leq 2}(\varepsilon/2 - \delta_X(\varepsilon)/t_n) > a/2$, for all $n \geq N$. We may then, for each $n \geq N$, take $0 < \varepsilon_n \leq 2$ such that $\varepsilon_n - a > 2\delta_X(\varepsilon_n)/t_n$. Since X is uniformly convex, $\delta_X(\varepsilon_n) > 0$, and so $\varepsilon_n > a$, for all $n \geq N$. On the other hand, we see that $t_n(\varepsilon_n - a) > 2\delta_X(\varepsilon_n)$, by construction of ε_n. The left-hand side of this inequality clearly tends to zero, as n tends to infinity, and then so does the right-hand side. But as δ_X is a monotonously increasing function, it follows that $\delta_X(\varepsilon) = 0$, for $\varepsilon \in (0, a)$. This contradicts that X is uniformly convex. Hence, we conclude that X^* is uniformly smooth.

Let now X be any Banach space such that X^* is not uniformly convex. Then we may find an $0 < \varepsilon_0 \leq 2$ such that $\delta_{X^*}(\varepsilon_0) = 0$. By Lemma A.3.3, it follows that

$$\frac{\rho_X(t)}{t} = \sup_{0<\varepsilon\leq 2} \left(\frac{\varepsilon}{2} - \frac{\delta_{X^*}(\varepsilon)}{t} \right) \geq \frac{\varepsilon_0}{2} - \frac{\delta_{X^*}(\varepsilon_0)}{t} = \frac{\varepsilon_0}{2}.$$

This holds for all $t > 0$, and so, $\rho_X(t)/t$ does not converge to zero. Hence, B is not uniformly smooth, and we conclude that uniform smoothness implies uniform convexity of the dual. □

A.4 Renormings of superreflexive Banach spaces

Uniform convex, respectively uniformly smooth, Banach spaces form a subclass of the class of superreflexive Banach spaces. We start by considering the former. Theorem A.4.1 below, stating that uniform convexity implies reflexivity, was proven independently by D. Milman [32] and B. J. Pettis [33]. We follow the shorter proof given by J. R. Ringrose in [39]. That uniformly convex spaces are, furthermore, superreflexive, is a corollary to this.

Theorem A.4.1 (Milman-Pettis)**.** *Every uniformly convex Banach space is reflexive.*

Proof. Let X be a uniformly convex Banach space, and suppose, for contradiction, that X is not reflexive. Let S_X and $S_{X^{**}}$ denote the closed unit spheres in X and X^{**}, respectively. Denote by $\widehat{S}_X$ the image of S_X in X^{**} and recall that $\widehat{S}_X$ is closed in norm. If X is not reflexive, we may find $\varphi \in X^{**}$ with $\|\varphi\| = 1$ and $\varepsilon > 0$ such that φ is at a distance of at least 2ε from $\widehat{S}_X$. By Goldstine's theorem, any element of $S_{X^{**}}$ is in the weak*-closure of $\widehat{S}_X$. Therefore, if $\varphi \in V \subset X^{**}$ is any weak*-neighborhood of φ, we see that φ is in the weak*-closure of $\widehat{S}_X \cap V$. In particular $\widehat{S}_X \cap V$ is non-empty.

Let $\delta = \delta_X(\varepsilon)$ be the modulus of convexity of X at ε. We may find $f \in X^*$ such that $|\varphi(f) - 1| < \delta$. Define $V = \{\, \psi \in X^{**} \mid |\psi(f) - 1| < \delta \,\}$. Then V is a weak*-neighborhood of φ, and so, it has non-empty intersection with $\widehat{S}_X$. Observe now that any $\psi \in V$ must, in particular, satisfy $\psi(f) > 1-\delta$. Thus, if ψ and ψ' are both in V, we see that $|\psi(f) + \psi'(f)| > 2(1-\delta)$, and therefore, $\|(\psi + \psi')/2\| > 1 - \delta$. As X is uniformly convex, this implies that $\|\psi - \psi'\| < \varepsilon$. Hence, $\widehat{S}_X \cap V \subset \varphi + \varepsilon S_{X^{**}}$. But, as the former set is non-empty, this contradicts that φ is at a distance of at least 2ε from $\widehat{S}_X$. Hence, we must have that X is reflexive. □

Corollary A.4.2. *Every uniformly convex Banach space is superreflexive.*

Proof. Let X be a uniformly convex Banach space. Then X is reflexive, by Theorem A.4.1, and then so is X^*. Let $\mathcal{U}$ be an ultrafilter on some index set. We aim to show that the ultrapower $X_{\mathcal{U}}$ is reflexive. Note first that isomorphisms lift to ultrapowers, and so, we have a canonical isomorphism $X_{\mathcal{U}} \cong (X^{**})_{\mathcal{U}}$. Let $j : (X^*)_{\mathcal{U}} \to (X_{\mathcal{U}})^*$ and $k : (X^{**})_{\mathcal{U}} \to ((X^*)_{\mathcal{U}})^*$ be the isometric isomorphisms of Theorem A.1.4 applied to X and X^*, respectively. Note that the adjoint of j is an isometric isomorphism $j^* : (X_{\mathcal{U}})^{**} \to ((X^*)_{\mathcal{U}})^*$. As usual, we have an isometric embedding of $X_{\mathcal{U}}$ into its double dual. Now, one can easily check that the following diagram commutes:

$$\begin{tikzcd} X_{\mathcal{U}} \arrow[r, hook] \arrow[d, "\wedge"] & (X_{\mathcal{U}})^{**} \arrow[d, "j^*"] \\ (X^{**})_{\mathcal{U}} \arrow[r, "k"] & ((X^*)_{\mathcal{U}})^* \end{tikzcd}$$

Hence, the canonical isometric embedding $X_{\mathcal{U}} \hookrightarrow (X_{\mathcal{U}})^{**}$ is an isometric isomorphism. □

As a corollary to the above corollary, we obtain that uniformly smooth Banach spaces are superreflexive, as well.

Corollary A.4.3. *Every uniformly smooth Banach space is superreflexive.*

Proof. Let X be a uniformly smooth Banach space. Then the dual X^* is uniformly convex, by Theorem A.3.4. Hence, X^* is superreflexive, by Corollary A.4.2, and so, X is superreflexive, by Corollary A.1.5. □

Superreflexive Banach spaces are not, in general, uniformly convex. Indeed, if superreflexivity implied uniform convexity, the class of superreflexive Banach spaces would be nothing but the class of reflexive Banach spaces, which is not the case. A counterexample is given by the closure of the direct sum $\bigoplus_{n\in\mathbb{N}} \ell_n^1$ in the 2-norm. Superreflexive Banach spaces are not, in general, uniformly smooth either. This is because uniform smoothness and uniform convexity are duals of each other. However, a deep result due to Enflo and Pisier, which we present below without proof, shows that superreflexivity can be characterized by having an equivalent norm which is uniformly convex and uniformly smooth.

Theorem A.4.4. *Let X be a Banach space. The following are equivalent:*

(i) X is superreflexive,

(ii) There is an equivalent uniformly convex norm on X,

(iii) There is an equivalent uniformly smooth norm on X,

(iv) There is an equivalent uniformly convex and uniformly smooth norm on X.

A.5 Superreflexivity of L^p

Lemma A.5.1. *Let $a, b \in \mathbb{R}$.*

(i) For every $1 \le p \le 2$, it holds that

$$|a+b|^p + |a-b|^p \ge 2^{p-1}\big(|a|^p + |b|^p\big). \tag{A.7}$$

(ii) For every $2 \le p < \infty$, it holds that

$$|a+b|^p + |a-b|^p \le 2^{p-1}\big(|a|^p + |b|^p\big). \tag{A.8}$$

Proof. We start by remarking, that for $p = 2$, equations (A.7) and (A.8), with the left- and right-hand side being equal, is the parallelogram identity for the Hilbert space $\mathbb{R}$.

For each $x, y \ge 0$ and each $0 < \alpha \le 1$, we claim that the following inequality hold:

$$(x+y)^\alpha \le x^\alpha + y^\alpha. \tag{A.9}$$

The case for $\alpha = 1$ is trivial, so assume $0 < \alpha < 1$. Pulling the larger of x and y out of the parenthesis, we see that it suffices to proof this claim for $y = 1$ and $0 \le x \le 1$. Consider the function $f(x) = 1 + x^\alpha - (1+x)^\alpha$, for $0 \le x \le 1$. The requirement that $0 < \alpha < 1$ ensures that the derivative of f is positive, and so, inequality (A.9) follows as $f(0) = 0$.

For any $1 \le q \le p < \infty$ and any $x, y \in \mathbb{R}$, it follows from inequality (A.9) that

$$\big(|x|^p + |y|^p\big)^{1/p} = \big(|x|^p + |y|^p\big)^{q/pq} \le \big(|x|^q + |y|^q\big)^{1/q}. \tag{A.10}$$

For $1 \le p < 2$, the parallelogram identity on $\mathbb{R}$ together with equation (A.10) yields

$$\big(|a+b|^p + |a-b|^p\big)^{1/p} \ge \big(|a+b|^2 + |a-b|^2\big)^{1/2} = 2^{1/2}\big(|a|^2 + |b|^2\big)^{1/2}.$$

Further, Hölders inequality for the space $\ell_2^{2/p}$ and its conjugate yields

$$|a|^p + |b|^p \le 2^{1-p/2}\big(|a|^2 + |b|^2\big)^{p/2}.$$

Putting this together and taking the p'th power yields inequality (A.7).

For $p \geq 2$, use Hölders inequality for the space $\ell_2^{p/2}$ and its conjugate to derive

$$|a|^2 + |b|^2 \leq 2^{1-2/p}\big(|a|^p + |b|^p\big)^{2/p}.$$

Inequality (A.8) is then obtained analogously to inequality (A.7) using equation (A.10) and the parallelogram identity on $\mathbb{R}$.

□

Using monotonicity of the integral, we obtain the following inequalities, named after J.A. Clarkson, as an immediate corollary to Lemma A.5.1.

Corollary A.5.2 (Clarkson's inequality)**.** *Let $f, g \in L^p(\mu)$.*

(i) For $1 \leq p \leq 2$, it holds that

$$\|f+g\|_p^p + \|f-g\|_p^p \geq 2^{p-1}\big(\|f\|_p^p + \|g\|_p^p\big). \tag{A.11}$$

(ii) For $2 \leq p < \infty$, it holds that

$$\|f+g\|_p^p + \|f-g\|_p^p \leq 2^{p-1}\big(\|f\|_p^p + \|g\|_p^p\big). \tag{A.12}$$

Theorem A.5.3. *For any $2 \leq p < \infty$, the space $L^p(\mu)$ is uniformly convex.*

Proof. We aim to show that, for $f, g \in L^p(\mu)$ with $\|f\|_p = \|g\|_p = 1$ and $\|f-g\|_p \geq \varepsilon$, it holds that $\|f+g\|_p \leq 2 - \delta(\varepsilon)$, for some $\delta(\varepsilon) > 0$ depending only on $\varepsilon > 0$. Let $0 < \varepsilon \leq 2$. For $f, g \in L^p(\mu)$ with the stated properties, we derive from Corollary A.5.2 the following inequality:

$$\|f+g\|_p^p \leq 2^p - \|f-g\|_p^p \leq 2^p - \varepsilon^p.$$

Set $\delta(\varepsilon) = 2 - (2^p - \varepsilon^p)^{1/p}$. Then $\|f+g\|_p \leq 2 - \delta(\varepsilon)$, as wanted. □

Corollary A.5.4. *The L^p-spaces are superreflexive, for all $1 < p < \infty$.*

Proof. For $2 \leq p < \infty$, superreflexivity is a consequence of Theorem A.5.3 together with Corollary A.4.2. Superreflexivity for $1 < p < 2$ follows from Corollary A.1.5. □

Remark A.5.5. It can also be shown that all L^p-spaces with $1 < p < 2$ are uniformly convex (see [10] and [23]). With this result at hand, superreflexivity of the L^p-spaces follows directly from Corollary A.4.2, for all $1 < p < \infty$.

Appendix B

Complex interpolation

We give in this appendix a brief overview of basics of the theory of complex interpolation. It is not our intention that this appendix constitutes a self-sustained introduction. Instead we merely aim to provide the tools needed in Section 5.3. Most proofs are omitted and the reader is referred to [7] and [28] for more thorough introductions, and to [24] for an introduction to complex interpolation of Bochner spaces.

B.1 Complex interpolation

Let $(X_0, \|\cdot\|_{X_0})$ and $(X_1, \|\cdot\|_{X_1})$ be Banach spaces over $\mathbb{C}$. The pair (X_0, X_1) is called *compatible* if there exists a Hausdorff topological vector space V and $\mathbb{C}$-linear continuous embeddings $\iota_j : X_j \hookrightarrow V$, $j = 0, 1$. We may in this case view X_0 and X_1 as vector subspaces of V by identifying them with the images of their respective embeddings. For a compatible pair of Banach spaces (X_0, X_1), we define their *intersection space*, respectively, their *sum space* as the vector supspaces of V given by

$$\begin{aligned} X_0 \cap X_1 &:= \iota_0(X_0) \cap \iota_1(X_1), \\ X_0 + X_1 &:= \iota_0(X_0) + \iota_1(X_1). \end{aligned}$$

We equip these vector spaces with the following norms:

$$\begin{aligned} \|x\|_{X_0 \cap X_1} &= \max\{\|x\|_{X_0}, \|x\|_{X_1}\}, \\ \|y\|_{X_0 + X_1} &= \inf\left\{ \|a_0\|_{X_0} + \|a_1\|_{X_1} \;\middle|\; y = \iota_0(a_0) + \iota_1(a_1) \right\}, \end{aligned}$$

for each $x \in X_0 \cap X_1$ and each $y \in X_0 + X_1$.

Proposition B.1.1. *The sum space and the intersection space of two compatible Banach spaces are Banach spaces.*

B.1.1 Analytic functions on the strip

Denote by S the strip in the complex plane, i.e., $S = \{ z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1 \}$, and by S° its interior. A central tool in the complex interpolation method, which we present in a moment, is the three lines lemma attributed to J. Hadamard in [22]. Following [1], we shall proof the three lines lemma as a corollary to a generalisation of the maximum-modulus principle due to E. Phragmén and E. Lindelöf in [34].

Theorem B.1.2 (Phragmén-Lindelöf)**.** *Let X be a complex Banach space and suppose that $f : S \to X$ is continuous and bounded on S and analytic on S°. Then*

$$\sup_{z \in S} \|f(z)\|_X \leq \max\left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_X, \sup_{t \in \mathbb{R}} \|f(1 + it)\|_X \right\}. \tag{B.1}$$

Proof. Suppose that $X = \mathbb{C}$ and assume that $f(z) \to 0$ as $\mathrm{Im}(z)$ tends to plus or minus infinity. Let $h : S \to \mathbb{C}$ be the map given by

$$h(z) = \frac{e^{i\pi z} - i}{e^{i\pi z} + i}, \qquad z \in S.$$

Note that h is the composition of the analytic map $z \mapsto e^{i\pi z}$ with the Cayley transformation. The map $z \mapsto e^{i\pi z}$ maps S bijectively to the closed upper half plane excluding $\{0\}$ (the interior of the strip is mapped to the open upper half plane). Hence, h maps S bijectively to $D = \overline{B}_{\mathbb{C}}(0,1) \backslash \{\pm 1\}$. The composition $g = f \circ h^{-1}$ is therefore bounded and continuous on D and analytic on the interior of D. Further, the assumption on the asymptotic properties of f ensures us that $\lim_{z \to \pm 1} g(z) = 0$. We may therefore extend g continuously to $\overline{B}_{\mathbb{C}}(0,1)$. The maximum modulus principle then yields that g attains its maximum modulus on the boundary of its domain. Unless g is equal zero, the maximum modulus is not attained at ± 1. The remaining boundary of D corresponds exactly to the boundary of S under the transformation h. Hence, for each $z \in S$,

$$|f(z)| \leq \sup_{z \in \overline{D}} |g(z)| = \max_{|z|=1} |g(z)| = \max \left\{ \sup_{t \in \mathbb{R}} |f(it)|, \sup_{t \in \mathbb{R}} |f(1+it)| \right\}.$$

Next, let $f : S \to \mathbb{C}$ be any function satisfying the assumptions of the theorem. For fixed $z_0 \in S^\circ$ and $\delta > 0$, let $f_{z_0,\delta} : S \to \mathbb{C}$ be the function given by $f_{z_0,\delta}(z) = e^{\delta(z-z_0)^2} f(z)$, for all $z \in S$. For each $z \in S$, we may write $z - z_0 = a + ib$, for some $a \in [-1,1]$ and some $b \in \mathbb{R}$. In terms of a and b, we see that $\left| e^{\delta(z-z_0)^2} \right| = e^{\delta(a^2-b^2)}$. Thus, $f_{z_0,\delta}(z) \to 0$ as $\mathrm{Im}(z)$ tends to plus or minus infinity. Moreover, it is clear that $\left| e^{\delta}(z-z_0)^2 \right| \leq e^{\delta}$, so that $|f_{z_0,\delta}(z)| = e^{\delta} |f(z)|$. It then follows from the above preparations that

$$\begin{aligned} |f(z_0)| = |f_{\delta,z_0}(z_0)| &\leq \max \left\{ \sup_{t \in \mathbb{R}} |f_{\delta,z_0}(it)|, \sup_{t \in \mathbb{R}} |f_{\delta,z_0}(1+it)| \right\} \\ &\leq e^{\delta} \max \left\{ \sup_{t \in \mathbb{R}} |f(it)|, \sup_{t \in \mathbb{R}} |f(1+it)| \right\}. \end{aligned}$$

For each $z_0 \in S^\circ$, this inequality holds for all $\delta > 0$, and so, letting $\delta \to 0$, we obtain inequality (B.1) in the case where $A = \mathbb{C}$.

Finally, let X be any Banach space and let $f : S \to X$ be as in the assumptions. For $z_0 \in S_0$ such that $f(z_0) \neq 0$, we may, by the Hahn-Banach theorem, find $x^* \in X^*$ with $\|x^*\| = 1$ such that $\|f(z_0)\| = |x^*(f(z_0))|$. Applying our above preparations to $x^* \circ f$ yields

$$\begin{aligned} \|f(z_0)\| = |x^* \circ f(z_0)| &\leq \max \left\{ \sup_{t \in \mathbb{R}} |x^* \circ f(it)|, \sup_{t \in \mathbb{R}} |x^* \circ f(1+it)| \right\} \\ &\leq \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_X, \sup_{t \in \mathbb{R}} \|f(1+it)\|_X \right\}. \end{aligned}$$

Since $z_0 \in S_0$ was arbitrary, this ends the proof of the theorem. $\square$

Lemma B.1.3 (Hadamard's three lines lemma)**.** *Let X be a complex Banach space and suppose that $f : S \to X$ is continuous and bounded on S and analytic on S°. For each $0 \leq \theta \leq 1$, let $M_\theta = \sup_{t \in \mathbb{R}} \|f(\theta + it)\|_X$. Then*

$$M_\theta \leq M_0^{1-\theta} M_1^{\theta}, \tag{B.2}$$

for all $0 \leq \theta \leq 1$.

Proof. For $\lambda \in \mathbb{R}$, define $F_\lambda : S \to X$ by setting $F_\lambda(z) = e^{\lambda z} f(z)$, for each $z \in S$. Then

$$\|F_z(z)\|_A \leq \max\{M_0, e^\lambda M_1\},$$

by Theorem B.1.2. Hence,

$$\|f(\theta + it)\|_X = e^{-\lambda\theta} \|F_\lambda(\theta + it)\|_X \leq \max\{e^{-\lambda\theta} M_0, e^{\lambda(1-\theta)} M_1\},$$

for all $t \in \mathbb{R}$. For $\lambda = \log(M_0/M_1)$, this is equation (B.2). □

B.1.2 The complex interpolation method

Let (X_0, X_1) be a compatible pair of Banach spaces. We denote by $\mathscr{F}$ the set of functions $f : S \to X_0 + X_1$ that are continuous and bounded on S, analytic on S°, and which, moreover, satisfy the following conditions:

- $f(it) \in X_0$ and $f(1 + it) \in X_1$, for all $t \in \mathbb{R}$,
- $f(it)$ and $f(1 + it)$ converges to 0 as $|t|$ tends to infinity.

The set $\mathscr{F}$ is a vector space over $\mathbb{C}$ with addition and scalar multiplication given pointwise.

Proposition B.1.4. *For any compatible pair of Banach spaces (X_0, X_1), the associated function space $\mathscr{F}$ is a Banach space under the norm*

$$\|f\|_{\mathscr{F}} = \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1 + it)\|_{X_1}\right\}, \qquad f \in \mathscr{F}. \tag{B.3}$$

Definition B.1.5. Let $0 \leq \theta \leq 1$. The *complex interpolation space (of exponent θ)* between compatible Banach spaces X_0 and X_1 is the complex vector space

$$[X_0, X_1]_\theta = \{\, x \in X_0 + X_1 \mid x = f(\theta), \text{ for some } f \in \mathscr{F} \,\},$$

equipped with the norm

$$\|x\|_\theta = \inf\{\, \|f\|_{\mathscr{F}} \mid x = f(\theta), f \in \mathscr{F} \,\}.$$

Proposition B.1.6. *For a compatible pair of Banach spaces, (X_0, X_1), we have the following continuous inclusions:*

$$X_0 \cap X_1 \subset [X_0, X_1]_\theta \subset X_0 + X_1.$$

Proposition B.1.7. *The complex interpolation space of any exponent between two compatible Banach spaces is itself a Banach space.*

B.1.3 Properties of interpolation spaces

Proposition B.1.8. *Any Banach space X is compatible with itself. Moreover, $[X, X]_\theta = X$, for any exponent $0 \leq \theta \leq 1$.*

Proposition B.1.9. *Let (X_0, X_1) be a pair of compatible Banach spaces. Their complex interpolation spaces have the following properties:*

(i) $[X_0, X_1]_\theta = [X_1, X_0]_{1-\theta}$, *for all* $0 \leq \theta \leq 0$.

(ii) *If* $X_0 \subset X_1$, *then* $[X_0, X_1]_{\theta_0} \subset [X_0, X_1]_{\theta_1}$, *whenever* $\theta_0 < \theta_1$.

Proposition B.1.10. *The intersection space $X_0 \cap X_1$ of a pair (X_0, X_1) of compatible Banach spaces is dense in their complex interpolation space $[X_0, X_1]_\theta$ of any exponent $0 \leq \theta \leq 1$.*

Theorem B.1.11 (The duality theorem)**.** *If at least one of the compatible Banach spaces X_0 and X_1 is reflexive, then*

$$\big([X_0, X_1]_\theta\big)^* \cong [X_0^*, X_1^*]_\theta,$$

for all $0 \leq \theta \leq 1$, where the isomorphism is isometric.

B.2 Interpolation of L^p-spaces

For a measure space (Ω, μ), let $\mathcal{M}_\mu(\Omega; \mathbb{C})$ denote the set of all (equivalence classes of) measurable complex valued functions. We say that a sequence $(f_n)_{n\geq 1}$ in $\mathcal{M}_\mu(\Omega; \mathbb{C})$ *converges in measure* to $f \in \mathcal{M}_\mu(\Omega; \mathbb{C})$ if, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\left(\{\, \omega \in \Omega \mid |f_n(\omega) - f(\omega)| \geq \varepsilon \,\}\right) = 0.$$

This induces a topology on $\mathcal{M}_\mu(\Omega; \mathbb{C})$: the *topology of convergence in measure.* Observe that, for any $1 \leq p \leq \infty$, convergence in p-norm implies convergence in measure. Hence, every L^p-space embeds continuously into the corresponding topological vector space of all measurable functions. Any pair of L^p-spaces on the same measure space are therefore compatible.

Theorem B.2.1. *Let $1 \leq p_1 < p_0 \leq \infty$, let (Ω, μ) be a measure space with a σ-finite measure, and consider the compatible pair of Banach spaces $L^{p_0}(\Omega, \mu)$ and $L^{p_1}(\Omega, \mu)$. For each exponent $0 < \theta < 1$,*

$$[L^{p_0}(\Omega, \mu), L^{p_1}(\Omega, \mu)]_\theta \cong L^{p_\theta}(\Omega, \mu), \qquad \textit{for} \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

where the isomorphism is isometric.

We state and proof in Theorem B.2.3 a celebrated result in the theory of complex interpolation concerning interpolation of operators on L^p-spaces. The theorem was originally stated by M. Riesz in [38] when $p_j \geq q_j$ and proved using results from convexity theorems for bilinear forms. It was then extended by G. O. Thorin in [49] to hold for all $1 \leq q_0, q_1 \leq \infty$. The modern proof, which we present below, uses Hadamard's three lines lemma and is due to J. D. Tamarkin, and A. Zygmund in [48].

Lemma B.2.2. *Let (Ω, μ) be a measure space. Let $1 \leq q_0, q_1 \leq \infty$ with $q_0 \neq q_1$ and define, for each $0 < \theta < 1$,*

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then $L^{q_0}(\Omega, \mu) \cap L^{q_1}(\Omega, \mu) \subset L^{q_\theta}(\Omega, \mu)$, and the following norm-inequality holds:

$$\|f\|_{q_\theta} \leq \|f\|_{q_0}^{1-\theta} \|f\|_{q_1}^{\theta},$$

for all $f \in L^{q_0}(\Omega, \mu) \cap L^{q_1}(\Omega, \mu)$.

Theorem B.2.3 (Riesz-Thorin)**.** *Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and let (Ω, μ) and (Ω', μ') be measure spaces. Suppose that $T : L^{p_j}(\Omega, \mu) \to L^{q_j}(\Omega', \mu')$ is bounded and linear, for $j = 0, 1$. Then, for each $0 < \theta < 1$, $T : L^{p_\theta}(\Omega, \mu) \to L^{q_\theta}(\Omega', \mu')$ is bounded and linear, where*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \textit{and} \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{B.4}$$

Moreover, the following norm-inequality holds:

$$\|T\|_{L^{p_\theta}(\Omega,\mu)\to L^{q_\theta}(\Omega',\mu')} \leq \|T\|_{L^{p_0}(\Omega,\mu)\to L^{q_0}(\Omega',\mu')}^{1-\theta} \|T\|_{L^{p_1}(\Omega,\mu)\to L^{q_1}(\Omega',\mu')}^{\theta}. \tag{B.5}$$

Proof. Assume that $p_0 = p_1 = p$. If also $q_0 = q_1$, there is nothing to show, so assume this is not the case. Let $f \in L^p(\Omega, \mu)$. By Lemma B.2.2, $Tf \in L^{q_\theta}(\Omega, \mu)$, and we have that

$$\begin{aligned} \|Tf\|_{q_\theta} &\leq \|Tf\|_{q_0}^{1-\theta} \|Tf\|_{q_1}^{\theta} \\ &\leq \|T\|_{L^p(\Omega,\mu)\to L^{q_0}(\Omega',\mu')}^{1-\theta} \|T\|_{L^p(\Omega,\mu)\to L^{q_1}(\Omega',\mu')}^{\theta} \|f\|_p. \end{aligned}$$

The statement for the case when $p_0 = p_1$ follows.

Assume that $p_0 \neq p_1$. Then $p_\theta < \infty$, and so, the simple functions, $S(\Omega)$, are dense in $L^{p_\theta}(\Omega, \mu)$. For each number $1 \leq q \leq \infty$, we shall denote by q' its conjugate. Recall that, for every $a \in L^{p_\theta}(\Omega, \mu)$,

$$\|Ta\|_{q_\theta} = \sup\left\{ \left| \int_{\Omega'} (Ta) b \, \mathrm{d}\mu' \right| \;\middle|\; b \in S(\Omega'), \|b\|_{q'_\theta} = 1 \right\},$$

It therefore suffices to proof that

$$\left| \int_{\Omega'} (Ta) b \, \mathrm{d}\mu' \right| \leq \|T\|^{1-\theta}_{L^{p_0}(\Omega,\mu) \to L^{q_0}(\Omega',\mu')} \|T\|^{\theta}_{L^{p_1}(\Omega,\mu) \to L^{q_1}(\Omega',\mu')} \|a\|_{p_\theta},$$

for all $a \in L^{p_\theta}(\Omega, \mu)$ and all $b \in S(\Omega')$ with $\|b\|_{q'_\theta} = 1$. By renormalization and by density of the simple functions in $L^{p_\theta}(\Omega, \mu)$, it even suffices to proof this for $a \in S(\Omega)$ with $\|a\|_{p_\theta} = 1$.

Let $a \in S(\Omega)$ with $\|a\|_{p_\theta} = 1$ and $b \in S(\Omega')$ with $\|b\|_{q'_\theta} = 1$. Define two functions $\alpha, \beta : S \to \mathbb{C}$ by

$$\alpha(z) = \frac{1-z}{p_0} + \frac{z}{p_1}, \qquad \text{and} \qquad \beta(z) = \frac{1-z}{q'_0} + \frac{z}{q'_1}.$$

Further, define $f : S \to L^{p_0}(\Omega, \mu) \cap L^{p_1}(\Omega, \mu)$ by

$$f(z)(\omega) = \begin{cases} |a(\omega)|^{p_\theta \alpha(z)} \frac{a(\omega)}{|a(\omega)|} & \text{if } a(\omega) \neq 0, \\ 0, & \text{if } a(\omega) = 0 \end{cases}.$$

If $q'_\theta \neq \infty$, define $g : S \to L^{q'_0}(\Omega', \mu') \cap L^{q'_1}(\Omega', \mu')$ by

$$g(z)(\omega) = \begin{cases} |b(\omega)|^{q'_\theta \beta(z)} \frac{b(\omega)}{|b(\omega)|} & \text{if } b(\omega) \neq 0, \\ 0, & \text{if } b(\omega) = 0 \end{cases}.$$

As a and b are simple, it is clear that $f(z)$ lie in $L^r(\Omega, \mu)$ and and $g(z)$ lie in $L^r(\Omega', \mu')$, for all $z \in S$ and all $1 \leq r \leq \infty$. Hence, f and g are well-defined. Let $\Phi : S \to \mathbb{C}$ be given by

$$\Phi(z) = \int_{\Omega'} Tf(z) g(z) \, \mathrm{d}\mu',$$

and observe that $\Phi(\theta)$ is the integral we wish to estimate. By Hölder's inequality, Φ is well-defined and we have, for $j = 0, 1$, that

$$|\Phi(z)| \leq \|Tf(z)\|_{q_j} \|g(z)\|_{q'_j}, \tag{B.6}$$

Since a and b are simple, one easily checks that Φ is bounded and continuous on S and holomorphic on S°. Hence, the three lines lemma, Lemma B.1.3, applies, and we obtain that

$$\left| \int_{\Omega} (Ta) b \, \mathrm{d}\mu \right| = |\Phi(\theta)| \leq \sup_{t \in \mathbb{R}} |\Phi(it)|^{1-\theta} \sup_{t \in \mathbb{R}} |\Phi(1+it)|^{\theta}. \tag{B.7}$$

Note that, putting together equations (B.6) and (B.7), we are already quite close to the norm-inequality we aim to show. We need only a few more norm estimations.

For each $t, r \in \mathbb{R}$ and $j = 0, 1$, we have

$$\|f(it+r)\|^{p_j}_{p_j} = \int_{\Omega} \left| |a|^{p_\theta \left(\frac{1-it-r}{p_0} + \frac{it+r}{p_1} \right)} \right|^{p_j} \mathrm{d}\mu = \int_{\Omega} |a|^{p_\theta p_j \left(\frac{1-r}{p_0} + \frac{r}{p_1} \right)} \mathrm{d}\mu,$$

$$\|g(it+r)\|^{q'_j}_{q'_j} = \int_{\Omega} \left| |b|^{q'_\theta \left(\frac{1-it-r}{q'_0} + \frac{it+r}{q'_1} \right)} \right|^{q'_j} \mathrm{d}\mu = \int_{\Omega} |b|^{q'_\theta q'_j \left(\frac{1-r}{q'_0} + \frac{r}{q'_1} \right)} \mathrm{d}\mu.$$

From this, we see that

$$\|f(it)\|_{p_0} = \|a\|_{p_\theta}^{p_\theta/p_0},$$
$$\|f(it+1)\|_{p_1} = \|a\|_{p_\theta}^{p_\theta/p_1},$$
$$\|g(it)\|_{q_0'} = \|b\|_{q_\theta'}^{q_\theta'/q_0'},$$
$$\|g(it+1)\|_{q_1'} = \|b\|_{q_\theta'}^{q_\theta'/q_1'}.$$

By assumption on a and b, these norms all equals 1. Plugging this into equation (B.6) yields the following bounds:

$$|\Phi(it)| \le \|T\|_{L^{p_0}(\Omega,\mu)\to L^{q_0}(\Omega',\mu')}$$
$$|\Phi(it+1)| \le \|T\|_{L^{p_1}(\Omega,\mu)\to L^{q_1}(\Omega',\mu')}.$$

equation (B.5) follows directly from equation (B.7) using the above estimates.

Finally, if $q_\theta' = \infty$, i.e., if $q_0 = q_1 = 1$, define $g(z) = b$. Then $\|g(z)\|_{q_j'} = \|b\|_{q_\theta'} = 1$, for $j = 0, 1$ and for all $z \in S$. We obtain equation (B.5) by the same argument as above. □

The Riesz-Thorin theorem was extended to families of operators by E. Stein in [46]. We present Stein's result in Theorem B.2.4 below. Let (Ω, μ) and (Ω', μ') be measure spaces. A family $(T_z)_{z\in S}$ of bounded and linear operators from the set of simple functions on Ω to the set of measurable functions on Ω' called *admissible* if, for every pair of simple functions $a \in S(\Omega)$ and $b \in S(\Omega')$, the map $S \to \mathbb{C}$ given by

$$z \mapsto \int_{\Omega'} (T_z a) b \, \mathrm{d}\mu'$$

is continuous on S, analytic on S°, and if there exists a constant $k < \pi$, which does not depend on a and b, such that

$$\sup_{z\in S} e^{-k|\mathrm{Im}\, z|} \log \left| \int_{\Omega'} (T_z a) b \, \mathrm{d}\mu' \right| < \infty.$$

Theorem B.2.4 (Stein's Interpolation Theorem)**.** *Let $1 \le p_0, p_1, q_0, q_1 \le \infty$ and let (Ω, μ) and (Ω', μ') be measure spaces. Suppose that $(T_z)_{z\in S}$ is an admissible family of linear operators $S(\Omega) \to \mathcal{M}(\Omega')$ satisfying*

$$\|T_{it} a\|_{q_0} \le M_0(t) \|a\|_{p_0} \qquad \text{and} \qquad \|T_{it+1} a\|_{q_1} \le M_1(t) \|a\|_{p_1},$$

for all simple functions $a \in S(\Omega)$ and where $M_j(t) > 0$ are independent of a, for $j = 0, 1$, and satisfy

$$\sup_{t\in\mathbb{R}} e^{-kt} \log M_j(t) < \infty,$$

for some $k < \pi$. Then, for each $0 < \theta < 1$, there exists a constant $M_\theta > 0$ such that

$$\|T_\theta a\|_{q_\theta} \le M_\theta \|a\|_{p_\theta}, \tag{B.8}$$

for every simple function $a \in S(\Omega)$, where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \text{and} \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

To proof Stein's interpolation theorem, we shall need the following extension of Hadamard's three lines lemma due to I. Hirschman. For a proof, we refer the reader to [47, Lemma 4.2].

Lemma B.2.5 (Hirschman)**.** *Let $\Phi : S \to \mathbb{C}$ be a continuous function and suppose that Φ is analytic on S° and satisfies that*

$$\sup_{z\in S} e^{-k|\operatorname{Im} z|}\log|\Phi(z)| < \infty,$$

for some konstant $k < \pi$. Then

$$\log|\Phi(\theta)| \leq \frac{\sin\pi\theta}{2}\int_{-\infty}^{\infty}\frac{\log|\Phi(it)|}{\cosh\pi t - \cos\pi\theta} + \frac{\log|\Phi(it+1)|}{\cosh\pi t + \cos\pi\theta}\,\mathrm{d}t,$$

for all $0 < \theta < 1$.

Proof of Theorem B.2.4. As in the proof of the Riesz-Thorin interpolation theorem, it suffices to proof the existence of a constant $M_\theta > 0$ such that

$$\left|\int_{\Omega'}(T_\theta a)b\,\mathrm{d}\mu'\right| \leq M_\theta \tag{B.9}$$

for all $a \in S(\Omega)$ with $\|a\|_{p_\theta} = 1$ and all $b \in S(\Omega')$ with $\|b\|_{q'_\theta} = 1$. So let a and b be such functions, and define $\alpha, \beta : S \to \mathbb{C}$ and $f : S \to S(\Omega)$ and $g : S \to S(\Omega')$ as in the proof of the Riesz-Thorin theorem. Let $\Phi : S \to \mathbb{C}$ be given by

$$\Phi(z) = \int_{\Omega'}\big(T_z f(z)\big)g(z)\,\mathrm{d}\mu'.$$

Observe that $\Phi(\theta)$ is the integral we aim to bound. As $(T_z)_{z\in S}$ is assumed to be an admissible family, the function Φ is continuous on S and analytic on S°. Moreover, there exists a constant $m < \pi$ such that

$$\sup_{z\in S} e^{-m|\operatorname{Im} z|}\log|\Phi(z)| < \infty.$$

We may then apply Lemma B.2.5 to obtain the following bound:

$$\log\left|\int_{\Omega'}(T_\theta a)b\,\mathrm{d}\mu'\right| \leq \frac{\sin\pi\theta}{2}\int_{-\infty}^{\infty}\frac{\log|\Phi(it)|}{\cosh\pi t - \cos\pi\theta} + \frac{\log|\Phi(it+1)|}{\cosh\pi t + \cos\pi\theta}\,\mathrm{d}t \tag{B.10}$$

For each $1 \leq r \leq \infty$, denote by r' its conjugate. We obtain the following bounds from Hölder's inequality together with our assumptions:

$$|\Phi(it)| \leq \|T_{it}f(it)\|_{q_0}\|g(it)\|_{q'_0} \leq M_0(t)\|f(it)\|_{p_0}\|g(it)\|_{q'_0},$$
$$|\Phi(it+1)| \leq \|T_{it+1}f(it+1)\|_{q_1}\|g(it+1)\|_{q'_1} \leq M_1(t)\|f(it+1)\|_{p_1}\|g(it+1)\|_{q'_1}.$$

As in the proof of the Riesz-Thorin theorem, we have

$$\|f(it)\|_{p_0} = \|g(it)\|_{q'_0} = \|f(it+1)\|_{p_1} = \|g(it+1)\|_{q'_1} = 1.$$

So the above bounds reduces to $|\Phi(it)| \leq M_0(t)$ and $|\Phi(it+1)| \leq M_1(t)$. Set, for $j = 0, 1$,

$$M_j = \sup_{t\in\mathbb{R}} e^{-kt}\log M_j(t) < \infty.$$

From our estimates on $\Phi(it)$ and $\Phi(it+1)$, we obtain the following bounds:

$$\begin{aligned}
\int_{-\infty}^{\infty}\frac{\log|\Phi(it)|}{\cosh\pi t - \cos\pi\theta}\,\mathrm{d}t &\leq \int_{-\infty}^{\infty}\frac{\log M_0(t)}{\cosh\pi t - \cos\pi\theta}\,\mathrm{d}t \\
&\leq M_0\int_{-\infty}^{\infty}\frac{e^{kt}}{\cosh\pi t - \cos\pi\theta}\,\mathrm{d}t \\
\int_{-\infty}^{\infty}\frac{\log|\Phi(it+1)|}{\cosh\pi t + \cos\pi\theta}\,\mathrm{d}t &\leq \int_{-\infty}^{\infty}\frac{\log M_1(t)}{\cosh\pi t + \cos\pi\theta}\,\mathrm{d}t \\
&\leq M_1\int_{-\infty}^{\infty}\frac{e^{kt}}{\cosh\pi t + \cos\pi\theta}\,\mathrm{d}t
\end{aligned}$$

These integrals converge, and so, the constant $M_\theta > 0$ defined by

$$\log M_\theta = \frac{\sin \pi\theta}{2} \int_{-\infty}^{\infty} \frac{M_0 e^{kt}}{\cosh \pi t - \cos \pi\theta} + \frac{M_1 e^{kt}}{\cosh \pi t + \cos \pi\theta} \,\mathrm{d}t$$

is finite, and $\log M_\theta$ provides a bound on the integral on the right-hand side of equation (B.10). Equation (B.9) follows directly. □

B.3 Interpolation of Bochner spaces

In the previous section, we discussed a few classical results for interpolation of L^p-spaces. These results apply, more generally, to Bochner spaces. The proofs are analogous to the ones in the classical setting. We refer the interested reader to [24, Section 2.2] for details. For a thorough introduction to Bochner spaces, see [24, Chapter 1].

Theorem B.3.1 (Complex interpolation of Bochner spaces)**.** *Let $1 \le p_1 < p_0 \le \infty$ and let (Ω, μ) be a measure space with a σ-finite measure. For a compatible pair of Banach spaces (X_0, X_1), the Banach spaces $L^{p_0}(\Omega, \mu; X_0)$ and $L^{p_1}(\Omega, \mu; X_1)$ are compatible, as well. Moreover, for each exponent $0 < \theta < 1$,*

$$[L^{p_0}(\Omega, \mu; X_0), L^{p_1}(\Omega, \mu; X_1)]_\theta \cong L^{p_\theta}(\Omega, \mu; [X_0, X_1]_\theta), \qquad \textit{for} \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

where the isomorphism is isometric.

Theorem B.3.2 (Riesz-Thorin for Bochner spaces)**.** *Let $1 \le p_0, p_1, q_0, q_1 \le \infty$, let (Ω, μ) and (Ω', μ') be measure spaces and let (X_0, X_1) and (Y_0, Y_1) be compatible pairs of Banach spaces. Suppose that $T : L^{p_j}(\Omega, \mu; X_j) \to L^{q_j}(\Omega', \mu'; Y_j)$ is bounded and linear, for $j = 0, 1$. Then, for each $0 < \theta < 1$, $T : L^{p_\theta}(\Omega, \mu; [X_0, X_1]_\theta) \to L^{q_\theta}(\Omega', \mu'; [Y_0, Y_1]_\theta)$ is bounded and linear, where*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \textit{and} \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Moreover, the following norm-inequality holds:

$$\|T\|_{L^{p_\theta}(\Omega,\mu;[X_0,X_1]_\theta)\to L^{q_\theta}(\Omega',\mu';[Y_0,Y_1]_\theta)} \le \|T\|^{1-\theta}_{L^{p_0}(\Omega,\mu;X_0)\to L^{q_0}(\Omega',\mu';Y_0)} \|T\|^{\theta}_{L^{p_1}(\Omega,\mu;X_1)\to L^{q_1}(\Omega',\mu';Y_1)} . \tag{B.11}$$

Theorem B.3.3 (Stein's Interpolation Theorem for Bochner spaces)**.** *Let $1 \le p_0, p_1, q_0, q_1 \le \infty$, let (Ω, μ) and (Ω', μ') be measure spaces and let X and Y be Banach spaces. Suppose that $(T_z)_{z\in S}$ is an admissible family of linear operators $\mathcal{S}(\Omega; X) \to \mathcal{M}_\mu(\Omega'; Y)$ satisfying*

$$\|T_{it}a\|_{q_0} \le M_0(t) \|a\|_{p_0} \qquad \textit{and} \qquad \|T_{it+1}a\|_{q_1} \le M_1(t) \|a\|_{p_1} ,$$

for all simple functions $a \in S(\Omega; X)$ and where $M_j(t) > 0$ are independent of a, for $j = 0, 1$, and satisfy

$$\sup_{t\in\mathbb{R}} e^{-kt} \log M_j(t) < \infty,$$

for some $k < \pi$. Then, for each $0 < \theta < 1$, there exists a constant $M_\theta > 0$ such that

$$\|T_\theta a\|_{q_\theta} \le M_\theta \|a\|_{p_\theta} , \tag{B.12}$$

for every simple function $a \in S(\Omega; X)$, where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \textit{and} \qquad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

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