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The  
**CONNES EMBEDDING PROBLEM**  
Sofic groups and the QWEP Conjecture

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## Abstract

In 1976 Connes casually remarked that every von Neumann algebra type  $II_1$ -factor ought to embed into an ultrapower of the hyperfinite  $II_1$ -factor. This remark has become a long-standing open problem, known as the Connes Embedding Problem, on which numerous attempts have been made to provide an answer. In the first part of this thesis we discuss several equivalent formulations of this problem, proved by Kirchberg, including the QWEP conjecture. The second part of the thesis concerns hyperlinear and sofic groups, and among other results we prove that the group von Neumann algebra of a sofic group embed into an ultrapower of the hyperfinite  $II_1$ -factor.

## Resumé

I 1976 kom Connes med en bemærkning om, at enhver von Neumann algebra faktor af type  $II_1$  burde kunne indlejres i en ultrapotent af den hyperendeligt  $II_1$ -faktor. Med tiden er denne bemærkning blevet til et stort uløst problem, kaldet Connes Indlejnings Problem, som der er blevet gjort utallige forsøg på at afgøre. I den første del af dette speciale vil vi give flere ækvivalente formuleringer af dette problem, bevist af Kirchberg, inklusiv QWEP Formodningen. Anden del af specialet omhandler hyperlineære og sofiske grupper, og vi viser blandt andet, at gruppe von Neumann algebreren til en sofisk gruppe kan indlejres i en ultrapotent af den hyperendelige  $II_1$ -faktor.

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**Author’s note.** This present version of the document is a corrected version of the original one. With many pages comes many small mistakes. Besides a number of typos and misspellings, there were a few actual errors, which have, of course, been corrected in this version.

Copenhagen, March 2013

# Introduction

Central to this thesis is the Connes Embedding Problem. It arose in 1976, when Alain Connes, in his famous paper [Con76], remarked that it ought to be the case, that all von Neumann algebra factors of type  $II_1$  with separable predual embed into an ultrapower of the hyperfinite  $II_1$ -factor. To be precise, Connes wrote:<sup>1</sup>

We now construct an approximate imbedding of  $N$  in  $R$ . Apparently such an imbedding ought to exist for all  $II_1$  factors because it does for the regular representation of free groups. [Con76, page 105].

Since 1976 this problem has gotten a considerable amount of attention, and it remains open, despite the numerous attempts to solve it. Along the way, it has been proved that Connes' problem has deep and interesting connections to many areas of mathematics.

Another important problem in this thesis is Eberhard Kirchberg's QWEP Conjecture. It can be formulated as follows: all  $C^*$ -algebras are QWEP. In 1993 Kirchberg proved, [Kir93], a vast amount of equivalences between various open problems in operator algebras. In particular, he proved several equivalent formulations of the QWEP conjecture. He showed that this conjecture is equivalent to an affirmative answer to the Connes Embedding Problem. To mention some of these equivalent statements, the QWEP Conjecture is equivalent to the statement that  $C^*(\mathbb{F}_\infty)$  is QWEP, or to the statement that there is a unique  $C^*$ -norm on the algebraic tensor product  $C^*(\mathbb{F}_\infty) \odot C^*(\mathbb{F}_\infty)$ .

In the above mentioned paper Kirchberg took on an investigation of when there is a unique  $C^*$ -norm on certain tensor products, and in the process he gave tensorial characterizations of the weak expectation property (WEP) and of the local lifting property (LLP). These characterizations build on an earlier result from [Kir94], where he proved that for a free group  $\mathbb{F}$  and a Hilbert space  $\mathcal{H}$  there is a unique  $C^*$ -norm on the algebraic tensor product  $B(\mathcal{H}) \odot C^*(\mathbb{F})$ .

In the quest for an answer to the Connes Embedding Problem, it is natural to ask the weaker question, whether the group von Neumann algebras associated to discrete countable groups with infinite conjugacy classes embed into an ultrapower of the hyperfinite  $II_1$ -factor. These form an important class of von Neumann algebra  $II_1$ -factors with separable predual. Indeed, this class often provides examples and counterexamples. This so-called "Connes Embedding Problem for Groups" is more accessible, since group von Neumann algebras of countable discrete groups are easier to handle than general von Neumann algebras. To put a name on the groups that provide an affirmative answer to the Connes Embedding Problem for Groups, they are the hyperlinear groups (more specifically, the countable and discrete ones with infinite conjugacy classes).

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<sup>1</sup>At this point in Connes' paper  $N$  refers to a von Neumann algebra  $II_1$ -factor satisfying certain conditions, and  $R$  to the hyperfinite  $II_1$ -factor

In recent years, starting with work of Mikhail Gromov in 1999, an interesting class of groups have been exhibited, namely the so-called sofic groups. These groups have drawn more and more interest over the last years, since it turns out that they have deep connections to many areas of mathematics, and indeed many long-standing open problems have recently been solved for sofic groups. Among these are Gottschalk's Surjunctivity Conjecture [Gro99] (see also [Wei00]), Kaplansky's Finiteness Conjecture [ES04] and others.<sup>2</sup> Most importantly for the purpose of this thesis, sofic groups do satisfy the Connes Embedding Conjecture for groups. In other words, sofic groups are hyperlinear. It is still not known how large the class of sofic groups is. It is not known whether, in fact, all groups are sofic.

In this thesis we will first prove the equivalence of the Connes Embedding Problem and the QWEP Conjecture, and establish other equivalent formulations of the QWEP Conjecture. Second, we make an investigation of hyperlinear and sofic groups. More precisely, we characterize these in terms of certain metric ultraproducts of groups, we prove that sofic groups are in fact hyperlinear and then we discuss permanence properties for sofic groups and examples of such.

Let us give a more detailed description of the content of this thesis. Chapter 1 contains preliminaries on  $C^*$ -algebras, von Neumann algebra, tensor products of  $C^*$ -algebras and filters. The preliminaries on  $C^*$ -algebras, tensor products and filters will be brief and not contain many proofs, while the preliminaries on von Neumann algebras will be given in more details. More precisely, a thorough exposition of von Neumann algebras with separable predual, as well as a careful presentation of the concept of universal enveloping von Neumann algebras, will be made.

In Chapter 2 we give an introduction to the weak expectation property, QWEP, the lifting property and the local lifting property. This chapter starts with a section on conditional expectations, and it is followed by a section on the weak expectation property. The latter starts with the notion of relative weak injectivity, a concept due to Kirchberg. After this comes an introductory section on QWEP, where certain permanence properties are proved. The chapter ends with a section on the lifting property and the local lifting property, containing several important results, including two famous lifting theorems, by Choi-Effros and Effros-Haagerup, respectively.

Chapter 3 contains the main results related to the weak expectation property, QWEP and the local lifting property. As mentioned above, the  $C^*$ -algebra  $C^*(\mathbb{F}_\infty)$  plays an important role in connection to QWEP, so the first section of the chapter is devoted to this  $C^*$ -algebra. The next section contains a proof Kirchberg's result stating that there is a unique  $C^*$ -norm on the algebraic tensor product  $B(\mathcal{H}) \otimes C^*(\mathbb{F})$ , mentioned before. The proof is due to Gilles Pisier, taken from [Pis96], and it is extremely elegant. After this, we give tensorial characterizations of the property of being relatively weakly injective in a  $C^*$ -algebra, the weak expectation property and the local lifting property. The chapter ends with collecting the different equivalent formulations of the QWEP conjecture, as well as proving its equivalence to an affirmative answer to the Connes Embedding Problem.

After Chapter 3 we turn away from the QWEP conjecture, and fix our attention upon the Connes Embedding Problem for Groups. In Chapter 4 we give the construction of the metric ultraproduct of groups and the tracial ultraproduct of von Neumann algebras. We spend some time proving certain properties of the latter, namely, that it is a von Neumann algebra under certain conditions. Besides this, we give sufficient conditions for when this tracial ultraproduct is a von Neumann algebra factor, and

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<sup>2</sup>For the precise statement of these two conjectures, see page 99.

under these conditions determine which type of factor it is. The chapter ends with a section on tensor product of ultrafilters. This concept has, in spite of its name, nothing to do with tensor products, but nonetheless it is an interesting and helpful tool when working with filters in an analytic setting.

With the background on ultraproducts in order, we turn our attention to hyperlinear groups in Chapter 5. Therein we establish equivalent conditions for a group to be hyperlinear, the most important of which concerns embeddability of the associated group von Neumann algebra into ultrapowers of the hyperfinite  $II_1$ -factor.

The last chapter, Chapter 6, concerns sofic groups. Naturally, we start with the definition of a sofic group. There is an abundance of choices for the definition of a sofic group, and we have picked the one which serves our purpose best. Immediately afterwards, we give the connection to the Connes Embedding Problem, by proving that sofic groups are, in fact, hyperlinear. Before continuing with examples of sofic groups, and permanence properties of such, we take a moment to introduce the terminology of local embeddability. Once this is done, we proceed to give examples of sofic groups and prove certain permanence properties of these. The chapter ends with a summary of our investigation of sofic groups.

The thesis also contains two appendices, the first of which is devoted to explaining a few results which are needed in the last section of Chapter 3. The second appendix deals with operator spaces and operator systems. Since these concepts are very central to the first part of the thesis, a thorough presentation of these is made, and most of the proofs are included.

In reality, this thesis consists of two rather distinct and disjoint parts. The one concerning the QWEP conjecture and the other concerning sofic groups. There is, of course, a lot more to be said on the Connes Embedding Conjecture than presented in this thesis. Also on the QWEP conjecture there are results not mentioned here. Besides this, the presentation of sofic groups is a narrow one. There is much more to be said about these, and we have restricted our attention to the connection to the Connes Embedding Problem, since this obviously is the central point in this thesis.





# Chapter 1

## Preliminaries

This chapter contains preliminaries which will be used throughout the thesis. The reader is assumed to be familiar with general functional analysis, basic  $C^*$ -algebra theory, and a reasonable good knowledge of von Neumann algebras.

Before we start let us make a note on notation. We will use the symbols  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  to denote the natural numbers (not including zero), the integers, the real numbers and the complex numbers, respectively. The complex unit circle will be denoted by  $\mathbb{T}$ . Given topological space  $\mathfrak{X}$  and  $\mathfrak{Y}$ , we will denote the continuous functions from  $\mathfrak{X}$  to  $\mathfrak{Y}$  by  $C(\mathfrak{X}; \mathfrak{Y})$ . In particular,  $C(\mathfrak{X}; \mathbb{C})$  will denote the continuous complex-valued functions on  $\mathfrak{X}$ . In the case where  $\mathfrak{X}$  is, in fact, a normed space, we will let  $\mathfrak{X}^*$  denote the dual space of  $\mathfrak{X}$ , that is, the Banach space bounded linear functionals on  $\mathfrak{X}$ .

Notation related to matrices with either complex entries or entries in a  $C^*$ -algebra is introduced in the beginning of Appendix B, but we mention that  $\mathbb{M}_n$  denotes the  $n \times n$  complex matrices, and that  $E_{i,j}$  denotes the standard matrix units in this algebra.

Groups will mostly be denoted by  $G$  and  $H$ , but sometimes also  $\Gamma$ . The standard separable Hilbert space will be denoted by  $\ell_2$ , and for  $n \in \mathbb{N}$ , we denote  $\mathbb{C}^n$  equipped with the supremum norm by  $\ell_\infty^n$ .

### 1.1 $C^*$ -algebras

This section mostly serves the purpose of fixing the notation concerning the basic results on  $C^*$ -algebras that we need. We will also include a few results, some of which we prove.

We will use the symbols  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  for  $C^*$ -algebras, and in some cases just for  $*$ -algebras. *In this thesis  $C^*$ -algebras are never assumed to be unital nor separable, unless explicitly stated.* For  $*$ -homomorphisms we will mostly use the symbols  $\pi$  and  $\rho$ , but in some cases also other symbols. We will denote the real and imaginary part of an element  $x$  in a  $C^*$ -algebra by  $\operatorname{Re} x$  and  $\operatorname{Im} x$ , respectively. That is,  $\operatorname{Re} x = \frac{1}{2}(x + x^*)$  and  $\operatorname{Im} x = \frac{1}{2i}(x - x^*)$ .

We will denote the unitization of a  $C^*$ -algebra  $\mathcal{A}$  by  $\tilde{\mathcal{A}}$ , and by unitization we shall mean the algebra with a unit adjoint, no matter if the  $C^*$ -algebra is already unital. A subset  $\mathfrak{X}$  of a  $*$ -algebra is called self-adjoint, if  $x^* \in \mathfrak{X}$  whenever  $x \in \mathfrak{X}$ . Given a  $C^*$ -algebra  $\mathcal{A}$  and a positive linear functional  $\phi$  on  $\mathcal{A}$ , we let  $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$  denote the GNS-construction corresponding to  $\phi$ . That is,  $\mathcal{H}_\phi$  is the Hilbert space constructed from  $\mathcal{A}$  using the sesquilinear form  $(x, y) \mapsto \phi(y^*x)$ ,  $x, y \in \mathcal{A}$ ,  $\pi_\phi: \mathcal{A} \rightarrow B(\mathcal{H}_\phi)$  is the representation induced by left multiplication and  $\xi_\phi$  is the cyclic vector corresponding

to the unit if  $\mathcal{A}$  is unital, or to the limit of an approximate unit if  $\mathcal{A}$  is not unital, satisfying  $\langle \pi_\phi(x)\xi_\phi \mid \xi_\phi \rangle = \phi(x)$ ,  $x \in \mathcal{A}$ .

For a  $C^*$  algebra  $\mathcal{A}$  we will denote the set of self-adjoint elements in  $\mathcal{A}$  and the set of positive elements in  $\mathcal{A}$  by  $\mathcal{A}_{sa}$  and  $\mathcal{A}_+$ , respectively. Also for the dual space  $\mathcal{A}^*$  we will let  $(\mathcal{A}^*)_+$  denote the positive linear functionals. The closed unit ball in  $\mathcal{A}$  and  $\mathcal{A}^*$  will be denoted by  $\mathcal{A}_1$  and  $(\mathcal{A}^*)_1$ , respectively.

Let us introduce a few notions and prove some statements about representations of  $C^*$ -algebras.

**Definition 1.1.1.** Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra and that  $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$  is a representation of  $\mathcal{A}$  on some Hilbert space  $\mathcal{H}$ . The representation  $\pi$  is said to be **non-degenerate** if  $\pi(\mathcal{A})\mathcal{H} = \{\pi(a)\xi : a \in \mathcal{A}, \xi \in \mathcal{H}\}$  spans a dense subset of  $\mathcal{H}$ . ◀

**Definition 1.1.2.** Suppose that  $\mathcal{H}$  is a Hilbert space and that  $\mathcal{A} \subseteq B(\mathcal{H})$  is a  $C^*$ -algebra. A vector  $\xi \in \mathcal{H}$  is called **cyclic** for  $\mathcal{A}$  if  $\mathcal{A}\xi = \{x\xi : x \in \mathcal{A}\}$  is norm dense in  $\mathcal{H}$ , respectively, **separating** for  $\mathcal{A}$  if  $x\xi = 0$  implies that  $x = 0$ , for  $x \in \mathcal{A}$ . ◀

It is not hard to show that a vector is cyclic for  $\mathcal{A}$  if and only if it is separating for  $\mathcal{A}'$ . This is proved in more generality later, namely, proved for cyclic and separating subsets (see Definition 1.3.2 and Proposition 1.3.3).

**Definition 1.1.3.** Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra and that  $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$  is a representation of  $\mathcal{A}$  on some Hilbert space  $\mathcal{H}$ . A vector  $\xi \in \mathcal{H}$  is called **cyclic** and **separating** for the representation  $\pi$ , if  $\xi$  is cyclic and separating for  $\pi(\mathcal{A})$ , respectively. A representation with a cyclic vector is called a **cyclic representation**. ◀

Cyclic representations are indeed non-degenerate, and, in fact, every non-degenerate representation can be decomposed into a direct sum of cyclic representations. Let us first recall the definition of the direct sum of representations.

**Definition 1.1.4.** Let  $\mathcal{A}$  be a  $C^*$ -algebra. Given an index set  $I$  and for each  $i \in I$  a representation  $(\pi_i, \mathcal{H}_i)$  of  $\mathcal{A}$ , we define the **direct sum** of these representations as follows: let  $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$  denote the direct sum Hilbert space and let  $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$  be the representation defined by  $\pi(x)(\xi_i)_{i \in I} = (\pi_i(x)\xi_i)_{i \in I}$ , when  $(\xi_i)_{i \in I} \in \mathcal{H}$ . We will denote this representation by  $\bigoplus_{i \in I} \pi_i$ . ◀

**Theorem 1.1.5.** *Every non-degenerate representation of a  $C^*$ -algebra can be decomposed into a direct sum of cyclic representations.*

*Proof.* Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra and  $(\pi, \mathcal{H})$  a representation of  $\mathcal{A}$ . Let  $\mathcal{F}$  denote the collection of all subsets  $F \subseteq \mathcal{H}$  such that  $\pi(\mathcal{A})\xi$  is orthogonal to  $\pi(\mathcal{A})\eta$ , for every two distinct elements  $\xi, \eta \in F$ . This is clearly a partially ordered set, and every linearly ordered subset has a majorant, namely the union. Thus by Zorn's Lemma there exists a maximal element  $\{\xi_i\}_{i \in I} \in \mathcal{F}$ . It follows from the maximality of  $\{\xi_i\}_{i \in I}$  that  $\mathcal{H} = \overline{\bigoplus_{i \in I} \pi(\mathcal{A})\xi_i}$ . If we let  $\mathcal{H}_i = \overline{\pi(\mathcal{A})\xi_i}$  and denote by  $\pi_i$  the representation  $\mathcal{A} \rightarrow B(\mathcal{H}_i)$  defined by  $\pi_i(x)\eta = \pi(x)\eta$ , for all  $\eta \in \mathcal{H}_i$ , then clearly  $\pi_i$  is cyclic with cyclic vector  $\xi_i$ . Since  $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_i$ , we get that  $\pi = \bigoplus_{i \in I} \pi_i$ . Hence we have a decomposition of  $\pi$  into a direct sum of cyclic representations. ◻

Let us recall an important tool of  $C^*$ -algebras, namely, that approximate identities exists. Even that quasi-central approximate identities exists.

**Definition 1.1.6.** Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra and  $\mathcal{I}$  a closed two-sided ideal in  $\mathcal{A}$ . An increasing net  $(e_\lambda)_{\lambda \in \Lambda}$  of positive operators in the closed unit ball of  $\mathcal{A}$  is called an **approximate identity** (or **approximate unit**) for  $\mathcal{I}$  if  $\lim_{\lambda \in \Lambda} e_\lambda x = \lim_{\lambda \in \Lambda} x e_\lambda = x$  for each  $x \in \mathcal{I}$ . Such an approximate identity is called **quasi-central** if in addition  $\lim_{\lambda \in \Lambda} (y e_\lambda - e_\lambda y) = 0$ , for each  $y \in \mathcal{A}$ .  $\blacktriangleleft$

Such quasi-central approximate units always exists by [Dav96, Theorem I.9.16], so we will use this fact without mentioning. Also, we recall the following proposition (for a proof see [Dav96, proof of Theorem I.5.4]):

**Proposition 1.1.7.** *Suppose that  $\mathcal{I}$  is an ideal in a  $C^*$ -algebra  $\mathcal{A}$ , and that  $(e_\lambda)_{\lambda \in \Lambda}$  is an approximate unit for  $\mathcal{I}$ . If  $\pi: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{I}$  denotes the canonical surjection, then  $\|\pi(x)\| = \lim_{\lambda \in \Lambda} \|x - e_\lambda x\|$ , for all  $x \in \mathcal{A}$ .*

An easy application of the existence of approximate units is that for a representation  $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$  of a  $C^*$ -algebra  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$ , the strong operator closure of  $\pi(\mathcal{A})$  contains a largest projection, which acts as identity for the strong operator closure of  $\pi(\mathcal{A})$ . Indeed, an approximate unit in  $\pi(\mathcal{A})$  must necessarily converge in strong operator topology to such a projection. In particular, if the representation is non-degenerate, then this projection must be the identity in  $B(\mathcal{H})$ .

Let us end this section with a results on  $*$ -homomorphisms and unitizations.

**Proposition 1.1.8.** *Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are  $C^*$ -algebras, and that  $\pi: \mathcal{A} \rightarrow \mathcal{B}$  is a  $*$ -homomorphism. Then  $\pi$  extends uniquely to a unital  $*$ -homomorphism  $\pi_1: \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{B}}$ . If in addition we know that  $\mathcal{B}$  is unital, then  $\pi$  also extends uniquely to a  $*$ -homomorphism  $\pi_2: \tilde{\mathcal{A}} \rightarrow \mathcal{B}$ .*

This proposition is straightforward to prove. Indeed, it is obvious how the maps  $\pi_1$  and  $\pi_2$  should be defined on  $\tilde{\mathcal{A}}$ , so knowing this, one just checks that these choices are in fact  $*$ -homomorphisms.

## 1.2 Von Neumann algebras

A von Neumann algebra is a self-adjoint algebra of bounded linear operators on a Hilbert space, which contains the identity and is closed in the weak operator topology. We will use the letters  $\mathcal{M}$  and  $\mathcal{N}$  to denote von Neumann algebras, and  $\tau$  to denote traces. For a group  $\Gamma$  we will denote the group von Neumann algebra associated to  $\Gamma$  by  $L\Gamma$ .

Let us start by setting the notation and terminology for the many interesting locally convex topologies on the set of bounded linear operators on a Hilbert space.

- ◊ the **weak operator topology** is generated by the seminorms  $x \mapsto |\langle x\xi \mid \eta \rangle|$ , for  $\xi, \eta \in \mathcal{H}$ . This means that a net  $x_\alpha$  converges to  $x$  in weak operator topology if and only if  $\langle x_\alpha \xi \mid \eta \rangle$  converge to  $\langle x\xi \mid \eta \rangle$ , for all  $\xi, \eta \in \mathcal{H}$ ;
- ◊ the **strong operator topology** is generated by the seminorms  $x \mapsto \|x\xi\|$ , for  $\xi, \eta \in \mathcal{H}$ . This means that a net  $x_\alpha$  converges to  $x$  in strong operator topology if and only if  $x_\alpha \xi$  converge to  $x\xi$ , for all  $\xi \in \mathcal{H}$ ;
- ◊ the **strong\* operator topology** is generated by the seminorms  $x \mapsto (\|x\xi\|^2 + \|x^* \xi\|^2)^{1/2}$ , for  $\xi \in \mathcal{H}$ . This means that a net  $x_\alpha$  converge to  $x$  in strong\* operator topology if and only if  $x_\alpha$  and  $x_\alpha^*$  converge to  $x$  and  $x^*$  in strong operator topology, respectively;

- ◇ the **ultraweak operator topology** is generated by the family of seminorms  $x \mapsto \sum_{n=1}^{\infty} |\langle x\xi_n \mid \eta_n \rangle|$ , for sequences in  $\mathcal{H}$  satisfying  $\sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| < \infty$ . This means that a  $x_\alpha$  converge to  $x$  in ultraweak operator topology if and only if  $\sum_{n=1}^{\infty} \langle x_\alpha \xi_n \mid \eta_n \rangle$  converge to  $\sum_{n=1}^{\infty} \langle x \xi_n \mid \eta_n \rangle$ , for all sequences  $(\xi_n)_{n \in \mathbb{N}}$  and  $(\eta_n)_{n \in \mathbb{N}}$  as above;
- ◇ the **ultrastrong operator topology** is generated by the family of seminorms  $x \mapsto (\sum_{n=1}^{\infty} \|x\xi_n\|^2)^{1/2}$ , for sequences in  $\mathcal{H}$  satisfying  $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ . This means that a  $x_\alpha$  converge to  $x$  in ultrastrong operator topology if and only if  $(\sum_{n=1}^{\infty} \|(x_\alpha - x)\xi_n\|^2)^{1/2}$  converge to zero, for all sequences  $(\xi_n)_{n \in \mathbb{N}}$  and  $(\eta_n)_{n \in \mathbb{N}}$  as above;
- ◇ the **ultrastrong\* operator topology** is generated by the family of seminorms  $x \mapsto (\sum_{n=1}^{\infty} \|x\xi_n\|^2 + \|x^*\xi_n\|^2)^{1/2}$ , for sequences in  $\mathcal{H}$ , which satisfy that  $\sum_{n=1}^{\infty} \|\xi_n\|^2 < \infty$ . This means that a  $x_\alpha$  converge to  $x$  in ultrastrong\* operator topology if and only if  $x_\alpha$  and  $x_\alpha^*$  converge to  $x$  and  $x^*$  in ultrastrong operator topology, respectively.

The ultraweak, the ultrastrong and the ultrastrong\* operator topologies are also known to some as the  $\sigma$ -weak, the  $\sigma$ -strong and the  $\sigma$ -strong\* operator topologies, respectively.

There are obviously some relations between these topologies. Namely, the weak operator topology is weaker than both the strong operator topology and the ultraweak operator topology; the strong operator topology is weaker than the strong\* operator topology and the ultrastrong operator topology; the strong\* operator topology is weaker than the ultrastrong\* operator topology; the ultraweak operator topology is weaker than the ultrastrong operator topology, which is weaker than the ultrastrong\* operator topology. Also, all these topologies are weaker than the uniform topology, that is, the norm topology in  $B(\mathcal{H})$ .

In general, many of these topologies are different. In fact, if they all agree, then the Hilbert space is finite dimensional. This can be deduced from the fact that the unit ball of  $B(\mathcal{H})$  is compact in the weak operator topology. There are a lot of connections between these topologies. For example, each of the following topologies: the weak operator, the strong operator and the strong\* operator topology agrees on bounded sets with the ultraweak operator, the ultrastrong operator and the ultrastrong\* operator topology, respectively. The weak operator, the strong operator and the strong\* operator closures of a convex set agree, and the ultraweak operator, the ultrastrong operator and the ultrastrong\* operator closures of a convex set agree. This last fact is a consequence of a more general statement saying that the linear functionals which are continuous in the weak operator, the strong operator and the strong\* operator topology are the same, and the linear functionals which are continuous in the ultraweak operator, the ultrastrong operator and the ultrastrong\* operator topology are also the same. Let us recall what these functionals look like.

**Proposition 1.2.1.** *If  $\mathcal{M}$  is a von Neumann algebra on a Hilbert space  $\mathcal{H}$ , then a linear functional  $f$  on  $\mathcal{M}$  is continuous in the ultraweak operator topology if and only there exist  $\xi_n, \eta_n \in \mathcal{H}$  ( $n = 1, 2, 3, \dots$ ), so that*

$$\sum_{n=1}^{\infty} \|\xi_n\| \|\eta_n\| < \infty \quad \text{and} \quad f(x) = \sum_{n=1}^{\infty} \langle x\xi_n \mid \eta_n \rangle, \quad x \in \mathcal{M}.$$

Moreover, a linear functional  $f$  is weak operator continuous if and only if it has the above form, but with  $\xi_n = \eta_n = 0$  for all except finitely many  $n \in \mathbb{N}$ .

A proof of this can be found in [Tak02, Theorem II.2.6].

One can also show that if  $\mathcal{A}$  is a self-adjoint subalgebra of  $B(\mathcal{H})$ , then the closures of  $\mathcal{A}$  in the six operator topologies mentioned above agree. In particular,  $\mathcal{A}$  is a von Neumann algebra if and only if it contains the identity of  $B(\mathcal{H})$  and is closed in one of these six topologies.

Given a subset  $\mathfrak{X}$  of  $B(\mathcal{H})$  we denote by  $\mathfrak{X}'$  its commutant, that is, the set  $\{y \in B(\mathcal{H}) : yx = xy, \text{ for all } x \in \mathfrak{X}\}$ . Naturally, we also denote the double commutant of  $\mathfrak{X}$  by  $\mathfrak{X}''$ , that is, the commutant of  $\mathfrak{X}'$ .

Let us recall the type decomposition of a von Neumann algebra. We start by reviewing the concepts of abelian, finite and infinite projections.

**Definition 1.2.2.** Let  $\mathcal{M}$  be a von Neumann algebra, and  $p \in \mathcal{M}$  a projection. We say that  $p$  is **abelian** if  $p\mathcal{M}p$  is abelian, and we say that  $p$  is **finite** if whenever  $q \in \mathcal{M}$  is a projection equivalent to  $p$  with  $q \leq p$ , then  $q = p$ . A projection which is not finite is called **infinite**. The von Neumann algebra  $\mathcal{M}$  is called **finite**, if the identity in  $\mathcal{M}$  is finite, and **properly infinite**, if it does not contain any central non-zero finite projections. ◀

We now define the type of a von Neumann algebra.

**Definition 1.2.3.** Let  $\mathcal{M}$  be a von Neumann algebra. We say that  $\mathcal{M}$  is: of **type I** if every non-zero projection majorizes a non-zero abelian projection; of **type II** if it does not contain any non-zero abelian projection and every non-zero projection majorizes a non-zero finite projection; of **type III** if it does not contain any non-zero finite projections. ◀

The type decomposition then says the following:

**Proposition 1.2.4.** Every von Neumann algebra  $\mathcal{M}$  can be written uniquely as a direct sum  $\mathcal{M}_I \oplus \mathcal{M}_{II} \oplus \mathcal{M}_{III}$ , where  $\mathcal{M}_I$ ,  $\mathcal{M}_{II}$  and  $\mathcal{M}_{III}$  are either zero or of type I, II and III, respectively.

Also, every von Neumann algebra  $\mathcal{M}$  can be written uniquely as a direct sum  $\mathcal{M}_f \oplus \mathcal{M}_\infty$ , where  $\mathcal{M}_f$  is finite or zero and  $\mathcal{M}_\infty$  is properly infinite or zero.

Combining the two statements above one obtains, that every von Neumann algebra can be written uniquely as a direct sum of five<sup>1</sup> von Neumann algebras or zero, which are finite of type I, properly infinite of type I, finite of type II, properly infinite of type II and of type III, respectively.

By a von Neumann algebra factor we shall understand a von Neumann algebra, whose center consists only of scalar multiples of the identity. Clearly, a factor is exactly of one of the three types mentioned. It is also either finite or properly infinite. Finite von Neumann algebras of type II are called type  $II_1$  von Neumann algebras. Recall also that if  $\mathcal{M}$  is a finite von Neumann algebra factor, then either  $\mathcal{M}$  is isomorphic to  $\mathbb{M}_n$  for some  $n \in \mathbb{N}$ , in which case we say that  $\mathcal{M}$  is of type  $I_n$ , or it is of type  $II_1$ . The latter happens if and only if  $\mathcal{M}$  has infinite linear dimension.

We will frequently use the fact that a finite von Neumann algebra factor has a unique faithful normal trace, see [KR83, Proposition 8.5.3]. In fact, we will prove a

<sup>1</sup>Five, because type III von Neumann algebras automatically are properly infinite.

theorem below describing existence and uniqueness of tracial states on von Neumann algebras, see Theorem 1.3.6.

Later we shall need the notion of semi-finite von Neumann algebras, which we now recall.

**Definition 1.2.5.** A von Neumann algebra is said to be **semi-finite**, if it does not contain any type III summand. ◀

**Proposition 1.2.6.** *Let  $\mathcal{M}$  be a semi-finite von Neumann algebra. Then the identity can be decomposed into a sum of orthogonal finite projections. In particular, there exists an increasing net of finite von Neumann subalgebras whose union is strong operator dense in  $\mathcal{M}$ .*

*Proof.* This is a standard Zorn's Lemma argument. The set of families of orthogonal finite projections is a directed set with inclusion. It must necessarily have a maximal element, that is, a family  $(p_\alpha)_{\alpha \in A}$  maximal with respect to being an orthogonal family of finite projections. If  $\sum_{\alpha \in A} p_\alpha \neq \mathbf{1}$ , then since  $\mathcal{M}$  is semi-finite,  $\mathbf{1} - \sum_{\alpha \in A} p_\alpha$  would majorize a non-zero finite projection, thus contradicting the maximality.

Let  $\mathcal{F}$  denote the set of finite subsets of  $A$  ordered by inclusion, and for each  $F \in \mathcal{F}$ , let  $p_F$  denote the projection  $\sum_{\alpha \in F} p_\alpha$ . Also, for each  $F \in \mathcal{F}$ , let  $\mathcal{N}_F$  denote the von Neumann algebra  $p_F \mathcal{M} p_F + (I - p_F)\mathbb{C}$ , all of which are von Neumann subalgebras of  $\mathcal{M}$ . Now,  $(\mathcal{N}_F)_{F \in \mathcal{F}}$  is an increasing net of von Neumann subalgebras of  $\mathcal{M}$ , which are all finite since the projections  $p_F$ ,  $F \in \mathcal{F}$ , are all finite. Let  $\mathcal{N} = \bigcup_{F \in \mathcal{F}} \mathcal{N}_F$ . Since  $p_F x p_F \rightarrow x$  in strong operator topology, for all  $x \in \mathcal{M}$ , we get that  $\mathcal{N}$  is strong operator dense in  $\mathcal{M}$ . ◻

It is not hard to see that the above necessary criteria for being semi-finite must also be sufficient. For if  $\mathbf{1} = \sum_{\alpha \in A} p_\alpha$  is the decomposition of the identity as an orthogonal sum of finite projections, and  $z_{\text{III}}$  denotes the central projection in  $\mathcal{M}$  satisfying  $z_{\text{III}} \mathcal{M} z_{\text{III}} = \mathcal{M}_{\text{III}}$ , then for each  $\alpha \in A$  we have that  $p z_{\text{III}}$  is a finite projection in  $\mathcal{M}_{\text{III}}$ , and hence zero. But then clearly  $z_{\text{III}}$  must be zero.

Let us turn to the concept of approximately finite von Neumann algebras.

**Definition 1.2.7.** A von Neumann algebra  $\text{II}_1$ -factor  $\mathcal{M}$  is called **approximately finite**, if there exist an ascending sequence  $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \dots$  of von Neumann subalgebras of  $\mathcal{M}$  such that  $\bigcup_{k=1}^{\infty} \mathcal{N}_k$  is weak operator-dense in  $\mathcal{M}$  and  $\mathcal{N}_k$  a type  $\text{I}_{n_k}$ -factor for some increasing sequence  $(n_1, n_2, n_3, \dots)$  of natural numbers. ◀

It is a classical theorem of Murray and von Neumann that all approximately finite type  $\text{II}_1$ -factors are isomorphic—see [MvN43, Theorem XIV]—and this unique  $\text{II}_1$ -factor is referred to as the **hyperfinite  $\text{II}_1$ -factor**.<sup>2</sup> As customary, we denote this hyperfinite  $\text{II}_1$ -factor by  $\mathcal{R}$ . In this thesis we will not deal with the uniqueness of  $\mathcal{R}$ . We shall nonetheless mention that the hyperfinite  $\text{II}_1$ -factor satisfy the following:

**Proposition 1.2.8.** *The following von Neumann algebras are all isomorphic:*

$$\mathcal{R}, \quad \mathcal{R} \otimes \mathbb{M}_2, \quad \text{M}_2(\mathcal{R}) \quad \text{and} \quad \overline{\mathcal{R} \otimes \mathcal{R}}.$$

<sup>2</sup>For those who find [MvN43] a bit hard to read, we point out that Murray and von Neumann introduce four types of approximately finite, namely, approximately finite of type  $[p_1, p_2, p_3, \dots]$ , approximately finite (A), approximately finite (B) and approximately finite (C). These are given in Definition 4.1.1, Definition 4.3.1, Definition 4.5.2 and Definition 4.6.1 in [MvN43], respectively. In Theorem XII they prove that all these types of approximately finite are the same, and in Theorem XIV they prove that all approximately finite type  $\text{II}_1$ -factors have the same algebraic type, meaning that they are all  $*$ -isomorphic.

There are several ways to prove this. One might consider taking the infinite tensor product  $\overline{\bigotimes}_{n=1}^{\infty} \mathbb{M}_2$  as a model of  $\mathcal{R}$ , in which case the statement is fairly trivial.<sup>3</sup> If one is not comfortable with the infinite tensor product, then one can think of  $\mathcal{R}$  as the group von Neumann algebra of one's favorite countable discrete amenable group with infinite conjugacy classes. Indeed, Connes proved in his article [Con76], that a discrete countable group with infinite conjugacy classes is amenable if and only if its group von Neumann algebra is the hyperfinite  $\text{II}_1$ -factor.

When Murray and von Neumann proved uniqueness of the hyperfinite  $\text{II}_1$ -factor in [MvN43, Theorem XIV], they also proved the following:

**Proposition 1.2.9.** *For each sequence  $(n_1, n_2, n_3, \dots)$  of natural numbers converging to infinity, with  $n_k$  dividing  $n_{k+1}$ , for all  $k \in \mathbb{N}$ , there exist an ascending sequence  $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \dots$  of von Neumann subalgebras of  $\mathcal{R}$ , whose union is strong operator dense in  $\mathcal{R}$ , with  $\mathcal{N}_k$  a factor of type  $\text{I}_{n_k}$ .*

In particular, for each  $k \in \mathbb{N}$ , the hyperfinite  $\text{II}_1$ -factor contains a von Neumann subalgebra, which is a factor of type  $\text{I}_k$ .

Let us now consider an important object related to a von Neumann algebra, namely the predual. It turns out that a von Neumann algebra  $\mathcal{M}$ , in a natural way, can be identified with the dual space of the set of ultraweakly continuous linear functionals on  $\mathcal{M}$ .<sup>4</sup>

**Definition 1.2.10.** Let  $\mathcal{M}$  be a von Neumann algebra. The predual of  $\mathcal{M}$  is the subset  $\mathcal{M}_*$  of  $\mathcal{M}^*$  consisting of all the ultraweakly continuous linear functionals on  $\mathcal{M}$ . ◀

The predual is actually a Banach space, and if  $\mathcal{M}$  is a von Neumann algebra, then there is a canonical map from  $\mathcal{M}$  to the dual space of  $\mathcal{M}_*$ , namely the one mapping an element  $x \in \mathcal{M}$  to the map that evaluates at  $x$ . Our goal is to prove that this map is a surjective linear isometry.

**Theorem 1.2.11.** *Let  $\mathcal{M}$  be a von Neumann algebra on a Hilbert space  $\mathcal{H}$ . The point-evaluation map  $\mathcal{M} \rightarrow (\mathcal{M}_*)^*$  is a surjective linear isometry. In particular  $\mathcal{M} \cong (\mathcal{M}_*)^*$ , as Banach spaces.*

*Proof.* Let  $\Phi$  denote the point-evaluation map, that is,  $\Phi(x)(f) = f(x)$ , for all  $x \in \mathcal{M}$ ,  $f \in \mathcal{M}_*$ . Clearly  $\Phi$  is linear, and we start by proving that it is also an isometry. Suppose that  $x \in \mathcal{M}$  and  $f \in \mathcal{M}_*$ . Then  $|\Phi(x)(f)| = |f(x)| \leq \|f\| \|x\|$ , which shows that  $\|\Phi(x)\| \leq \|x\|$ . For the other inequality, suppose that  $x \in \mathcal{M}$  and let  $\xi \in \mathcal{H}$  with  $\|\xi\| = 1$ . If  $x\xi = 0$  then clearly  $\|\Phi(x)\| \geq \|x\xi\|$ . If  $x\xi \neq 0$ , then we let  $g$  denote the linear functions  $y \mapsto \|x\xi\|^{-1} \langle y\xi \mid x\xi \rangle$ . Now,  $g$  is an ultraweakly continuous linear functional of norm less than or equal to one. Since  $\Phi(x)(g) = \|x\xi\|$ , we get that  $\|\Phi(x)\| \geq \|x\xi\|$ . In any case, since  $\xi \in \mathcal{H}$  was arbitrary of norm one, we conclude that  $\|\Phi(x)\| \geq \|x\|$ . Hence  $\Phi$  is an isometry. Let us prove that  $\Phi$  is surjective. Suppose that  $\phi \in (\mathcal{M}_*)^*$ , and for  $\xi, \eta \in \mathcal{H}$  let  $f_{\xi, \eta}$  denote the linear functional on  $\mathcal{M}$  given by  $y \mapsto \langle y\xi \mid \eta \rangle$ . Consider the bilinear map

$$\hat{\phi}: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C} \quad \text{defined by} \quad (\xi, \eta) \mapsto \phi(f_{\xi, \eta}), \quad \xi, \eta \in \mathcal{H}$$

<sup>3</sup>We will not explain how the infinite tensor product of von Neumann algebras is constructed, nor will we prove that this particular tensor product turns out to be the hyperfinite  $\text{II}_1$ -factor.

<sup>4</sup>In fact, von Neumann algebras can be characterized as  $C^*$ -algebras having a unique predual, but we will not treat this uniqueness here.

Clearly  $\hat{\phi}$  is a bounded bilinear map bounded by  $\|\phi\|$ , and the Riesz Representation Theorem asserts that there exists some  $x \in B(\mathcal{H})$  so that

$$\hat{\phi}(\xi, \eta) = \langle x\xi \mid \eta \rangle, \quad \text{for all } \xi, \eta \in \mathcal{H}.$$

We want to show that  $x \in \mathcal{M}$ . Suppose that  $y \in \mathcal{M}'$  is a self-adjoint element. Then, for all  $z \in \mathcal{M}$  and  $\xi, \eta \in \mathcal{H}$ , we have

$$f_{\xi, y\eta}(z) = \langle z\xi \mid y\eta \rangle = \langle zy\xi \mid \eta \rangle = f_{y\xi, \eta}(z),$$

which shows that  $f_{\xi, y\eta} = f_{y\xi, \eta}$  for all  $\xi, \eta \in \mathcal{H}$ .<sup>5</sup> Now, by definition of this bilinear map we get that

$$\langle xy\xi \mid \eta \rangle = \phi(f_{y\xi, \eta}) = \phi(f_{\xi, y\eta}) = \langle yx\xi \mid \eta \rangle, \quad \text{for all } \xi, \eta \in \mathcal{H}.$$

Thus  $xy = yx$ . Since  $y \in \mathcal{M}'$  was an arbitrary self-adjoint element, and these span  $\mathcal{M}'$ , we get that  $xy = yx$ , for all  $y \in \mathcal{M}'$ . This shows that  $x \in \mathcal{M}$ , by von Neumann's Double Commutant Theorem. Now, notice that  $\Phi(x)(f_{\xi, \eta}) = \phi(f_{\xi, \eta})$ , for all  $\xi, \eta \in \mathcal{H}$ . Since the set  $\text{span}\{f_{\xi, \eta} : \xi, \eta \in \mathcal{H}\}$ , which actually consists of all the weak operator continuous linear functionals on  $\mathcal{M}$ , is dense in  $\mathcal{M}_*$ , continuity ensures that  $\Phi(x) = \phi$ . This proves surjectivity of  $\Phi$ , and the proof is complete.  $\square$

The following two proposition will be mentioned, as we will need them later:

**Proposition 1.2.12.** *If  $\mathcal{H}$  and  $\mathcal{K}$  are Hilbert spaces, then the map  $B(\mathcal{H}) \rightarrow B(\mathcal{H} \otimes \mathcal{K})$  given by  $x \mapsto x \otimes \mathbf{1}_{\mathcal{K}}$  is ultraweak operator-to-weak operator continuous.*

The above proposition is not hard to prove, but can be a bit tedious.

**Proposition 1.2.13.** *Suppose that  $\mathcal{M}$  is a von Neumann algebra and  $\mathcal{I} \subseteq \mathcal{M}$  an ultraweakly closed two-sided ideal. Then  $\mathcal{I}$  is complemented in  $\mathcal{M}$ , that is, there exists an ideal  $\mathcal{J} \subseteq \mathcal{M}$  such that  $\mathcal{M} = \mathcal{I} \oplus \mathcal{J}$ . More precisely, there exists a central projection  $p \in \mathcal{I}$  such that  $\mathcal{M} = p\mathcal{M} + (1 - p)\mathcal{M}$ .*

*Proof.* Let  $(e_\lambda)_\lambda$  be a quasi-central approximate identity for  $\mathcal{I}$  in  $\mathcal{M}$ . Then  $(e_\lambda)_\lambda$  converges strongly to a central projection  $p$  in  $\mathcal{M}$  which acts as the identity on  $\mathcal{I}$ . In particular, since  $\mathcal{I}$  is an ideal, we have  $\mathcal{I} = p\mathcal{M}$ . Now, with  $\mathcal{J} = (1 - p)\mathcal{M}$  we have  $\mathcal{M} = \mathcal{I} \oplus \mathcal{J}$ .  $\square$

### 1.3 Algebras with separable predual

It is a common practice in von Neumann algebra theory to make the assumption of separability of the predual. It is not obvious at first what the consequences are of this assumption. Let us quickly recall that, when it comes to  $C^*$ -algebras, it is often customary to assume norm separability. This custom is not a useful one for von Neumann algebras, since, as we shall see, a norm separable von Neumann algebra is, in fact, finite dimensional. In this section we will discuss in detail the meaning of separability of the predual for a von Neumann algebra.

<sup>5</sup>Here it is of great importance that  $f_{\xi, \eta}$  is considered a linear functional on  $\mathcal{M}$ , and not on the whole of  $B(\mathcal{H})$ , in which case the statement is not necessarily true.



**Lemma 1.3.1.** *Suppose that  $\mathcal{A}$  is a  $*$ -subalgebra of bounded operators on a Hilbert space  $\mathcal{H}$ , and that  $\mathfrak{X}$  is a subset of  $\mathcal{H}$ . Then the projection onto the closed linear span of  $\mathcal{A}\mathfrak{X}$  belongs to  $\mathcal{A}'$ .*

*Proof.* Denote the closed linear span of  $\mathcal{A}\mathfrak{X}$  by  $\mathcal{K}$ , and let  $p$  denote the projection onto  $\mathcal{K}$ . Let  $x \in \mathcal{A}$ . Clearly the linear span of  $\mathcal{A}\mathfrak{X}$  is reducing for  $x$ , that is, it is invariant under  $x$  and  $x^*$ . In particular, since the linear span of  $\mathcal{A}\mathfrak{X}$  is dense in  $\mathcal{K}$ , also  $\mathcal{K}$  is reducing for  $x$ . This means that  $x$  commutes with  $p$ , so since  $x \in \mathcal{A}$  was arbitrary, we conclude that  $p \in \mathcal{A}'$ .  $\square$

The following definition generalizes the concepts of cyclic and separating vectors.

**Definition 1.3.2.** Suppose that  $\mathcal{A}$  is a set of bounded operators on a Hilbert space  $\mathcal{H}$ , and that  $\mathfrak{X}$  is a subset of  $\mathcal{H}$ . The subset  $\mathfrak{X}$  is called **separating** for  $\mathcal{A}$  if, for each  $x \in \mathcal{A}$ ,  $x\xi = 0$ , whenever  $x\xi = 0$  for all  $\xi \in \mathfrak{X}$ , and it is called **cyclic** for  $\mathcal{A}$  if  $\mathcal{A}\mathfrak{X} = \{x\xi : x \in \mathcal{A}, \xi \in \mathfrak{X}\}$  spans a dense subspace of  $\mathcal{H}$ .  $\blacktriangleleft$

As with separating and cyclic vectors, where there is a natural relationship between these, we also have a relationship between cyclic and separating subsets.

**Proposition 1.3.3.** *Suppose that  $\mathcal{A}$  is a  $*$ -algebra of bounded operators on a Hilbert space, which contains the unit. A subset  $\mathfrak{X}$  is cyclic for  $\mathcal{A}$  if and only if  $\mathfrak{X}$  is separating for  $\mathcal{A}'$ .*

*Proof.* Suppose that  $\mathfrak{X}$  is cyclic for  $\mathcal{A}$ . Let  $y \in \mathcal{A}'$ , with  $y\xi = 0$ , for all  $\xi \in \mathfrak{X}$ . Fix  $\eta \in \mathcal{H}$ . For each  $x \in \mathcal{A}$  and  $\xi \in \mathfrak{X}$  we have

$$\langle y^*\eta \mid x\xi \rangle = \langle \eta \mid xy\xi \rangle = 0,$$

so since  $\mathcal{A}\mathfrak{X}$  spans a dense subset of  $\mathcal{H}$ , we get by continuity, that  $y^*\eta = 0$ . Since  $\eta$  was arbitrary, this shows that  $y^* = 0$ , and indeed also  $y = 0$ . Hence  $\mathfrak{X}$  is separating for  $\mathcal{A}$ .

Suppose instead that  $\mathfrak{X}$  is separating for  $\mathcal{A}'$ . Let  $p$  denote the projection onto the closed linear span of  $\mathcal{A}\mathfrak{X}$ . What we need to prove is that  $p$  is indeed the identity operator. By Lemma 1.3.1 we know that  $p \in \mathcal{A}'$ . Since  $\mathcal{A}$  contains the identity operator the closed linear span of  $\mathcal{A}\mathfrak{X}$  contains  $\mathfrak{X}$ . In particular  $(1-p)\xi = 0$ , for all  $\xi \in \mathfrak{X}$ . Now, since  $1-p \in \mathcal{A}'$ , and  $\mathfrak{X}$  is separating for  $\mathcal{A}'$ , it follows that  $1-p = 0$ . Hence  $\mathfrak{X}$  is cyclic for  $\mathcal{A}$ .  $\square$

**Definition 1.3.4.** A projection  $p$  in a von Neumann algebra  $\mathcal{M}$  is called **countably decomposable** if every orthogonal family of non-zero subprojections of  $p$  is countable. The von Neumann algebra  $\mathcal{M}$  is called **countably decomposable** if the identity is countably decomposable.  $\blacktriangleleft$

A countably decomposable projection is also called  $\sigma$ -**finite** in the literature, and likewise a countably decomposable von Neumann algebra is also called  $\sigma$ -**finite** in the literature.

Notice that, as with many other properties of projections, being countably decomposable is a property relative to the von Neumann algebra in question. The next proposition gives equivalent formulations of countable decomposability:

**Proposition 1.3.5.** *For a von Neumann algebra  $\mathcal{M}$  on a Hilbert space  $\mathcal{H}$  the following are equivalent:*

- (i)  $\mathcal{M}$  is countably decomposable;
- (ii) there is a countable subset of  $\mathcal{H}$ , which is separating for  $\mathcal{M}$ ;
- (iii)  $\mathcal{M}$  admits a normal faithful state.

Moreover, the countable separating set can be chosen orthogonal.

*Proof.* Suppose that (i) holds and let us prove (ii). By Zorn's Lemma, we choose a family of unit vectors  $\{\xi_\alpha : \alpha \in A\}$  in  $\mathcal{H}$ , which is maximal with respect to the property that  $\mathcal{M}'\xi_\alpha$  is orthogonal to  $\mathcal{M}'\xi_\beta$ , when  $\alpha, \beta \in A$  with  $\alpha \neq \beta$ . Let us argue why this is possible. Clearly the set of such families is non-empty, since every unit vector constitutes such a family, and if we are given a linearly ordered subset of such families—the set is ordered by inclusion of course—then their union will again be such a family. Hence by Zorn's Lemma there exist a family  $\{\xi_\alpha : \alpha \in A\}$ , which is maximal with this property. Denote this family by  $\mathfrak{X}$ . Now, by maximality the set  $\mathfrak{X}$  must be cyclic for  $\mathcal{M}'$ , for if not, then there exists some unit vector  $\xi$ , which is orthogonal to  $\mathcal{M}'\mathfrak{X}$ , but then  $\langle x\xi \mid y\xi_\alpha \rangle = \langle \xi \mid x^*y\xi_\alpha \rangle = 0$ , for all  $\alpha$  and  $x, y \in \mathcal{M}'$ , which shows that the family  $\mathfrak{X} \cup \{\xi\}$  is strictly larger than  $\mathfrak{X}$  with the property described above, thus contradicting the maximality of  $\mathfrak{X}$ . Now, by Proposition 1.3.3 and von Neumann's Double Commutant Theorem (see [KR83, Theorem 5.3.1]) we get that  $\mathfrak{X}$  is separating for  $\mathcal{M}$  since it is cyclic for  $\mathcal{M}'$ . For each  $\alpha \in A$ , let  $p_\alpha$  denote the projection onto the closure of  $\mathcal{M}'\xi_\alpha$ . Then  $p_\alpha$  belongs to  $\mathcal{M}$ , for all  $\alpha \in A$ , by Lemma 1.3.1. Since  $\xi_\alpha \in \mathcal{M}'\xi_\alpha$  we get that  $p_\alpha$  is non-zero, for all  $\alpha \in A$ , so, in particular, since the projections  $(p_\alpha)_{\alpha \in A}$  are all orthogonal, we get by assumption that  $A$  is countable. Thus  $\mathfrak{X}$  is a countable separating subset for  $\mathcal{M}$ , which proves (ii).

Next, assume that (ii) holds. Let  $\{\xi_n : n \in \mathbb{N}\}$ , be a subset of  $\mathcal{H}$ , which is separating for  $\mathcal{M}$ . By normalizing, we may assume that all the vectors are unit vectors. Define now a linear functional  $\phi$  by

$$\phi(x) = \sum_{n=1}^{\infty} 2^{-n} \langle x\xi_n \mid \xi_n \rangle, \quad x \in \mathcal{M}.$$

This linear functional is a well-defined normal state since

$$\sum_{n=1}^{\infty} 2^{-n} \|\xi_n\|^2 = \sum_{n=1}^{\infty} 2^{-n} = 1,$$

and it is clearly faithful, since the set  $\{\xi_n : n \in \mathbb{N}\}$  is separating for  $\mathcal{M}$ .

Last, suppose that (iii) holds, and let  $\phi$  be a faithful normal state on  $\mathcal{M}$ . Let  $(p_\alpha)_{\alpha \in A}$  be an orthogonal family of non-zero projections in  $\mathcal{M}$ . Let  $p = \sum_{\alpha \in A} p_\alpha$ . By normality of  $\phi$ , we get that  $\phi(p) = \sum_{\alpha \in A} \phi(p_\alpha)$ . Since  $\phi(p) = \sum_{\alpha \in A} \phi(p_\alpha)$  is a convergent sum of non-negative numbers, we conclude that  $\phi(p_\alpha)$  can be non-zero for only countably many  $\alpha \in A$ . Since  $\phi$  is faithful, we know that  $\phi(p_\alpha) > 0$ , for all  $\alpha \in A$ . Thus we conclude that  $A$  is countable, which shows that (i) holds.  $\square$

Now we will prove a result about traces on von Neumann algebras. We will later use the existence part of this result.

**Theorem 1.3.6.** *A von Neumann algebra has a faithful normal tracial state if and only if it is finite and countably decomposable. This tracial state is unique if and only if the von Neumann algebra is a factor. Moreover, any finite von Neumann algebra has a separating family of normal tracial states.*

*Proof.* Clearly a von Neumann algebra with a faithful normal trace is finite, and it is also countably decomposable by Proposition 1.3.5. So the rest of the proof we will assume that we are dealing with a finite von Neumann algebra.

Suppose that  $\mathcal{M}$  is a finite von Neumann algebra and let  $\mathcal{C}$  denote its center. We will let  $T$  denote the center-valued trace on  $\mathcal{M}$ , that is, the unique linear map  $T: \mathcal{M} \rightarrow \mathcal{C}$  satisfying  $T(x) = x$ , for all  $x \in \mathcal{C}$  and  $T(xy) = T(yx)$ , for all  $x, y \in \mathcal{M}$ . This map automatically ultraweakly continuous and bounded of norm one, with the properties that  $T(x) > 0$  if  $x > 0$  and  $T(yx) = yT(x)$ , for all  $x \in \mathcal{M}$  and  $y \in \mathcal{C}$ . See [KR86, Theorem 8.2.8] for the existence and properties of the center-valued trace.

Suppose  $\mathcal{M}$  is countably decomposable. Then  $\mathcal{C}$  is countably decomposable as well. By Proposition 1.3.5 we may choose some faithful normal state  $\phi$  on  $\mathcal{C}$ . Now  $\phi \circ T$  will be a faithful normal tracial state on  $\mathcal{M}$ .

This proves the first statement, let us now prove the last statement. Every von Neumann algebra has a separating set of normal states, or in other words, the set of normal states is separating. Let  $S$  denote the set of normal states. Then  $\{\psi \circ T : \psi \in S\}$  is a separating family of tracial states.

Last we prove the statement of uniqueness. So suppose that  $\tau$  is a faithful normal trace on  $\mathcal{M}$ . Assume that  $\mathcal{C} = \mathbb{C}\mathbf{1}$ . There is a unique state on  $\mathcal{C}$ , and it is given by  $\lambda\mathbf{1} \mapsto \lambda$ , for  $\lambda \in \mathbb{C}$ . Denote this state by  $\phi$ . If we can prove that  $\tau = \phi \circ T$ , then we are done, since  $\tau$  was arbitrary. By uniqueness of  $T$  we get that the map  $g: \mathcal{M} \rightarrow \mathcal{C}$  given by  $x \mapsto \tau(x)\mathbf{1}$  must equal  $T$ , and so  $\tau = \phi \circ g = T \circ \phi$ . This proves uniqueness of  $\tau$ . Suppose conversely that  $\mathcal{C} \neq \mathbb{C}\mathbf{1}$ . It suffices to prove that there exist two distinct faithful normal states  $\psi_1$  and  $\psi_2$  on  $\mathcal{C}$ , because then  $\psi_1 \circ T$  and  $\psi_2 \circ T$  will be two distinct faithful tracial states on  $\mathcal{M}$ . Let  $p \in \mathcal{C}$  be a non-zero projection different from  $\mathbf{1}$ . Since  $\mathcal{C}$  is countably decomposable there exist a faithful normal state  $\phi_1$  on  $\mathcal{C}$ . Let  $t \in (0, 1)$ , with  $t \neq \phi_1(p)$ , and define a linear functional  $\phi_2: \mathcal{C} \rightarrow \mathbb{C}$  by

$$\phi_2(x) = \frac{t}{\phi_1(p)}\phi_1(px) + \frac{1-t}{1-\phi_1(p)}\phi_1((\mathbf{1}-p)x).$$

It is straight forward to check that  $\phi_2$  is a normal state on  $\mathcal{C}$ . Moreover  $\phi_2 \neq \phi_1$ , since for example  $\phi_2(p) = t \neq \phi_1(p)$ . This proves that the trace on  $\mathcal{M}$  is not unique, and thus concludes the proof.  $\square$

Recall that the **support** (or **carrier**) of a positive normal linear functional  $\phi$  on a von Neumann algebra  $\mathcal{M}$  is a projection  $p \in \mathcal{M}$ , such that  $\phi(q) = 0$ , for all projections  $q \in \mathcal{M}$  with  $q \leq \mathbf{1} - p$ , and  $\phi(q) > 0$ , for all non-zero projections  $q \in \mathcal{M}$  with  $q \leq p$ . In particular the support projection of  $\phi$  countably decomposable, since  $\phi$  is faithful on  $p\mathcal{M}p$ . See for example [KR86, Definition 7.1.1].

Next is an analog of Proposition 1.2.6 for countably decomposability instead of finiteness.

**Proposition 1.3.7.** *Let  $\mathcal{M}$  be any von Neumann algebra. Then the identity can be decomposed into a sum of orthogonal countably decomposable projections. In particular, there exists an increasing net of countably decomposable von Neumann subalgebras whose union is strong operator dense in  $\mathcal{M}$ .*

*Proof.* The proof is almost the same as the one of Proposition 1.2.6. In particular the second statement is proved in the exact same way, when noting that the sum of orthogonal countably decomposable projections is again countably decomposable,

and that if  $p$  is a countably decomposable projection, then  $p\mathcal{M}p + (\mathbf{1} - p)\mathbb{C}$  is a countably decomposable von Neumann subalgebra of  $\mathcal{M}$ . Hence we prove only the first statement.

The first statement is a standard Zorn's Lemma argument. The set of families of orthogonal countably decomposable projections is a directed set with inclusion. First note that this set is non-empty, since the support projection of any non-zero positive normal linear functional is a non-zero countably decomposable projection. Choose a maximal element by Zorn's Lemma, that is, a family  $(p_\alpha)_{\alpha \in A}$  maximal with respect to being an orthogonal family of countably decomposable projections. Let us show that  $\sum_{\alpha \in A} p_\alpha = \mathbf{1}$ . Suppose not, and denote  $\sum_{\alpha \in A} p_\alpha$  by  $p$ . Choose positive normal linear functional  $\phi$  on  $\mathcal{M}$  with  $\phi(\mathbf{1} - p) > 0$ . Now the positive normal linear functional on  $\mathcal{M}$  given by  $x \mapsto \phi((\mathbf{1} - p)x(\mathbf{1} - p))$  is non zero with support projection below  $\mathbf{1} - p$ , thus contradicting the maximality of  $(p_\alpha)_{\alpha \in A}$ . Hence  $\sum_{\alpha \in A} p_\alpha = \mathbf{1}$ .

As mentioned in the beginning, the proof of the second statement is the same as that in Proposition 1.2.6, using the notes made in the beginning.  $\square$

**Definition 1.3.8.** A subset  $\mathfrak{A}$  of a von Neumann algebra  $\mathcal{M}$  is called a **generating set** for  $\mathcal{M}$ , if  $\mathcal{M}$  is the smallest von Neumann algebra containing  $\mathfrak{A}$ . If the subset  $\mathfrak{A}$  is self-adjoint, then this amounts to  $\mathfrak{A}'' = \mathcal{M}$ . A von Neumann algebra is called **finitely generated**, if it has a finite generating set, and **countably generated**, if it has a countable generating set.  $\blacktriangleleft$

The von Neumann algebra generated by a subset  $\mathfrak{A}$ , that is, the weak operator closure of the  $*$ -algebra generated by  $\mathfrak{A}$  and the identity operator, is of course the smallest von Neumann algebra containing  $\mathfrak{A}$ . Hence the set  $\mathfrak{A}$  is a generating set for the von Neumann algebra generated by  $\mathfrak{A}$ . If this were not the case, then the terminology would have been very poorly chosen.

Let us prove the following easy proposition, which states that generating subsets are preserved under  $*$ -homomorphisms.

**Proposition 1.3.9.** *Suppose that  $\mathcal{M}$  and  $\mathcal{N}$  are von Neumann algebras, and that  $\psi: \mathcal{M} \rightarrow \mathcal{N}$  is a  $*$ -isomorphism. If  $\mathfrak{A}$  is a generating subset for  $\mathcal{M}$ , then  $\psi(\mathfrak{A})$  is a generating subset of  $\mathcal{N}$ .*

*Proof.* Let  $\mathcal{A}$  denote the unital  $C^*$ -algebra generated by  $\mathfrak{A}$ . Then  $\mathcal{A}$  is ultraweakly dense in  $\mathcal{M}$ , so  $\psi(\mathcal{A})$  is ultraweakly dense in  $\mathcal{N}$ , since  $\psi$  is an ultraweak operator-to-ultraweak operator homeomorphism. The  $C^*$ -algebra generated by  $\psi(\mathfrak{A})$  is  $\psi(\mathcal{A})$ , since  $\psi$  is isometric. Hence  $\psi(\mathfrak{A})$  generates  $\mathcal{N}$ .  $\square$

Now that we have introduced the necessary terminology, we are ready to prove the characterization of von Neumann algebras with separable predual. But first we need the following general result from topology:

**Proposition 1.3.10.** *A metric space  $\mathfrak{X}$  is separable if and only if it is second-countable, that is, the topology has a countable basis. In particular every subset of a separable metric space is itself separable.*

*Proof.* Suppose that  $\mathfrak{X}$  is separable, and let  $\{x_n : n \in \mathbb{N}\}$  be a dense subset of  $\mathfrak{X}$ . Denote by  $B(x, r)$  the ball in  $\mathfrak{X}$  with center  $x \in \mathfrak{X}$  and radius  $r > 0$ . It is straightforward to check that the collection  $B = \{B(x_n, m^{-1}) : n, m \in \mathbb{N}\}$  defines a countable basis for the topology.

Suppose instead that  $\mathfrak{X}$  is second-countable. Let  $\{U_n : n \in \mathbb{N}\}$  be a countable basis for the topology. For each  $n \in \mathbb{N}$ , choose some  $y_n \in U_n$ . Then  $\{y_n : n \in \mathbb{N}\}$  is dense in  $\mathfrak{X}$  since it meets every open set.

Now, the last assertion follows from the fact that a basis for the space  $\mathfrak{X}$  restricts to a basis for the subspace topology on a subset.  $\square$

If this was not already clear, we stress that, by separable predual, we mean that the predual is separable in the norm topology.

**Theorem 1.3.11.** *Suppose that  $\mathcal{M}$  is a von Neumann algebra. Then the following conditions are equivalent:*<sup>6</sup>

- (i)  $\mathcal{M}$  has separable predual;
- (ii)  $\mathcal{M}$  can be represented faithfully as a von Neumann algebra on a separable Hilbert space;
- (iii)  $\mathcal{M}$  is countably generated and countably decomposable;
- (iv)  $\mathcal{M}$  is countably decomposable and separable in one of the six locally convex operator topologies;
- (v)  $\mathcal{M}$  is countably decomposable and separable in all of the six locally convex operator topologies;
- (vi)  $\mathcal{M}$  is countably decomposable and  $\mathcal{M}_1$  separable in one of the six locally convex operator topologies;
- (vii)  $\mathcal{M}$  is countably decomposable and  $\mathcal{M}_1$  separable in all of the six locally convex operator topologies.

*Proof.* Suppose that (i) holds, and let us prove (vii). Choose a norm dense subset  $\{f_n : n \in \mathbb{N}\}$  of  $\mathcal{M}_*$ , and for each  $n, m \in \mathbb{N}$  choose some  $x_{n,m}$  in the closed unit ball of  $\mathcal{M}$ , with  $\operatorname{Re} f_n(x_{n,m}) \geq \|f_n\| - \frac{1}{m}$ . Let  $\mathfrak{X}$  denote the set of convex combinations of elements from  $\{x_{n,m} : n, m \in \mathbb{N}\}$  with rational coefficients, that is, the set

$$\bigcup_{k=1}^{\infty} \left\{ \sum_{n,m=1}^k \lambda_{n,m} x_{n,m} : \lambda_{n,m} \in \mathbb{Q} \cap [0, 1], n, m = 1, 2, \dots, k, \sum_{n,m=1}^k \lambda_{n,m} = 1 \right\}.$$

This is clearly countable and contains  $\{x_{n,m} : n, m \in \mathbb{N}\}$ . We now want to prove that  $\mathfrak{X}$  is ultrastrong\* operator dense in  $\mathcal{M}_1$ . Suppose towards a contradiction that this is not the case, and let  $y$  be an element in  $\mathcal{M}_1$  which is not in the ultrastrong\* closure of  $\mathfrak{X}$ . Since the ultrastrong\* topology is weaker than the norm topology, the ultrastrong\* closure of  $\mathfrak{X}$  and the ultrastrong\* closure of the norm closure of  $\mathfrak{X}$  must agree. The norm closure of  $\mathfrak{X}$  is clearly equal to the closed convex hull of  $\{x_{n,m} : n, m \in \mathbb{N}\}$ , so, in particular, the ultrastrong\* closure of  $\mathfrak{X}$  must be convex, since it is equal to the ultrastrong\* closure of a convex set, namely the norm closure of  $\mathfrak{X}$ . By the Hahn Banach Theorem (see [KR83, Corollary 1.2.12]) there exists an ultrastrong\* operator continuous linear functional  $g$  on  $\mathcal{M}$ , such that

$$\sup\{\operatorname{Re} g(x) : x \in \mathfrak{X}\} < \operatorname{Re} g(y).$$

<sup>6</sup>Note, that by the six locally convex operator topologies, we mean the weak, the strong, the strong\*, the ultraweak, the ultrastrong and the ultrastrong\* operator topology.

Since the ultrastrong\* operator continuous linear functionals and the ultraweak operator continuous linear functionals are the same we get that  $g \in \mathcal{M}_*$ . Fix  $n \in \mathbb{N}$ . Choose some  $k \in \mathbb{N}$ , so that  $\|f_k - g\| < \frac{1}{n}$ . Then  $\|f_k\| > \|g\| - \frac{1}{n}$ , and we see that

$$\begin{aligned} \operatorname{Re} g(x_{k,n}) &= \operatorname{Re} f_k(x_{k,n}) - \operatorname{Re}(f_k(x_{k,n}) - g(x_{k,n})) \\ &\geq \|f_k\| - n^{-1} - |f_k(x_{k,n}) - g(x_{k,n})| \\ &> \|g\| - 3n^{-1}. \end{aligned}$$

In particular, since  $n$  was arbitrary, we conclude that  $\sup\{\operatorname{Re} g(x) : x \in \mathfrak{X}\} \geq \|g\|$ . This gives rise to a contradiction, since  $y$  was in the unit ball of  $\mathcal{M}$ , so that

$$\|g\| \leq \sup\{\operatorname{Re} g(x) : x \in \mathfrak{X}\} < \operatorname{Re} g(y) \leq |g(y)| \leq \|g\|.$$

It follows that  $\mathfrak{X}$  is ultrastrong\* dense in the unit ball of  $\mathcal{M}$ . Since the ultrastrong\* operator topology is the strongest of the six locally convex operator topologies, we conclude that  $\mathfrak{X}$  is dense in the unit ball of  $\mathcal{M}$  with respect to all of them. What we need now, is to show that  $\mathcal{M}$  is countably decomposable. Since  $\mathcal{M}_*$  is a separable, we get by Proposition 1.3.10 that the set of states in  $\mathcal{M}_*$  is norm separable. Let  $\{\phi_n : n \in \mathbb{N}\}$  be a norm dense subset of the states in  $\mathcal{M}_*$ . Now, let  $\phi$  denote the linear functional  $\sum_{n=1}^{\infty} 2^{-n} \phi_n$ , and let us show that  $\phi$  is faithful. Let  $x \in \mathcal{M}$  be a non-zero positive element. Choose some normal state  $\psi$  on  $\mathcal{M}$ , such that  $\psi(x) > 0$ , for example a vector state corresponding to a unit vector not in the kernel of  $x$ . Let  $n \in \mathbb{N}$  be so that  $\|\phi_n - \psi\| < 2^{-1} \psi(x)$ . Then  $\phi_n(x) \geq 2^{-1} \psi(x) > 0$ , so in particular  $\phi(x) \geq \phi_n(x) > 0$ . Hence  $\phi$  is faithful, and by Proposition 1.3.5 we conclude (vii).

Clearly (vii) implies (vi) and (v) implies (iv), so let us show that (vii) also implies (v). If (vii) holds, then we can choose some ultrastrong\* dense subset  $\mathfrak{Y}$  of  $\mathcal{M}_1$ . The set  $\mathbb{Q}\mathfrak{Y} = \{sx : s \in \mathbb{Q}, x \in \mathfrak{Y}\}$  will then be ultrastrong\* dense in  $\mathcal{M}$ , and since the ultrastrong\* operator topology is the strongest of the six locally convex operator topologies, we conclude that  $\mathbb{Q}\mathfrak{Y}$  is also dense in  $\mathcal{M}$  with respect to the five others locally convex operator topologies. Thus we conclude (v). The same argument, with the ultrastrong\* operator topology replaced by one of the other topologies, can be used to prove that (vi) implies (iv).

Assume now that (iv) holds, and let us prove (iii). All we need to show is that  $\mathcal{M}$  is countably generated. By assumption we can find a countable set  $\mathfrak{Y}$  which is weak operator dense in  $\mathcal{M}$ , since a set which is dense with respect to one of the six topologies, must necessarily be weak operator dense. By von Neumann's Double Commutant Theorem  $\mathcal{M}$  contains  $\mathfrak{Y}''$ , and since the latter set is weak operator closed, it must contain the weak operator closure of  $\mathfrak{Y}$ , which was  $\mathcal{M}$ . Thus the set  $\mathfrak{Y}$  generates  $\mathcal{M}$ . Hence (iii) holds.

Let us then prove that (iii) implies (ii). Let  $\mathfrak{X}$  be a countable generating set for  $\mathcal{M}$ . We may assume that it is self-adjoint and contains the unit, since we can always replace it with the countable set  $\mathfrak{X} \cup \mathfrak{X}^* \cup \{1\}$ . By Proposition 1.3.5 there exists a countable subset  $\{\xi_1, \xi_2, \xi_3, \dots\}$  of  $\mathcal{H}$ , which is separating for  $\mathcal{M}$ . Let us argue that we can find a countable strong operator dense subset of  $\mathcal{M}$ . The set

$$\mathfrak{A} = \{x_1 x_2 \cdots x_n : n \in \mathbb{N}, x_k \in \mathfrak{X}, k = 1, 2, \dots, n\}$$

of finite products of elements from  $\mathfrak{X}$  is clearly countable, and the linear span of this set is a \*-algebra since  $\mathfrak{X}$  was assumed to be self-adjoint containing the unit. The set of linear combinations of elements in  $\mathfrak{A}$  with rational coefficients is countable, and we

denote it by  $\mathfrak{A}_{\mathbb{Q}}$ . By construction, the strong operator closure of  $\mathfrak{A}_{\mathbb{Q}}$  contains a strong operator closed  $*$ -algebra, which, in turn, contains  $\mathfrak{X}$ . Thus, since  $\mathfrak{X}''$  is the smallest such algebra, we conclude that the strong operator closure of  $\mathfrak{A}_{\mathbb{Q}}$  contains  $\mathfrak{X}''$ , but  $\mathfrak{X}$  was a generating set for  $\mathcal{M}$ , so  $\mathfrak{X}'' = \mathcal{M}$ . This shows that  $\mathfrak{A}_{\mathbb{Q}}$  is strong operator dense in  $\mathcal{M}$ . From this it follows that the countable set consisting of linear combinations of elements from the set

$$\{x\xi_n : x \in \mathfrak{A}_{\mathbb{Q}}, n \in \mathbb{N}\}$$

with rational coefficients, is dense in the closed linear span of  $\mathcal{M}\{\xi_1, \xi_2, \xi_3, \dots\}$ , which is then separable. Let  $p$  denote the projection onto this closed subspace. Then  $p$  commutes with  $\mathcal{M}$  by Lemma 1.3.1, so  $p\mathcal{H}$  is a separable reducing subspace for  $\mathcal{M}$ . The representation  $\rho: \mathcal{M} \rightarrow B(p\mathcal{H})$  defined by  $\rho(x) = px$ , is then a weak operator continuous  $*$ -homomorphism, which is injective, since  $p\mathcal{H}$  contains the separating set  $\{\xi_1, \xi_2, \xi_3, \dots\}$ . Thus the image is a von Neumann algebra, and we have proved (ii).

Suppose that (ii) holds, and let us prove (i). Let  $\mathcal{K}$  be a separable Hilbert space, and let  $\pi: \mathcal{M} \rightarrow B(\mathcal{K})$  be a faithful representation, with  $\pi(\mathcal{M})$  a von Neumann algebra. Since isomorphisms of von Neumann algebras are automatically ultraweakly continuous, the restriction of  $\pi^*$  to the predual  $\pi(\mathcal{M})_*$  of  $\pi(\mathcal{M})$ , is an isometric isomorphism of Banach spaces. Hence it suffices to show that  $\pi(\mathcal{M})$  has separable predual. This though should be clear from the fact that  $\mathcal{H}$  is separable, and the fact that the weak operator continuous linear functionals are norm dense in the set of ultraweakly continuous linear functionals.  $\square$

Given the theorem above, the assumption of a separable predual suddenly seems very reasonable, and definitely a desirable assumption to make.

**Proposition 1.3.12.** *If  $\mathcal{M}$  is a von Neumann algebra which is separable in the weak operator topology, then the Hilbert space  $\mathcal{H}_\phi$  from the GNS-construction corresponding to a normal positive linear functional  $\phi$  on  $\mathcal{M}$  is separable.*

*Proof.* Let  $\phi$  be a normal positive linear functional, and let  $j: \mathcal{M} \rightarrow \mathcal{H}_\phi$  denote the natural map sending an element to its equivalence class. It is not hard to see, by arguing as in the proof of Theorem 1.3.11, that  $\mathcal{M}$  is indeed separable in the ultrastrong topology, as well. Choose some countable subset  $\mathfrak{A}$  of  $\mathcal{M}$ , which is ultrastrong operator dense in  $\mathcal{M}$ . Let  $\tilde{\mathfrak{A}}$  denote the closure of  $j(\mathfrak{A})$  in  $\mathcal{H}_\phi$ . We want to show that  $\tilde{\mathfrak{A}}$  contains  $j(\mathcal{M})$ , because then we are done, since  $j(\mathcal{M})$  is dense in  $\mathcal{H}_\phi$ . Let  $x \in \mathcal{M}$ . Since  $\mathfrak{A}$  is ultrastrongly dense in  $\mathcal{M}$ , we can choose some net  $(x_\alpha)_{\alpha \in A}$ , which converges to  $x$  in ultrastrong operator topology. Then  $(x_\alpha - x)^*(x_\alpha - x)$  converges to zero in the ultraweak operator topology. Since  $\phi$  is normal, we get that  $\phi((x_\alpha - x)^*(x_\alpha - x))$  converges to zero, but this precisely means that the square of the distance between  $j(x_\alpha)$  and  $j(x)$  goes to zero in  $\mathcal{H}_\phi$ . Hence  $j(x) \in \tilde{\mathfrak{A}}$ , and the proof is complete.  $\square$

It is natural to ask whether the countable decomposability assumption in conditions (iii), (iv), (v), (vi) and (vii) is necessary for Theorem 1.3.11 to hold. The answer to this question is yes. There are examples of von Neumann algebras without separable predual, which are separable in all the six operator topologies. Let us give an example of such. Since we do not introduce the double dual of a  $C^*$ -algebra  $\mathcal{A}$  before next section, let us mention that the  $\mathcal{A}^{**}$  has a natural structure as a von Neumann algebra, such that the weak $*$  topology becomes the ultraweak operator topology (which

makes  $\mathcal{A}^*$  the predual). With respect to this structure, the closed unit ball of  $\mathcal{A}$  is dense in the closed unit ball of  $\mathcal{A}^{**}$  with respect to all the six operator topologies. More about this is explained in the following section. Let us give the example.

Consider the  $C^*$ -algebra  $\mathcal{A}$  of continuous functions on the unit interval, that is,  $\mathcal{A} = C([0, 1]; \mathbb{C})$ . This  $C^*$ -algebra  $\mathcal{A}$  is clearly separable, since the polynomials are dense, by the Weierstrass Approximation Theorem. The dual space, however, is not separable, as we shall see. The dual space is naturally characterized by the Riesz Representation Theorem, but we do not need this for proving non-separability of  $\mathcal{A}^*$ . For each  $s \in [0, 1]$ , we may consider the state on  $\mathcal{A}$  given by  $\phi_s(f) = f(s)$ , which corresponds to the Dirac measure at  $s$ . If  $s, t \in [0, 1]$  are distinct, then  $\|\phi_s - \phi_t\| = 2$ . Hence we have found an uncountable family of elements, namely the family  $\{\phi_s : s \in [0, 1]\}$ , whose distances to each other are equal to two, which shows that  $\mathcal{A}^*$  cannot be separable. Thus we have proved that countable decomposability is not ensured, when assuming separability in one of the six operator topologies.

Let us give an examples of von Neumann algebras which will occur many times in the thesis having non-separable predual, Namely, ultrapowers of the hyperfinite  $\text{II}_1$ -factor. The reader may consult Chapter 4 for the necessary background on ultra-products.

**Lemma 1.3.13.** *There exists an uncountable set  $C$  of infinite subsets of  $\mathbb{N}$ , so that every two sets from  $C$  have only finitely many elements in common.*

*Proof.* For each  $t \in (0, 1)$  let  $\sum_{n=0}^{\infty} a_n^{(t)} \frac{1}{10^n}$  be the base 10 expansion of  $t$ , that is,  $a_k^{(t)} \in \{0, 1, 2, \dots, 9\}$ , for all  $k \in \mathbb{N}$ , and  $t = \sum_{n=0}^{\infty} a_n^{(t)} \frac{1}{10^n}$ . Now, for each  $n \in \mathbb{N}$ , let  $b_n^{(t)}$  be defined by  $b_n^{(t)} = 10^{n+1} + \sum_{k=0}^n a_k^{(t)} \cdot 10^{n-k}$ . In other words,  $b_n^{(t)}$  is the number  $1a_0^{(t)}a_1^{(t)}a_2^{(t)} \dots a_n^{(t)}$ , where this should be understood as a number with digits  $1, a_0^{(t)}, a_1^{(t)}, a_2^{(t)}, \dots, a_n^{(t)}$  instead of a product. Note that, by construction, the natural number  $b_n^{(t)}$  has  $n+1$  digits, for all  $n \in \mathbb{N}$  and  $t \in (0, 1)$ , that is,  $10^{n+1} \leq b_n^{(t)} < 10^{n+2}$ , for all  $n \in \mathbb{N}$  and  $t \in (0, 1)$ . In particular, these elements are all distinct for fixed  $t \in (0, 1)$ . The claim is now that the set  $C = \{\{b_n^{(t)} : n \in \mathbb{N}\} : t \in (0, 1)\}$  satisfies the criteria. Clearly the set is uncountable, so we only need to check that any two distinct sets in  $C$  only have finitely many elements in common. Suppose that  $s, t \in (0, 1)$ , and that the sets  $\{b_n^{(t)} : n \in \mathbb{N}\}$  and  $\{b_n^{(s)} : n \in \mathbb{N}\}$  have infinitely many elements in common. If  $b_n^{(t)} = b_m^{(s)}$ , for some  $n, m \in \mathbb{N}$ , then  $n = m$ , since the two numbers have a different amount of digits when  $n \neq m$ . Thus, it must be the case that  $b_n^{(t)} = b_n^{(s)}$  for infinitely many  $n \in \mathbb{N}$ . Now if  $n \in \mathbb{N}$  with  $b_n^{(t)} = b_n^{(s)}$  then

$$10^{n+1} + \sum_{k=0}^n a_k^{(t)} 10^k = 10^{n+1} + \sum_{k=0}^n a_k^{(s)} 10^k,$$

but this must mean that  $a_k^{(t)} = a_k^{(s)}$  for all  $k = 0, 1, 2, \dots, n$ . Since  $b_n^{(t)} = b_n^{(s)}$  for arbitrarily large  $n \in \mathbb{N}$ , we conclude that  $a_k^{(t)} = a_k^{(s)}$  for all  $k \in \mathbb{N}$ . Hence,  $s = t$  since they have the same base 10 expansion.  $\square$

**Proposition 1.3.14.** *If  $\omega$  is a free ultrafilter on  $\mathbb{N}$ , then there exists an uncountable set of unitaries in  $\mathcal{R}^\omega$  which have  $\sqrt{2}$  distance to each other in the trace norm.*

*Proof.* Let  $\tau$  denote the trace on  $\mathcal{R}$  and  $\tau_\omega$  the trace on  $\mathcal{R}^\omega$ . Choose an infinite sequence of unitaries  $(u_n)_{n \in \mathbb{N}}$  in  $\mathcal{R}$  so that  $\tau(u_n^* u_k) = 0$  when  $n \neq k$ . Such



a sequence exists, and if one thinks of the hyperfinite  $\text{II}_1$ -factor as the group von Neumann algebra of an amenable countable discrete group with infinite conjugacy classes, then one can just take the canonical unitaries corresponding to a sequence of distinct group elements. Now let  $C$  be an uncountable set of infinite subsets of  $\mathbb{N}$  from Lemma 1.3.13. We may assume that the elements of  $C$  are sequences, that is, they are ordered in some way. This particular order plays no role. For each  $b \in C$ , say  $b = (b_n)_{n \in \mathbb{N}}$  let  $u_b = (u_{b_n})_{n \in \mathbb{N}}$ . Now for  $a, b \in C$ , say  $b = (b_n)_{n \in \mathbb{N}}$   $a = (a_n)_{n \in \mathbb{N}}$ , we have that  $b_n = a_n$  for only finitely many  $n \in \mathbb{N}$ , by choice of  $C$ , and so  $\tau_n((u_{b_n} - u_{a_n})^*(u_{b_n} - u_{a_n})) = 2$  for all but finitely many  $n \in \mathbb{N}$ . Now since  $\omega$  is a free ultrafilter this implies that  $\{n \in \mathbb{N} : \tau_n((u_{b_n} - u_{a_n})^*(u_{b_n} - u_{a_n})) = 2\} \in \omega$ , and thus it follows that  $\tau_\omega((u_b - u_a)^*(u_b - u_a)) = 2$ . Hence  $\{u_b : b \in C\}$  is an uncountable family of unitaries with  $\|u_b - u_a\|_{\tau_\omega} = \sqrt{2}$  for all  $a \neq b$ .  $\square$

**Corollary 1.3.15.** *If  $\omega$  is a free ultrafilter, then the ultrapower  $\mathcal{R}^\omega$  does not have separable predual.*

*Proof.* Let  $\tau_\omega$  denote the trace on  $\mathcal{R}^\omega$ . There is an uncountable family of unitaries in  $\mathcal{R}^\omega$  whose distance to each other is  $\sqrt{2}$  in the Hilbert space  $\mathcal{H}_{\tau_\omega}$  from the GNS-construction corresponding to  $\tau_\omega$ . Hence  $\mathcal{H}_{\tau_\omega}$  cannot be separable, and it follows Theorem 1.3.11 and Proposition 1.3.12 that  $\mathcal{R}^\omega$  does not have separable predual.  $\square$

This proves non-separability of the predual of  $\mathcal{R}^\omega$  for a free ultrafilter  $\omega$  on  $\mathbb{N}$ . Let us now move on to proving that a norm separable von Neumann algebra is finite dimensional, as announced in the beginning of this section. First, recall that a projection in a von Neumann algebra is called **minimal** if it is non-zero and has no proper non-zero subprojections.

**Proposition 1.3.16.** *If a von Neumann algebra is infinite dimensional, then it contains an infinite family of non-zero pairwise orthogonal projections.*

*Proof.* Let  $\mathcal{M}$  be an infinite dimensional von Neumann algebra. Suppose that there exist minimal projections  $p_1, p_2, \dots, p_n$  in  $\mathcal{M}$  with  $\mathbf{1} = p_1 + p_2 + \dots + p_n$ . It is general structure theory (see [KR83, Chapter 6]), that we can then find natural numbers  $k_1, k_2, \dots, k_n \in \mathbb{N}$ , with  $k_1 + k_2 + \dots + k_n = n$ , and von Neumann subalgebras  $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_n$  of  $\mathcal{M}$ , with  $\mathcal{N}_i$  a type  $\text{I}_{k_i}$ -factor,  $i = 1, 2, \dots, n$ , such that  $\mathcal{M} = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \dots \oplus \mathcal{N}_n$ . In particular the dimension of  $\mathcal{M}$  must be  $k_1^2 + k_2^2 + \dots + k_n^2$ , which contradicts the fact that  $\mathcal{M}$  is infinite dimensional.

Now, if we can write the identity as a sum of pairwise orthogonal non-zero projection, then the sum is necessarily infinite by the above argument. Hence we can just choose countably many of these infinitely many pairwise orthogonal non-zero projections.

If it is not the case that we can write the identity as a sum of pairwise orthogonal non-zero projection, then it follows that there must be a non-zero projection  $p \in \mathcal{M}$  that does not have any minimal subprojections. If this was not the case, then a Zorn's Lemma argument would imply that a maximal family of orthogonal minimal projections would sum up to the identity. This we just saw cannot happen. Hence, there exist a non-zero projection  $p \in \mathcal{M}$  with no minimal subprojections. Let us now construct a sequence  $p_1, p_2, p_3, \dots$  of non-zero pairwise orthogonal projections in  $\mathcal{M}$ . Denote  $p$  by  $p_0$  for notational reasons. Choose a proper non-zero subprojection  $p_1$  of  $p_0$ . Since  $p_0 - p_1$  is non-zero and cannot be minimal, there exist a proper subprojection  $p_2$  of  $p_0 - p_1$ . Continuing like this successively, we may, for each  $k \in \mathbb{N}$ , choose a

non-zero proper subprojection  $p_{k+1}$  of  $p_{k-1} - p_k$ . In this way we construct the desired sequence  $p_1, p_2, p_3, \dots$  of non-zero pairwise orthogonal projections.  $\square$

**Corollary 1.3.17.** *If a von Neumann algebra is norm separable, then it is finite dimensional.*

*Proof.* Let us show, that if a von Neumann algebra  $\mathcal{M}$  is infinite dimensional, then it is non-separable, in the norm topology. Let  $p_1, p_2, p_3, \dots$  be non-zero pairwise orthogonal projection. For each subset  $A \subseteq \mathbb{N}$ , let  $q_A$  denote the projection  $\sum_{k \in A} p_k$ . For distinct subsets  $A$  and  $B$  of  $\mathbb{N}$ , we have  $\|p_A - p_B\| \in \{1, 2\}$ ,<sup>7</sup> so since the power set of  $\mathbb{N}$  is uncountable, we have constructed an uncountable family of elements in  $\mathcal{M}$ , which have distance greater then or equal to one to each other. In particular,  $\mathcal{M}$  cannot be separable in the norm topology.  $\square$

## 1.4 Universal enveloping von Neumann algebra

Let us introduce another important concept, namely, the universal enveloping von Neumann algebra. In the previous section we saw that a von Neumann algebra can be thought of as a dual space in a particularly nice way. In this section, we shall see that the double dual of a  $C^*$ -algebra can be given the structure of a von Neumann algebra, in a natural way.

First we introduce the notion of a universal representation.

**Definition 1.4.1.** Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra and  $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$  a representation of  $\mathcal{A}$  on some Hilbert space  $\mathcal{H}$ . The representation  $\pi$  is called **universal** if  $\pi$  is non-degenerate, and satisfies the following universal property: given another non-degenerate representation  $\rho$  of  $\mathcal{A}$  on some Hilbert space  $\mathcal{K}$ , there exists a surjective  $*$ -homomorphism  $\tilde{\rho} : \pi(\mathcal{A})'' \rightarrow \rho(\mathcal{A})''$ , which is ultraweakly operator-to-ultraweak operator continuous, such that  $\tilde{\rho} \circ \pi = \rho$ . In other words, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\pi} & \pi(\mathcal{A})'' \\ & \searrow \rho & \downarrow \tilde{\rho} \\ & & \rho(\mathcal{A})'' \end{array}$$

It is not hard to show that such a universal representation is unique in the sense that, if  $\pi_1 : \mathcal{A} \rightarrow B(\mathcal{H}_1)$  and  $\pi_2 : \mathcal{A} \rightarrow B(\mathcal{H}_2)$  are two universal representations, then there exist a  $*$ -isomorphism  $\rho : \pi_1(\mathcal{A})'' \rightarrow \pi_2''(\mathcal{A})$ , which is also an ultraweak operator-to-ultraweak operator homeomorphism, such that  $\rho \circ \pi_1 = \pi_2$ .

We want to show that the universal representation of a  $C^*$ -algebra exists, but not only that, we also want to show that it relates to the double dual of the  $C^*$ -algebra in question.

First we prove the following theorem, which shows the dual space of a  $C^*$ -algebra is spanned by its states:

**Theorem 1.4.2.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra. Every Hermitian linear functional  $\phi$  on  $\mathcal{A}$  has the form  $\phi = \phi_+ - \phi_-$  for positive linear functionals  $\phi_+$  and  $\phi_-$  on  $\mathcal{A}$ , with  $\|\phi\| = \|\phi_+\| + \|\phi_-\|$ . In particular, every element in  $\mathcal{A}^*$  is a linear combination of at most four states.*

<sup>7</sup>The distance is 1 if  $A \subseteq B$  or  $B \subseteq A$ , and 2 otherwise.

*Proof.* Assume at first that  $\mathcal{A}$  is unital, and assume that  $\|\phi\| = 1$ . Since  $\mathcal{A}$  is unital, the state space  $S(\mathcal{A})$  of  $\mathcal{A}$ , that is, the set of states on  $\mathcal{A}$ , is weak\*-compact. Let  $S = S(\mathcal{A}) \cup (-S(\mathcal{A}))$ , and note that since  $S(\mathcal{A})$  is convex

$$\text{conv } S = \{\lambda\phi_1 - \mu\phi_2 : \lambda, \mu \geq 0, \lambda + \mu = 1, \phi_1, \phi_2 \in S(\mathcal{A})\}.$$

Hence  $\text{conv } S$  is the image of the map

$$(\phi_1, \phi_2, \lambda) \mapsto \lambda\phi_1 - (1 - \lambda)\phi_2, \quad S(\mathcal{A}) \times S(\mathcal{A}) \times [0, 1] \rightarrow \mathcal{A}^*.$$

This map is clearly continuous when  $\mathcal{A}^*$  and  $S$  is given the weak\*-topology and  $S(\mathcal{A}) \times S(\mathcal{A}) \times [0, 1]$  the product topology, but the latter set is compact, and thus its image  $\text{conv } S$  is weak\*-compact. Now our goal is to prove that  $\phi \in \text{conv } S$ . Suppose towards a contradiction that  $\phi \notin \text{conv } S$ . By the Hahn-Banach Theorem there exist some  $x \in \mathcal{A}$  and  $\mu \in \mathbb{R}$  so that

$$\text{Re } \phi(x) > \mu \geq \text{Re } \psi(x)$$

for all  $\psi \in \text{conv } S$ . Let  $y = \text{Re } x$ , then  $\phi(y) = \text{Re } \phi(x) > \mu \geq \text{Re } \psi(x) = \psi(y)$  for all  $\psi \in \text{conv } S$ , since  $\psi(\text{Re } x) = \text{Re } \psi(x)$ , for all Hermitian functionals  $\psi$  on  $\mathcal{A}$ . Since  $y$  is self-adjoint  $\|y\| = \sup\{\psi(y) : \psi \in S\}$ , so in particular  $\mu \geq \|y\|$ . We assumed that  $\|\phi\| \leq 1$ , so that  $\phi(y) \leq \|y\| \leq \mu$ , which contradicts the fact that  $\phi \notin \text{conv } S$ . Thus there exist states  $\phi'_+$  and  $\phi'_-$ , and non-negative real numbers  $\lambda$  and  $\mu$ , with  $\lambda + \mu = 1$ , so that  $\phi = \lambda\phi'_+ - \mu\phi'_-$ . Now, letting  $\phi_+ = \lambda\phi'_+$  and  $\phi_- = \mu\phi'_-$ , we see that  $\phi = \phi_+ - \phi_-$  and

$$\|\phi\| = 1 = \lambda + \mu = \lambda\|\phi'_+\| + \mu\|\phi'_-\| = \|\phi_+\| + \|\phi_-\|.$$

This proves that  $\phi$  has the desired decomposition. Clearly the zero functional has this decomposition, and when  $\|\phi\| \neq 0$  the result can be obtained by scaling.

Suppose now that  $\mathcal{A}$  is not unital, and that  $\phi$  is a Hermitian linear functional on  $\mathcal{A}$ . Define a linear functional  $\tilde{\phi}$  on  $\tilde{\mathcal{A}}$  given by  $\tilde{\phi}(a + \lambda\mathbf{1}) = \phi(a)$ ,  $a \in \mathcal{A}$  and  $\lambda \in \mathbb{C}$ . Clearly  $\tilde{\phi}$  is a bounded Hermitian linear functional extending  $\phi$ , with the same norm. By the previous part we can write  $\tilde{\phi} = \tilde{\phi}_+ - \tilde{\phi}_-$ , for positive linear functionals  $\tilde{\phi}_+$  and  $\tilde{\phi}_-$  on  $\tilde{\mathcal{A}}$ . If we let  $\phi_+ = \tilde{\phi}_+|_{\mathcal{A}}$  and  $\phi_- = \tilde{\phi}_-|_{\mathcal{A}}$ , then clearly  $\phi = \phi_+ - \phi_-$ . Let us check that the condition on the norms also hold. We see that

$$\|\phi\| = \|\tilde{\phi}\| = \|\tilde{\phi}_+\| + \|\tilde{\phi}_-\| \geq \|\phi_+\| + \|\phi_-\|.$$

The other inequality is trivial, so we get  $\|\phi\| = \|\phi_+\| + \|\phi_-\|$ .

Now suppose that  $\phi \in \mathcal{A}$  is just any linear functional. If we let  $\phi_1$  and  $\phi_2$  denote the linear functionals on  $\mathcal{A}$  given by

$$\phi_1(x) = \frac{1}{2}(\phi(x) + \overline{\phi(x^*)}) \quad \text{and} \quad \phi_2(x) = \frac{1}{2i}(\phi(x) - \overline{\phi(x^*)})$$

then it is straightforward to check that  $\phi_1$  and  $\phi_2$  are Hermitian with  $\phi = \phi_1 + i\phi_2$ . By the first part of this theorem  $\phi_1$  and  $\phi_2$  can both be written as a linear combination of at most two states. Thus  $\phi$  can be written as a linear combination of at most four states.  $\square$

**Proposition 1.4.3.** *Suppose that  $\pi: \mathcal{A} \rightarrow \mathcal{B}$  is a \*-homomorphism between  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ . Then  $\pi$  maps the open unit ball of  $\mathcal{A}$  onto the open unit ball of  $\pi(\mathcal{A})$ .*

*Proof.* Since  $*$ -isomorphisms are isometric we may assume that  $\mathcal{B} = \mathcal{A}/\mathcal{I}$  for some ideal in  $\mathcal{A}$ , and that  $\pi$  is the quotient map. Suppose that  $x \in \mathcal{A}/\mathcal{I}$  with  $\|x\| < 1$ , and choose  $a \in \mathcal{A}$  with  $\pi(a) = x$ . Since  $\|x\| = \inf\{\|a + b\| : b \in \mathcal{I}\}$ , we may choose  $b \in \mathcal{I}$  so that  $\|x\| \leq \|a + b\| < 1$ . Thus conclude that  $x$  is in the image of the open unit ball of  $\mathcal{A}$ , since  $\pi(a + b) = x$ .  $\square$

Now, the following proposition is the key ingredient in proving that the double dual of a  $C^*$ -algebra has a natural structure as a von Neumann algebra:

**Proposition 1.4.4.** *Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra and  $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$  a non-degenerate representation of  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$ . Then  $\pi$  extends uniquely to a weak\*-to-ultraweak continuous map  $\tilde{\pi}: \mathcal{A}^{**} \rightarrow \pi(\mathcal{A})''$ , that is,  $\tilde{\pi}$  is weak\*-to-ultraweakly continuous and the diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\pi} & \pi(\mathcal{A})'' \\ \downarrow \iota & \nearrow \tilde{\pi} & \\ \mathcal{A}^{**} & & \end{array}$$

*commutes. Here  $\iota$  denotes the natural inclusion. Moreover,  $\tilde{\pi}$  maps the closed unit ball of  $\mathcal{A}^{**}$  onto the closed unit ball of  $\pi(\mathcal{A})''$ , so in particular it is surjective.*

*Proof.* Let us for short denote  $\pi(\mathcal{A})''$  by  $\mathcal{M}$ . First consider the adjoint of  $\pi$ . This is a map  $\pi^*: \mathcal{M}^* \rightarrow \mathcal{A}^*$ , and we let  $\rho$  denote its restriction to the predual  $\mathcal{M}_*$  of  $\mathcal{M}$ , that is,  $\rho = \pi^*|_{\mathcal{M}_*}$ . Taking the adjoint again we get a map  $\rho^*: \mathcal{A}^{**} \rightarrow (\mathcal{M}_*)^*$ , and so composing with the inverse of the canonical isomorphism  $\kappa: \mathcal{M} \rightarrow (\mathcal{M}_*)^*$ , we obtain a map  $\tilde{\pi}: \mathcal{A}^{**} \rightarrow \mathcal{M}$ . Let us check that this map extends  $\pi$ , that is  $\kappa(\tilde{\pi}(x)) = \rho^*(\iota(x))$ . So suppose that  $x \in \mathcal{A}$ , and let  $\phi \in \mathcal{M}_*$ . Then

$$\rho^*(\iota(x))(\phi) = \iota(x)(\rho(\phi)) = \rho(\phi)(x) = \phi(\pi(x)) = \kappa(\pi(x))(\phi),$$

so since  $\phi \in \mathcal{M}_*$  was arbitrary, this shows that  $\kappa(\pi(x)) = \rho^*(\iota(x))$ . Hence the map  $\tilde{\pi}$  extends  $\pi$ .

Now that  $\tilde{\pi}$  is weak\*-to-ultraweak continuous follows from the fact that the ultraweak operator topology on  $\mathcal{M}$  is the weak\* topology on  $\mathcal{M}$  considered as the dual space of  $\mathcal{M}_*$  and the fact that the conjugate of a bounded linear map is always weak\*-to-weak\* continuous. This also proves uniqueness since  $\mathcal{A}$  is weak\*-dense in  $\mathcal{A}^{**}$ .

Let us show that  $\tilde{\pi}$  maps the closed unit ball of  $\mathcal{A}^{**}$  onto the closed unit ball of  $\mathcal{M}$ . Let  $S$  denote the image of the closed unit ball of  $\mathcal{A}^{**}$ , that is,  $S = \tilde{\pi}((\mathcal{A}^{**})_1)$ . By Goldstine's Theorem (see [Woj91, page 31]) we get that  $(\mathcal{A}^{**})_1$  is the weak\* closure of  $(\mathcal{A})_1$ , and so since  $\tilde{\pi}$  is weak\*-to-ultraweak continuous  $S$  is contained in the ultraweak closure of  $\pi(\mathcal{A}_1)$  in  $\mathcal{M}$ , which by Kaplansky's Density Theorem is  $\mathcal{M}_1$  (see [KR83, Theorem 5.3.5]). In this last use of Kaplansky's Density Theorem it is important that  $\pi(\mathcal{A})$  is ultraweakly dense in  $\pi(\mathcal{A})''$ , since  $\pi$  is non-degenerate. Indeed, Kaplansky's Theorem also applies even if  $\pi(\mathcal{A})$  does not contain the unit, as long as the unit lies in the weak closure of  $\pi(\mathcal{A})$ . We know already by Proposition 1.4.3 that  $S$  contains the open unit ball of  $\pi(\mathcal{A})$ , since  $\tilde{\pi}$  extends  $\pi$ . By Banach-Alaoglu's Theorem (see [Zhu93, Theorem 1.4])  $(\mathcal{A}^{**})_1$  is weak\* compact, and therefore  $S$  must be ultraweakly compact by continuity. In particular  $S$  is ultraweakly closed, and contains the ultraweak closure of  $\pi(\mathcal{A})_1$ , that is,  $\mathcal{M}_1 \subseteq S$ . Thus  $S = \mathcal{M}_1$ , which was the last thing we needed to prove.  $\square$

**Remark 1.4.5.** Notice that the only place where we used that the representation in Proposition 1.4.4 was non-degenerate, was when proving surjectivity, that is, when proving that the extension maps the closed unit ball surjectively onto the closed unit ball.

Now we are ready to show that universal representations exist, and give the connection to the double dual.

Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra, and for each element  $\phi$  in the state space  $S(\mathcal{A})$  of  $\mathcal{A}$ , let  $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$  be the GNS-construction corresponding to  $\phi$ . We let  $\pi_u$  denote the direct sum of all these representations. So  $\pi_u = \bigoplus_{\phi \in S(\mathcal{A})} \pi_\phi$ , and it is a representation on the Hilbert space  $\mathcal{H}_u = \bigoplus_{\phi \in S(\mathcal{A})} \mathcal{H}_\phi$ . This representation is non-degenerate, since each representation  $\pi_\phi$ ,  $\phi \in S(\mathcal{A})$ , is so. It is well-known that this representation is faithful, since the states on  $\mathcal{A}$  separate points. Next, we will show that this representation is actually universal, and that  $\pi_u(\mathcal{A})''$  can be identified with  $\mathcal{A}^{**}$  in a natural way.

**Theorem 1.4.6.** *Let  $\mathcal{A}$  be a  $C^*$ -algebra. With the notation above, the representation  $\pi_u$  is a universal representation of  $\mathcal{A}$ . Moreover,  $\pi_u$  extends to a surjective isometry  $\tilde{\pi}_u: \mathcal{A}^{**} \rightarrow \pi_u(\mathcal{A})''$ , which is also a weak\*-to-ultraweak operator topology homeomorphism.*

*Proof.* We start by proving the second statement, and then afterwards return to the universality of  $\pi_u$ . From Proposition 1.4.4 we know that  $\pi_u$  extends uniquely to a surjective weak\*-to-ultraweak continuous map  $\tilde{\pi}_u: \mathcal{A}^{**} \rightarrow \pi_u(\mathcal{A})''$ , so let us start by recalling how this map was obtained. Let us for short denote  $\pi_u(\mathcal{A})''$  by  $\mathcal{M}$ . We started by considering the map  $\pi_u^*$ , and letting  $\rho$  denote the restriction of this map to  $\mathcal{M}_*$ , that is,  $\rho = \pi_u^*|_{\mathcal{M}_*}$ . If  $\kappa$  denotes the canonical isomorphism  $\mathcal{M} \rightarrow (\mathcal{M}_*)^*$ , then the map  $\tilde{\pi}_u$  was defined by  $\tilde{\pi}_u = \kappa^{-1} \circ \rho^*$ . Our aim is to prove that  $\rho$  is a surjective isometry. So let us start by proving that  $\rho$  is an isometry. For  $\phi \in \mathcal{M}_*$

$$\|\rho(\phi)\| = \|\phi \circ \pi_u\| = \sup\{|\phi(\pi_u(x))| : x \in \mathcal{A}, \|x\| \leq 1\}.$$

Since  $\pi_u$  is a faithful representation it is an isometry. Hence

$$\sup\{|\phi(\pi_u(x))| : x \in \mathcal{A}, \|x\| \leq 1\} = \sup\{|\phi(y)| : y \in \pi_u(\mathcal{A}), \|y\| \leq 1\}.$$

Since  $\pi_u$  is non-degenerate,  $\pi_u(\mathcal{A})$  is ultraweakly dense in  $\mathcal{M}$  so since  $\phi$  is ultraweakly continuous

$$\sup\{|\phi(y)| : y \in \pi_u(\mathcal{A}), \|y\| \leq 1\} = \sup\{|\phi(y)| : y \in \mathcal{M}, \|y\| \leq 1\} = \|\phi\|.$$

This shows that  $\rho$  is in fact an isometry. Let us also show that  $\rho$  is surjective. Let  $\phi \in \mathcal{A}^*$ . By Theorem 1.4.2 we can write  $\phi$  as a linear combination of at most states, that is, we can write  $\phi = \sum_{i=1}^4 \lambda_i \phi_i$  with  $\lambda_i \in \mathbb{C}$  and  $\phi_i \in S(\mathcal{A})$  for  $i = 1, 2, 3, 4$ . With the notation set-forth above let  $\xi, \eta \in \mathcal{H}_u$  be given by  $\xi = \sum_{i=1}^4 \lambda_i \xi_{\phi_i}$  and  $\eta = \sum_{i=1}^4 \xi_{\phi_i}$ , where the sums are taken inside  $\mathcal{H}_u = \bigoplus_{\psi \in S(\mathcal{A})} \mathcal{H}_\psi$ . Consider now the linear functional  $\psi \in \mathcal{M}_*$  given by  $\psi(a) = \langle a\xi \mid \eta \rangle$  for  $a \in \mathcal{M}$ . For  $x \in \mathcal{A}$  we see that

$$\rho(\psi)(x) = \langle \pi_u(x)\xi \mid \eta \rangle = \sum_{i=1}^4 \lambda_i \langle \pi_u(x)\xi_{\phi_i} \mid \xi_{\phi_i} \rangle = \sum_{i=1}^4 \lambda_i \phi_i(x) = \phi(x),$$

so since  $\phi$  was arbitrary this shows that  $\rho$  is surjective.<sup>8</sup> Since  $\rho$  is a bijective isometry, we get that its adjoint  $\rho^*$  is also a surjective isometry. Since the adjoint of a bounded map is weak\*-to-weak\* continuous and  $(\rho^*)^{-1} = (\rho^{-1})^*$ ,  $\rho^*$  must be a weak\*-to-weak\* homeomorphism. Now,  $\kappa$  is an isometry and a weak\*-to-ultra-weak homeomorphism, so since  $\tilde{\pi}_u = \kappa^{-1} \circ \rho^*$  we get that  $\tilde{\pi}_u$  is an isometry which is weak\*-to-ultra-weak homeomorphism.

This proves the second statement. The universality of  $\pi_u$  follows directly from Proposition 1.4.4, since we know that  $\mathcal{A}^{**} \cong \pi_u(\mathcal{A})''$ .  $\square$

The von Neumann algebra  $\pi_u(\mathcal{A})''$  from Theorem 1.4.6 is called the **universal enveloping von Neumann algebra**. The theorem justify that we may identify  $\mathcal{A}^{**}$  with this universal enveloping von Neumann algebra, and so in the following we not distinguish between the two spaces. It is also referred to as the **double dual**.

Since a Banach space is weak\* dense in its double dual, we also get from Theorem 1.4.6 that  $\mathcal{A}$  is ultraweakly dense in  $\mathcal{A}^{**}$ . Clearly it is then dense in all the six operator topologies, since they all have the same closure on \*-algebras.

**Remark 1.4.7.** Suppose that  $\mathcal{B}$  is a  $C^*$ -algebra and  $\mathcal{A}$  a  $C^*$ -subalgebra in  $\mathcal{B}$ . Recall that we say that  $\mathcal{A}$  is **hereditary** in  $\mathcal{B}$ , if for each  $x \in \mathcal{A}$  and  $y \in \mathcal{B}$ , we have  $x^*yx \in \mathcal{A}$ . Since the product is ultrastrongly continuous in each variable separately it follows easily that  $\mathcal{A}^{**}$  is hereditary in  $\mathcal{B}^{**}$ , since  $\mathcal{A}$  and  $\mathcal{B}$  are ultrastrongly dense in  $\mathcal{A}^{**}$  and  $\mathcal{B}^{**}$ , respectively.

At first it might not seem so magical that the double dual of a  $C^*$ -algebra can be given this structure as a von Neumann algebra, but it turns out to be a powerful tool. In particular, it has the advantage that it is exceptionally easy to extend maps between  $C^*$ -algebras to their double duals, namely, that can be done just by taking adjoint of the map twice. We shall see that taking adjoint of a map between  $C^*$ -algebras preserves many properties of the map.

**Remark 1.4.8.** For a  $C^*$ -algebras  $\mathcal{A}$  there is a natural way of identifying  $M_n(\mathcal{A}^{**})$  with  $M_n(\mathcal{A})^{**}$ . If  $\mathcal{B}$  is another  $C^*$ -algebra and  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  is a linear map, then with this identification,  $(\phi^{**})_n = (\phi_n)^{**}$ .

**Theorem 1.4.9.** *Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are  $C^*$ -algebras,  $\phi : \mathcal{A} \rightarrow \mathcal{B}$  a linear map and  $\phi^{**} : \mathcal{A}^{**} \rightarrow \mathcal{B}^{**}$  its second conjugate. Then: (i) the map  $\phi^{**}$  is ultraweakly continuous, that is continuous when both  $\mathcal{A}^{**}$  and  $\mathcal{B}^{**}$  are equipped with their ultra-weak operator topologies; (ii) the norm of  $\phi^{**}$  is the same as that of  $\phi$ ; (iii) if  $\phi$  is a homomorphism, then so is  $\phi^{**}$ ; (iv) if  $\phi$  is Hermitian, then so is  $\phi^{**}$ ; (v) if  $\phi$  is positive, then so is  $\phi^{**}$ ; (vi) if  $\phi$  is unital then  $\phi^{**}$  is also unital; (vii) if  $\phi$  is completely bounded, then so is  $\phi^{**}$ , with the same completely bounded norm; (viii) if  $\phi$  is completely positive, then so is  $\phi^{**}$ .*

*Proof.* That (i) and (ii) holds follows just from general Banach space theory. Now, that (iii), (iv) and (v) holds follows from Kaplansky's Density Theorem and ultraweak continuity of  $\phi^{**}$  together with the fact that the product in  $\mathcal{A}$  is ultraweakly continuous on bounded sets. Obviously (vi) is true, and (vii) and (viii) follows from Remark 1.4.8, which states that  $(\phi^{**})_n = (\phi_n)^{**}$ , together with (ii) and (v).  $\square$

<sup>8</sup>The computation with inner products is justified, because for  $i \neq j$ , we have that  $\xi_{\phi_i}$  and  $\xi_{\phi_j}$  are orthogonal in the direct sum.

Before we end this section, let us prove the following small result, which we will need a couple of times.

**Proposition 1.4.10.** *Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are  $C^*$ -algebras and  $\pi: \mathcal{B} \rightarrow \mathcal{A}$  a surjective  $*$ -homomorphism. Let  $\mathcal{I} = \ker \pi$ , and let  $p$  denote the central projection in  $\mathcal{B}^{**}$  so that  $p\mathcal{B}^{**} = \mathcal{I}^{**}$ . If  $\phi: \mathcal{B}^{**} \rightarrow \mathcal{I}^{**}$  denotes multiplication by  $p$ , then the map  $\phi \oplus \pi^{**}: \mathcal{B}^{**} \rightarrow \mathcal{I}^{**} \oplus \mathcal{A}^{**}$  is an isomorphism. In particular,  $\tilde{\mathcal{A}}^{**}$  is canonically isomorphic to  $\mathcal{A}^{**} \oplus \mathbb{C}$ .*

*Proof.* This is actually just a corollary to Proposition 1.2.13, and with this proposition in mind, all we need to verify is that  $\mathcal{I}^{**}$  is an ultraweakly closed two-sided ideal in  $\mathcal{B}$ . Since  $\iota^{**}$  is ultraweak-to-ultraweak continuous  $\mathcal{I}^{**}$  is a von Neumann algebra, and in particular ultraweakly closed. Now, that  $\mathcal{I}^{**}$  is a two-sided ideal follows just from the fact that  $\mathcal{B}$  is ultraweakly dense in  $\mathcal{B}^{**}$  and the multiplication is ultraweakly continuous in each variable separately.  $\square$

## 1.5 Tensor products

This section contains, just about, the necessary results on tensor products of  $C^*$ -algebras needed for this thesis. Most of the proofs are omitted, and the results can be found in the literature. For example, the results on the maximal and minimal tensor products are all contained in [BO08, Chapter 3], and so are the last few notes on tensor product, see also [BO08, Appendix B]. Tensor product of operators are discussed in detail in [KR83, Section 2.6].

The reader is assumed to be familiar with the algebraic tensor product. We denote the algebraic tensor product of vector spaces  $V$  and  $W$  by  $V \odot W$ , and elementary tensors in the algebraic tensor product, are denoted by  $v \otimes w$ , for  $v \in V$  and  $w \in W$ .

The reader is also expected to be familiar with the tensor product of Hilbert spaces and operators on Hilbert spaces. For Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ , we denote by  $\mathcal{H} \otimes \mathcal{K}$  their tensor product. For bounded linear operators  $x$  and  $y$  on  $\mathcal{H}$  and  $\mathcal{K}$ , respectively, we denote by  $x \otimes y$  the tensor product operator on  $\mathcal{H} \otimes \mathcal{K}$ . It is uniquely determined by acting on elementary tensors by  $(x \otimes y)(\xi \otimes \eta) = (x\xi) \otimes (y\eta)$ ,  $\xi \in \mathcal{H}$ ,  $\eta \in \mathcal{K}$ . For a Hilbert space  $\mathcal{K}$ , we denote by  $\mathcal{H}^{\otimes n}$  the  $n$ -fold tensor product  $\mathcal{H} \otimes \mathcal{H} \otimes \dots \otimes \mathcal{H}$ .

If  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are  $C^*$ -algebras, and  $\pi_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{C}$  and  $\pi_{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{C}$  are  $*$ -homomorphisms with commuting ranges, then we denote by  $\pi_{\mathcal{A}} \times \pi_{\mathcal{B}}$  the  $*$ -homomorphism

$$\pi_{\mathcal{A}} \times \pi_{\mathcal{B}}: \mathcal{A} \odot \mathcal{B} \rightarrow \mathcal{C} \quad \text{defined by} \quad (\pi_{\mathcal{A}} \times \pi_{\mathcal{B}})(a \otimes b) = \pi_{\mathcal{A}}(a)\pi_{\mathcal{B}}(b),$$

for  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ .

**Proposition 1.5.1.** *Given  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ , a Hilbert space  $\mathcal{H}$  and a  $*$ -homomorphism  $\pi: \mathcal{A} \odot \mathcal{B} \rightarrow B(\mathcal{H})$ , there exist  $*$ -homomorphisms  $\pi_{\mathcal{A}}: \mathcal{A} \rightarrow B(\mathcal{H})$  and  $\pi_{\mathcal{B}}: \mathcal{B} \rightarrow B(\mathcal{H})$  with commuting ranges, such that  $\pi = \pi_{\mathcal{A}} \times \pi_{\mathcal{B}}$ .*

The maps  $\pi_{\mathcal{A}}$  and  $\pi_{\mathcal{B}}$  from the above proposition are called the **restrictions** of  $\pi$ .

Given  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  together with representations  $\pi_{\mathcal{A}}: \mathcal{A} \rightarrow B(\mathcal{H})$  and  $\pi_{\mathcal{B}}: \mathcal{B} \rightarrow B(\mathcal{K})$  on Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ , respectively, we get a  $*$ -representation

$$\pi_{\mathcal{A}} \odot \pi_{\mathcal{B}}: \mathcal{A} \odot \mathcal{B} \rightarrow B(\mathcal{H} \otimes \mathcal{K}),$$

which is defined on elementary tensors by

$$(\pi_{\mathcal{A}} \odot \pi_{\mathcal{B}})(a \otimes b) = \pi_{\mathcal{A}}(a) \otimes \pi_{\mathcal{B}}(b).$$

One can define several  $C^*$ -norms on the algebraic tensor product of  $C^*$ -algebras. The most important are the maximal norm and the minimal norm, whose definitions we now recall.

**Definition 1.5.2.** Given  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ , the **maximal tensor product** of  $\mathcal{A}$  and  $\mathcal{B}$  is the completion of  $\mathcal{A} \odot \mathcal{B}$ , with respect to the norm

$$\|x\|_{\max} = \sup\{\|\pi(x)\| : \pi : \mathcal{A} \odot \mathcal{B} \rightarrow B(\mathcal{H}) \text{ is a } * \text{-representation}\},$$

for  $x \in \mathcal{A} \odot \mathcal{B}$ , and it is denoted by  $\mathcal{A} \otimes_{\max} \mathcal{B}$ .  $\blacktriangleleft$

The maximal tensor norm is well-defined, and  $\mathcal{A} \otimes_{\max} \mathcal{B}$  is a  $C^*$ -algebra. The maximal norm turns out to be the largest possible  $C^*$ -norm on  $\mathcal{A} \odot \mathcal{B}$ .

**Definition 1.5.3.** Given  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ , the **minimal tensor product** of  $\mathcal{A}$  and  $\mathcal{B}$  is the completion of  $\mathcal{A} \odot \mathcal{B}$ , with respect to the norm

$$\left\| \sum_{k=1}^n a_k \otimes b_k \right\|_{\min} = \left\| \sum_{k=1}^n \pi(a_k) \otimes \rho(b_k) \right\|_{B(\mathcal{H} \otimes \mathcal{K})},$$

for  $a_1, \dots, a_n \in \mathcal{A}$  and  $b_1, \dots, b_n \in \mathcal{B}$ , and some choice of faithful representations  $\pi : \mathcal{A} \rightarrow B(\mathcal{H})$  and  $\rho : \mathcal{B} \rightarrow B(\mathcal{K})$ . This completion is denoted by  $\mathcal{A} \otimes_{\min} \mathcal{B}$ .  $\blacktriangleleft$

The minimal tensor product is also called the **spatial tensor product**, and it is not right away clear why this is well-defined. It can be shown that  $\otimes_{\min}$  is independent of the choice of faithful representation. A famous theorem of Takesaki states that the minimal tensor norm is, in fact, the smallest  $C^*$ -algebra norm on  $\mathcal{A} \odot \mathcal{B}$ .

The maximal and minimal tensor product norms are both cross-norms, meaning that  $\|x \otimes y\| = \|x\| \|y\|$  holds for all elementary tensors  $x \otimes y$ .

Since the maximal norm and the minimal norm are the largest and the smallest  $C^*$ -algebra norms on  $\mathcal{A} \odot \mathcal{B}$ , respectively, we obtain for any other  $C^*$ -algebra norm  $\|\cdot\|_{\alpha}$ , canonical surjective  $*$ -homomorphisms

$$\mathcal{A} \otimes_{\max} \mathcal{B} \longrightarrow \mathcal{A} \otimes_{\alpha} \mathcal{B} \longrightarrow \mathcal{A} \otimes_{\min} \mathcal{B},$$

where  $\mathcal{A} \otimes_{\alpha} \mathcal{B}$  denotes the completion of  $\mathcal{A} \odot \mathcal{B}$  with respect to the norm  $\|\cdot\|_{\alpha}$ . By canonical maps we mean that they restrict to the identity on the algebraic tensor product. We deduce that there is a unique  $C^*$ -norm on  $\mathcal{A} \odot \mathcal{B}$  if and only if  $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$ .

Later in the thesis, we will be interested in cases where  $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$ . A particular case where this happens, is if  $\mathcal{A}$  or  $\mathcal{B}$  is equal to  $\mathbb{M}_n$ , for some  $n \in \mathbb{N}$ .

The maximal tensor product has the following universal property:

**Proposition 1.5.4.** *Suppose that  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  are  $C^*$ -algebras. Given a  $*$ -homomorphism  $\pi : \mathcal{A} \odot \mathcal{B} \rightarrow \mathcal{C}$ , there exists a unique map  $\mathcal{A} \otimes_{\max} \mathcal{B} \rightarrow \mathcal{C}$  extending  $\pi$ . In particular, if  $\pi_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{C}$  and  $\pi_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{C}$  are  $C^*$ -algebras with commuting ranges, then they induce a unique  $*$ -homomorphism*

$$\pi_{\mathcal{A}} \times \pi_{\mathcal{B}} : \mathcal{A} \otimes_{\max} \mathcal{B} \rightarrow \mathcal{C}$$

Since a map  $\pi_{\mathcal{A}} \times \pi_{\mathcal{B}}$  always extends from the algebraic tensor product to the maximal tensor product, we will use the symbol  $\pi_{\mathcal{A}} \times \pi_{\mathcal{B}}$  to mean both these things.



Since restrictions always exist, every  $*$ -homomorphism  $\pi$  going out of the maximal tensor product has the form  $\pi = \pi_{\mathcal{A}} \times \pi_{\mathcal{B}}$ .

Now, let us turn our attention to maps between tensor products, and continuity properties of such.

**Theorem 1.5.5.** *Suppose that  $\mathcal{A}_i$  and  $\mathcal{B}_i$ ,  $i = 1, 2$ , are  $C^*$ -algebras, together with completely positive maps  $\phi_i: \mathcal{A}_i \rightarrow \mathcal{B}_i$ ,  $i = 1, 2$ . Then the map*

$$\mathcal{A}_1 \odot \mathcal{A}_2 \rightarrow \mathcal{B}_1 \odot \mathcal{B}_2 \quad \text{given by} \quad (a_1 \otimes a_2) \mapsto \phi_1(a_1) \otimes \phi_2(a_2)$$

*extends to a completely positive map*

$$\phi_1 \otimes_{\max} \phi_2: \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 \rightarrow \mathcal{B}_1 \otimes_{\max} \mathcal{B}_2,$$

*and it extends to a completely positive map*

$$\phi_1 \otimes_{\min} \phi_2: \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \rightarrow \mathcal{B}_1 \otimes_{\min} \mathcal{B}_2.$$

*Moreover, these satisfy  $\|\phi_1 \otimes_{\max} \phi_2\| = \|\phi_1 \otimes_{\min} \phi_2\| = \|\phi_1\| \|\phi_2\|$ .*

A particular case of the above theorem is when  $\phi_1$  and  $\phi_2$  are  $*$ -homomorphisms. In this case, continuity ensures that both  $\phi_1 \otimes_{\max} \phi_2$  and  $\phi_1 \otimes_{\min} \phi_2$  are again  $*$ -homomorphisms.

In the above theorem we used the notation  $\phi_1 \otimes_{\max} \phi_2$  and  $\phi_1 \otimes_{\min} \phi_2$ , for these specific tensor product maps, but later we will use  $\phi_1 \otimes \phi_2$  as a generic symbol for most tensor product maps. It should be clear from the context, which maps are we talking about.

At this point, let us make some comments on a particular kind of maps, namely, the inclusion of a  $C^*$ -subalgebra into a  $C^*$ -algebra. The following proposition follows directly from the fact that the minimal tensor norm is independent of the choice of faithful representation:

**Proposition 1.5.6.** *Given  $C^*$ -algebras  $\mathcal{B}_i$  and  $C^*$ -subalgebras  $\mathcal{A}_i \subseteq \mathcal{B}_i$ ,  $i = 1, 2$ , the minimal tensor norm on  $\mathcal{B}_1 \odot \mathcal{B}_2$  restricts to the minimal tensor norm on  $\mathcal{A}_1 \odot \mathcal{A}_2$ . Hence, the inclusion of the algebraic tensor products induces an isometric inclusion*

$$\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \subseteq \mathcal{B}_1 \otimes_{\min} \mathcal{B}_2.$$

There is no analogue of the above proposition for the maximal tensor product. There always exist a map

$$\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2 \rightarrow \mathcal{B}_1 \otimes_{\max} \mathcal{B}_2,$$

and it maps surjectively onto the closure of the algebraic tensor product  $\mathcal{A}_1 \odot \mathcal{A}_2$  inside  $\mathcal{B}_1 \odot \mathcal{B}_2$ , with respect to the maximal tensor norm on  $\mathcal{B}_1 \odot \mathcal{B}_2$ , but we are not guaranteed that it is injective. We will see later that, for a  $C^*$ -algebra  $\mathcal{B}$  and a  $C^*$ -subalgebras  $\mathcal{A} \subseteq \mathcal{B}$ , the inclusion  $\mathcal{A} \otimes_{\max} \mathcal{C} \subseteq \mathcal{B} \otimes_{\max} \mathcal{C}$  being isometric for all  $C^*$ -algebras  $\mathcal{C}$ , is equivalent to what will be called  $\mathcal{A}$  being relatively weakly injective in  $\mathcal{B}$ .

Besides inclusions, another special kind of maps are quotient maps, and in this respect, one would like to know how the tensor products behave with respect to exact sequences. Unlike the case of inclusions, in this case there is an easy answer for the maximal tensor product:

**Proposition 1.5.7.** *Given a  $C^*$ -algebra  $\mathcal{A}$  and an ideal  $\mathcal{I}$  in  $\mathcal{A}$ , the sequence*

$$0 \longrightarrow \mathcal{I} \otimes_{\max} \mathcal{B} \longrightarrow \mathcal{A} \otimes_{\max} \mathcal{B} \longrightarrow \mathcal{A}/\mathcal{I} \otimes_{\max} \mathcal{B} \longrightarrow 0$$

*is exact for all  $C^*$ -algebras  $\mathcal{B}$ , where all the maps are the obvious ones.*

However, the answer for the minimal tensor product is not as easy. In this case, among other results we have the following proposition:

**Proposition 1.5.8.** *Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are  $C^*$ -algebras, and that  $\mathcal{I}$  is an ideal in  $\mathcal{A}$ . If there is a unique norm on  $(\mathcal{A}/\mathcal{I}) \otimes \mathcal{B}$ , then the sequence*

$$0 \longrightarrow \mathcal{I} \otimes_{\min} \mathcal{B} \longrightarrow \mathcal{A} \otimes_{\min} \mathcal{B} \longrightarrow \mathcal{A}/\mathcal{I} \otimes_{\min} \mathcal{B} \longrightarrow 0$$

*is exact, where all the maps are the obvious ones.*

Before ending this section we will talk about some other tensor products, namely the von Neumann algebra tensor product, and the tensor product of operator spaces.

Suppose that we are given von Neumann algebras  $\mathcal{M}$  and  $\mathcal{N}$  on Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ . The von Neumann algebra tensor product of  $\mathcal{M}$  and  $\mathcal{N}$ , denoted by  $\mathcal{M} \overline{\otimes} \mathcal{N}$ , is the set

$$\mathcal{M} \overline{\otimes} \mathcal{N} = \{x \otimes y : x \in \mathcal{M}, y \in \mathcal{N}\}'' ,$$

which is also the strong operator closure of the set  $\{x \otimes y : x \in \mathcal{M}, y \in \mathcal{N}\}$  in  $B(\mathcal{H} \otimes \mathcal{K})$ . For a von Neumann algebra  $\mathcal{M}$  we will denote by  $\mathcal{M}^{\overline{\otimes} n}$  the  $n$ -fold tensor product  $\mathcal{M} \overline{\otimes} \mathcal{M} \overline{\otimes} \dots \overline{\otimes} \mathcal{M}$ .

Let us define the tensor product of operator spaces. The tensor product of operator spaces is the minimal tensor product. For operator spaces  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , with ambient  $C^*$ -algebras  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , we define the tensor product of  $\mathcal{M}_1$  and  $\mathcal{M}_2$  to be the closure of  $\mathcal{M}_1 \odot \mathcal{M}_2$  inside  $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ , and we denote it by  $\mathcal{M}_1 \otimes_{\min} \mathcal{M}_2$ . Notice that this is well defined, that is, it does not depend on the choice of ambient  $C^*$ -algebras, since the minimal tensor product behaves nicely with respect to  $C^*$ -subalgebras. Indeed, if  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are two other ambient  $C^*$ -algebras of  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , respectively, then, with  $\mathcal{C}_1$  and  $\mathcal{C}_2$  denoting the  $C^*$ -algebras generated by  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , respectively, the inclusions

$$\mathcal{C}_1 \otimes_{\min} \mathcal{C}_2 \rightarrow \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \quad \text{and} \quad \mathcal{C}_1 \otimes_{\min} \mathcal{C}_2 \rightarrow \mathcal{B}_1 \otimes_{\min} \mathcal{B}_2$$

are both isometric. Hence the choice does not matter. We will need the following two propositions about tensor product of maps between operator spaces.

**Proposition 1.5.9.** *Suppose that  $\mathcal{M}_i$  and  $\mathcal{M}'_i$  are operator spaces,  $i = 1, 2$ , together with completely bounded maps  $\phi_i: \mathcal{M}_i \rightarrow \mathcal{M}'_i$ . Then the tensor product map  $\phi_1 \otimes \phi_2$  extends uniquely to a completely bounded map*

$$\phi_1 \otimes \phi_2: \mathcal{M}_1 \otimes_{\min} \mathcal{M}_2 \rightarrow \mathcal{M}'_1 \otimes_{\min} \mathcal{M}'_2.$$

*Moreover,  $\|\phi_1 \otimes \phi_2\|_{\text{cb}} = \|\phi_1\|_{\text{cb}} \|\phi_2\|_{\text{cb}}$ . In particular, if both  $\phi_1$  and  $\phi_2$  are completely contractive, then  $\phi_1 \otimes \phi_2$  is completely contractive.*

**Proposition 1.5.10.** *If  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are operator spaces,  $\phi: \mathcal{M}_1 \rightarrow \mathcal{M}_2$  a completely bounded map and  $\mathcal{H}$  an infinite dimensional Hilbert space, then*

$$\|\phi\|_{\text{cb}} = \|\phi \otimes \text{id}_{B(\mathcal{H})}\|.$$

The subtlety of the above proposition is that the norm on the right hand side is not the completely bounded norm, but the usual operator norm on the set of bounded linear operator from  $\mathcal{M}_1 \otimes_{\min} B(\mathcal{H})$  to  $\mathcal{M}_2 \otimes_{\min} B(\mathcal{H})$ , which in some cases are easier to compute. The intuitive idea behind the above proposition is that  $B(\mathcal{H})$  contains sufficiently large matrix algebras.

## 1.6 Filters and ultrafilters

This section is a *very short* introduction to filters. The purpose of this section is to set the terminology and state a number of results on filters that will be used frequently throughout the thesis. With only a few exceptions, filters will be used exclusively in the chapters 4, 5 and 6. If the reader is not already familiar with filters, then it might be a good idea to read a more thorough introduction to filters before reading these chapters, for example [CSC10, Appendix J].

**Definition 1.6.1.** Suppose that  $I$  is an some index set. A family  $\mathcal{F}$  of subsets of  $I$  is called a **filter**, if it satisfies the following three conditions:

- (i) the empty set is not in  $\mathcal{F}$ ;
- (ii) if  $A \in \mathcal{F}$  and  $B \subseteq I$  with  $A \subseteq B$ , then  $B \in \mathcal{F}$ ;
- (iii) if  $A, B \in \mathcal{F}$  then their intersection is also in  $\mathcal{F}$ .

If in addition to the conditions above, the set  $\mathcal{F}$  satisfies:

- (iv) for each  $A \subseteq I$ , either  $A \in \mathcal{F}$  or  $I \setminus A \in \mathcal{F}$ ,

then  $\mathcal{F}$  is called an **ultrafilter**. ◀

If  $I$  is a set and  $J$  a collection of subsets of  $I$  such that  $J$  has the finite intersection property, then there is a filter containing  $J$ , namely the set of all subsets  $I_0$  of  $I$  such that there exist  $I_1, \dots, I_n \in J$  with  $I_1 \cap \dots \cap I_n \subseteq I_0$ . This is called the filter **generated** by  $J$ .

It is straightforward to check that for any non-empty set  $A \subset I$ , the set

$$\mathcal{F} = \{B \subseteq I : A \subseteq B\}$$

is a filter on  $I$ . Such a filter is called a **principal filter** on  $I$ . An ultrafilter which is not principal is called **free** (or **non-principal**). If  $\mathcal{F}$  is a principal ultrafilter, then  $A$  must necessarily be a singleton, that is,  $A$  only has one point.

The proofs of the following theorem is omitted. It can be found in almost all literature on filters, see for example [CSC10, Appendix J].

**Theorem 1.6.2.** *Every filter is contained in an ultrafilter.*

One application of the above theorem, is that one can construct ultrafilters containing certain specified sets. This is made precise in the following proposition:

**Proposition 1.6.3.** *Let  $I$  be a set, and let  $\mathfrak{A}$  be a non-empty collection of subsets of  $I$ , which have the finite intersection property, that is, any two elements from  $\mathfrak{A}$  have non-empty intersection. Then there exists an ultrafilter  $\omega$  on  $I$ , which contains  $\mathfrak{A}$ .*

*Proof.* Let  $\mathcal{F}$  denote the filter generated by  $\mathfrak{A}$ . Then by the above theorem there exists an ultrafilter  $\omega$  on  $I$  containing  $\mathcal{F}$ . This ultrafilter is the desired ultrafilter, since clearly  $\mathfrak{A} \subseteq \mathcal{F} \subseteq \omega$ .  $\square$

**Definition 1.6.4.** Suppose that  $\mathfrak{X}$  is a topological space,  $I$  an index set and  $\mathcal{F}$  a filter on  $I$ . An indexed family  $(x_i)_{i \in I}$  of elements in  $\mathfrak{X}$  is said to **converge along the filter**  $\mathcal{F}$  to some  $x \in \mathfrak{X}$  if

$$\{i \in I : x_i \in U\} \in \mathcal{F}$$

for all open neighbourhoods  $U$  of  $x$ . This is written  $\lim_{i \rightarrow \mathcal{F}} x_i = x$ .  $\blacktriangleleft$

It is straightforward to check that if the topological space is Hausdorff, then a potential limit along a filter is unique. This follows from the fact that a filter cannot contain the empty set.

The following theorem is probably the main reason that we, in this thesis, prefer ultrafilters, in contrast to just filters.

**Theorem 1.6.5.** *Suppose that  $\mathfrak{X}$  is a compact topological space,  $I$  an index set and  $\omega$  an ultrafilter on  $I$ . Then every subset of  $\mathfrak{X}$  indexed by  $I$  converges along  $\omega$ , that is, for every indexed subset  $(x_i)_{i \in I}$  of  $\mathfrak{X}$  the limit  $\lim_{i \rightarrow \omega} x_i$  exists.*

This is an analogue of the theorem in topology which says that a net in a compact topological space has a cluster point, or equivalently, a convergent subnet. In general, filters can be used—instead of nets—in a way to generalize the well-known results about describing topological properties in metric spaces, such as closedness and continuity, in terms of convergent sequences. Then following two propositions are an example of this.

**Proposition 1.6.6.** *Suppose that  $f: \mathfrak{X} \rightarrow \mathfrak{Y}$  is a continuous map between topological spaces  $\mathfrak{X}$  and  $\mathfrak{Y}$ . Let  $I$  be a set and  $\mathcal{F}$  a filter on  $I$ . If  $(x_i)_{i \in I}$  is an indexed family of elements in  $\mathfrak{X}$  which converges to some  $x \in \mathfrak{X}$  along  $\mathcal{F}$ , then  $\lim_{i \rightarrow \mathcal{F}} f(x_i) = f(x)$ .*

*Proof.* Let  $U$  be an open neighbourhood of  $f(x)$  in  $\mathfrak{Y}$ . Then  $V = f^{-1}(U)$  is an open neighbourhood of  $x$  in  $\mathfrak{X}$  since  $f$  is continuous. Hence

$$\{i \in I : f(x_i) \in U\} = \{i \in I : x_i \in V\} \in \mathcal{F}$$

since  $(x_i)_{i \in I}$  converges to  $x$  along  $\mathcal{F}$ . In this way it follows that  $(f(x_i))_{i \in I}$  converges to  $f(x)$  along  $\mathcal{F}$ .  $\square$

**Proposition 1.6.7.** *Suppose that  $\mathfrak{X}$  is a topological space and  $A$  is a subset of  $\mathfrak{X}$ . Let  $I$  be an index set,  $\mathcal{F}$  a filter on  $I$  and  $(x_i)_{i \in I}$  a subset of  $A$  that converges to some  $x \in \mathfrak{X}$  along  $\mathcal{F}$ . Then  $x$  is in the closure of  $A$  in  $\mathfrak{X}$ . In particular if  $A$  is closed then  $x \in A$ .*

*Proof.* Let  $U$  be an open neighbourhood of  $x$  in  $\mathfrak{X}$ . Since  $\lim_{i \rightarrow \mathcal{F}} x_i = x$  the set  $\{i \in I : x_i \in U\}$  is in  $\mathcal{F}$ . In particular, it is non-empty, and we may choose an element  $j \in \{i \in I : x_i \in U\}$ . Then  $x_j \in U$ , so it follows that  $A \cap U \neq \emptyset$ . Since  $U$  was an arbitrary open neighbourhood of  $x$  in  $\mathfrak{X}$ , we conclude that  $x$  is in the closure of  $A$  in  $\mathfrak{X}$ .  $\square$

## Chapter 2

# An introduction to WEP, QWEP, LP and LLP

This chapter contains an introduction to the concept of the weak expectation property (abbreviated as WEP), QWEP, the lifting property (abbreviated as LP) and the local lifting property (abbreviated as LLP). See definitions 2.2.5, 2.3.1 and 2.4.13, respectively.

The weak expectation property was introduced by E. Christopher Lance in his article [Lan73] of 1973. The purpose for which Lance introduced the weak expectation property was to investigate nuclearity of  $C^*$ -algebras.

Before discussing the central concepts in this chapter, we introduce the notion of conditional expectations, and prove a nifty theorem due to Tomiyama.

### 2.1 Conditional expectations

Let us start by exploring the concepts of bimodule maps and multiplicative domains. After this, we prove Tomiyama's theorem, and define conditional expectations.

**Definition 2.1.1.** A linear map  $\phi: \mathcal{B} \rightarrow \mathcal{C}$  between  $C^*$ -algebras  $\mathcal{B}$  and  $\mathcal{C}$  is called an  $\mathcal{A}$ -bimodule map, for some  $C^*$ -subalgebra  $\mathcal{A} \subseteq \mathcal{B}$ , if  $\phi(ab) = \phi(a)\phi(b)$  and  $\phi(ba) = \phi(b)\phi(a)$ , for all  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ . ◀

**Proposition 2.1.2.** Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are  $C^*$ -algebras. If  $\phi: \mathcal{A} \rightarrow \mathcal{B}$  is a contractive completely positive map, then

- (i) for each  $a \in \mathcal{A}$  we have the inequality  $\phi(a)^*\phi(a) \leq \phi(a^*a)$ , which is called the Schwarz inequality;
- (ii) If  $a \in \mathcal{A}$  such that  $\phi(a^*a) = \phi(a)^*\phi(a)$  and  $\phi(aa^*) = \phi(a)\phi(a)^*$ , then  $\phi(ax) = \phi(a)\phi(x)$  and  $\phi(xa) = \phi(x)\phi(a)$ , for all  $x \in \mathcal{A}$ ;
- (iii) The set  $\{a \in \mathcal{A} : \phi(a^*a) = \phi(a)^*\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a)^*\}$  is a  $C^*$ -subalgebra of  $\mathcal{A}$ .

*Proof.* First of all, we may assume that  $\mathcal{B} \subseteq B(\mathcal{H})$  for a Hilbert space  $\mathcal{H}$ . Let  $(\pi, V, \mathcal{K})$  be a Stinespring representation for  $\phi$  with  $\|V\| \leq 1$ . If  $\mathcal{A}$  is unital, then this representation exists by Stinespring's Dilation Theorem, and if  $\mathcal{A}$  is not unital, then it exists by Corollary B.2.2. In either case,  $\mathbf{1} - VV^*$  is a positive operator on  $\mathcal{K}$ .

Let us start by proving (i). For  $a \in \mathcal{A}$  we have

$$\phi(a^*a) - \phi(a)^*\phi(a) = V^*\pi(a)^*(\mathbf{1}_{\mathcal{K}} - VV^*)\pi(a)V,$$

so since the operator  $\mathbf{1}_{\mathcal{K}} - VV^*$  is positive, it follows that  $\phi(a^*a) - \phi(a)^*\phi(a)$  is positive. This proves (i).

Now let us prove (ii). Suppose that  $a \in \mathcal{A}$  satisfies  $\phi(a^*a) = \phi(a)^*\phi(a)$  and  $\phi(aa^*) = \phi(a)\phi(a)^*$ . Then

$$V^*\pi(a)^*(\mathbf{1}_{\mathcal{K}} - VV^*)\pi(a)V = \phi(a^*a) - \phi(a)^*\phi(a) = 0,$$

which shows that  $(\mathbf{1}_{\mathcal{K}} - VV^*)^{1/2}\pi(a)V = 0$ , and so  $(\mathbf{1}_{\mathcal{K}} - VV^*)\pi(a)V = 0$ . Thus we have  $\pi(a)V = VV^*\pi(a)V$ , and by replacing  $a$  with  $a^*$  and taking adjoints we also get  $V^*\pi(a) = V^*\pi(a)VV^*$ . Now, for  $x \in \mathcal{A}$  we see that

$$\phi(ax) = V^*\pi(a)\pi(x)V = V^*\pi(a)VV^*\pi(x)V = \phi(a)\phi(x),$$

and similarly,  $\phi(xa) = \phi(x)\phi(a)$ . Hence (ii) holds.

Last, let us prove (iii). Let  $\mathcal{C}$  denote the set

$$\{a \in \mathcal{A} : \phi(a^*a) = \phi(a)^*\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a)^*\},$$

which is clearly self-adjoint. Suppose that  $a, b \in \mathcal{C}$ , then using (ii) we see that

$$\phi((ab)^*(ab)) = \phi((ab)^*a)\phi(b) = \phi((ab)^*)\phi(a)\phi(b) = \phi(ab)^*\phi(ab)$$

and likewise  $\phi((ab)(ab)^*) = \phi(ab)\phi(ab)^*$ . Thus  $ab \in \mathcal{C}$ . In the same manner it is easy to show that  $\mathcal{C}$  is a subspace using (ii), and so  $\mathcal{C}$  is a  $*$ -subalgebra. The fact that  $\mathcal{C}$  is closed follows from continuity of  $\phi$ .  $\square$

**Corollary 2.1.3.** *Suppose that  $\mathcal{B}$  and  $\mathcal{C}$  are  $C^*$ -algebras and that  $\mathcal{A} \subseteq \mathcal{B}$  is a  $C^*$ -subalgebra of  $\mathcal{B}$ . If  $\phi : \mathcal{B} \rightarrow \mathcal{C}$  is a contractive completely positive map, so that the restriction of  $\phi$  to  $\mathcal{A}$  is a  $*$ -homomorphism, then  $\phi$  is an  $\mathcal{A}$ -bimodule map.*

Clearly the  $C^*$ -algebra from point (iii) in the above theorem is the largest subset for which the map is a bimodule map. This leads to the following definition:

**Definition 2.1.4.** In the setting Proposition 2.1.2, the  $C^*$ -algebra from (iii) is called the **multiplicative domain** of  $\phi$ .  $\blacktriangleleft$

Now, the next thing we are interested in is Tomiyama's theorem, which characterizes a particular kind of bimodule maps. First we need a definition, and then a lemma, which serves as a tool to determine whether a map is completely positive.

**Definition 2.1.5.** A surjective linear map  $\phi : \mathcal{B} \rightarrow \mathcal{A}$  from a  $C^*$ -algebra  $\mathcal{B}$  to a  $C^*$ -subalgebra  $\mathcal{A}$  of  $\mathcal{B}$  is called a **projection** if  $\phi^2 = \phi$ , that is, if  $\phi|_{\mathcal{A}} = \text{id}_{\mathcal{A}}$ .  $\blacktriangleleft$

**Lemma 2.1.6.** *A map  $\phi : \mathcal{S} \rightarrow \mathcal{A}$  from an operator system  $\mathcal{S}$  to a  $C^*$ -algebra  $\mathcal{A}$  is completely positive if and only if  $\rho \circ \phi$  is completely positive for all cyclic representations  $\rho$  of  $\mathcal{A}$ .*

*Proof.* Clearly  $\phi$  is completely positive if and only if  $\pi \circ \phi$  is completely positive for some faithful non-degenerate representation of  $\mathcal{A}$ . So let  $(\pi, \mathcal{H})$  be any faithful non-degenerate representation of  $\mathcal{A}$ . Write  $\pi = \sum_{\alpha \in A} \pi_\alpha$  as a direct sum of cyclic representations  $(\pi_\alpha, \mathcal{H}_\alpha, \xi_\alpha)$  with  $\mathcal{H} = \bigoplus_{\alpha \in A} \mathcal{H}_\alpha$ . This is possible by Theorem 1.1.5. Suppose that  $[x_{i,j}] \in M_n(\mathcal{S})$  is positive, then we want to show that  $(\pi \circ \phi)_n([x_{i,j}])$  is positive. Let  $\xi = (\xi^1, \dots, \xi^n) \in \mathcal{H}^{\oplus n}$  and write  $\xi^j = \sum_{\alpha \in A} \xi_\alpha^j$  with  $\xi_\alpha^j \in \mathcal{H}_\alpha$  for  $\alpha \in A$ . Then

$$\begin{aligned} \langle (\pi \circ \phi)_n([x_{i,j}])\xi \mid \xi \rangle &= \sum_{i,j=1}^n \langle \pi(\phi(x_{i,j}))\xi^j \mid \xi^i \rangle \\ &= \sum_{i,j=1}^n \sum_{\alpha \in A} \langle \pi_\alpha(\phi(x_{i,j}))\xi_\alpha^j \mid \xi_\alpha^i \rangle \\ &= \sum_{\alpha \in A} \langle (\pi_\alpha \circ \phi)_n([x_{i,j}])\xi_\alpha \mid \xi_\alpha \rangle, \end{aligned}$$

where  $\xi_\alpha = (\xi_\alpha^1, \dots, \xi_\alpha^n)$  for all  $\alpha \in A$ . Now since  $\pi_\alpha \circ \phi$  is completely positive by assumption, the last sum is a sum of non-negative numbers, and therefore is itself non-negative. Hence  $\langle (\pi \circ \phi)_n([x_{i,j}])\xi \mid \xi \rangle \geq 0$ , which shows that  $\pi \circ \phi$  is completely positive.  $\square$

The following—rather surprising—theorem is due to Jun Tomiyama, and will be referred to as Tomiyama's theorem.

**Theorem 2.1.7.** *Suppose that  $\mathcal{B}$  is a  $C^*$ -algebra and  $\mathcal{A}$  a  $C^*$ -subalgebra. For a projection  $\phi : \mathcal{B} \rightarrow \mathcal{A}$  the following are equivalent:*

- (i)  $\phi$  is a positive  $\mathcal{A}$ -bimodule map;
- (ii)  $\phi$  is contractive completely positive;
- (iii)  $\phi$  is contractive.

*Proof.* It follows from Theorem 1.4.9, that  $\phi$  is contractive completely positive if and only if  $\phi^{**}$  is contractive completely positive, and that  $\phi$  is contractive if and only if  $\phi^{**}$  is contractive. Now, it is easy to see that  $\phi$  is an  $\mathcal{A}$ -bimodule map if and only if  $\phi^{**}$  is an  $\mathcal{A}^{**}$ -bimodule map. This follows from the fact that  $\phi^{**}$  is ultraweakly continuous by Theorem 1.4.9 and that the product is ultraweakly continuous in each variable separately. Thus  $\phi$  satisfies one of the conditions above if and only if  $\phi^{**}$  satisfies the corresponding conditions with  $\mathcal{B}$  and  $\mathcal{A}$  replaced by  $\mathcal{B}^{**}$  and  $\mathcal{A}^{**}$ , respectively. Therefore we may assume that  $\mathcal{A}$  and  $\mathcal{B}$  are von Neumann algebras, since we could otherwise pass to the double dual.<sup>1</sup> Suppose that  $\phi$  is contractive, and let us show that  $\phi$  is an  $\mathcal{A}$  bimodule map. We let  $\mathbf{1}_\mathcal{A}$  and  $\mathbf{1}_\mathcal{B}$  denote the unit of  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. Suppose that  $p \in \mathcal{A}$  is a projection and let  $p^\perp$  denote the projection  $\mathbf{1}_\mathcal{B} - p$ . Fix some

<sup>1</sup>There is a slight problem here, namely that  $\mathcal{A}^{**}$  and  $\mathcal{B}^{**}$  are not necessarily von Neumann algebras on the same Hilbert space (the inclusion  $\mathcal{A}^{**} \subseteq \mathcal{B}^{**}$  is not necessarily unital). We are going to plainly ignore this problem, or in other words, we are going to prove the theorem where  $\mathcal{A}$  and  $\mathcal{B}$  are von Neumann algebras, and the inclusion of  $\mathcal{A}$  in  $\mathcal{B}$  is replaced with an injective  $*$ -homomorphism which is an ultraweak-to-ultraweak-homeomorphism onto its image, and then suppress the actual injection in the proof.

$x \in \mathcal{B}$  and  $t \in \mathbb{R}$ . Since  $p\phi(p^\perp x) \in \mathcal{A}$ , we have  $p\phi(p^\perp x) = \phi(p\phi(p^\perp x))$ , so

$$\begin{aligned} (1+t)^2 \|p\phi(p^\perp x)\|^2 &= \|p\phi(p^\perp x + tp\phi(p^\perp x))\|^2 \\ &\leq \|p^\perp x + tp\phi(p^\perp x)\|^2 \\ &\leq \|p^\perp x\|^2 + t^2 \|p\phi(p^\perp x)\|^2, \end{aligned}$$

where the first inequality follows from the fact that  $\|p\| \leq 1$  and  $\|\phi\| \leq 1$ , and the second inequality follows from the fact that  $p$  and  $p^\perp$  have orthogonal ranges. Now by rearranging the terms we obtain the inequality

$$\|p\phi(p^\perp x)\|^2 + 2t\|p\phi(p^\perp x)\|^2 \leq \|p^\perp x\|^2,$$

so since this holds for all  $t \in \mathbb{R}$  we get  $p\phi(p^\perp x) = 0$ . The same argument, with  $p$  replaced by  $\mathbf{1}_{\mathcal{A}}$  and  $\mathbf{1}_{\mathcal{A}} - p$ , shows that  $\phi(\mathbf{1}_{\mathcal{A}}^\perp x) = 0$  and  $(\mathbf{1}_{\mathcal{A}} - p)\phi((\mathbf{1}_{\mathcal{A}} - p)^\perp x) = 0$ , respectively. Since  $(\mathbf{1}_{\mathcal{A}} - p)^\perp = \mathbf{1}_{\mathcal{A}}^\perp + p$ , we get  $\phi((\mathbf{1}_{\mathcal{A}} - p)^\perp x) = \phi(px)$  and

$$\phi(px) - p\phi(px) = (\mathbf{1}_{\mathcal{A}} - p)\phi((\mathbf{1}_{\mathcal{A}} - p)^\perp x) = 0.$$

This shows that  $\phi(px) = p\phi(px)$ , so using again that  $p\phi(p^\perp x) = 0$  we get

$$p\phi(x) = p\phi(px + p^\perp x) = p\phi(px) + p\phi(p^\perp x) = \phi(px)$$

Since  $\mathcal{A}$  is the closed linear span of its projections and  $\phi$  bounded, this shows that  $\phi$  is an  $\mathcal{A}$ -bimodule map. The only thing left is to show that  $\phi$  is positive. By Proposition B.1.10 it suffices to show that  $\phi$  is unital, since it is a contraction. So let us now show that  $\phi$  is unital. For each  $a \in \mathcal{A}$  we have

$$a\phi(\mathbf{1}_{\mathcal{B}}) = \phi(a\mathbf{1}_{\mathcal{B}}) = \phi(a) = a,$$

and similarly  $\phi(\mathbf{1}_{\mathcal{B}})a = a$ . Since  $a \in \mathcal{A}$  was arbitrary, this shows that  $\phi(\mathbf{1}_{\mathcal{B}}) = \mathbf{1}_{\mathcal{A}}$ . Thus  $\phi$  is unital and therefore positive.

Suppose now instead that  $\phi$  is a positive  $\mathcal{A}$ -bimodule map, and let us show that  $\phi$  is completely positive. By Lemma 2.1.6 it suffices to show that  $\rho \circ \phi$  is completely positive for any cyclic representation  $(\rho, \mathcal{K}, \xi)$  of  $\mathcal{A}$ . So suppose that  $(\rho, \mathcal{K}, \xi)$  is such a representation. It is straightforward to check that since  $\xi$  is cyclic for  $\rho$ , the vector  $\xi_n = (\xi, \xi, \dots, \xi)$  ( $n$  times) is cyclic for  $\rho_n$ , that is,  $(\rho_n, \mathcal{K}^{\oplus n}, \xi_n)$  is a cyclic representation. In particular,  $\rho_n(\mathcal{A})\xi_n$  is dense in  $\mathcal{K}^{\oplus n}$ . Suppose that  $[x_{i,j}] \in M_n(\mathcal{B})$  is positive, and that  $a_1, \dots, a_n \in \mathcal{A}$ . Then with  $\eta = (\rho(a_1)\xi, \dots, \rho(a_n)\xi)$ , we have

$$\begin{aligned} \langle [\rho(\phi(x_{i,j}))]\eta \mid \eta \rangle &= \sum_{i,j=1}^n \langle \rho(\phi(x_{i,j}))\rho(a_j)\xi \mid \rho(a_i)\xi \rangle \\ &= \sum_{i,j=1}^n \langle \rho(a_i^* \phi(x_{i,j}) a_j) \xi \mid \xi \rangle \\ &= \left\langle \rho \left( \phi \left( \sum_{i,j=1}^n a_i^* x_{i,j} a_j \right) \right) \xi \mid \xi \right\rangle \end{aligned}$$

Since  $\phi$  is positive and  $\sum_{i,j=1}^n a_i^* x_{i,j} a_j$  is positive, the right hand side is positive. This shows that  $[\rho(\phi(x_{i,j}))]$  is positive, since  $a_1, \dots, a_n \in \mathcal{A}$  were arbitrary and  $\rho_n(\mathcal{A})\xi_n$  is dense in  $\mathcal{K}^{\oplus n}$ . Hence  $\rho \circ \phi$  is completely positive, and so is  $\phi$ . Now, since  $\phi$  is unital completely positive, it is contractive completely positive.

The last implication, that is, the fact that  $\phi$  is contractive if  $\phi$  is contractive and completely positive is trivial, so we have proved the equivalence.  $\square$



**Definition 2.1.8.** A projection  $\phi: \mathcal{B} \rightarrow \mathcal{A}$  from a  $C^*$ -algebra  $\mathcal{B}$  to a  $C^*$ -subalgebra  $\mathcal{A} \subseteq \mathcal{B}$  is called a **conditional expectation** if it satisfies the equivalent conditions in Tomiyama's theorem. ◀

## 2.2 The weak expectation property

Before defining the weak expectation property, we will introduce the concept of a  $C^*$ -algebra being relatively weakly injective in a larger  $C^*$ -algebra. This terminology is well suited, when dealing with the weak expectation property. The weak expectation property can be described via the concept of relatively weakly injectivity, and some results about the weak expectation property can be formulated more generally through relatively weak injectivity.

**Definition 2.2.1.** Suppose that  $\mathcal{B}$  is a  $C^*$ -algebra and  $\mathcal{A} \subseteq \mathcal{B}$  a  $C^*$ -subalgebra. Then we say that  $\mathcal{A}$  is **relatively weakly injective** in  $\mathcal{B}$ , if there exists a contractive completely positive map  $\phi: \mathcal{B} \rightarrow \mathcal{A}^{**}$  such that  $\phi(a) = a$ , for all  $a \in \mathcal{A}$ . ◀

We see that  $\mathcal{A}$  being relatively weakly injective in  $\mathcal{B}$ , is like requiring that there “almost” exists a conditional expectation from  $\mathcal{B}$  onto  $\mathcal{A}$ . Indeed, if there exists a conditional expectation from  $\mathcal{B}$  onto  $\mathcal{A}$ , then  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$ . This is a particularly nice way of being relatively weak injective.

The next proposition gives equivalent characterizations of relatively weakly injectivity, and to prove this we need the following easy lemma, which is entirely a functional analysis result:

**Lemma 2.2.2.** Suppose that  $\mathfrak{X}$  and  $\mathfrak{Y}$  are normed spaces. If  $\phi: \mathfrak{X} \rightarrow \mathfrak{Y}^*$  is a bounded linear map, then  $\phi$  extends to a weak\*-continuous bounded linear map  $\hat{\phi}: \mathfrak{X}^{**} \rightarrow \mathfrak{Y}^*$ , with the same norm.

*Proof.* Consider the restriction of the adjoint map  $\phi^*: \mathfrak{Y}^{**} \rightarrow \mathfrak{X}^*$  to  $\mathfrak{Y}$ . Taking the adjoint of this map, and calling it  $\hat{\phi}$ , that is  $\hat{\phi} = ((\phi^*)|_{\mathfrak{Y}})^*$ , we obtain a map from  $\mathfrak{X}^{**}$  to  $\mathfrak{Y}^*$ . Let us show that this map extends  $\phi$ . For each  $x \in \mathfrak{X}$  and  $y \in \mathfrak{Y}$  we have

$$\hat{\phi}(x)(y) = (\phi^*|_{\mathfrak{Y}})(y)(x) = \phi^*(y)(x) = \phi(x)(y).$$

Thus  $\hat{\phi}(x) = \phi(x)$ , which shows that  $\hat{\phi}$  extends  $\phi$ . Clearly  $\|\phi\| \leq \|\hat{\phi}\|$ , since  $\hat{\phi}$  extends  $\phi$ , and we also see that  $\|\hat{\phi}\| = \|\phi^*|_{\mathfrak{Y}}\| \leq \|\phi^*\| = \|\phi\|$ . The adjoint of a bounded linear map is weak\*-continuous, so  $\hat{\phi}$  is weak\*-continuous. ◻

**Proposition 2.2.3.** Suppose that  $\mathcal{B}$  is a  $C^*$ -algebra and  $\mathcal{A}$  is a  $C^*$ -subalgebra of  $\mathcal{B}$ . Then the following are equivalent:

- (i)  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$ ;
- (ii) there exists a conditional expectation  $\psi: \mathcal{B}^{**} \rightarrow \mathcal{A}^{**}$ ;
- (iii) for every finite dimensional subspace  $\mathcal{M} \subseteq \mathcal{B}$  and any  $\varepsilon > 0$ , there exist a linear contraction  $\psi: \mathcal{M} \rightarrow \mathcal{A}$  such that  $\|\psi|_{\mathcal{A} \cap \mathcal{M}} - \text{id}_{\mathcal{A} \cap \mathcal{M}}\| < \varepsilon$ .

*Proof.* Clearly (ii) implies (i), since the restriction of the conditional expectation to  $\mathcal{B}$  is a contractive completely positive map  $\mathcal{B} \rightarrow \mathcal{A}^{**}$  which restricts to the identity on  $\mathcal{A}$ . Also, if (i) holds, then there exists a contractive completely positive map  $\phi: \mathcal{B} \rightarrow \mathcal{A}^{**}$

which is the identity on  $\mathcal{A}$ , and by Lemma 2.2.2 this map extends to an ultraweakly continuous linear map  $\psi: \mathcal{B}^{**} \rightarrow \mathcal{A}^{**}$  of the same norm. Since  $\psi$  is ultraweakly continuous and  $\mathcal{A}$  is ultraweakly dense in  $\mathcal{A}^{**}$ , it must then be the identity on  $\mathcal{A}^{**}$ . We also know that  $\psi$  is a contraction since its norm agrees with that of  $\phi$ . Thus  $\psi$  is a conditional expectation, and we conclude (ii).

The fact that (i) implies (iii) follows from the Principle of Local Reflexivity for Banach spaces (see [Woj91, page 76]). So suppose that (iii) holds and let us prove that (ii) holds. For each finite dimensional subspace  $\mathcal{M} \subseteq \mathcal{A}$  and each  $n \in \mathbb{N}$ , choose a map  $\psi_{\mathcal{M},n}$  according to our assumptions with  $\varepsilon = n^{-1}$ . With the convention that  $\psi_{\mathcal{M},n}(x) = 0$  for  $x \in \mathcal{B} \setminus \mathcal{M}$ , we will consider  $\psi_{\mathcal{M},n}$  a map from  $\mathcal{B}$  to  $\mathcal{A}$ .<sup>2</sup> Let  $\mathcal{F}$  denote the set of finite dimensional subspaces of  $\mathcal{A}$ , and order  $\mathcal{F} \times \mathbb{N}$  by  $(\mathcal{M}, n) \leq (\mathcal{M}', n')$  if and only if  $\mathcal{M} \subseteq \mathcal{M}'$  and  $n \leq n'$ . Choose some cofinal ultrafilter  $\omega$  on  $\mathcal{F} \times \mathbb{N}$ , that is, an ultrafilter containing the set  $\{(\mathcal{M}', n') \in \mathcal{F} \times \mathbb{N} : \mathcal{M}' \subseteq \mathcal{M}, n' \leq n\}$  for all  $(\mathcal{M}, n) \in \mathcal{F} \times \mathbb{N}$ . For each  $x \in \mathcal{B}$ , the net  $(\psi_\alpha(x))_{\alpha \in \mathcal{F} \times \mathbb{N}}$  is bounded in  $\mathcal{A}^{**}$ , bounded by  $\|x\|$ . Hence the limit  $\lim_{\alpha \rightarrow \omega} \psi_\alpha(x)$  exists in the ultraweak topology, for all  $x \in \mathcal{B}$ . Denote this limit by  $\psi(x)$ . Now we have obtained a map  $\psi: \mathcal{B} \rightarrow \mathcal{A}^{**}$ . Let us check that this map is linear, completely positive and restricts to the identity on  $\mathcal{A}$ . First linearity: suppose  $x_1, x_2 \in \mathcal{B}$  and  $\lambda_1, \lambda_2 \in \mathbb{C}$ . Denote the linear span of  $x_1$  and  $x_2$  by  $\mathcal{M}_0$ , then

$$\psi_{\mathcal{M},n}(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \psi_{\mathcal{M},n}(x_1) + \lambda_2 \psi_{\mathcal{M},n}(x_2)$$

for all  $n \in \mathbb{N}$  and all finite dimensional subspace  $\mathcal{M} \subseteq \mathcal{B}$  containing  $\mathcal{M}_0$ . From this it follows that  $\psi$  is linear, by the choice of ultrafilter. In the same way we see that  $\psi$  is the identity on  $\mathcal{A}$ . Clearly  $\psi$  is contractive, since  $(\psi_\alpha(x))_{\alpha \in \mathcal{F} \times \mathbb{N}}$  is bounded by  $\|x\|$ . Now, by Lemma 2.2.2, we can extend  $\psi$  to a ultraweak-to-ultraweak continuous contraction  $\hat{\psi}: \mathcal{B}^{**} \rightarrow \mathcal{A}^{**}$ . By continuity  $\hat{\psi}$  must be the identity on  $\mathcal{A}^{**}$ , since  $\mathcal{A}$  is ultraweakly dense in  $\mathcal{A}^{**}$ . This means that  $\hat{\psi}$  is a conditional expectation, and so we conclude (ii).  $\square$

**Remark 2.2.4.** Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra and  $\mathcal{I}$  a closed two-sided ideal in  $\mathcal{A}$ . Then by Proposition 1.4.10 there exists a central projection  $p$  in  $\mathcal{A}^{**}$  so that  $\mathcal{I}^{**} = p\mathcal{A}^{**}$ . Hence the map  $\phi: \mathcal{A}^{**} \rightarrow \mathcal{I}^{**}$  given by  $\phi(x) = px$ ,  $x \in \mathcal{A}^{**}$ , is a unital completely positive map, which is clearly the identity on  $\mathcal{I}^{**}$ , that is,  $\phi$  is a conditional expectation. By the previous proposition, this means that  $\mathcal{I}$  is relatively weakly injective in  $\mathcal{A}$ . A particular application of this result is that  $\mathcal{A}$  is relatively weakly injective in its unitization  $\tilde{\mathcal{A}}$ .

If  $\mathcal{I}$  is not an ideal, but just a hereditary subalgebra, then  $\mathcal{I}$  is still relatively weakly injective. Here we just let  $p$  be the identity in  $\mathcal{I}^{**}$ , and let  $\phi$  be the conditional expectation defined by  $\phi(x) = pxp$ .  $\blacktriangleleft$

By now we are ready to define the weak expectation property.

**Definition 2.2.5.** A  $C^*$ -algebra  $\mathcal{A}$  is said to have the **weak expectation property** (abbreviated **WEP**) if there exists a faithful representation  $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$  of  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$  such that  $\pi(\mathcal{A})$  is relatively weakly injective in  $B(\mathcal{H})$ . Or equivalently, there exist a representation  $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$  and a contractive completely positive map  $\Phi: B(\mathcal{H}) \rightarrow \mathcal{A}^{**}$  such that  $(\Phi \circ \pi)(a) = a$ , for all  $a \in \mathcal{A}$ .  $\blacktriangleleft$

<sup>2</sup>Note, that these extensions are in all probability *not* linear, but they still satisfy that  $\|\psi_{\mathcal{M},n}(x)\| \leq \|x\|$ , for all  $x \in \mathcal{B}$ .

Obviously, a  $C^*$ -algebra has the weak expectation property if it is isomorphic to a  $C^*$ -algebra with the weak expectation property. An immediate example of a  $C^*$ -algebra with the WEP is  $B(\mathcal{H})$ , for any Hilbert spaces  $\mathcal{H}$ .

Naturally one might ask whether all  $C^*$ -algebras have the weak expectation property. This is not the case, and in fact, the weak expectation property also provides one more characterization in the abundance of equivalent formulations of amenability for discrete groups. More precisely, a discrete group is amenable if and only if its reduced group  $C^*$ -algebra has the weak expectation property (see [BO08, Proposition 3.6.9] for a proof). In particular, the existence of  $C^*$ -algebras without the weak expectation property follows from the existence of non-amenable groups.

This following proposition shows that the particular choice of Hilbert space in the definition of the WEP is of no importance.

**Proposition 2.2.6.** *Suppose that  $\mathcal{A}$  has the weak expectation property. Then, for each faithful representation  $\rho: \mathcal{A} \rightarrow B(\mathcal{K})$  of  $\mathcal{A}$  on a Hilbert space  $\mathcal{K}$ , there exists a contractive completely positive map  $\Psi: B(\mathcal{K}) \rightarrow \mathcal{A}^{**}$  such that  $(\Psi \circ \rho)(a) = a$ , for all  $a \in \mathcal{A}$ . If  $\mathcal{A}$  is unital, then  $\Psi$  may be chosen to be unital completely positive.*

*Proof.* Choose a faithful representation  $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$  of  $\mathcal{A}$  on some Hilbert space  $\mathcal{H}$  and a contractive completely positive map  $\Phi: B(\mathcal{H}) \rightarrow \mathcal{A}^{**}$  such that  $(\Phi \circ \pi)(a) = a$ , for all  $a \in \mathcal{A}$ . This is possible by the assumption that  $\mathcal{A}$  has the WEP. Since both  $\rho$  and  $\pi$  are faithful, the map from  $\rho(\mathcal{A})$  to  $\pi(\mathcal{A})$  given by  $\rho(a) \mapsto \pi(a)$  is a well-defined  $*$ -homomorphism. By Arveson's Extension Theorem (see Corollary B.3.7) this map extends to a contractive completely positive map  $\psi: B(\mathcal{K}) \rightarrow B(\mathcal{H})$ . Now, let  $\Psi$  be the composition of  $\psi$  and  $\Phi$ , then clearly  $\Psi$  is contractive completely positive, and for each  $a \in \mathcal{A}$  we have

$$\Psi(\rho(a)) = \Phi(\psi(\rho(a))) = \Phi(\pi(a)) = a,$$

as wanted.

Suppose furthermore, that  $\mathcal{A}$  is unital. Let  $p = \rho(\mathbf{1})$ . Then  $p$  is a projection and  $\rho(\mathcal{A}) \subseteq pB(\mathcal{K})p$ . The map  $\Psi': B(\mathcal{K}) \rightarrow \mathcal{A}^{**}$  given by  $\Psi'(x) = \Psi(pxp)$  is clearly also contractive completely positive, and  $\Psi'(\mathbf{1}_{\mathcal{K}}) = \Psi(p) = \mathbf{1}_{\mathcal{A}} = \mathbf{1}_{\mathcal{A}^{**}}$ . Hence  $\Psi'$  is unital completely positive.  $\square$

The weak expectation property does not, a priori, behave well with respect to subalgebras. In fact, if the weak expectation property did pass to subalgebras, then it would immediately follow that all  $C^*$ -algebras had the weak expectation property, since  $B(\mathcal{H})$  have the WEP, for all Hilbert spaces  $\mathcal{H}$ . However, the weak expectation property passes to certain subalgebras. A more precise statement is given in the following proposition:

**Proposition 2.2.7.** *If  $\mathcal{B}$  is a  $C^*$ -algebra with the WEP and  $\mathcal{A} \subseteq \mathcal{B}$  is a  $C^*$ -subalgebra which is relatively weakly injective in  $\mathcal{B}$ , then  $\mathcal{A}$  has the WEP.*

*Proof.* Let  $\pi: \mathcal{B} \rightarrow B(\mathcal{H})$  be a faithful representation, and let  $\Phi: B(\mathcal{H}) \rightarrow \mathcal{B}^{**}$  be a contractive completely positive map with  $(\Phi \circ \pi)(b) = b$ , for all  $b \in \mathcal{B}$ . By Proposition 2.2.3 there exists a conditional expectation  $\Psi: \mathcal{B}^{**} \rightarrow \mathcal{A}^{**}$ , since  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$  by assumption. Now, the restriction of  $\pi$  to  $\mathcal{A}$  is a faithful representation, the map  $\Psi \circ \Phi$  is contractive completely positive and  $(\Psi \circ \Phi \circ \pi)(a) = a$ , for all  $a \in \mathcal{A}$ . Thus  $\mathcal{A}$  has the WEP.  $\square$

Clearly the proposition above can be generalized a little, namely to say that the relation “is relatively weakly injective in” is transitive.

Let us next analyse how the weak expectation property behaves with respect to unitizations.

**Proposition 2.2.8.** *A  $C^*$ -algebra has the weak expectation property if and only if its unitization has the weak expectation property.*

*Proof.* Let  $\mathcal{A}$  be a  $C^*$ -algebra. Assume that  $\tilde{\mathcal{A}}$  has the weak expectation property. By Remark 2.2.4, we get that  $\mathcal{A}$  is relatively weakly injective in  $\tilde{\mathcal{A}}$  and by Proposition 2.2.7 we get that  $\mathcal{A}$  has the weak expectation property.

Suppose instead that  $\mathcal{A}$  has the weak expectation property, and let us show that so does  $\tilde{\mathcal{A}}$ . Choose some faithful representation  $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$  for a Hilbert space  $\mathcal{H}$ . Let  $\Phi: B(\mathcal{H}) \rightarrow \mathcal{A}^{**}$  be a contractive completely positive map such that  $\Phi(\pi(a)) = a$ , for all  $a \in \mathcal{A}$ . Let  $\mathbf{1}_{\mathcal{H}}$  and  $\mathbf{1}_{\tilde{\mathcal{A}}}$  denote the unit in  $B(\mathcal{H})$  and  $\tilde{\mathcal{A}}$ , respectively. Define a representation  $\tilde{\pi}: \tilde{\mathcal{A}} \rightarrow B(\mathcal{H} \oplus \mathbb{C})$  by

$$\tilde{\pi}(a + \lambda \mathbf{1}_{\tilde{\mathcal{A}}}) = \begin{bmatrix} \pi(a) + \lambda \mathbf{1}_{\mathcal{H}} & 0 \\ 0 & \lambda \end{bmatrix}, \quad \text{for } a \in \mathcal{A} \text{ and } \lambda \in \mathbb{C}.$$

Clearly this representation is faithful. Let  $p$  and  $p^\perp$  denote the projection of  $\mathcal{H} \oplus \mathbb{C}$  onto  $\mathcal{H}$  and the projection of  $\mathcal{H} \oplus \mathbb{C}$  onto  $\mathbb{C}$ , respectively. Define a map  $\tilde{\Phi}$  from  $B(\mathcal{H} \oplus \mathbb{C})$  to  $\tilde{\mathcal{A}}^{**}$  by

$$\tilde{\Phi}(x) = \Phi(pxp) + p^\perp xp^\perp (\mathbf{1}_{\tilde{\mathcal{A}}} - \Phi(\mathbf{1}_{\mathcal{H}})), \quad x \in B(\mathcal{H} \oplus \mathbb{C})$$

First of all, note that this definition makes sense, since  $p^\perp xp^\perp$  is a complex number for each  $x \in B(\mathcal{H} \oplus \mathbb{C})$ . Second, note that the map  $\tilde{\Phi}$  is actually completely positive, since both the maps  $x \mapsto \Phi(pxp)$  and  $x \mapsto p^\perp xp^\perp$  are completely positive, and  $\mathbf{1}_{\tilde{\mathcal{A}}} - \Phi(\mathbf{1}_{\mathcal{H}})$  is positive. Now, since  $\Phi$  unital, that is,  $\Phi(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}$ , it follows that  $\tilde{\Phi}$  is unital. Hence  $\tilde{\Phi}$  is unital completely positive. Let  $a \in \mathcal{A}$  and  $\lambda \in \mathbb{C}$ . Then  $p\tilde{\pi}(a + \lambda \mathbf{1}_{\tilde{\mathcal{A}}})p = \pi(a) + \lambda \mathbf{1}_{\mathcal{H}}$  and  $p^\perp \tilde{\pi}(a + \lambda \mathbf{1}_{\tilde{\mathcal{A}}})p^\perp = \lambda$ , so

$$\tilde{\Phi}(\tilde{\pi}(a + \lambda \mathbf{1}_{\tilde{\mathcal{A}}})) = \Phi(\pi(a) + \lambda \mathbf{1}_{\mathcal{H}}) + \lambda(\mathbf{1}_{\tilde{\mathcal{A}}} - \Phi(\mathbf{1}_{\mathcal{H}})) = a + \lambda \mathbf{1}_{\tilde{\mathcal{A}}},$$

This proves that  $\tilde{\mathcal{A}}$  has the weak expectation property.  $\square$

Now it is time for a lemma which states that, given a family of Hilbert spaces, the set of diagonal operators (with respect to their direct sum) is relatively weakly injective in the bounded operators on the direct sum Hilbert space.

**Lemma 2.2.9.** *Suppose that  $(\mathcal{H}_i)_{i \in I}$  is a family of Hilbert spaces, and let  $\mathcal{H}$  denote their direct sum  $\bigoplus_{i \in I} \mathcal{H}_i$ . Then  $\ell_\infty(I; B(\mathcal{H}_i))$  is relatively weakly injective in  $B(\mathcal{H})$  when its elements are regarded as operators on  $\mathcal{H}$  acting diagonally. In particular  $\ell_\infty(I; B(\mathcal{H}_i))$  has the WEP.*

*Proof.* It suffices to show that there exists a conditional expectation  $\mathcal{E}: B(\mathcal{H}) \rightarrow \ell_\infty(I; B(\mathcal{H}_i))$ . For each  $i \in I$ , let  $p_i$  denote the orthogonal projection of  $\mathcal{H}$  onto  $\mathcal{H}_i$ . Let  $\mathcal{F}$  denote the collection of finite subsets of  $I$ , and for each  $F \in \mathcal{F}$ , let  $\psi_F: B(\mathcal{H}) \rightarrow B(\mathcal{H})$  denote the map

$$\psi_F(x) = \sum_{j \in F} p_j x p_j, \quad x \in \mathcal{H}.$$

Clearly,  $\psi_F$  is completely positive for all  $F \in \mathcal{F}$ , since the map  $x \mapsto p_j x p_j$ ,  $x \in B(\mathcal{H})$ , is completely positive for each  $j \in F$ . Note that, if  $F \in \mathcal{F}$  and  $x \in B(\mathcal{H})$ , then since  $\|p_j x p_j\| \leq \|x\|$  and all the operators  $(p_j x p_j)_{j \in F}$  have orthogonal supports and ranges, it follows that  $\|\psi_F(x)\| \leq \|x\|$ . Thus  $(\psi_F)_{F \in \mathcal{F}}$  is a net of contractive completely positive maps. Let us argue that this net converges in point-weak operator topology. Suppose that  $x \in B(\mathcal{H})$ . For  $i, j \in I$ ,  $\xi \in \mathcal{H}_i$  and  $\eta \in \mathcal{H}_j$ , it holds that

$$\lim_{F \in \mathcal{F}} (\psi_F(x)\xi \mid \eta) = 0, \quad \text{if } i \neq j \quad \text{and} \quad \lim_{F \in \mathcal{F}} (\psi_F(x)\xi \mid \eta) = (x\xi \mid \eta), \quad \text{if } i = j.$$

In particular,  $\lim_{F \in \mathcal{F}} (\psi_F(x)\xi \mid \eta)$  exists whenever  $\xi, \eta \in \text{span} \bigcup_{i \in I} B(\mathcal{H}_i)$ . It is straightforward to check that, since the net  $(\psi_F(x))_{F \in \mathcal{F}}$  is bounded, the limit  $\lim_{F \in \mathcal{F}} (\psi_F(x)\xi \mid \eta)$  exists for all  $\xi, \eta \in \mathcal{H}$ . Thus the net  $(\psi_F(x))_{F \in \mathcal{F}}$  is weak operator convergent to some operator  $\mathcal{E}(x) \in B(\mathcal{H})$  of norm less than or equal to  $\|x\|$ . In particular, the net  $(\psi_F)_{F \in \mathcal{F}}$  is point-weak operator convergent to the bounded linear operator  $\mathcal{E}$ . By Proposition B.2.7 the point-weak operator topology on  $B(B(\mathcal{H}), B(\mathcal{H}))$  agrees on bounded sets with the weak\*-topology, so since the set of contractive completely positive maps from  $B(\mathcal{H})$  to  $B(\mathcal{H})$  is compact in this topology by Proposition B.2.8, it follows that  $\mathcal{E}$  is contractive completely positive. Now the only thing left to prove is that  $\mathcal{E}(a) = a$  for all  $a \in \ell_\infty(I; B(\mathcal{H}_i))$ . Since  $\ell_\infty(I; B(\mathcal{H}_i))$  acts diagonally on  $\mathcal{H}$  we have that  $p_i a p_j = 0$  when  $i, j \in I$  are distinct. Thus

$$\psi_F(a) = \sum_{i \in F} p_i a p_i = \left( \sum_{i \in F} p_i \right) a \left( \sum_{j \in F} p_j \right),$$

and since  $(\sum_{i \in F} p_i)_{F \in \mathcal{F}}$  is a bounded net converging to  $\mathbf{1}$  in weak operator topology, it follows that  $\mathcal{E}(a) = a$ , for all  $a \in \ell_\infty(I; B(\mathcal{H}_i))$ . Hence  $\mathcal{E}$  is a conditional expectation.  $\square$

With the conclusion of the above lemma we can now prove our first permanence property for the weak expectation property.

**Proposition 2.2.10.** *Suppose that  $(\mathcal{A}_i)_{i \in I}$  is a family of  $C^*$ -algebras with the WEP. Then  $\ell_\infty(I; \mathcal{A}_i)$  has the WEP.*

*Proof.* We may assume that  $\mathcal{A}_i \subseteq B(\mathcal{H}_i)$ , with  $\mathcal{H}_i$  a Hilbert space, for each  $i \in I$ . By Proposition 2.2.7 and Lemma 2.2.9 it suffices to show that  $\ell_\infty(I; \mathcal{A}_i)$  is relatively weakly injective in  $\ell_\infty(I; B(\mathcal{H}_i))$ . We do this by proving that condition (iii) of Proposition 2.2.3 is satisfied. Suppose that  $E \subseteq \ell_\infty(I; B(\mathcal{H}_i))$  is a finite dimensional subspace, and let  $\varepsilon > 0$ . For each  $i \in I$ , let  $E_i$  denote the projection of  $E$  onto the  $i$ 'th coordinate. Clearly  $E_i$  is a finite dimensional subspace of  $B(\mathcal{H}_i)$ , so by assumption and Proposition 2.2.3 there exists a linear contraction  $\psi_i: E_i \rightarrow \mathcal{A}_i$  so that  $\|\psi_i|_{E_i \cap \mathcal{A}_i} - \text{id}_{E_i \cap \mathcal{A}_i}\| < \varepsilon$ . Now, by construction  $E \subseteq \ell_\infty(I; E_i)$ , so we may define a map

$$\psi: E \rightarrow \ell_\infty(I; \mathcal{A}_i) \quad \text{by} \quad \psi((x_i)_{i \in I}) = (\psi_i(x_i))_{i \in I}.$$

This is a contraction, since all the maps  $\psi_i$ ,  $i \in I$  are contractions. If  $(x_i)_{i \in I} \in E \cap \ell_\infty(I; \mathcal{A}_i)$ , then  $x_i \in E_i \cap \mathcal{A}_i$  for each  $i \in I$ , and it follows that

$$\|\psi((x_i)_{i \in I}) - (x_i)_{i \in I}\| = \sup\{\|\psi_i(x_i) - x_i\| : i \in I\} \leq \varepsilon.$$

Hence  $\|\psi|_{E \cap \ell_\infty(I; \mathcal{A}_i)} - \text{id}_{E \cap \ell_\infty(I; \mathcal{A}_i)}\| \leq \varepsilon$ . By Proposition 2.2.3 this shows that  $\ell_\infty(I; \mathcal{A}_i)$  is relatively weakly injective in  $\ell_\infty(I; B(\mathcal{H}_i))$ , and so we conclude by Lemma 2.2.9 that  $\ell_\infty(I; \mathcal{A}_i)$  has the WEP.  $\square$

## 2.3 QWEP

In this section we introduce a new property, called QWEP, and prove, on one hand, certain permanence properties of QWEP, and on the other hand, some tools for determining whether a  $C^*$ -algebra is QWEP. Naturally, we begin with the definition.

**Definition 2.3.1.** If  $\mathcal{A}$  is the quotient of a  $C^*$ -algebra with WEP, in the sense that there exists a  $C^*$ -algebra  $\mathcal{B}$  with WEP and a surjective  $*$ -homomorphism  $\mathcal{B} \rightarrow \mathcal{A}$ , then we say that  $\mathcal{A}$  is **quotient of a  $C^*$ -algebra with the weak expectation property** (abbreviated **QWEP**). ◀

Obviously, being QWEP is preserved under  $*$ -isomorphisms. In fact, it must also necessarily be preserved under surjective  $*$ -homomorphisms, that is, a quotient of a  $C^*$ -algebra which is QWEP, is also QWEP.

As with the weak expectation property, it is not obvious that QWEP does pass to subalgebras. In fact, we know that it is not true for the weak expectation property, but we do not know whether it is true for QWEP.<sup>3</sup> Like with WEP we have a result saying that this does happen, if we require that the smaller algebra is relatively weakly injective in the larger one.

**Proposition 2.3.2.** *Suppose that  $\mathcal{B}$  is a  $C^*$ -algebra and  $\mathcal{A} \subseteq \mathcal{B}$  a  $C^*$ -subalgebra. If  $\mathcal{B}$  is QWEP and  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$ , then  $\mathcal{A}$  is QWEP.*

*Proof.* By Proposition 2.2.3 there exists a conditional expectation  $\phi: \mathcal{B}^{**} \rightarrow \mathcal{A}^{**}$ . Let  $\mathcal{B}'$  be a  $C^*$ -algebra with the WEP, and let  $\pi: \mathcal{B}' \rightarrow \mathcal{B}$  be a surjective  $*$ -homomorphism. Let  $\mathcal{I} = \ker \pi$  let  $\mathcal{A}'$  denote the  $C^*$ -subalgebra  $\pi^{-1}(\mathcal{A})$  of  $\mathcal{B}'$ . By Proposition 1.4.10 we have isomorphisms  $(\mathcal{B}')^{**} \cong \mathcal{I}^{**} \oplus \mathcal{B}^{**}$  and  $(\mathcal{A}')^{**} \cong \mathcal{I}^{**} \oplus \mathcal{A}^{**}$ , so we get a conditional expectation

$$(\mathcal{B}')^{**} \cong \mathcal{I}^{**} \oplus \mathcal{B}^{**} \xrightarrow{\text{id} \oplus \phi} \mathcal{I}^{**} \oplus \mathcal{A}^{**} \cong (\mathcal{A}')^{**} .^4$$

Thus  $\mathcal{A}'$  is relatively weakly injective in  $\mathcal{B}'$ , and by Proposition 2.2.3 we get that  $\mathcal{A}'$  is relatively weakly injective in  $\mathcal{B}$ , and by Proposition 2.2.7 we get that  $\mathcal{A}'$  has the WEP. Since  $\pi$  restricts to a surjective  $*$ -homomorphism  $\mathcal{A}' \rightarrow \mathcal{A}$  we conclude that  $\mathcal{A}$  is QWEP. ◻

Recalling that conditional expectations always exist in the setting of finite von Neumann algebras with separable predual, we get the following corollary:

**Corollary 2.3.3.** *Suppose that  $\mathcal{M}$  is a finite von Neumann algebra, with a faithful normal trace  $\tau$ , and that  $\mathcal{M}$  is QWEP. Then every von Neumann subalgebra of  $\mathcal{M}$  is also QWEP.*

*Proof.* Suppose that  $\mathcal{M}$  is a von Neumann algebra and  $\mathcal{N}$  a von Neumann subalgebra of  $\mathcal{M}$ . By [BO08, Lemma 1.5.11] there exist a trace-preserving conditional expectation  $\mathcal{E}: \mathcal{M} \rightarrow \mathcal{N}$ . Thus  $\mathcal{N}$  is relatively weakly injective in  $\mathcal{M}$ , which means that  $\mathcal{N}$  must be QWEP. ◻

<sup>3</sup>Obviously this is equivalent to an affirmative answer to the QWEP Conjecture, since  $B(\mathcal{H})$  is QWEP, for all Hilbert spaces  $\mathcal{H}$ .

<sup>4</sup>It is straight forward to check that this indeed defines a conditional expectation, using the concrete description of the isomorphisms in Proposition 1.4.10.

Let us, at this point, deal with non-unital technicalities related to QWEP. After this, we will turn to permanence properties, and prove a few.

**Proposition 2.3.4.** *A  $C^*$ -algebra is QWEP if and only if its unitization is QWEP. Moreover, each unital  $C^*$ -algebra which is QWEP is the quotient of a unital  $C^*$ -algebra with the WEP.*

*Proof.* Let  $\mathcal{A}$  be a  $C^*$ -algebra. Suppose that  $\tilde{\mathcal{A}}$  is QWEP. By Remark 2.2.4 we know that  $\mathcal{A}$  is relatively weakly injective in  $\tilde{\mathcal{A}}$ , so by Proposition 2.3.2 we get that  $\mathcal{A}$  is QWEP.

Now, suppose instead that  $\mathcal{A}$  is QWEP. Choose some  $C^*$ -algebra  $\mathcal{B}$  with the WEP, and a surjective  $*$ -homomorphism  $\pi: \mathcal{B} \rightarrow \mathcal{A}$ . By Proposition 1.1.8 we get that  $\pi$  extends to a surjective  $*$ -homomorphism  $\tilde{\pi}: \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{A}}$ . Since  $\tilde{\mathcal{B}}$  has the WEP by Proposition 2.2.8, we conclude that  $\tilde{\mathcal{A}}$  is QWEP. If, in addition, we know that  $\mathcal{A}$  is unital, then also by Proposition 1.1.8 we can also extend  $\pi$  to a map  $\pi': \tilde{\mathcal{B}} \rightarrow \mathcal{A}$ , which of course is still surjective. Hence, if a unital  $C^*$ -algebra is QWEP, then it is a quotient of a unital  $C^*$ -algebra with the WEP.  $\square$

In the following proposition is a reference to ultraproducts of  $C^*$ -algebras, and the reader may consider Chapter 4, but for now it suffices to know that ultraproducts are quotients of algebras on the form  $\ell_\infty(I; \mathcal{A}_i)$  for some choice of a family  $(\mathcal{A}_i)_{i \in I}$  of  $C^*$ -algebras.

**Proposition 2.3.5.** *If  $(\mathcal{A}_i)_{i \in I}$  is a family of  $C^*$ -algebras which are all QWEP, then the  $C^*$ -algebra  $\ell_\infty(I; \mathcal{A}_i)$  is QWEP. In particular, ultraproducts of  $C^*$ -algebras with QWEP are again QWEP.*

*Proof.* For each  $i \in I$ , choose a  $C^*$ -algebra  $\mathcal{B}_i$  with the WEP and a surjective  $*$ -homomorphism  $\pi_i: \mathcal{B}_i \rightarrow \mathcal{A}_i$ . Clearly, the natural projection  $\pi: \ell_\infty(I; \mathcal{B}_i) \rightarrow \ell_\infty(I; \mathcal{A}_i)$  given by  $\pi((x_i)_{i \in I}) = (\pi_i(x_i))_{i \in I}$  is a surjective  $*$ -homomorphism. Thus  $\ell_\infty(I; \mathcal{A}_i)$  is QWEP. The last statement follows from the fact that quotients of  $C^*$ -algebras which are QWEP are QWEP as well.  $\square$

We know from Kaplansky's Density Theorem that every element in the weak operator closure of a self-adjoint algebra of bounded operators on a Hilbert space can be approximated in weak operator topology by a bounded net in this self-adjoint algebra. The following lemma shows that one can choose a fixed directed set for this.

**Lemma 2.3.6.** *Suppose that  $\mathcal{H}$  is a Hilbert space. Then there exists a directed set  $A$  such that, whenever  $\mathcal{A} \subseteq B(\mathcal{H})$  is a self-adjoint subalgebra, and  $x$  in the weak operator closure of  $\mathcal{A}$ , there exists a bounded net  $(x_\alpha)_{\alpha \in A}$  in  $\mathcal{A}$  converging to  $x$  in the strong\* operator topology.*

*Proof.* Let  $A$  be a neighbourhood basis of zero in the strong\* topology on  $B(\mathcal{H})$ , ordered by reversed inclusion. Suppose that  $x$  is in the weak operator closure of  $\mathcal{A}$ . We may assume that  $\|x\| \leq 1$ . Then  $\{x + \mathcal{V} \cap \mathcal{A} : \mathcal{V} \in A\}$  is a neighbourhood basis of  $x$  in the strong\* topology on  $\mathcal{A}$ , so by Kaplansky's Density Theorem, and the fact that the weak operator and strong\* operator closures of a convex set agree, there exists some  $x_\mathcal{V} \in (x + \mathcal{V} \cap \mathcal{A}) \cap \mathcal{A}_1$ , for each  $\mathcal{V} \in A$ . By construction,  $(x_\alpha)_{\alpha \in A}$  is a bounded net converging to  $x$  in the strong\* operator topology.  $\square$

Now, by force of the above lemma, we may prove the following technical result, which, in turn, will enable us to prove Proposition 2.3.9, after just one more lemma. If the reader is not familiar with the use of filters, then he might benefit from either consulting Chapter 4 during the proof of the following lemma or entirely skipping the proof until after Chapter 4.

**Lemma 2.3.7.** *Suppose that  $\mathcal{H}$  is a Hilbert space, and  $(\mathcal{A}_i)_{i \in I}$  an increasing net of unital subalgebras of  $B(\mathcal{H})$ . Then there exist an index set  $J$ , a free ultrafilter  $\omega$  on  $J$  and a family of  $C^*$ -algebras  $(\mathcal{B}_j)_{j \in J}$  with  $\mathcal{B}_j \in \{\mathcal{A}_i : i \in I\}$  for all  $j \in J$ , such that the map*

$$\phi: \ell_\infty(J; \mathcal{B}_j) \rightarrow \left(\bigcup_{i \in I} \mathcal{A}_i\right)'' \quad \text{defined by} \quad \phi((x_j)_{j \in J}) = \lim_{j \rightarrow \omega} x_j$$

is a unital completely positive map, where the limit is taken in the weak operator topology. Moreover, for each  $x \in \left(\bigcup_{i \in I} \mathcal{A}_i\right)''$  there exist an element  $(x_j)_{j \in J} \in \ell_\infty(J; \mathcal{B}_j)$  so that  $x$  is the strong\* operator limit of  $(x_j)_{j \in J}$  along  $\omega$ . In particular the map is surjective.

*Proof.* Let  $A$  be the directed set from Lemma 2.3.6, and let  $J = A \times I$ . For each  $(\alpha, i) \in J$  let  $\mathcal{B}_{(\alpha, i)} = \mathcal{A}_i$ . Choose free ultrafilters  $\nu$  and  $\nu'$  on  $A$  and  $I$ , respectively, and set  $\omega = \nu \otimes \nu'$ , which is a free ultrafilter by Proposition 4.3.2. The map  $\phi$  is clearly well-defined, since the unit ball of  $B(\mathcal{H})$  is weak operator compact and  $\omega$  is an ultrafilter. Let us start by proving the last assertion about the strong\* operator limit. Let  $x \in \left(\bigcup_{i \in I} \mathcal{A}_i\right)''$ . The algebra  $\bigcup_{i \in I} \mathcal{A}_i$  is weak operator dense in  $\left(\bigcup_{i \in I} \mathcal{A}_i\right)''$ , since it is unital. Thus by Lemma 2.3.6, there exists a bounded net  $(x_\alpha)_{\alpha \in A}$  in  $\bigcup_{i \in I} \mathcal{A}_i$  converging to  $x$  in strong\* operator topology. Fix  $\alpha \in A$ . For  $i \in I$ , we let  $x_{(\alpha, i)} = x_\alpha$  if  $x_\alpha \in \mathcal{A}_i$  and  $x_{(\alpha, i)} = 0$  if  $x_\alpha \notin \mathcal{A}_i$ . Then the net  $(x_{(\alpha, i)})_{i \in I}$  is constantly equal to  $x_\alpha$  from a certain point on, so since the ultrafilter  $\nu'$  is free, we have that  $\lim_{i \rightarrow \nu'} x_{(\alpha, i)} = x_\alpha$  in the strong\* operator topology. Now, since the ultrafilter  $\nu$  was also free, and  $(x_\alpha)_{\alpha \in A}$  converges to  $x$  in strong\* operator topology, we get that  $\lim_{\alpha \rightarrow \nu} x_\alpha = x$ . In particular, by Proposition 4.3.3<sup>5</sup> we get

$$\lim_{j \rightarrow \omega} x_j = \lim_{(\alpha, i) \rightarrow \nu \otimes \nu'} x_{(\alpha, i)} = \lim_{\alpha \rightarrow \nu} \lim_{i \rightarrow \nu'} x_{(\alpha, i)} = \lim_{\alpha \rightarrow \nu} x_\alpha = x$$

in strong\* operator topology. This proves the assertion, since  $(x_j)_{j \in J}$  is an element of  $\ell_\infty(J; \mathcal{B}_j)$ . The only thing left to show is that  $\phi$  is unital completely positive. Clearly it is unital, so let us show that it is completely positive. Let  $n \in \mathbb{N}$ , and let  $[(x_j^{k, l})_{j \in J}]_{k, l} \in M_n(\ell_\infty(J; \mathcal{B}_j))$  be a positive element. For each  $j \in J$ , the matrix  $[x_j^{k, l}]_{k, l}$  is positive, and it is straightforward to check that the matrix  $[\phi((x_j^{k, l})_{j \in J})]_{k, l}$  in  $M_n\left(\left(\bigcup_{i \in I} \mathcal{A}_i\right)''\right)$  is the weak operator limit of the matrices  $[x_j^{k, l}]_{k, l}$  in  $M_n(B(\mathcal{H}))$  along the ultrafilter  $\omega$ . In particular,  $\phi_n([(x_j^{k, l})_{j \in J}]_{k, l})$  is positive, since it is the weak operator limit of a family of positive elements along an ultrafilter. Hence  $\phi_n$  is positive, and  $\phi$  must therefore be completely positive.  $\square$

Recalling the concept of a hereditary  $C^*$ -subalgebra from Remark 1.4.7, we prove the following lemma:

<sup>5</sup>The reader should not be alarmed that this is a reference forward in the thesis. The given proposition does not build on anything prior to Chapter 4, but it is placed in Chapter 4 for the sake of exposition.



**Lemma 2.3.8.** *Suppose that  $\mathcal{B}$  is a  $C^*$ -algebra,  $\mathcal{A} \subseteq \mathcal{B}$  a  $C^*$ -subalgebra and  $\mathcal{I} \subseteq \mathcal{A}$  a closed two-sided ideal in  $\mathcal{A}$  which is hereditary in  $\mathcal{B}$ . If the quotient map  $\pi: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{I}$  extends to a contractive completely positive map  $\phi: \mathcal{B} \rightarrow \mathcal{A}/\mathcal{I}$ , then  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$ .*

*Proof.* Our strategy is to find a conditional expectation of  $\mathcal{B}^{**}$  onto  $\mathcal{A}^{**}$ , since this is equivalent to  $\mathcal{A}$  being relatively weakly injective in  $\mathcal{B}$  by Proposition 2.2.3. Let  $p$  denote the central projection in  $\mathcal{A}^{**}$  such that  $p\mathcal{A}^{**} = \mathcal{I}^{**}$ , let  $p^\perp = \mathbf{1}_{\mathcal{B}^{**}} - p$  and let  $\psi: \mathcal{A}^{**} \rightarrow \mathcal{I}^{**}$  denote multiplication by  $p$ . By Proposition 1.4.10, the map

$$\psi \oplus \pi^{**}: \mathcal{A}^{**} \rightarrow \mathcal{I}^{**} \oplus (\mathcal{A}/\mathcal{I})^{**}$$

is an isomorphism. Since  $\mathcal{I}$  is hereditary in  $\mathcal{B}$  we get that  $\mathcal{I}^{**}$  is hereditary in  $\mathcal{B}^{**}$  by Remark 1.4.7, and so  $p\mathcal{B}^{**}p = \mathcal{I}^{**}$ . Hence we may define a map

$$\Phi: \mathcal{B}^{**} \rightarrow \mathcal{I}^{**} \oplus (\mathcal{A}/\mathcal{I})^{**}, \quad \text{by} \quad \Phi(x) = \begin{bmatrix} pxp \\ \phi^{**}(p^\perp xp^\perp) \end{bmatrix}$$

Let us argue that  $\Phi$  is contractive completely positive. Both the maps  $x \mapsto pxp$  and  $x \mapsto p^\perp xp^\perp$ ,  $x \in \mathcal{B}^{**}$  are completely positive, so their sum is as well. Their sum is also unital, and hence contractive completely positive. Composing this sum further with  $\text{id}_{\mathcal{I}^{**}} \oplus \phi^{**}$ , which is contractive completely positive, we get  $\Phi$ . Thus  $\Phi$  is contractive completely positive. We want to show that we, by composing with the inverse of  $\psi \oplus \pi^{**}$  obtain a conditional expectation onto  $\mathcal{A}^{**}$ , that is,  $(\psi \oplus \pi^{**})(x) = \Phi(x)$ , for  $x \in \mathcal{A}^{**}$ . Clearly the first coordinates match up, so we shall prove that  $\phi^{**}(p^\perp xp^\perp) = \pi(x)$ , for all  $x \in \mathcal{A}^{**}$ . So suppose that  $x \in \mathcal{A}^{**}$ . Since  $\mathbf{1}_{\mathcal{B}^{**}}$  acts as the identity on  $\mathcal{A}$ , we see that

$$p^\perp xp^\perp = (\mathbf{1}_{\mathcal{B}^{**}} - p)x(\mathbf{1}_{\mathcal{B}^{**}} - p) = (\mathbf{1}_{\mathcal{A}^{**}} - p)x(\mathbf{1}_{\mathcal{A}^{**}} - p).$$

Using that  $\phi^{**}$  extends  $\pi^{**}$  together with the fact that  $\pi^{**}$  is a  $*$ -homomorphism with  $\pi^{**}(p) = 0$ , we see that

$$\begin{aligned} \phi^{**}(p^\perp xp^\perp) &= \phi^{**}((\mathbf{1}_{\mathcal{A}^{**}} - p)x(\mathbf{1}_{\mathcal{A}^{**}} - p)) \\ &= \pi^{**}((\mathbf{1}_{\mathcal{A}^{**}} - p)x(\mathbf{1}_{\mathcal{A}^{**}} - p)) = \pi^{**}(x) \end{aligned}$$

Thus we have proved that  $\Phi$  composed with the inverse of  $\psi \oplus \pi^{**}$  is a conditional expectation of  $\mathcal{B}^{**}$  onto  $\mathcal{A}^{**}$ , and so we conclude that  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$ .  $\square$

**Proposition 2.3.9.** *Suppose that  $\mathcal{H}$  is a Hilbert space, and  $(\mathcal{A}_i)_{i \in I}$  an increasing net of  $C^*$ -subalgebras of  $B(\mathcal{H})$ , which are all QWEP. Then  $(\bigcup_{i \in I} \mathcal{A}_i)''$  is QWEP, as well.*

*Proof.* We may assume that all the  $C^*$ -algebras are unital, since the von Neumann algebra generated by their union is the same, whether we include the unit in each  $C^*$ -algebra or not, and since the  $C^*$ -algebras where the unit of  $B(\mathcal{H})$  is included is still QWEP by Proposition 2.3.4. Let  $J$ ,  $\omega$ ,  $(\mathcal{B}_j)_{j \in J}$  and  $\phi$  be as in Lemma 2.3.7. Let  $\mathcal{B}$  denote the  $C^*$ -algebra  $\ell_\infty(J; \mathcal{B}_j)$ , and let  $\mathcal{M}$  denote the von Neumann algebra  $(\bigcup_{i \in I} \mathcal{A}_i)''$ . Let  $\mathcal{A}$  denote the subset of  $\mathcal{B}$  consisting of all those  $(x_j)_{j \in J}$  for which the limit  $\lim_{j \rightarrow \omega} x_j$  exists in the strong\* operator topology. Now, let  $x = (x_j)_{j \in J}$  be an element in  $\mathcal{B}$ . By construction  $\phi(x)$  is the weak operator limit of  $(x_j)_{j \in J}$  along  $\omega$ , and if  $(x_j)_{j \in J}$  converges in strong\* topology, then the limit must necessarily be

$\phi(x)$ . Thus it follows that  $(x_j)_{j \in J}$  converges in strong\* operator topology if and only if  $(x_j - \phi(x))^*(x_j - \phi(x))$  and  $(x_j - \phi(x))(x_j - \phi(x))^*$  both converge to zero along  $\omega$  in the weak operator topology. It is straightforward to check, that the limits of  $(x_j - \phi(x))^*(x_j - \phi(x))$  and  $(x_j - \phi(x))(x_j - \phi(x))^*$  along  $\omega$  in the weak operator topology, are given by  $\phi(x^*x) - \phi(x)^*\phi(x)$  and  $\phi(xx^*) - \phi(x)\phi(x)^*$ , respectively. Hence

$$\mathcal{A} = \{y \in \mathcal{B} : \phi(y^*y) = \phi(y)^*\phi(y) \text{ and } \phi(yy^*) = \phi(y)\phi(y)^*\},$$

that is,  $\mathcal{A}$  is the multiplicative domain of  $\phi$ . In particular,  $\mathcal{A}$  is a  $C^*$ -algebra by Proposition 2.1.2, and the restriction of  $\phi$  to  $\mathcal{A}$ , which we denote by  $\pi$ , is a \*-homomorphism. We know from Lemma 2.3.7 that  $\pi$  is surjective, so if we can show that  $\mathcal{A}$  is QWEP, then  $\mathcal{M}$  must be QWEP as well. By Proposition 2.3.5 we know that  $\mathcal{B}$  is QWEP, since all the algebras  $(\mathcal{B}_j)_{j \in J}$  are QWEP. Thus it suffices to prove that  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$ . By Lemma 2.3.8, it suffices to prove that  $\ker \pi$  is hereditary in  $\mathcal{B}$ . So suppose that  $(y_j)_{j \in J} \in \mathcal{B}$  and  $(x_j)_{j \in J} \in \ker \pi$ , then we need to show that  $(x_j y_j x_j^*)_{j \in J} \in \ker \pi$ . Let  $\xi \in \mathcal{H}$ , then for  $j \in J$

$$\|x_j y_j x_j^* \xi\| \leq \|(x_i)_{i \in J}\| \|(y_i)_{i \in J}\| \|x_j^* \xi\|,$$

so since  $\|x_j \xi\|$  converges to zero along  $\omega$  we get that  $\|x_j y_j x_j^* \xi\|$  converges to zero along  $\omega$ . Likewise,  $\|x_j y_j^* x_j^* \xi\|$  converges to zero along  $\omega$ , but this precisely means that  $(x_j y_j x_j^*)_{j \in J}$  converges to zero along  $\omega$  in the strong\* operator topology, that is  $(x_j y_j x_j^*)_{j \in J} \in \ker \pi$ . Thus  $\ker \pi$  is hereditary in  $\mathcal{B}$ , and it follows from Lemma 2.3.8 that  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$ , which in turn shows that  $\mathcal{A}$  is QWEP.  $\square$

From this proposition, we obtain the following corollary, which in some cases allows one to restrict to von Neumann algebras:

**Corollary 2.3.10.** *A  $C^*$ -algebra is QWEP if and only if its double dual is QWEP.*

*Proof.* Let  $\mathcal{A}$  be a  $C^*$ -algebra. Obviously  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{A}^{**}$ , so by Proposition 2.3.2  $\mathcal{A}$  is QWEP if  $\mathcal{A}^{**}$  is QWEP. Now if  $\mathcal{A}$  is QWEP, then since  $\mathcal{A}$  is ultraweakly dense in  $\mathcal{A}^{**}$  it follows that  $\mathcal{A}^{**}$  is QWEP.  $\square$

**Remark 2.3.11.** The conclusion of Proposition 2.3.9 also holds when the double commutant is replaced by the norm closure. In this case the proof becomes considerably easier. Instead of using Lemma 2.3.8, one just choose free ultrafilters  $\omega$  and  $\nu$  on  $I$  and  $\mathbb{N}$ , and define  $\phi$  to be the map from  $\ell_\infty(I \times \mathbb{N}; \mathcal{A}_{i,n})$  to the norm closure of the union, that sends an element to its limit along  $\omega \otimes \nu$ . This map will be a surjective \*-homomorphism, proving that the norm closed union is QWEP.

## 2.4 The lifting property and the local lifting property

Here we introduce the last of the central concepts in this chapter, namely, the lifting property and the local lifting property. Before defining these notions, we will explore the point-norm topology and the concept of liftable maps. This will include a theorem by William Arveson on liftable maps and a lifting theorem due to Man-Duen Choi and Edward George Effros. We prove the latter theorem using Arveson's ideas from [Arv77] involving his above-mentioned theorem on liftable maps.

**Definition 2.4.1.** Suppose that  $\mathfrak{X}$  and  $\mathfrak{Y}$  are normed spaces. The **point-norm topology** on  $B(\mathfrak{X}, \mathfrak{Y})$  is the locally convex topology determined by the family of seminorms  $\{\|\cdot\|_x : x \in \mathfrak{X}\}$ , where  $\|T\|_x = \|Tx\|$ , for all  $x \in \mathfrak{X}$ . ◀

The point-norm topology is, as the name suggests, the topology of pointwise convergence. A special case of this is the weak\*-topology on a dual space  $\mathfrak{X}^*$ , which is the point-norm topology on  $B(\mathfrak{X}, \mathbb{C})$ .

In the general setting it turns out that if the first normed space is separable, then the point-norm topology is metrizable on bounded sets, and in particular first-countable.

**Proposition 2.4.2.** *If  $\mathfrak{X}$  is a separable normed space, then the point-norm topology on  $B(\mathfrak{X}, \mathfrak{Y})$  is metrizable on bounded sets, for any normed space  $\mathfrak{Y}$ . More precisely, if  $\mathfrak{A} = \{x_n : n \in \mathbb{N}\}$  is a dense subset of the closed unit ball of  $\mathfrak{X}$ , then*

$$d_{\mathfrak{A}}(\phi, \psi) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|\phi(x_n) - \psi(x_n)\|, \quad \phi, \psi \in B(\mathfrak{X}, \mathfrak{Y}),$$

*defines a metric on  $B(\mathfrak{X}, \mathfrak{Y})$ , which induces the point-norm topology on bounded sets. In particular, the point-norm topology on  $B(\mathfrak{X}, \mathfrak{Y})$  is first-countable on bounded sets.*

*Proof.* First, note that  $d_{\mathfrak{A}}$  is well-defined, since  $\|x_n\| \leq 1$  for all  $n \in \mathbb{N}$ . Clearly  $d_{\mathfrak{A}}$  is always non-negative. If  $\phi, \psi \in B(\mathfrak{X}, \mathfrak{Y})$  with  $d_{\mathfrak{A}}(\phi, \psi) = 0$ , then  $\phi$  and  $\psi$  must agree on a dense subset of the closed unit ball, namely  $\mathfrak{A}$ . Hence  $\phi = \psi$ . Now, that  $d_{\mathfrak{A}}$  is symmetric is obvious, and that the triangle inequality is satisfied is also easy to see. Thus  $d_{\mathfrak{A}}$  defines a metric on  $B(\mathfrak{X}, \mathfrak{Y})$ .

Let us check that this metric induces the point-norm topology on bounded sets. Suppose that  $(\phi_\alpha)_{\alpha \in A}$  is a net in  $B(\mathfrak{X}, \mathfrak{Y})$ , and that  $\phi \in B(\mathfrak{X}, \mathfrak{Y})$ . First we prove that that the topology induced by  $d_{\mathfrak{A}}$  is stronger than the point-norm topology. For this we do not need any boundedness assumptions. So suppose that  $d_{\mathfrak{A}}(\phi_\alpha, \phi)$  converges to zero. For each  $n \in \mathbb{N}$ , we have

$$\|\phi_\alpha(x_n) - \phi(x_n)\| \leq 2^n d_{\mathfrak{A}}(\phi_\alpha, \phi).$$

This shows that  $(\phi_\alpha)_{\alpha \in A}$  converge to  $\phi$  pointwise on the set  $\mathfrak{A}$ . Since the latter set is dense in the closed unit ball of  $\mathfrak{X}$ , a standard argument shows that  $(\phi_\alpha)_{\alpha \in A}$  converges to  $\phi$  pointwise on the closed unit ball. Thus, the net converges honestly to  $\phi$  in the point-norm topology.

This shows that the topology induced by  $d_{\mathfrak{A}}$  is stronger than the norm topology. Suppose now instead that  $(\phi_\alpha)_{\alpha \in A}$  is a bounded net which converges to  $\phi$  in the point-norm topology, bounded by  $M \in (0, \infty)$  say. Let  $\varepsilon > 0$  be given. Choose  $k \in \mathbb{N}$  so that  $\sum_{n=k}^{\infty} 2^{-n} < \varepsilon(4M)^{-1}$ , and choose then  $\alpha_0 \in A$ , so that  $\sum_{n=1}^{k-1} 2^{-n} \|\phi_\alpha(x_n) - \phi(x_n)\| < \varepsilon 2^{-1}$ , for all  $\alpha \in A$  with  $\alpha \geq \alpha_0$ . Then it follows that

$$d_{\mathfrak{A}}(\phi_\alpha, \phi) \leq \sum_{n=1}^{k-1} 2^{-n} \|\phi_\alpha(x_n) - \phi(x_n)\| + (\|\phi_\alpha\| + \|\phi\|) \sum_{n=k}^{\infty} 2^{-n} \|x_n\| < \varepsilon,$$

for all  $\alpha \geq \alpha_0$ . Since  $\varepsilon > 0$  was arbitrary this shows that  $(\phi_\alpha)_{\alpha \in A}$  converge to  $\phi$  in the topology induced by  $d_{\mathfrak{A}}$ .

This proves that the two topologies agree on bounded sets, so in particular the point norm topology is first-countable on bounded sets. ◻

**Remark 2.4.3.** It is not hard to see that if both  $\mathfrak{X}$  and  $\mathfrak{Y}$  are Banach spaces, then all closed bounded sets of  $B(\mathfrak{X}, \mathfrak{Y})$  are actually a complete metric space, with respect to the metric  $d_{\mathfrak{X}}$  from Proposition 2.4.2. Indeed, since  $\mathfrak{A}$  is dense in the closed unit ball of  $\mathfrak{X}$ , a sequence which is Cauchy with respect to  $d_{\mathfrak{X}}$  must necessarily be pointwise Cauchy, and hence pointwise convergent to a linear map from  $\mathfrak{X}$  to  $\mathfrak{Y}$ , by completeness of  $\mathfrak{Y}$ . By the Uniform Boundedness Principle (see [KR83, Theorem 1.8.9]), we get that the sequence in question must be uniformly bounded in operator norm, and therefore the limit must be so, as well. The Uniform Boundedness Principle was applicable, since  $\mathfrak{X}$  was assumed to be complete.

Actually, the above argument did not rely on the metric at all, but only on the fact that the sequence in question is pointwise Cauchy. Thus, when  $\mathfrak{X}$  and  $\mathfrak{Y}$  are Banach spaces, it holds that every sequence in  $B(\mathfrak{X}, \mathfrak{Y})$  which is pointwise Cauchy, is actually convergent. ◀

Now, returning to operator spaces, operator systems, completely contractive maps and completely positive maps, we have the following result about the point-norm topology:

**Proposition 2.4.4.** *For operator spaces  $\mathcal{M}_1$  and  $\mathcal{M}_2$ , operator systems  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , and  $r \in [0, \infty]$ , the sets*

$$\{\phi \in \text{CB}(\mathcal{M}_1, \mathcal{M}_2) : \|\phi\|_{\text{cb}} \leq r\} \quad \text{and} \quad \{\phi \in \text{CP}(\mathcal{S}_1, \mathcal{S}_2) : \|\phi\|_{\text{cb}} \leq r\}$$

*are closed in  $B(\mathcal{M}_1, \mathcal{M}_2)$  and  $B(\mathcal{S}_1, \mathcal{S}_2)$ , respectively, with respect to the point-norm topology. In particular the set of completely contractive maps from  $\mathcal{M}_1$  to  $\mathcal{M}_2$ , and the set of contractive completely positive maps from  $\mathcal{S}_1$  to  $\mathcal{S}_2$  are closed in the point-norm topology. Moreover, the same holds when completely positive maps are replaced by unital completely positive maps.*

*Proof.* If  $(\phi_\alpha)_{\alpha \in A}$  is a net of maps in  $B(\mathcal{M}_1, \mathcal{M}_2)$  converging to  $\phi \in B(\mathcal{M}_1, \mathcal{M}_2)$ , in point-norm topology, then the net  $((\phi_\alpha)_n)_{\alpha \in A}$ , obtained by taking  $n$ 'th inflations, converges to  $\phi_n$  in  $B(\mathcal{M}_n(\mathcal{M}_1), \mathcal{M}_n(\mathcal{M}_2))$ , with respect to the point-norm topology. The same conclusion holds if  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are replaced by  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , respectively. Realizing this, the conclusion of the proposition is immediate, since the pointwise limit of contractive maps is contractive and the pointwise limit of positive maps is positive. ◻

Let us define what we shall mean by a liftable map. This notion is, of course, central in the definition of the lifting property and the local lifting property.

**Definition 2.4.5.** Let  $\mathcal{S}$  be an operator system,  $\mathcal{B}$  a  $C^*$ -algebra and  $\mathcal{I}$  a closed two-sided ideal in  $\mathcal{B}$ . Denote the quotient map  $\mathcal{B} \rightarrow \mathcal{B}/\mathcal{I}$  by  $\pi$ . A contractive completely positive map  $\phi: \mathcal{S} \rightarrow \mathcal{B}/\mathcal{I}$  is said to be **liftable** if there exists a contractive completely positive map  $\psi: \mathcal{S} \rightarrow \mathcal{B}$  so that  $\pi \circ \psi = \phi$ . The map  $\psi$  is called a **lift** of  $\phi$ . A contractive completely positive map  $\phi: \mathcal{S} \rightarrow \mathcal{B}/\mathcal{I}$  is said to be **locally liftable**, if for each finite dimensional operator system  $\tilde{\mathcal{S}} \subseteq \mathcal{S}$ , the restriction of  $\phi$  to  $\tilde{\mathcal{S}}$  is liftable. ◀

**Remark 2.4.6.** If a map  $\phi: \mathcal{S} \rightarrow \mathcal{B}/\mathcal{I}$  is unital and liftable, then the lift can be chosen to be unital completely positive. This can be accomplished by choosing a state  $\theta$  on  $\mathcal{S}$  and then replacing the lift  $\psi$  by the unital completely positive lift given by  $\psi + (\mathbf{1} - \psi(\mathbf{1}))\theta$ . ◀

After the following technical lemma, we will prove the theorem by Arveson on liftable maps, which, in turn, we will use to prove the Choi-Effros lifting Theorem.

**Lemma 2.4.7.** *Suppose that  $\mathcal{A}$  is a unital  $C^*$ -algebra,  $\mathcal{I}$  an ideal in  $\mathcal{A}$  and  $(e_\lambda)_{\lambda \in \Lambda}$  a quasi-central approximate unit for  $\mathcal{I}$  in  $\mathcal{A}$ . Given  $x, y \in \mathcal{A}$ , it holds that*

$$\lim_{\lambda \in \Lambda} \left\| (\mathbf{1} - e_\lambda)^{1/2} x (\mathbf{1} - e_\lambda)^{1/2} + e_\lambda^{1/2} y e_\lambda^{1/2} - y \right\| = \|\pi(x - y)\|,$$

where  $\pi: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{I}$  denotes the quotient map.

*Proof.* It is not hard to show that for every element  $a \in \mathcal{A}$  and every polynomial  $p$ , it holds that  $\lim_{\lambda \in \Lambda} \|p(e_\lambda)a - ap(e_\lambda)\| = 0$ , and so by a standard approximation argument we get that, for every  $f \in C([0, 1]; \mathbb{C})$ , we have  $\lim_{\lambda \in \Lambda} \|f(e_\lambda)a - af(e_\lambda)\| = 0$ . It follows that

$$\begin{aligned} \lim_{\lambda \in \Lambda} \left\| (\mathbf{1} - e_\lambda)^{1/2} y (\mathbf{1} - e_\lambda)^{1/2} + e_\lambda^{1/2} y e_\lambda^{1/2} - y \right\| \\ = \lim_{\lambda \in \Lambda} \left\| (\mathbf{1} - e_\lambda)y + e_\lambda y - y \right\| = 0. \end{aligned}$$

Also, using that  $\|\pi(x - y)\| = \lim_{\lambda \in \Lambda} \|(\mathbf{1} - e_\lambda)(x - y)\|$  by Proposition 1.1.7, we conclude that

$$\lim_{\lambda \in \Lambda} \left\| (\mathbf{1} - e_\lambda)^{1/2} (x - y) (\mathbf{1} - e_\lambda)^{1/2} \right\| = \lim_{\lambda \in \Lambda} \|(\mathbf{1} - e_\lambda)(x - y)\| = \|\pi(x - y)\|.$$

By adding these two relations, we obtain the desired conclusion.  $\square$

We now prove a theorem due to Arveson, which asserts that the set of liftable maps is closed in the point-norm topology. This was proved by Arveson in [Arv77], wherein he gave a simplified proof of the Choi-Effros lifting theorem, using quasi-central approximate identities.

**Theorem 2.4.8.** *Suppose that  $\mathcal{S}$  is a separable operator system,  $\mathcal{B}$  a  $C^*$ -algebra and  $\mathcal{I}$  a closed two-sided ideal in  $\mathcal{B}$ . Then the set of liftable contractive completely positive maps  $\mathcal{S} \rightarrow \mathcal{B}/\mathcal{I}$  is closed in the point-norm topology.*

*Proof.* First, note that the set of contractive completely positive maps is closed in  $B(\mathcal{S}, \mathcal{B}/\mathcal{I})$ , by Proposition 2.4.4, and the point-norm topology is first-countable on this set, by Proposition 2.4.2. Hence we can make do with sequences instead of nets.

Fix some dense subset  $\{x_n : n \in \mathbb{N}\}$  of  $\mathcal{S}$ , and let  $(e_\lambda)_{\lambda \in \Lambda}$  be a quasi-central approximate unit for  $\mathcal{I}$  in  $\mathcal{B}$ . Also, suppose first that the  $C^*$ -algebra  $\mathcal{B}$  is unital.

Let  $\phi: \mathcal{S} \rightarrow \mathcal{B}/\mathcal{I}$  be a contractive completely positive map, and assume that  $(\psi'_n)_{n \in \mathbb{N}}$  is a sequence of contractive completely positive maps from  $\mathcal{S}$  to  $\mathcal{B}$  such that  $(\pi \circ \psi'_n)_{n \in \mathbb{N}}$  converges to  $\phi$  in the point-norm topology, where  $\pi: \mathcal{B} \rightarrow \mathcal{B}/\mathcal{I}$  denotes the quotient map. By passing to a subsequence, we may assume that

$$\|\pi \circ \psi'_n(x_k) - \phi(x_k)\| < 2^{-n}, \quad \text{for all } k = 1, 2, \dots, n.$$

Our strategy is to construct a new sequence  $(\psi_n)_{n \in \mathbb{N}}$  in  $B(\mathcal{S}, \mathcal{B})$  of contractive completely positive maps, which converges in the point-norm topology. This point-norm limit will be the contractive completely positive lift of  $\phi$ . More precisely, we construct this sequence successively, with the properties

$$(1) \quad \|\pi \circ \psi_n(x_k) - \phi(x_k)\| < 2^{-n}, \quad k = 1, 2, \dots, n;$$

$$(2) \|\psi_{n+1}(x_k) - \psi_n(x_k)\| < 2^{1-n}, k = 1, 2, \dots, n;$$

$$(3) \pi \circ \psi_n = \pi \circ \psi'_n.$$

First, let  $\psi_1 = \psi'_1$ . Suppose that we have constructed  $\psi_1, \psi_2, \dots, \psi_n$  satisfying the three criteria above. By Lemma 2.4.7, we know that, for all  $k \in \mathbb{N}$ ,

$$\begin{aligned} \lim_{\lambda \in \Lambda} \left\| (\mathbf{1} - e_\lambda)^{1/2} \psi'_{n+1}(x_k) (\mathbf{1} - e_\lambda)^{1/2} + e_\lambda^{1/2} \psi_n(x_k) e_\lambda^{1/2} - \psi_n(x_k) \right\| \\ = \left\| \pi \circ \psi'_{n+1}(x_k) - \pi \circ \psi_n(x_k) \right\|, \end{aligned}$$

and for  $k = 1, 2, \dots, n$  we have

$$\begin{aligned} \left\| \pi \circ \psi'_{n+1}(x_k) - \pi \circ \psi_n(x_k) \right\| \\ \leq \left\| \pi \circ \psi'_{n+1}(x_k) - \phi(x_k) \right\| + \left\| \phi(x_k) - \pi \circ \psi_n(x_k) \right\| \\ < 2^{-n} + 2^{-n} = 2^{1-n}. \end{aligned}$$

So we can choose some  $\lambda_0 \in \Lambda$ , such that (with  $e = e_{\lambda_0}$ ) the following holds

$$\left\| (\mathbf{1} - e)^{1/2} \psi'_{n+1}(x_k) (\mathbf{1} - e)^{1/2} + e^{1/2} \psi_n(x_k) e^{1/2} - \psi_n(x_k) \right\| < 2^{1-n},$$

for all  $k = 1, 2, \dots, n$ . Hence, if we let  $\psi_{n+1}$  be the map defined by

$$\psi_{n+1}(x) = (\mathbf{1} - e)^{1/2} \psi'_{n+1}(x) (\mathbf{1} - e)^{1/2} + e^{1/2} \psi_n(x) e^{1/2}, \quad x \in \mathcal{S}$$

then (2) is satisfied by the choice of  $e$ . The fact that (1) is satisfied, actually follows directly from (3), which we see is satisfied, since  $\pi(e^{1/2}) = \pi(e)^{1/2} = 0$  and  $\pi((\mathbf{1} - e)^{1/2}) = (\mathbf{1} - \pi(e))^{1/2} = \mathbf{1}$ .

Now that we have constructed this new sequence, we want to argue that it converges to a lift of  $\phi$  in point-norm topology. Because of (2), and the fact that  $\{x_n : n \in \mathbb{N}\}$  is dense in  $\mathcal{S}$ , we get that  $(\psi_n(x))_{n \in \mathbb{N}}$  is a Cauchy sequence, for all  $x \in \mathcal{S}$ . Hence it converges in point-norm topology to a bounded map  $\psi : \mathcal{S} \rightarrow \mathcal{B}$  by Remark 2.4.3. Since the set of contractive completely positive maps is closed in the point-norm topology by Proposition 2.4.4, we get that  $\psi$  is contractive completely positive. The fact that  $\psi$  is a lift of  $\phi$  follows from (1), and again the fact that  $\{x_n : n \in \mathbb{N}\}$  is dense.

Now suppose that  $\mathcal{B}$  is non-unital, and let  $\phi : \mathcal{S} \rightarrow \mathcal{B}/\mathcal{I}$  be a contractive completely positive map which can be approximated by liftable maps. Extend the quotient map to a surjective  $*$ -homomorphism  $\tilde{\pi} : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}}/\tilde{\mathcal{I}}$ . By the first part of the proof, we can lift  $\phi$  to a contractive completely positive map  $\psi : \mathcal{S} \rightarrow \tilde{\mathcal{B}}$ . All we need to check is that the image of  $\psi$  is contained in  $\mathcal{B}$ . Since  $\tilde{\pi}^{-1}(\mathcal{B}/\mathcal{I}) = \mathcal{B}$ , it follows that the image of  $\psi$  must be contained in  $\mathcal{B}$ , because the image of  $\phi$  is contained in  $\mathcal{B}/\mathcal{I}$ .  $\square$

In the definition of a liftable map we required that the lift of a contractive completely positive map be contractive completely positive, but it turns out to be sufficient to have a completely positive lift (which is not necessarily contractive).

**Proposition 2.4.9.** *Suppose that  $\mathcal{S}$  is an operator system,  $\mathcal{B}$  a unital  $C^*$ -algebra and  $\mathcal{I}$  a closed two-sided ideal in  $\mathcal{B}$ . A contractive completely positive map  $\phi : \mathcal{S} \rightarrow \mathcal{B}/\mathcal{I}$  is liftable if and only if there exists a completely positive map  $\psi : \mathcal{S} \rightarrow \mathcal{B}$  such that  $\pi \circ \psi = \phi$ , where  $\pi : \mathcal{B} \rightarrow \mathcal{B}/\mathcal{I}$  denotes the quotient map.*

*Proof.* We may assume that  $\|\phi\| = 1$ . Let  $\pi: \mathcal{B} \rightarrow \mathcal{B}/\mathcal{I}$  denote the quotient map. Suppose first that  $\mathcal{B}$ , and therefore also  $\mathcal{B}/\mathcal{I}$ , is unital. Let  $(e_\lambda)_{\lambda \in \Lambda}$  be an approximate unit in  $\mathcal{I}$ . Since  $\psi$  is completely positive, so is the map  $\psi_\lambda: \mathcal{S} \rightarrow \mathcal{B}$  given by  $\psi_\lambda(x) = (\mathbf{1}_\mathcal{B} - e_\lambda)\psi(x)(\mathbf{1}_\mathcal{B} - e_\lambda)$ . Note that  $\pi \circ \psi_\lambda = \phi$ , for all  $\lambda \in \Lambda$ , since  $\pi(e_\lambda) = 0$ . By Corollary B.1.8 we get

$$\|\psi_\lambda\| = \|\psi_\lambda(\mathbf{1})\| = \|(\mathbf{1} - e_\lambda)\psi(\mathbf{1})(\mathbf{1} - e_\lambda)\| = \|(\mathbf{1} - e_\lambda)\psi(\mathbf{1})^{1/2}\|^2.$$

Thus  $\lim_{\lambda \in \Lambda} \|\psi_\lambda\| = \|\pi(\psi(\mathbf{1})^{1/2})\|^2 = \|\phi(\mathbf{1})^{1/2}\|^2 = 1$ . Since we have  $\|\phi\| \leq \|\pi\|\|\psi_\lambda\| = \|\psi_\lambda\|$ , for all  $\lambda \in \Lambda$ , we get that  $\|\psi_\lambda\|^{-1}\phi$  is a contractive completely positive map, with contractive completely positive lift  $\|\psi_\lambda\|^{-1}\psi_\lambda$ . Now for each  $x \in \mathcal{S}$  we have  $\lim_{\lambda \in \Lambda} \|\psi_\lambda\|^{-1}\phi(x) = \phi(x)$ , so since the set of liftable contractive completely positive maps  $\mathcal{S} \rightarrow \mathcal{B}/\mathcal{I}$  is closed in the point-norm topology by Theorem 2.4.8, we deduce that  $\phi$  is liftable.

Now suppose that  $\mathcal{B}$  is non-unital. The quotient map  $\pi$  extends to a unital  $*$ -homomorphism  $\tilde{\pi}: \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}}/\tilde{\mathcal{I}}$ —which is, of course, still surjective—by Proposition B.2.9. By the previous part of the proof, the map  $\phi: \mathcal{S} \rightarrow \tilde{\mathcal{B}}/\tilde{\mathcal{I}}$  has a contractive completely positive lift  $\psi: \mathcal{S} \rightarrow \tilde{\mathcal{B}}$ , so all we need to check is that the image of  $\psi$  is in fact contained in  $\mathcal{B}$ . This is, obvious since  $\tilde{\pi}(\psi(\mathcal{S})) = \phi(\mathcal{S}) \subseteq \mathcal{B}/\mathcal{I}$ , so that  $\psi(\mathcal{S}) \subseteq \tilde{\pi}^{-1}(\mathcal{B}/\mathcal{I}) = \mathcal{B}$ .  $\square$

Let us prove a lifting theorem for finite dimensional  $C^*$ -algebras, which Arveson attributes to Choi (see [Arv77, page 349]).

**Proposition 2.4.10.** *Suppose that  $\mathcal{A}$  is a finite dimensional  $C^*$ -algebra,  $\mathcal{B}$  any  $C^*$ -algebra and  $\mathcal{I}$  a closed two-sided ideal in  $\mathcal{B}$ . Then all contractive completely positive maps from  $\mathcal{A}$  to  $\mathcal{B}/\mathcal{I}$  are liftable. In particular, all maps from  $\mathbb{M}_n$  to  $\mathcal{B}/\mathcal{I}$  are liftable.*

*Proof.* By representing  $\mathcal{A}$  on a finite dimensional Hilbert space and extending the map  $\mathcal{A} \rightarrow \mathcal{B}/\mathcal{I}$  using Arveson's Extension Theorem (Theorem B.3.6) we can ensure that we have a contractive completely positive map  $\mathbb{M}_n \rightarrow \mathcal{B}/\mathcal{I}$ , for some  $n \in \mathbb{N}$ . Assume that  $\phi$  is such a map, and let us prove that  $\phi$  is liftable. By Theorem B.3.1 the matrix  $\phi_n([E_{i,j}]_{i,j})$  is positive in  $M_n(\mathcal{B}/\mathcal{I})$ . Let  $\pi: \mathcal{B} \rightarrow \mathcal{B}/\mathcal{I}$  denote the quotient map. Then  $\pi_n$  is surjective, so we can lift  $\phi_n([E_{i,j}]_{i,j})$  to a positive element  $x \in M_n(\mathcal{B})$ . Let  $\psi: \mathbb{M}_n \rightarrow \mathcal{B}$  denote the completely positive map associated to  $x$  via the correspondence set-forth in Theorem B.3.1, that is,  $\psi_n([E_{i,j}]_{i,j}) = x$ . Now we have  $(\pi \circ \psi)_n([E_{i,j}]_{i,j}) = \pi_n(x) = \phi_n([E_{i,j}]_{i,j})$ , so by Theorem B.3.1 we infer that  $\pi \circ \psi = \phi$ . By Proposition 2.4.9 we conclude that the map  $\phi$  is liftable.  $\square$

**Definition 2.4.11.** Suppose that  $\theta: \mathcal{A} \rightarrow \mathcal{B}$  is a linear between  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ . Then  $\theta$  is said to be **nuclear** if there exist a directed set  $A$ , natural numbers  $(n(\alpha))_{\alpha \in A}$  and contractive completely positive maps  $\phi_\alpha: \mathcal{A} \rightarrow \mathbb{M}_{n(\alpha)}$  and  $\psi_\alpha: \mathbb{M}_{n(\alpha)} \rightarrow \mathcal{B}$ , for each  $\alpha \in A$ , such that  $\psi_\alpha \circ \phi_\alpha$  converges to  $\theta$  in the point-norm topology. A  $C^*$ -algebra is called **nuclear** if the identity map is nuclear.  $\blacktriangleleft$

A famous theorem states that a  $C^*$ -algebra  $\mathcal{A}$  is nuclear if and only if  $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$ , for all  $C^*$ -algebras  $\mathcal{B}$ . See [BO08, Theorem 3.8.7] for a proof.

Now we are ready to prove the Choi-Effros Lifting Theorem, originally established in [CE76]. As already mentioned, we follow the proof from [Arv77]. The original lifting theorem proved by Choi and Effros is slightly stronger, in the sense that it asserts that one can choose the lift to be nuclear. As explained in [Arv77, page 351],

Arveson's proof can be modified to cover the original stronger statement, by showing that the set of liftable *nuclear* maps is closed in point-norm topology, as well. This though we do not prove.

**Theorem 2.4.12.** *Every nuclear contractive completely positive map from a separable  $C^*$ -algebra  $\mathcal{A}$  into the quotient  $\mathcal{B}/\mathcal{I}$  of a  $C^*$ -algebra  $\mathcal{B}$  by a closed two-sided ideal  $\mathcal{I}$  is liftable.*

*Proof.* Let  $\theta: \mathcal{A} \rightarrow \mathcal{B}/\mathcal{I}$  be a nuclear contractive completely positive map. Choose some directed set  $A$ , natural numbers  $\{n(\alpha) : \alpha \in A\}$  and contractive completely positive maps  $\phi_\alpha: \mathcal{A} \rightarrow \mathbb{M}_{n(\alpha)}$  and  $\psi_\alpha: \mathbb{M}_{n(\alpha)} \rightarrow \mathcal{B}/\mathcal{I}$  for each  $\alpha \in A$ , such that  $\psi_\alpha \circ \phi_\alpha$  converges to  $\theta$  in the point-norm topology. By Proposition 2.4.10 the maps  $(\psi_\alpha)_{\alpha \in A}$  are all liftable, so for each  $\alpha \in A$  we let  $\tilde{\psi}_\alpha$  be a contractive completely positive lift of  $\psi_\alpha$ . Now for each  $\alpha \in A$  the map  $\psi_\alpha \circ \phi_\alpha$  is liftable with lift  $\tilde{\psi}_\alpha \circ \phi_\alpha$ . Thus  $\theta$  is liftable, since the set of liftable maps is closed in the point-norm topology by Theorem 2.4.8.  $\square$

Let us give the definition of the lifting property and the local lifting property before we give a corollary to the Choi-Effros Lifting Theorem.

**Definition 2.4.13.** Let  $\mathcal{A}$  be a  $C^*$ -algebra. Then  $\mathcal{A}$  is said to have the **lifting property**, if for each  $C^*$ -algebra  $\mathcal{B}$  and each ideal  $\mathcal{I}$  in  $\mathcal{B}$ , every contractive completely positive map  $\phi: \mathcal{A} \rightarrow \mathcal{B}/\mathcal{I}$  is liftable. Also,  $\mathcal{A}$  is said to have the **local lifting property**, if for each  $C^*$ -algebra  $\mathcal{B}$  and each ideal  $\mathcal{I}$  in  $\mathcal{B}$ , every contractive completely positive map  $\phi: \mathcal{A} \rightarrow \mathcal{B}/\mathcal{I}$  is locally liftable. The lifting property and the local lifting property are abbreviated as **LP** and **LLP**, respectively.

With this defined, we can read the following corollary straight out of the Choi-Effros Lifting Theorem, if we remember that all contractive completely positive maps to of from a nuclear  $C^*$ -algebra are themselves nuclear:

**Corollary 2.4.14.** *All separable nuclear  $C^*$ -algebras have the lifting property.*

Another important lifting theorem is the Effros-Haagerup Lifting Theorem, which we state here, but do not prove. A proof can be found in [BO08, Appendix C].

**Theorem 2.4.15.** *Suppose that  $\mathcal{B}$  is a unital  $C^*$ -algebra and  $\mathcal{I}$  a closed two-sided ideal in  $\mathcal{B}$ . Then the following statements are equivalent:*

(i) *for any  $C^*$ -algebra  $\mathcal{A}$  the sequence*

$$0 \longrightarrow \mathcal{A} \otimes_{\min} \mathcal{I} \xrightarrow{\text{id}_{\mathcal{A}} \otimes \iota} \mathcal{A} \otimes_{\min} \mathcal{B} \xrightarrow{\text{id}_{\mathcal{A}} \otimes \pi} \mathcal{A} \otimes_{\min} (\mathcal{B}/\mathcal{I}) \longrightarrow 0$$

*is exact, where  $\iota: \mathcal{I} \rightarrow \mathcal{B}$  denotes the inclusion map and  $\pi: \mathcal{B} \rightarrow \mathcal{B}/\mathcal{I}$  denotes the quotient map;*

(ii) *the same as (i), but only with  $\mathcal{A} = B(\ell_2)$ ;*

(iii) *for any finite dimensional operator system  $\mathcal{S} \subseteq \mathcal{B}/\mathcal{I}$ , the inclusion of  $\mathcal{S}$  into  $\mathcal{B}/\mathcal{I}$  is liftable, that is, the identity map on  $\mathcal{B}/\mathcal{I}$  is locally liftable.*

A natural thing to ask is whether the local lifting property implies the lifting property. In general this is not the case (see [Oza04, page 510]). It is though still open, whether the local lifting property implies the lifting property, for separable  $C^*$ -algebras.



## Chapter 3

# The QWEP Conjecture

### 3.1 The free group on countably many generators

A group which plays an important role in the following sections, is the free group on countably infinitely many generators. More precisely, we will see that the full group  $C^*$ -algebra of this group plays an important role.

Let us start with some notation. We will use the symbol  $\mathbb{F}$  for a generic free group. In a situation where we need to specify the generators, the generators being an index set  $I$  say, we let  $\mathbb{F}_I$  denote the free group with generators  $I$ . This group of course only depends on—up to isomorphism—the cardinality of  $I$ . In the case where  $I$  is finite, with  $n$  elements say, we denote  $\mathbb{F}_I$  by  $\mathbb{F}_n$ , and in the case where  $I$  is countably infinite, we denote it by  $\mathbb{F}_\infty$ . Moreover, we will denote the complex group ring of a discrete group  $\Gamma$  by  $\mathbb{C}\Gamma$ .

We begin by exploring maps out of the full group  $C^*$ -algebra of a free group.

**Proposition 3.1.1.** *Suppose that  $\Gamma$  is a discrete group and  $\mathcal{A}$  a unital  $C^*$ -algebra. Let  $\varphi: \Gamma \rightarrow \mathcal{U}(\mathcal{A})$  be a group homomorphism. Then  $\varphi$  extends uniquely to a  $*$ -homomorphism  $C^*(\Gamma) \rightarrow \mathcal{A}$ .*

*Proof.* We may assume that  $\mathcal{A} \subseteq B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ . Clearly the map extends uniquely to a homomorphism  $\tilde{\varphi}: \mathbb{C}\Gamma \rightarrow \mathcal{A}$ . This map must necessarily be contractive, when  $\mathbb{C}\Gamma$  is equipped with the norm of  $C^*(\Gamma)$ , since it is a representation of  $\mathbb{C}\Gamma$ . Thus, by continuity, it extends uniquely to a unital  $*$ -homomorphism from  $C^*(\Gamma)$  to  $\mathcal{A}$ .  $\square$

One of the reasons why the full group  $C^*$ -algebras of free groups are of much interest, is that it is incredibly easy to construct  $*$ -homomorphisms from these into other  $C^*$ -algebras. Indeed, if one combines the above proposition with the universal property of the free groups, then one can construct such  $*$ -homomorphisms just by specifying the value on the generating unitaries.

**Remark 3.1.2.** Suppose that  $I$  is an index set, and  $J \subseteq I$  a non-empty subset. Let  $\mathcal{A}$  be a  $C^*$ -algebra, and let  $\pi: C^*(\mathbb{F}_J) \rightarrow \mathcal{A}$  be a  $*$ -homomorphism, with  $\mathcal{B}$  denoting the image of  $\pi$ . The map  $\pi$  restricts to a group homomorphism from  $\mathbb{F}_J$  to the unitary group of  $\mathcal{B}$ —which is unital since  $C^*(\mathbb{F}_J)$  is unital. This group homomorphism extends to a group homomorphism from  $\mathbb{F}_I$  to  $\mathcal{U}(\mathcal{B})$ , by the universal property of the free groups, and so by Proposition 3.1.1, we get a  $*$ -homomorphism from  $C^*(\mathbb{F}_I)$  to

$\mathcal{B}$ , which clearly extends  $\pi$ .<sup>1</sup> In this manner we have extended the  $*$ -homomorphism  $\pi$  to a  $*$ -homomorphism from  $C^*(\mathbb{F}_I)$  to  $\mathcal{A}$ . In particular, every representation of  $\mathbb{C}\mathbb{F}_J$  extends to a representation of  $\mathbb{F}_I$ , so it follows that the natural map from  $C^*(\mathbb{F}_J)$  to  $C^*(\mathbb{F}_I)$  is injective. In this way we may consider  $C^*(\mathbb{F}_J)$  as a  $C^*$ -subalgebra of  $C^*(\mathbb{F}_I)$ .

Now if we are given two index sets  $I_1$  and  $I_2$ , and a map  $j: I_1 \rightarrow I_2$ , then this naturally induces a map between the associated groups  $\mathbb{F}_{I_1}$  and  $\mathbb{F}_{I_2}$ , by mapping a generator  $i \in I_1$  to the generator  $j(i) \in I_2$ . Identifying the group  $\mathbb{F}_{I_2}$  with a subgroup of the unitary group of  $C^*(\mathbb{F}_{I_2})$  in the canonical way, we get a homomorphism from  $\mathbb{F}_{I_1}$  to  $\mathcal{U}(C^*(\mathbb{F}_{I_2}))$ . Thus, in a canonical way, we obtain a  $*$ -homomorphism  $\tilde{j}: C^*(\mathbb{F}_{I_1}) \rightarrow C^*(\mathbb{F}_{I_2})$ . Now, if  $j$  is surjective, then the induced map  $\mathbb{C}\mathbb{F}_{I_1} \rightarrow \mathbb{C}\mathbb{F}_{I_2}$  will also be surjective. Thus the image of  $\tilde{j}$  is a  $C^*$ -subalgebra of  $C^*(\mathbb{F}_{I_2})$ , which contains  $\mathbb{C}\mathbb{F}_{I_2}$ , and must therefore be equal to the whole of  $C^*(\mathbb{F}_{I_2})$ . So, in the case where  $j$  is surjective, so is the map  $\tilde{j}$ . Now, if  $j$  is injective, then by the previous part, we get an isometric inclusion of  $C^*(\mathbb{F}_{I_1})$  into  $C^*(\mathbb{F}_{I_2})$ . Hence, if  $j$  is injective, then so is  $\tilde{j}$ . ◀

One application of the universality of the  $C^*$ -algebra  $C^*(\mathbb{F}_\infty)$  is that every separable  $C^*$ -algebra is isomorphic to a quotient  $C^*(\mathbb{F}_\infty)$ .

**Proposition 3.1.3.** *If  $\mathcal{A}$  is a  $C^*$ -algebra and  $\mathfrak{X} \subseteq \mathcal{A}$  is a separable subset, then there exists a  $*$ -homomorphism  $\varphi: C^*(\mathbb{F}_\infty) \rightarrow \mathcal{A}$ , such that the image of  $\varphi$  contains  $\mathfrak{X}$ . In particular, every separable  $C^*$ -algebra is a quotient of  $C^*(\mathbb{F}_\infty)$ .*

*Proof.* Let  $\{x_n : n \in \mathbb{N}\}$  be a countable dense subset of  $\mathfrak{X}$ . Since  $x_n$  is a linear combination of at most four unitaries, we can choose a countable set of unitaries  $\{u_n : n \in \mathbb{N}\}$  in  $\mathcal{A}$  such that their span contains  $\{x_n : n \in \mathbb{N}\}$ . In particular the  $C^*$ -algebra generated by  $\{u_n : n \in \mathbb{N}\}$  contains  $\mathfrak{X}$ . Now, let  $\{g_n : n \in \mathbb{N}\}$  be a set of generators of  $\mathbb{F}_\infty$ . Then the association  $g_n \mapsto u_n$  for  $n \in \mathbb{N}$  defines uniquely a homomorphism  $\mathbb{F}_\infty \rightarrow \mathcal{U}(\mathcal{A})$  by the universal property of the free groups. By Proposition 3.1.1 this homomorphism extends uniquely to a  $*$ -homomorphism  $\varphi: C^*(\mathbb{F}_\infty) \rightarrow \mathcal{A}$ . Since the image of  $\varphi$  is a  $C^*$ -algebra containing  $\{u_n : n \in \mathbb{N}\}$ , the image must contain  $\mathfrak{X}$ , as well.

Now suppose that  $\mathcal{A}$  is separable. Then by the first part, we can find a  $*$ -homomorphism  $\varphi: C^*(\mathbb{F}_\infty) \rightarrow \mathcal{A}$ , whose image contains  $\mathcal{A}$ . Hence  $\mathcal{A}$  is a quotient of  $C^*(\mathbb{F}_\infty)$ . ◻

Recalling that  $C^*(\mathbb{F}_\infty)$  is separable, we obtain the following interesting corollary:

**Corollary 3.1.4.** *The following statements are equivalent:*

- (i) every separable  $C^*$ -algebra is QWEP;
- (ii) the  $C^*$ -algebra  $C^*(\mathbb{F}_\infty)$  is QWEP.

We now prove a proposition, which states that a surjective  $*$ -homomorphism between  $C^*$ -algebras maps the closed unit ball of the domain onto the closed unit ball of the target. This result will be used in the proof of the subsequent proposition.

**Proposition 3.1.5.** *Suppose that  $\pi: \mathcal{B} \rightarrow \mathcal{A}$  is a surjective  $*$ -homomorphism between  $C^*$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$ . Then  $\pi$  maps the closed unit ball of  $\mathcal{B}$  onto the closed unit ball of  $\mathcal{A}$ .*

<sup>1</sup>By all probability this extension is not unique, but existence and not uniqueness is the point here.

*Proof.* Suppose that  $x \in \mathcal{A}$  is a self-adjoint element of norm less than or equal to one, and choose a self-adjoint  $y \in \mathcal{B}$  with  $\pi(y) = x$ . Define  $g \in C(\sigma(x); \mathbb{C})$  by

$$g(x) = \begin{cases} x & \text{if } |x| \leq 1 \\ x|x|^{-1} & \text{if } |x| \geq 1 \end{cases}, \quad x \in \sigma(x)$$

Then  $g(y)$  is self-adjoint element of norm less than or equal to one, with  $\pi(g(y)) = g(\pi(y)) = g(x) = x$ , since  $\|x\| \leq 1$ .

Now suppose that  $x \in \mathcal{A}$  is a not necessarily self-adjoint element of norm less than or equal to one. The map  $(\pi)_2: M_2(\mathcal{B}) \rightarrow M_2(\mathcal{A})$  is also surjective, so the self-adjoint element

$$\begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix} \quad \text{lifts to an element} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in  $M_2(\mathcal{B})$  of norm less than or equal to one. Clearly  $b$  is a lift of  $x$ , and since the matrix on the right has norm less than or equal to one, we get by the standard matrix inequalities that  $\|b\| \leq 1$ .  $\square$

Kirchberg proved that  $C^*(\mathbb{F})$  has the lifting property, when  $\mathbb{F}$  is a free group on a countable (possibly finite) set of generators. Using this, it easily follows that  $C^*(\mathbb{F}_I)$  has the local lifting property, for all index sets  $I$ . We do not intend to prove this result by Kirchberg, since the proof is rather involved, but merely state it in Theorem 3.1.7 below. Before we even state the result, we will discuss part of its proof, which amounts to the proposition below. Sometimes one can make do with this weaker result.

**Proposition 3.1.6.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $C^*$ -algebras and  $\pi: \mathcal{B} \rightarrow \mathcal{A}$  a surjective  $*$ -homomorphism. For a free group  $\mathbb{F}$ , any  $*$ -homomorphism  $\phi: C^*(\mathbb{F}) \rightarrow \mathcal{A}$  lifts to a contractive completely positive map  $\tilde{\phi}: C^*(\mathbb{F}) \rightarrow \mathcal{B}$ . If  $\mathcal{B}$  is unital (in which case  $\mathcal{A}$  is also unital) and  $\phi$  is unital, then the lift can be chosen to be unital completely positive.*

*Proof.* Let  $\mathcal{B}'$  denote the unitization  $\tilde{\mathcal{B}}$  of  $\mathcal{B}$  if  $\mathcal{B}$  is non-unital, and  $\mathcal{B}' = \mathcal{B}$  otherwise. Note that, in either case,  $\mathcal{B}$  is an ideal in  $\mathcal{B}'$ .

For each  $u \in I$ , we can choose some lift  $x_u \in \mathcal{B}$  of  $\phi(u) \in \mathcal{A}$  of norm one. Fix some  $u \in I$ , and let  $y_u \in M_2(\mathcal{B}')$  denote the element

$$y_u = \begin{bmatrix} x_u & (\mathbf{1}_{\mathcal{B}'} - x_u x_u^*)^{-1/2} \\ (\mathbf{1}_{\mathcal{B}'} - x_u^* x_u)^{-1/2} & -x_u^* \end{bmatrix}.$$

Since  $x_u p(x_u^* x_u) = p(x_u x_u^*) x_u$  for all polynomials  $p$ , we get by a continuity argument that  $x_u (\mathbf{1}_{\mathcal{B}'} - x_u^* x_u)^{-1/2} = (\mathbf{1}_{\mathcal{B}'} - x_u x_u^*)^{-1/2} x_u$ . Using this it is straightforward to check that  $y_u$  is unitary. Thus we get a unique  $*$ -homomorphism  $\psi: C^*(\mathbb{F}_I) \rightarrow M_2(\mathcal{B}')$  satisfying  $\psi(u) = y_u$ . Now, let  $\tilde{\psi}: C^*(\mathbb{F}_I) \rightarrow \mathcal{B}'$  denote the upper left corner of  $\psi$ , that is, the composition of  $\psi$  with the map

$$M_2(\mathcal{B}') \longrightarrow \mathcal{B}' \quad \text{defined by} \quad \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \longmapsto x_{11}.$$

The above-defined map is clearly contractive completely positive, so its composition with  $\psi$ , that is,  $\tilde{\psi}$ , must also be contractive completely positive. Since the upper left corner of  $y_u$  is  $x_u$  we see that  $\tilde{\psi}(u) = x_u$  for all  $u \in I$ . At first, this map  $\tilde{\psi}$  is our candidate for a lift of  $\phi$ , and if  $\mathcal{B}$  is unital, then it is actually a lift of  $\phi$ , but if not,

then it might happen that the image of  $\tilde{\psi}$  is not contained in  $\mathcal{B}$ . So let us fix this. Let  $b \in \mathcal{B}$  be a lift of  $\phi(\mathbf{1})$  of norm one, where  $\mathbf{1}$  denotes the identity on  $C^*(\mathbb{F}_I)$ . Now define the map  $\tilde{\phi}: C^*(\mathbb{F}_I) \rightarrow \mathcal{B}$  by  $\tilde{\phi}(x) = b^* \tilde{\psi}(x) b$ ,  $x \in C^*(\mathbb{F}_I)$ . Clearly this map is contractive completely positive, and it is well-defined since  $\mathcal{B}$  is an ideal in  $\mathcal{B}'$ , which ensures that  $b^* b' b \in \mathcal{B}$  for all  $b' \in \mathcal{B}'$ . By construction

$$(\pi \circ \tilde{\phi})(u) = \pi(b^* \tilde{\psi}(u) b) = \pi(b)^* \pi(x_u) \pi(b) = \phi(\mathbf{1})^* \phi(u) \phi(\mathbf{1}) = \phi(u), \quad u \in I.$$

Thus  $\tilde{\phi}$  is a contractive completely positive lift of  $\phi$ . Now suppose that  $\mathcal{B}$ ,  $\mathcal{A}$  and  $\phi$  are all unital. Let  $\theta$  be a state on  $C^*(\mathbb{F}_I)$ , then the map  $\tilde{\phi} + (\mathbf{1}_{\mathcal{B}} - \tilde{\phi}(\mathbf{1}))\theta$ , that is, the map from  $C^*(\mathbb{F}_I)$  to  $\mathcal{B}$  given by  $x \mapsto \tilde{\phi}(x) + (\mathbf{1}_{\mathcal{B}} - \tilde{\phi}(\mathbf{1}))\theta(x)$  will be a unital completely positive lift of  $\phi$ .  $\square$

The above proof displays a standard trick, namely, the unitary dilation of a contractive element. This trick will also be used in the proof of Lemma 3.2.3 in the quest for proving Kirchberg's Theorem.

Here we state the previously mentioned result on the lifting property due to Kirchberg.

**Theorem 3.1.7.** *The  $C^*$ -algebra  $C^*(\mathbb{F})$  has the lifting property, when  $\mathbb{F}$  is a free group on at most countable many generators.*

A proof of this can be found in [Kir94, Lemma 3.3] and [Oza04, Theorem 3.8]. From this we get that following important corollary:

**Corollary 3.1.8.** *For every free group  $\mathbb{F}$ , the  $C^*$ -algebra  $C^*(\mathbb{F})$  has the local lifting property.*

*Proof.* Suppose that  $\mathcal{S} \subseteq C^*(\mathbb{F})$  is a finite dimensional operator system. Then one can choose a  $*$ -homomorphism  $\pi: C^*(\mathbb{F}_\infty) \rightarrow C^*(\mathbb{F})$ , such that the image of  $\pi$  contains  $\mathcal{S}$ . This can be done by taking a basis of  $\mathcal{S}$  and writing the elements as linear combinations of unitaries from  $C^*(\mathbb{F})$ —these unitaries do not necessarily lie in  $\mathcal{S}$ , but this matters little. Now completely positive maps from  $\mathcal{S}$  can be lifted, using the lifting property of  $C^*(\mathbb{F}_\infty)$ .  $\square$

Now we move on to the subject of tensor products. More precisely, our next goal is to analyze how  $C^*(\mathbb{F})$  behaves with respect to tensor products, when  $\mathbb{F}$  is a free group. This investigation will come in handy, when proving tensorial characterizations of WEP and LLP in section 3.3.

**Theorem 3.1.9.** *Suppose that we are given a  $C^*$ -algebra  $\mathcal{B}$  and a  $C^*$ -subalgebra  $\mathcal{A}$  of  $\mathcal{B}$ . Then*

- (i) *if  $I$  is some index set and  $I_0 \subseteq I$  a non-empty subset, then the norm on  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I)$  restricts to the norm on  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_{I_0})$ , that is, the canonical map  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_{I_0}) \rightarrow \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I)$  is isometric.*
- (ii) *if the canonical map  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty) \rightarrow \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_\infty)$  is isometric, then the canonical map  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I) \rightarrow \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_I)$  is isometric, as well, for each non-empty set  $I$ .*
- (iii) *if  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty) = \mathcal{A} \otimes_{\min} C^*(\mathbb{F}_\infty)$ , then  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I) = \mathcal{A} \otimes_{\min} C^*(\mathbb{F}_I)$ , for each non-empty set  $I$ .*

*Proof.* We start by proving (i). By the definition of the maximal tensor product, it suffices to show that every representation of  $\mathcal{A} \odot C^*(\mathbb{F}_{I_0})$  extends to a representation of  $\mathcal{A} \odot C^*(\mathbb{F}_I)$ . So let  $\mathcal{H}$  be a Hilbert space and  $\pi: \mathcal{A} \odot C^*(\mathbb{F}_{I_0}) \rightarrow B(\mathcal{H})$  a representation of  $\mathcal{A} \odot C^*(\mathbb{F}_{I_0})$ . Let  $\pi_1: \mathcal{A} \rightarrow B(\mathcal{H})$  and  $\pi_2: C^*(\mathbb{F}_{I_0}) \rightarrow B(\mathcal{H})$  denote its restrictions (see Section 1.5). As mentioned in Remark 3.1.2, the representation  $\pi_2$  extends to a representation  $\tilde{\pi}_2: C^*(\mathbb{F}_I) \rightarrow \pi_2(C^*(\mathbb{F}_{I_0}))$ . Clearly  $\pi_1$  and  $\tilde{\pi}_2$  have commuting ranges, since this was true for  $\pi_1$  and  $\pi_2$ , and by construction the representation  $\pi_1 \times \tilde{\pi}_2: \mathcal{A} \odot C^*(\mathbb{F}_I) \rightarrow B(\mathcal{H})$  extends  $\pi$ , since  $\pi = \pi_1 \times \pi_2$ . Thus we have proved (i).

Let us prove (ii). Assume that  $I$  is finite, with  $n$  elements, say. We have an inclusion  $C^*(\mathbb{F}_n) \subseteq C^*(\mathbb{F}_\infty)$ . By part (i) we know that both the canonical maps  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_n) \rightarrow \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty)$  and  $\mathcal{B} \otimes_{\max} C^*(\mathbb{F}_n) \rightarrow \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_\infty)$  are isometric. Hence the map  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_n) \rightarrow \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_n)$  is isometric, since it is just the restriction of the map  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty) \rightarrow \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_\infty)$ , which is isometric by assumption.

Suppose instead that  $I$  is infinite. To show that the map  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I) \rightarrow \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_I)$  is isometric, it suffices to show that it is isometric on  $\mathcal{A} \odot C^*(\mathbb{F}_I)$ . So let  $a \in \mathcal{A} \odot C^*(\mathbb{F}_I)$ , and write  $a = \sum_{k=1}^n a_k \otimes x_k$ , with  $a_k \in \mathcal{A}$  and  $x_k \in C^*(\mathbb{F}_I)$ , for  $k = 1, 2, \dots, n$ . Since  $\mathbb{C}\mathbb{F}_I$  is dense in  $C^*(\mathbb{F}_I)$  we may choose some countable (and infinite) set  $J \subseteq I$ , say  $J = \{u_m : m \in \mathbb{N}\}$ , such that  $x_k = \lim_{m \rightarrow \infty} \sum_{j=1}^m \lambda_{j,m}^k u_j$ , for each  $k \in \{1, 2, \dots, n\}$ , with  $\lambda_{j,m}^k \in \mathbb{C}$ , for all  $m, j \in \mathbb{N}$ . Now we see that

$$a = \lim_{m \rightarrow \infty} \sum_{k=1}^n \sum_{j=1}^m \lambda_{j,m}^k \lambda_{j,m}^k a_k \otimes u_j.$$

So in particular  $a \in C^*(\mathbb{F}_J)$ . Now, by choosing a bijection  $j: \mathbb{N} \rightarrow J$  we get a map  $\tilde{j}: C^*(\mathbb{F}_\infty) \rightarrow C^*(\mathbb{F}_J)$ , mapping  $C^*(\mathbb{F}_\infty)$  isometrically onto  $C^*(\mathbb{F}_J)$  as in Remark 3.1.2. Now, by again applying part (i), and drawing the diagram

$$\begin{array}{ccc} \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty) & \longrightarrow & \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_\infty) , \\ \text{id}_{\mathcal{A}} \otimes \tilde{j} \downarrow & & \downarrow \text{id}_{\mathcal{B}} \otimes \tilde{j} \\ \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I) & \longrightarrow & \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_I) \end{array}$$

we realize that the top map as well as the vertical maps are isometric. By construction,  $a$  is in the image of  $\text{id}_{\mathcal{A}} \otimes \tilde{j}$ , it follows that the norm of  $a$  is the same as the norm of the image of  $a$  under the bottom map. Thus it follows that the map  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I) \rightarrow \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_I)$  is isometric on  $\mathcal{A} \odot C^*(\mathbb{F}_I)$ , and hence isometric on  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I)$ .

The proof of part (iii) is similar to the one of part (ii). More precisely, one proves the statement for the algebraic tensor product, using the same trick as above.  $\square$

The last results we discuss in this section concerns non-unital technicalities related to the maximal tensor product with  $C^*(\mathbb{F})$ , for a free group  $\mathbb{F}$ .

**Lemma 3.1.10.** *Suppose that  $\mathcal{A}_i$  is a  $C^*$ -algebra and  $\mathcal{I}_i \subseteq \mathcal{A}_i$  an ideal in  $\mathcal{A}_i$ , for  $i = 1, 2$ . For a  $C^*$ -algebra  $\mathcal{B}$  and three vertical maps making the diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{I}_1 \otimes_{\max} \mathcal{B} & \longrightarrow & \mathcal{A}_1 \otimes_{\max} \mathcal{B} & \longrightarrow & (\mathcal{A}_1/\mathcal{I}_1) \otimes_{\max} \mathcal{B} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{I}_2 \otimes_{\max} \mathcal{B} & \longrightarrow & \mathcal{A}_2 \otimes_{\max} \mathcal{B} & \longrightarrow & (\mathcal{A}_2/\mathcal{I}_2) \otimes_{\max} \mathcal{B} \longrightarrow 0 \end{array}$$

commute, where the rows are exact (due to exactness of the maximal tensor product) it holds that, if the two outer vertical maps are injective, then so is the middle one.

*Proof.* The exactness of the rows is from Proposition 1.5.7 and the statement of the lemma is a standard argument using diagram chasing.  $\square$

**Lemma 3.1.11.** *If  $\mathcal{A}$  and  $\mathcal{B}$  are  $C^*$ -algebras, then the norm on  $\mathcal{A} \otimes_{\max} \tilde{\mathcal{B}}$  restricts to the norm on  $\mathcal{A} \otimes_{\max} \mathcal{B}$ , where  $\tilde{\mathcal{B}}$  denotes the unitization of  $\mathcal{B}$ .*

*Proof.* This follows from the fact that the sequence

$$0 \longrightarrow \mathcal{A} \otimes_{\max} \mathcal{B} \longrightarrow \mathcal{A} \otimes_{\max} \tilde{\mathcal{B}} \longrightarrow \mathcal{A} \otimes_{\max} \mathbb{C} \longrightarrow 0,$$

is exact by Proposition 1.5.7.  $\square$

**Lemma 3.1.12.** *Let  $\mathcal{B}$  and  $\mathcal{C}$  be  $C^*$ -algebras and let  $\mathcal{A}$  be a  $C^*$ -subalgebra of  $\mathcal{B}$ . Then the following are equivalent: (i) the map  $\mathcal{A} \otimes_{\max} \mathcal{C} \rightarrow \mathcal{B} \otimes_{\max} \mathcal{C}$  is isometric; (ii) the map  $\mathcal{A} \otimes_{\max} \tilde{\mathcal{C}} \rightarrow \mathcal{B} \otimes_{\max} \tilde{\mathcal{C}}$  is isometric; (iii) the map  $\tilde{\mathcal{A}} \otimes_{\max} \mathcal{C} \rightarrow \tilde{\mathcal{B}} \otimes_{\max} \mathcal{C}$  is isometric; (iv) the map  $\tilde{\mathcal{A}} \otimes_{\max} \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{B}} \otimes_{\max} \tilde{\mathcal{C}}$  is isometric.*

*Proof.* The implications (iv)  $\Rightarrow$  (iii), (iv)  $\Rightarrow$  (ii), (iii)  $\Rightarrow$  (i) and (ii)  $\Rightarrow$  (i) follows from Lemma 3.1.11. To prove that (i)  $\Rightarrow$  (iv) it suffices to prove that (i)  $\Rightarrow$  (iii) and (i)  $\Rightarrow$  (ii). The implication (i)  $\Rightarrow$  (iii) follows from Lemma 3.1.10 applied to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{A} \otimes_{\max} \mathcal{C} & \longrightarrow & \tilde{\mathcal{A}} \otimes_{\max} \mathcal{C} & \longrightarrow & \mathbb{C} \otimes_{\max} \mathcal{C} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{B} \otimes_{\max} \mathcal{C} & \longrightarrow & \tilde{\mathcal{B}} \otimes_{\max} \mathcal{C} & \longrightarrow & \mathbb{C} \otimes_{\max} \mathcal{C} \longrightarrow 0 \end{array}$$

where the left vertical map is injective by assumption and the right vertical map is an equality. The implication (i)  $\Rightarrow$  (ii) follows from Lemma 3.1.10 applied to the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{A} \otimes_{\max} \mathcal{C} & \longrightarrow & \mathcal{A} \otimes_{\max} \tilde{\mathcal{C}} & \longrightarrow & \mathcal{A} \otimes_{\max} \mathbb{C} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{B} \otimes_{\max} \mathcal{C} & \longrightarrow & \mathcal{B} \otimes_{\max} \tilde{\mathcal{C}} & \longrightarrow & \mathcal{B} \otimes_{\max} \mathbb{C} \longrightarrow 0 \end{array}$$

where the left vertical map is injective by assumption and the right vertical map is the injection of  $\mathcal{A}$  into  $\mathcal{B}$ .  $\square$

### 3.2 Pisier's proof of Kirchberg's theorem

This section is devoted to a famous theorem of Kirchberg, from his 1993 article [Kir93], stating that there is only one  $C^*$ -algebra norm on the algebraic tensor product of  $B(\mathcal{H})$  with  $C^*(\mathbb{F})$ , where  $\mathcal{H}$  is a Hilbert space and  $\mathbb{F}$  a free group.

We follow the elegant proof of Gilles Pisier presented in his 1996 article [Pis96]. The idea of Pisier's proof is, in a sense, to first reduce the problem to a statement about a certain operator space spanned by unitaries, and then to describe the minimal norm of such elements in a (sort of) algebraic way.

We start with three lemmas, the first two of which are fairly elementary, and the last of which is more complicated. The latter forms an important part of the theorem of Kirchberg.

**Lemma 3.2.1.** *Suppose that we are given two Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ , a unitary element  $u \in B(\mathcal{K})$  and an isometry  $V: \mathcal{H} \rightarrow \mathcal{K}$ . If  $V^*uV$  is unitary, then  $u$  commutes with  $VV^*$ .*

*Proof.* Let  $P$  denote the projection  $VV^*$ . For  $\xi \in \mathcal{H}$  we get

$$\|PuP\xi\| = \|VV^*uVV^*\xi\| = \|V^*\xi\| = \|VV^*\xi\| = \|P\xi\|,$$

since both  $V$  and  $V^*uV$  are assumed to be isometries. Hence,

$$\|(\mathbf{1} - P)uP\xi\|^2 = \|uP\xi\|^2 - \|PuP\xi\|^2 = 0,$$

which shows that  $PuP = uP$ . Now, replacing  $u$  with  $u^*$  and repeating the argument we get  $Pu^*P = u^*P$ . Combining these two identities we conclude that  $u$  commutes with  $P$ .  $\square$

**Lemma 3.2.2.** *If we are given bounded operators  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  on a Hilbert space  $\mathcal{H}$ , then*

$$\left\| \sum_{k=1}^n a_k b_k \right\| \leq \left\| \sum_{k=1}^n a_k a_k^* \right\|^{1/2} \left\| \sum_{k=1}^n b_k^* b_k \right\|^{1/2}.$$

*Proof.* The inequality follows by noticing that

$$\begin{bmatrix} \sum_{k=1}^n a_k b_k & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} b_1 & 0 & \dots & 0 \\ b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & \dots & 0 \end{bmatrix},$$

and calculating the norm of the matrices on the right hand side using the identities

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \dots & a_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}^* = \begin{bmatrix} \sum_{k=1}^n a_k a_k^* & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} b_1 & 0 & \dots & 0 \\ b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & \dots & 0 \end{bmatrix}^* \begin{bmatrix} b_1 & 0 & \dots & 0 \\ b_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_n & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^n b_k^* b_k & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

and nothing more.  $\square$

**Lemma 3.2.3.** *Let  $\mathbb{F}$  be a free group and  $\mathcal{H}$  an infinite dimensional Hilbert space. Given a set  $u_1, u_2, \dots, u_n$  of the canonical generators of  $\mathbb{F}$ , all distinct, and bounded operators  $x_0, x_1, \dots, x_n$  on  $\mathcal{H}$ , the following are equivalent (with  $u_0 = \mathbf{1}_{C^*(\mathbb{F})}$  and  $x = (x_0, x_1, \dots, x_n)$ ):*

- (i) the linear operator  $T_x: \ell_\infty^{n+1} \rightarrow B(\mathcal{H})$  given by  $T_x(\lambda_0, \lambda_1, \dots, \lambda_n) = \lambda_0 x_0 + \lambda_1 x_1 + \dots + \lambda_n x_n$  is completely bounded with  $\|T_x\|_{\text{cb}} < 1$ ;
- (ii) the element  $z = \sum_{k=0}^n u_k \otimes x_k$  in  $C^*(\mathbb{F}) \odot B(\mathcal{H})$  satisfies  $\|z\|_{\text{min}} < 1$ ;
- (iii) there exist bounded linear operators  $a_0, a_1, \dots, a_n$  and  $b_0, b_1, \dots, b_n$  on  $\mathcal{H}$  so that  $x_k = a_k b_k$ , for all  $k = 0, 1, \dots, n$ , and

$$\left\| \sum_{k=0}^n a_k a_k^* \right\| < 1 \quad \text{and} \quad \left\| \sum_{k=0}^n b_k^* b_k \right\| < 1$$

*Proof.* Let us start by proving that  $\|T_x\|_{\text{cb}} = \|z\|_{\text{min}}$ . This will clearly imply the equivalence of (i) and (ii). Denote the linear span of  $u_0, u_1, \dots, u_n$  by  $E$ .

Let  $\delta_0, \delta_1, \dots, \delta_n$  denote the standard basis of  $\ell_\infty^{n+1}$ , that is,  $\delta_k$  is the indicator function of  $\{k\}$ ,  $k = 0, 1, \dots, n$ . Consider the element  $w$  of  $E \odot \ell_\infty^{n+1}$  given by  $w = \sum_{k=0}^n u_k \otimes \delta_k$ . It is straightforward to check that this element is unitary, and it satisfies

$$(\text{id}_E \otimes T_x)(w) = \sum_{k=0}^n (\text{id}_E \otimes T_x)(u_k \otimes \delta_k) = \sum_{k=0}^n u_k \otimes x_k = z.$$

By Proposition 1.5.9, we get that  $\text{id}_E \otimes T_x$  is completely bounded and that

$$\|\text{id}_E \otimes T_x\| \leq \|\text{id}_E \otimes T_x\|_{\text{cb}} = \|\text{id}_E\|_{\text{cb}} \|T_x\|_{\text{cb}} = \|T_x\|_{\text{cb}},$$

so in particular, we get that

$$\|z\|_{\text{min}} = \|(\text{id}_E \otimes T_x)(w)\|_{\text{min}} \leq \|\text{id}_E \otimes T_x\| \|w\|_{\text{min}} \leq \|T_x\|_{\text{cb}}.$$

This proves the first inequality. The other one is a bit more involved. The strategy is as follows: first of all, since  $\|T_x\|_{\text{cb}} = \|\text{id}_{B(\mathcal{H})} \otimes T_x\|$  by Proposition 1.5.10, we want to show that  $\|\text{id}_{B(\mathcal{H})} \otimes T_x\| \leq \|z\|_{\text{min}}$ . Second, for  $a \in B(\mathcal{H}) \odot \ell_\infty^{n+1}$  with  $\|a\| \leq 1$ , our strategy then is to construct a completely contractive map  $\theta_a: C^*(\mathbb{F}) \rightarrow B(\mathcal{H})$  so that  $(\text{id}_{B(\mathcal{H})} \otimes T_x)(a) = (\theta_a \otimes \text{id}_{B(\mathcal{H})})(z)$ , because then

$$\begin{aligned} \|(\text{id}_{B(\mathcal{H})} \otimes T_x)(a)\|_{\text{min}} &= \|(\theta_a \otimes \text{id}_{B(\mathcal{H})})(z)\|_{\text{min}} \\ &\leq \|\theta_a \otimes \text{id}_{B(\mathcal{H})}\| \|z\|_{\text{min}} \leq \|z\|_{\text{min}}, \end{aligned}$$

and since  $a$  was arbitrary with  $\|a\| \leq 1$ , the conclusion follows. We now perform the desired construction. Let  $a \in B(\mathcal{H}) \odot \ell_\infty^{n+1}$  be given with  $\|a\| \leq 1$ , and write  $a = \sum_{k=0}^n a_k \otimes \delta_k$ , with  $a_k \in B(\mathcal{H})$  and  $k = 0, 1, \dots, n$ . Since  $\|a\| = \max\{\|a_0\|, \|a_1\|, \dots, \|a_n\|\}$  we get that  $\|a_k\| \leq 1$ , for all  $k = 0, 1, \dots, n$ . Let  $\hat{a}_k$  denote the unitary dilation of  $a_k$  given by

$$\hat{a}_k = \begin{bmatrix} a_k & (\mathbf{1} - a_k a_k^*)^{1/2} \\ (\mathbf{1} - a_k^* a_k)^{1/2} & -a_k^* \end{bmatrix}, \quad \text{for } k = 0, 1, \dots, n.$$

Now, the prescription  $u_k \mapsto \hat{a}_0^* \hat{a}_k$  for each  $k = 1, 2, \dots, n$  and  $v \mapsto \mathbf{1}_{M_2(B(\mathcal{H}))}$ , for each canonical unitary generator of  $C^*(\mathbb{F})$  besides  $u_1, u_2, \dots, u_n$ , defines a uniquely a  $*$ -homomorphism  $\psi: C^*(\mathbb{F}) \rightarrow M_2(B(\mathcal{H}))$ . The map  $M_2(B(\mathcal{H})) \rightarrow M_2(B(\mathcal{H}))$  given by  $x \mapsto \hat{a}_0 x$  is completely contractive, since  $\hat{a}_0$  is unitary. Hence by letting  $\phi: M_2(B(\mathcal{H})) \rightarrow B(\mathcal{H})$  denote the contractive completely positive map taking an



element to its upper left corner, we get a completely contractive map  $\theta_a : C^*(\mathbb{F}) \rightarrow B(\mathcal{H})$  given by  $\theta_a(x) = \phi(\hat{a}_0\psi(x))$ . It is straightforward to check that  $\theta_a(u_k) = a_k$ , for all  $k = 0, 1, \dots, n$ , so in particular,

$$(\theta_a \otimes \text{id}_{B(\mathcal{H})})(z) = \sum_{k=0}^n (\theta_a \otimes \text{id}_{B(\mathcal{H})})(u_k \otimes x_k) = \sum_{k=0}^n (a_k \otimes x_k).$$

Now, we also have

$$(\text{id}_{B(\mathcal{H})} \otimes T_x)(a) = \sum_{k=0}^n (\text{id}_{B(\mathcal{H})} \otimes T_x)(a_k \otimes \delta_k) = \sum_{k=0}^n (a_k \otimes x_k),$$

so we conclude that  $\theta_a$  is a completely contractive map with the desired property. As explained above, it follows that  $\|T_x\|_{\text{cb}} \leq \|z\|_{\min}$ . So we conclude  $\|T_x\|_{\text{cb}} = \|z\|_{\min}$ , and therefore also the equivalence of (i) and (ii).

Let us now assume that (iii) and then prove (ii). Let  $\pi : C^*(\mathbb{F}) \rightarrow B(\mathcal{K})$  be a faithful representation of  $C^*(\mathbb{F})$  on a Hilbert space  $\mathcal{K}$ . By definition of the minimal tensor product norm, the map  $\pi \otimes \text{id}_{B(\mathcal{H})} : C^*(\mathbb{F}) \otimes_{\min} B(\mathcal{H}) \rightarrow B(\mathcal{H} \otimes \mathcal{K})$  is a faithful representation, and we see that

$$\|z\|_{\min} = \|(\pi \otimes \text{id}_{B(\mathcal{H})})(z)\| = \left\| \sum_{k=0}^n \pi(u_k) \otimes x_k \right\| = \left\| \sum_{k=0}^n \pi(u_k) \otimes a_k b_k \right\|.$$

By Lemma 3.2.2 we obtain that

$$\begin{aligned} \left\| \sum_{k=0}^n \pi(u_k) \otimes a_k b_k \right\| &= \left\| \sum_{k=0}^n (\pi(u_k) \otimes a_k) (\mathbf{1}_{C^*(\mathbb{F})} \otimes b_k) \right\| \\ &\leq \left\| \sum_{k=0}^n (\mathbf{1}_{C^*(\mathbb{F})} \otimes b_k^* b_k) \right\|^{1/2} \left\| \sum_{k=0}^n (\mathbf{1}_{C^*(\mathbb{F})} \otimes a_k a_k^*) \right\|^{1/2} \\ &= \left\| \mathbf{1}_{C^*(\mathbb{F})} \otimes \sum_{k=0}^n b_k^* b_k \right\|^{1/2} \left\| \mathbf{1}_{C^*(\mathbb{F})} \otimes \sum_{k=0}^n a_k a_k^* \right\|^{1/2} \\ &= \left\| \sum_{k=0}^n b_k^* b_k \right\|^{1/2} \left\| \sum_{k=0}^n a_k a_k^* \right\|^{1/2} < 1, \end{aligned}$$

which proves that  $\|z\|_{\min} < 1$ , and hence (ii).

Now assume instead that (i) holds and let us prove (iii). By Theorem B.4.5 there exist a representation  $\rho : \ell_\infty^{n+1} \rightarrow B(\mathcal{K})$  on a Hilbert space  $\mathcal{K}$  and bounded linear operators  $V_1$  and  $V_2$  from  $\mathcal{H}$  to  $\mathcal{K}$ , satisfying that

$$T_x(y) = V_1^* \rho(y) V_2, \quad \text{for all } y \in \ell_\infty^{n+1},$$

together with  $\|T_x\|_{\text{cb}} = \|V_1\| \|V_2\|$ . By scaling  $V$  and  $W$  we may assume that  $\|V_1\| = \|V_2\| = \|T_x\|_{\text{cb}}^{1/2}$ . We may also assume that  $\mathcal{K}$  has a greater dimension than  $\mathcal{H}$ , since we can replace  $\mathcal{K}$  with  $\mathcal{K} \oplus \mathcal{H}$  and extend each operator in question by zero. Since  $\mathcal{H}$  is infinite dimensional, the closed linear span of the set

$$\{\rho(\delta_k) V_j \xi : \xi \in \mathcal{H}, j = 1, 2, k = 0, 1, \dots, n\}$$

in  $\mathcal{K}$  has at most the same dimension as that of  $\mathcal{H}$ .<sup>2</sup> Thus we can choose a partial isometry  $W: \mathcal{K} \rightarrow \mathcal{H}$ , whose support contains the closed linear span in question. Or, in other words, we can choose a partial isometry  $W$  so that  $W^*W\rho(\delta_k)V_j = \rho(\delta_k)V_j$ , for all  $k = 0, 1, \dots, n$  and  $j = 1, 2$ . Now, let

$$a_k = V_1^*\rho(\delta_k)W^* \quad \text{and} \quad b_k = W\rho(\delta_k)V_2,$$

for  $k = 0, 1, \dots, n$ , and let us justify that these satisfy the conditions specified in (iii). By construction

$$x_k = V_1^*\rho(\delta_k)V_2 = V_1^*\rho(\delta_k)\rho(\delta_k)V_2 = V_1^*\rho(\delta_k)W^*W\rho(\delta_k)V_2 = a_k b_k,$$

for all  $k = 0, 1, \dots, n$ , and if we use that  $\sum_{k=0}^n \delta_k = \mathbf{1}_{\ell_\infty^{n+1}}$ , we get that

$$\sum_{k=0}^n b_k^* b_k = \sum_{k=0}^n V_2^*\rho(\delta_k)^*W^*W\rho(\delta_k)V_2 = V_2^*\left(\sum_{k=0}^n \rho(\delta_k)\right)V_2 = V_2^*V_2$$

By similar calculations we see that  $\sum_{k=0}^n a_k a_k^* = V_1^*V_1$ . In particular, recalling that  $\|T_x\|_{\text{cb}} < 1$  we see that

$$\|V_2^*V_2\| = \|V_2\|^2 = \|T_x\|_{\text{cb}} < 1 \quad \text{and} \quad \|V_1^*V_1\| = \|V_1\|^2 = \|T_x\|_{\text{cb}} < 1.$$

So we have proved that

$$\left\| \sum_{k=0}^n a_k a_k^* \right\| < 1 \quad \text{and} \quad \left\| \sum_{k=0}^n b_k^* b_k \right\| < 1,$$

which shows that (iii) holds.

This proves the last implications, and so the three conditions are equivalent.  $\square$

Before we prove an important theorem of Pisier, which is crucial in his proof of Kirchberg's theorem, we need the following technical propositions on extensions of completely positive maps.

**Proposition 3.2.4.** *Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are unital  $C^*$ -algebras. Let  $(u_i)_{i \in I}$  be a family of unitaries generating  $\mathcal{A}$  as a unital  $C^*$ -algebra, and let  $E$  denote the linear span of this family of unitaries and the identity in  $\mathcal{A}$ . If  $T: E \rightarrow \mathcal{B}$  is a unital completely contractive map such that  $Tu_i$  is unitary, for all  $i \in I$ , then  $T$  extends to a  $*$ -homomorphism  $\tilde{T}: \mathcal{A} \rightarrow \mathcal{B}$ .*

*Proof.* We may assume that  $\mathcal{B} \subseteq B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ . By Wittstock's Extension Theorem the map  $T$  extends to a completely contractive map  $\tilde{T}: \mathcal{A} \rightarrow B(\mathcal{H})$ . Since unital completely contractive maps on  $C^*$ -algebras are automatically unital completely positive by Corollary B.1.11, we get that  $\tilde{T}$  is unital completely positive. Hence Stinespring's Dilation Theorem is applicable, and we may choose a representation  $\pi$  of  $\mathcal{A}$  on a Hilbert space  $\mathcal{K}$  and an isometry  $V: \mathcal{H} \rightarrow \mathcal{K}$ , such that  $\tilde{T}x = V^*\pi(x)V$ , for all  $x \in \mathcal{A}$ . By construction  $V^*\pi(u_i)V$  is unitary, for each  $i \in I$ , since  $V$  is an isometry and  $\pi(u_i)$  is unitary. Because of this we get from Lemma 3.2.1, that  $\pi(u_i)$  commutes with  $VV^*$ , for all  $i \in I$ . In particular, given  $i, j \in I$  we have

$$\tilde{T}(u_i u_j) = V^*\pi(u_i)\pi(u_j)V = V^*\pi(u_i)V V^*\pi(u_j)V = \tilde{T}(u_i)\tilde{T}(u_j).$$

<sup>2</sup>It might be that the dimension gets  $2(n+1)$  times larger, since we let  $j$  and  $k$  vary in  $\{1, 2\}$  and  $\{0, 1, \dots, n\}$ , respectively. So unless  $\mathcal{H}$  is infinite dimensional, the statement is not true.

So since  $\tilde{T}$  is unital and the family  $(u_i)_{i \in I}$  generates  $\mathcal{A}$  as a unital  $C^*$ -algebra, we deduce that  $\tilde{T}$  must be multiplicative, and hence a  $*$ -homomorphism. Now, the only thing left to check is that the image of  $\tilde{T}$  is contained in  $\mathcal{B}$ , but this follows from the fact that  $Tu_i \in \mathcal{B}$ , for all  $i \in I$ , and  $(u_i)_{i \in I}$  generates  $\mathcal{A}$  as a unital  $C^*$ -algebra.  $\square$

The following theorem of Pisier forms an important part of his proof of Kirchberg's theorem. It reduces the problem of checking whether the maximal and minimal tensor product of two  $C^*$ -algebras agree, to checking this on the tensor product of operator spaces spanned in a nice way by unitaries.

**Theorem 3.2.5.** *Suppose that we are given unital  $C^*$ -algebras  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , together with families  $(u_i)_{i \in I}$  and  $(v_j)_{j \in J}$  of unitaries generating  $\mathcal{A}_1$  and  $\mathcal{A}_2$  as unital  $C^*$ -algebras, respectively. Let  $E_1$  and  $E_2$  denote the linear span of  $(u_i)_{i \in I}$  plus the unit of  $\mathcal{A}_1$ , and  $(v_j)_{j \in J}$  plus the unit of  $\mathcal{A}_2$ , respectively. Then the following are equivalent:*

- (i) *the inclusion  $E_1 \odot E_2 \rightarrow \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$  is completely contractive when  $E_1 \odot E_2$  is equipped with the minimal tensor norm;*
- (ii) *we have  $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 = \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$  canonically.*

*Proof.* It should be clear that (ii) implies (i), since the inclusion map  $E_1 \odot E_2 \rightarrow \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$  is completely contractive by Proposition 1.5.9, when  $E_1 \odot E_2$  is equipped with the minimal tensor norm.

Suppose that (i) holds, and let us prove (ii). Denote  $E_1 \odot E_2$  by  $E$ . First, let us convince ourselves that the set  $L$  given by

$$L = \{u_i \otimes v_j : i \in I, j \in J\} \cup \{u_i \otimes \mathbf{1}_{\mathcal{A}_2} : i \in I\} \cup \{\mathbf{1}_{\mathcal{A}_1} \otimes v_j : j \in J\}$$

together with the unit spans  $E$ , and moreover, generates  $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$  as a unital  $C^*$ -algebra. That  $L$  plus the unit  $\mathbf{1}_{\mathcal{A}_1} \otimes \mathbf{1}_{\mathcal{A}_2}$  spans  $E$  should be clear. Since  $(u_i)_{i \in I}$  generates  $\mathcal{A}_1$  as a unital  $C^*$ -algebra, the unital  $C^*$ -algebra generated by  $L$  must contain  $\mathcal{A}_1 \otimes \mathbf{1}_{\mathcal{A}_2}$ . By a similar argument it must contain  $\mathbf{1}_{\mathcal{A}_1} \otimes \mathcal{A}_2$ , so it follows that  $L$  must generate  $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$  as a unital  $C^*$ -algebra, since it generates a unital  $C^*$ -algebra containing  $\mathcal{A}_1 \odot \mathcal{A}_2$ . Second, let us apply Proposition 3.2.4 to what we already know. The proposition applies to the inclusion  $E_1 \odot E_2 \rightarrow \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$  which is completely contractive by assumption, obviously unital and maps every element of  $L$  to a unitary in  $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$ , since all the elements of  $L$  are unitary. Thus, since  $E$  is the linear span of  $L$  and the identity, and  $L$  generates  $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$  as a unital  $C^*$ -algebra, the conclusion of Proposition 3.2.4 is that the inclusion  $E_1 \odot E_2 \rightarrow \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$  extends to a  $*$ -homomorphism  $\pi : \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \rightarrow \mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$ . All we are left with to check, is that  $\pi$  is actually the identity on  $\mathcal{A}_1 \odot \mathcal{A}_2$ . Since  $E_1 \otimes \mathbf{1}_{\mathcal{A}_2}$  generates  $\mathcal{A}_1 \otimes \mathbf{1}_{\mathcal{A}_2}$  and  $\pi$  is the identity on  $E_1 \otimes \mathbf{1}_{\mathcal{A}_2}$  we conclude that  $\pi$  is the identity on  $\mathcal{A}_1 \otimes \mathbf{1}_{\mathcal{A}_2}$ . Similarly  $\pi$  is the identity on  $\mathbf{1}_{\mathcal{A}_1} \otimes \mathcal{A}_2$ , and therefore also on the algebraic tensor product  $\mathcal{A}_1 \odot \mathcal{A}_2$ . Thus we have proved that  $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$  and  $\mathcal{A}_1 \otimes_{\max} \mathcal{A}_2$  are canonically isomorphic.  $\square$

Now, with all this work done, we are ready to prove the theorem of Kirchberg.

**Theorem 3.2.6.** *Given a free group  $\mathbb{F}$  and a Hilbert space  $\mathcal{H}$ , it holds that*

$$C^*(\mathbb{F}) \otimes_{\min} B(\mathcal{H}) = C^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H}).$$

*Proof.* This will be an application of Lemma 3.2.3 and Theorem 3.2.5. First of all, note that if  $\mathcal{H}$  is finite dimensional, then the statement is trivially satisfied. Thus we may assume that  $\mathcal{H}$  is infinite dimensional. Let  $(u_i)_{i \in I}$  denote the canonical unitary generators of  $C^*(\mathbb{F})$ , and let  $E_1$  denote the linear span of these unitaries together with the identity in  $C^*(\mathbb{F})$ . Now, if we let  $E_2 = B(\mathcal{H})$ , then  $E_2$  is the linear span of the family  $\mathcal{U}(B(\mathcal{H}))$  of unitaries in  $B(\mathcal{H})$ . Clearly  $(u_i)_{i \in I}$  and  $\mathcal{U}(B(\mathcal{H}))$  generate  $C^*(\mathbb{F})$  and  $B(\mathcal{H})$  as unital  $C^*$ -algebras, we obtain by Theorem 3.2.5 that

$$C^*(\mathbb{F}) \otimes_{\min} B(\mathcal{H}) = C^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H})$$

if and only if the inclusion  $E_1 \odot E_2 \rightarrow C^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H})$  is completely contractive, when  $E_1 \odot E_2$  is equipped with the minimal tensor norm. So suppose that  $z \in E_1 \odot E_2$ , that is, there are some  $n \in \mathbb{N}$  and distinct elements  $v_1, v_2, \dots, v_n$  of the family  $(u_i)_{i \in I}$  and bounded linear operators  $x_0, x_1, \dots, x_n$  on  $\mathcal{H}$ , so that

$$z = \sum_{k=0}^n v_k \otimes x_k,$$

where  $v_0 = \mathbf{1}_{C^*(\mathbb{F})}$ . Assume that  $\|z\|_{\min} < 1$ . Then by Lemma 3.2.3 there exist bounded operators  $a_0, a_1, \dots, a_n$  and  $b_0, b_1, \dots, b_n$  on  $\mathcal{H}$ , such that  $x_k = a_k b_k$ , for each  $k = 0, 1, \dots, n$ , with

$$\left\| \sum_{k=0}^n a_k a_k^* \right\| < 1 \quad \text{and} \quad \left\| \sum_{k=0}^n b_k^* b_k \right\| < 1.$$

Now, let  $\pi : C^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H}) \rightarrow B(\mathcal{K})$  be a faithful representation of  $C^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H})$  on some Hilbert space  $\mathcal{K}$ , and let  $\pi_1$  and  $\pi_2$  denote its restrictions to  $C^*(\mathbb{F})$  and  $B(\mathcal{H})$ , respectively. Then

$$\|z\|_{\max} = \|\pi(z)\| = \left\| \sum_{k=0}^n \pi_1(v_k) \pi_2(a_k) \pi_2(b_k) \right\| = \left\| \sum_{k=0}^n \pi_2(a_k) \pi_1(v_k) \pi_2(b_k) \right\|,$$

and by applying Lemma 3.2.2 we obtain that

$$\begin{aligned} \|z\|_{\max} &\leq \left\| \sum_{k=0}^n \pi_2(a_k) \pi_1(v_k) \pi_1(v_k)^* \pi_2(a_k)^* \right\|^{1/2} \left\| \sum_{k=0}^n \pi_2(b_k)^* \pi_2(b_k) \right\|^{1/2} \\ &= \left\| \pi_2 \left( \sum_{k=0}^n a_k a_k^* \right) \right\|^{1/2} \left\| \pi_2 \left( \sum_{k=0}^n b_k^* b_k \right) \right\|^{1/2} < 1. \end{aligned}$$

So, just to summarize, we have proved that if  $z \in E_1 \odot E_2$ , with  $\|z\|_{\min} < 1$ , then  $\|z\|_{\max} < 1$ . In particular, if we let  $z \in E_1 \odot E_2$  and let  $\delta > \|z\|_{\min}$ , then by applying this result to the element  $\delta^{-1}z$ , we get that  $\|z\|_{\max} < \delta$ . Since  $\delta > \|z\|_{\min}$  was arbitrary, we conclude that  $\|z\|_{\max} \leq \|z\|_{\min}$ . Hence the inclusion  $E_1 \odot E_2 \rightarrow C^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H})$  is contractive, when  $E_1 \odot E_2$  is equipped with the minimal tensor norm. Let us show that it is in fact completely contractive. Since  $\mathcal{H}$  is infinite dimensional, we can find a  $*$ -isomorphism  $\psi : B(\mathcal{H}) \otimes \mathbb{M}_n \rightarrow B(\mathcal{H})$ . Consider the

following commutative diagram

$$\begin{array}{ccc}
M_n(E_1 \odot B(\mathcal{H})) & \longrightarrow & M_n(C^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H})) \\
\downarrow & & \downarrow \\
E_1 \odot B(\mathcal{H}) \odot M_n & \longrightarrow & C^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H}) \otimes_{\max} M_n \\
\downarrow \text{id}_{E_1} \otimes \psi & & \downarrow \text{id}_{C^*(\mathbb{F})} \otimes \psi \\
E_1 \odot B(\mathcal{H}) & \longrightarrow & C^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H})
\end{array}$$

where all the unmarked arrows are the obvious ones. We know that the vertical maps are all isometric, and we have proved that the bottom map is contractive. Hence the top map, that is, the  $n$ 'th inflation of the inclusion, must also be contractive. This finishes the argument that the inclusion of  $E_1 \odot B(\mathcal{H})$  (with the minimal tensor norm) into  $C^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H})$  is completely contractive. As explained earlier, this shows that

$$C^*(\mathbb{F}) \otimes_{\min} B(\mathcal{H}) = C^*(\mathbb{F}) \otimes_{\max} B(\mathcal{H}),$$

and we are done.  $\square$

With this grand conclusion we end this section, and hold our breath for its consequences, which are sure to come in the following section.

### 3.3 Tensorial characterizations

In this section, we give tensorial characterizations of the weak expectation property and the local lifting property. That is, we formulate these properties in terms of properties of tensor products; primarily, as statements about when the maximal and the minimal tensor products agree.

We start with a tensorial characterization of the property of being *relatively weakly injective*.

**Theorem 3.3.1.** *Suppose that  $\mathcal{B}$  is a  $C^*$ -algebra and  $\mathcal{A}$  a  $C^*$ -subalgebra of  $\mathcal{B}$ . Then the following are equivalent:*

- (i)  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$ ;
- (ii) for each representation  $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$  of  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$ , there exists a contractive completely positive map  $\phi: \mathcal{B} \rightarrow \pi(\mathcal{A})''$  extending  $\pi$ ;
- (iii) the inclusion  $\mathcal{A} \otimes_{\max} \mathcal{C} \rightarrow \mathcal{B} \otimes_{\max} \mathcal{C}$  is isometric for every  $C^*$ -algebra  $\mathcal{C}$ ;
- (iv) the inclusion  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty) \rightarrow \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_\infty)$  is isometric.

*Proof.* Suppose (i), and that  $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$  is a representation of  $\mathcal{A}$  on the Hilbert space  $\mathcal{H}$ . By Remark 1.4.5, the representation  $\pi$  extends to a representation  $\tilde{\pi}: \mathcal{A}^{**} \rightarrow \pi(\mathcal{A})''$ . Since  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$ , there exists a contractive completely positive map  $\Phi: \mathcal{B} \rightarrow \mathcal{A}^{**}$  which restricts to the identity on  $\mathcal{A}$ . Now, the map  $\tilde{\pi} \circ \Phi$  is a contractive completely positive map from  $\mathcal{B}$  to  $\pi(\mathcal{A})''$  extending  $\pi$ . Thus (ii) holds.

Condition (i) is just condition (ii) applied to the universal representation of  $\mathcal{A}$ , so clearly (i) implies (ii).

Suppose that (i) holds, and let us prove (iii). Let  $\mathcal{C}$  be a  $C^*$ -algebra, and choose some faithful representation  $\pi$  of  $\mathcal{A} \otimes_{\max} \mathcal{C}$  on a Hilbert space  $\mathcal{H}$ . Let

$$\pi_{\mathcal{A}}: \mathcal{A} \rightarrow B(\mathcal{H}) \quad \text{and} \quad \pi_{\mathcal{C}}: \mathcal{C} \rightarrow B(\mathcal{H})$$

be the restrictions of  $\pi$ . Then these two representation have commuting ranges, and  $\pi = \pi_{\mathcal{A}} \times \pi_{\mathcal{C}}$ . By Remark 1.4.5, we can extend the representation  $\pi_{\mathcal{A}}$  to a representation  $\tilde{\pi}_{\mathcal{A}}: \mathcal{A}^{**} \rightarrow \pi_{\mathcal{A}}(\mathcal{A})''$ . Since  $\pi_{\mathcal{C}}(\mathcal{C}) \subseteq \pi_{\mathcal{A}}(\mathcal{A})'$  we conclude that  $\tilde{\pi}_{\mathcal{A}}(\mathcal{A}^{**}) \subseteq \pi_{\mathcal{A}}(\mathcal{A})'' \subseteq \pi_{\mathcal{C}}(\mathcal{C})'$ . Hence  $\tilde{\pi}_{\mathcal{A}}$  and  $\pi_{\mathcal{C}}$  have commuting ranges, so we obtain a representation  $\tilde{\pi}_{\mathcal{A}} \times \pi_{\mathcal{C}}: \mathcal{A}^{**} \otimes_{\max} \mathcal{C} \rightarrow B(\mathcal{H})$ , which extends  $\pi_{\mathcal{A}} \times \pi_{\mathcal{C}}$ , since restrictions are unique. By assumption  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$ , so there exists a contractive completely positive map  $\Phi: \mathcal{B} \rightarrow \mathcal{A}^{**}$  which restricts to the identity on  $\mathcal{A}$ . So using this, we now have a contractive completely positive map  $\Phi \otimes \text{id}_{\mathcal{C}}: \mathcal{B} \otimes_{\max} \mathcal{C} \rightarrow \mathcal{A}^{**} \otimes_{\max} \mathcal{C}$ , which is the identity on  $\mathcal{A} \odot \mathcal{C}$ . To summarize we now have the commutative diagram

$$\begin{array}{ccc} \mathcal{B} \otimes_{\max} \mathcal{C} & \xrightarrow{\Phi \otimes \text{id}_{\mathcal{C}}} & \mathcal{A}^{**} \otimes_{\max} \mathcal{C} \\ \uparrow \iota & & \downarrow \tilde{\pi}_{\mathcal{A}} \times \pi_{\mathcal{C}} \\ \mathcal{A} \otimes_{\max} \mathcal{C} & \xrightarrow{\pi} & B(\mathcal{H}) \end{array}$$

where  $\iota$  denote the canonical map from  $\mathcal{A} \otimes_{\max} \mathcal{C}$  to  $\mathcal{B} \otimes_{\max} \mathcal{C}$ . Since  $\pi$  is injective, commutativity of the diagram implies that  $\iota$  is injective, and hence isometric. This proves (iii).

Clearly (iii) implies (iv), so assume (iv) and let us prove (i). Let  $\pi_u: \mathcal{A} \rightarrow B(\mathcal{H}_u)$  denote the universal representation of  $\mathcal{A}$ , and let  $I = \mathcal{U}(\pi_u(\mathcal{A})')$ . Consider the free group with generators  $I$ . There is a natural choice of a unitary representation of  $\mathbb{F}_I$  on  $\mathcal{H}_u$ , namely the one that maps a generator—which is a unitary operator on  $\mathcal{H}_u$ —to itself. By Proposition 3.1.1, this unitary representation extends uniquely to a  $*$ -homomorphism  $\rho: C^*(\mathbb{F}_I) \rightarrow \pi_u(\mathcal{A})'$ . Clearly  $\rho$  is surjective, since the image contains  $\mathcal{U}(\pi_u(\mathcal{A})')$ . By assumption, the map  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I) \rightarrow \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_I)$  is isometric, and so by Theorem 3.1.9 the map  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I) \rightarrow \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_I)$  is also isometric.

Assume first that the  $C^*$ -algebra  $\mathcal{A}$  is unital. Combining the maps  $\pi_u$  and  $\rho$ , whose ranges commute, we obtain a representation

$$\pi_u \times \rho: \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I) \rightarrow B(\mathcal{H}).$$

Since the norm on  $\mathcal{B} \otimes_{\max} C^*(\mathbb{F}_I)$  restricts to that of  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I)$  we get by Corollary B.3.7, that this  $*$ -homomorphism extends to a contractive completely positive map  $\phi: \mathcal{B} \otimes_{\max} C^*(\mathbb{F}_I) \rightarrow B(\mathcal{H})$ . Now, define a map

$$\Phi: \mathcal{B} \rightarrow B(\mathcal{H}) \quad \text{by} \quad \Phi(b) = \phi(b \otimes \mathbf{1}), \quad b \in \mathcal{B}.$$

Clearly this map is contractive completely positive map extending  $\pi$ , so all we need to check is that its image is actually contained in  $\mathcal{A}^{**}$ . Since  $\phi$  restricts to a  $*$ -homomorphism on  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I)$ , we deduce by Corollary 2.1.3 that  $\phi$  is an  $(\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_I))$ -bimodule map. So for  $b \in \mathcal{B}$  and  $x \in C^*(\mathbb{F}_I)$  we see that

$$\Phi(b)\rho(x) = \phi(b \otimes \mathbf{1})\phi(\mathbf{1}_{\mathcal{A}} \otimes x) = \phi(b \otimes x) = \phi(\mathbf{1}_{\mathcal{A}} \otimes x)\phi(b \otimes \mathbf{1}) = \rho(x)\Phi(b).$$

Hence  $\Phi(\mathcal{B}) \subseteq \rho(C^*(\mathbb{F}_I))' = \pi_u(\mathcal{A})'' = \mathcal{A}^{**}$ . This proves that  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$ .

Now, let us drop the assumption that  $\mathcal{A}$  is unital. By Proposition 1.1.8 we may extend the map  $\pi_u$  to a unital  $*$ -homomorphism  $\tilde{\pi}: \tilde{\mathcal{A}} \rightarrow \pi_u(\mathcal{A})''$ . By Lemma 3.1.12 the inclusion of  $\tilde{\mathcal{A}} \otimes_{\max} C^*(\mathbb{F}_I)$  into  $\tilde{\mathcal{B}} \otimes_{\max} C^*(\mathbb{F}_I)$  is also isometric, so by the same arguments as in the unital case, we obtain a contractive completely positive map  $\tilde{\Phi}: \tilde{\mathcal{B}} \rightarrow \pi_u(\mathcal{A})'' = \mathcal{A}^{**}$ , which restricts to  $\tilde{\pi}$  on  $\tilde{\mathcal{A}}$ . In particular,  $\tilde{\Phi}$  restricts to  $\pi$  on  $\mathcal{A}$ , since  $\tilde{\pi}$  extends  $\pi_u$ . Thus  $\tilde{\Phi}|_{\mathcal{B}}: \mathcal{B} \rightarrow \mathcal{A}^{**}$  is a contractive completely positive map which restricts to  $\pi_u$  on  $\mathcal{A}$ , that is,  $\mathcal{A}$  is relatively weakly injective in  $\mathcal{B}$ .  $\square$

With this theorem and the help of Theorem 3.2.6, we can prove the following characterization of the weak expectation property:

**Proposition 3.3.2.** *Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra. Then  $\mathcal{A}$  has the weak expectation property if and only if  $C^*(\mathbb{F}_\infty) \otimes_{\max} \mathcal{A} = C^*(\mathbb{F}_\infty) \otimes_{\min} \mathcal{A}$ .*

*Proof.* Let  $\mathcal{A}$  be a  $C^*$ -algebra. We may assume that  $\mathcal{A} \subseteq B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ . Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty) & \longrightarrow & B(\mathcal{H}) \otimes_{\max} C^*(\mathbb{F}_\infty) \\ \downarrow & & \downarrow \\ \mathcal{A} \otimes_{\min} C^*(\mathbb{F}_\infty) & \longrightarrow & B(\mathcal{H}) \otimes_{\min} C^*(\mathbb{F}_\infty) \end{array}$$

where all the maps are the natural ones. The bottom map is always isometric, and we know from Theorem 3.2.6 that the right map is also isometric. Hence, it follows that either both the two maps left—the top one and the left one—are isometric, or none of them are.

If we assume that  $C^*(\mathbb{F}_\infty) \otimes_{\max} \mathcal{A} = C^*(\mathbb{F}_\infty) \otimes_{\min} \mathcal{A}$ , then the left map, and hence also the top map, is isometric. By Theorem 3.3.1, this means that  $\mathcal{A}$  is relatively weakly injective in  $B(\mathcal{H})$ , that is,  $\mathcal{A}$  has the weak expectation property.

If we assume that  $\mathcal{A}$  has the weak expectation property, then by Proposition 2.2.6 we get that  $\mathcal{A}$  is relatively weakly injective in  $B(\mathcal{H})$ . Thus by Theorem 3.3.1, the top map is isometric, and it follows that left map is also isometric. Hence  $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty) = \mathcal{A} \otimes_{\min} C^*(\mathbb{F}_\infty)$ .  $\square$

This proposition describes in a nice way when a  $C^*$ -algebra has the weak expectation property. Seen in another light, it also gives a way to determine whether the two tensor products agree, if this is the point of interest.

Before we go on to prove a similar tensorial characterization of the local lifting property, we establish two propositions relating the local lifting property of a  $C^*$ -algebra to a statement which only involves maps into quotients of  $C^*(\mathbb{F}_\infty)$ .

**Proposition 3.3.3.** *A  $C^*$ -algebra  $\mathcal{A}$  has the lifting property if there exists a surjective  $*$ -homomorphism  $\pi: C^*(\mathbb{F}_\infty) \rightarrow \mathcal{A}$ , such that the identity map on  $\mathcal{A}$  is liftable.*

*Proof.* Suppose that  $\pi: C^*(\mathbb{F}_\infty) \rightarrow \mathcal{A}$  is such a  $*$ -homomorphism, and that  $\psi: \mathcal{A} \rightarrow C^*(\mathbb{F}_\infty)$  is a lift of the identity map on  $\mathcal{A}$ . Then we want to show that  $\mathcal{A}$  has the lifting property. Let  $\mathcal{B}$  be a  $C^*$ -algebra,  $\mathcal{I}$  a closed two-sided ideal in  $\mathcal{B}$  and  $\phi: \mathcal{A} \rightarrow \mathcal{B}/\mathcal{I}$  be a contractive completely positive map. The map  $\phi \circ \pi$  is contractive completely positive. Since  $C^*(\mathbb{F}_\infty)$  has the lifting property by Theorem 3.1.7, this map has a lift  $\psi'$ . Clearly now  $\psi' \circ \psi$  is a lift of  $\phi$ . Hence  $\mathcal{A}$  has the lifting property.  $\square$

The existence of such a surjective  $*$ -homomorphism in the above proposition clearly implies that  $\mathcal{A}$  is separable, and it is not hard to see that the reverse implication is also true, if we require that  $\mathcal{A}$  is separable. With the above result we can prove the following proposition:

**Proposition 3.3.4.** *A  $C^*$ -algebra  $\mathcal{A}$  has the local lifting property if and only if there exist a free group  $\mathbb{F}$  and a surjective  $*$ -homomorphism  $\pi: C^*(\mathbb{F}) \rightarrow \mathcal{A}$ , such that the identity map on  $\mathcal{A}$  is locally liftable.*

*Proof.* Suppose that  $\mathcal{A}$  has the local lifting property. Clearly, there exists a free group  $\mathbb{F}$ , such that  $C^*(\mathbb{F})$  maps surjectively onto  $\mathcal{A}$ , via a  $*$ -homomorphism. For example, one could take the free group with generators  $\mathcal{U}(\mathcal{A})$ . Since  $\mathcal{A}$  has the local lifting property, the identity map is clearly locally liftable with respect to this surjection.

Suppose now that there exist a free group  $\mathbb{F}$  and a surjective  $*$ -homomorphism  $\pi: C^*(\mathbb{F}) \rightarrow \mathcal{A}$ , such that the identity map on  $\mathcal{A}$  is locally liftable. Suppose further that  $\mathcal{B}$  is a  $C^*$ -algebra,  $\mathcal{I}$  a closed two-sided ideal in  $\mathcal{B}$  and  $\phi: \mathcal{A} \rightarrow \mathcal{B}/\mathcal{I}$  a contractive completely positive map. Let  $\mathcal{S} \subseteq \mathcal{A}$  be a finite dimensional operator system. Choose a lift  $\psi: \mathcal{S} \rightarrow C^*(\mathbb{F})$  of the inclusion  $\mathcal{S} \rightarrow \mathcal{A}$ . This is possible by assumption. Let  $\mathcal{S}' \subseteq C^*(\mathbb{F})$  be a finite dimensional operator system containing  $\psi(\mathcal{S})$ . The map  $\phi \circ \pi: C^*(\mathbb{F}) \rightarrow \mathcal{B}/\mathcal{I}$  is contractive completely positive, so since  $C^*(\mathbb{F})$  has the local lifting property by Corollary 3.1.8, we can choose a lift  $\psi': \mathcal{S}' \rightarrow \mathcal{B}$  of the map  $\phi \circ \pi|_{\mathcal{S}'}$ . Now the map  $\psi' \circ \psi: \mathcal{S} \rightarrow \mathcal{B}$  is a lift of  $\phi|_{\mathcal{S}}$ . Hence  $\mathcal{A}$  has the local lifting property.  $\square$

We are now ready to prove the following tensorial characterization of the local lifting property. This proposition is a consequence of the Effros-Haagerup Lifting Theorem and the theorem of Kirchberg.

**Proposition 3.3.5.** *Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra and  $\mathcal{H}$  an infinite dimensional Hilbert space. Then  $\mathcal{A}$  has the local lifting property if and only if  $B(\mathcal{H}) \otimes_{\max} \mathcal{A} = B(\mathcal{H}) \otimes_{\min} \mathcal{A}$ .*

*Proof.* Suppose that  $B(\mathcal{H}) \otimes_{\max} \mathcal{A} = B(\mathcal{H}) \otimes_{\min} \mathcal{A}$ . Since  $\mathcal{H}$  is infinite dimensional, we may choose some isometry  $V: \ell^2 \rightarrow \mathcal{H}$ , and this induces a faithful representation  $\pi: B(\ell^2) \rightarrow B(\mathcal{H})$ , given by  $\pi(x) = VxV^*$ . Since  $B(\ell^2)$  has the weak expectation property, the image of  $\pi$  is relatively weakly injective in  $B(\mathcal{H})$ , so by Theorem 3.3.1 we get that the map  $\pi \otimes \text{id}_{\mathcal{A}}: B(\ell^2) \otimes_{\max} \mathcal{A} \rightarrow B(\mathcal{H}) \otimes_{\max} \mathcal{A}$  is isometric. Hence  $B(\ell^2) \otimes_{\max} \mathcal{A} = B(\ell^2) \otimes_{\min} \mathcal{A}$ , since  $B(\mathcal{H}) \otimes_{\max} \mathcal{A} = B(\mathcal{H}) \otimes_{\min} \mathcal{A}$ . Choose some free group  $\mathbb{F}$  and a surjective  $*$ -homomorphism  $\pi: C^*(\mathbb{F}) \rightarrow \mathcal{A}$ . Then by Proposition 1.5.8 the sequence

$$0 \longrightarrow B(\ell^2) \otimes_{\min} \ker \pi \longrightarrow B(\ell^2) \otimes_{\min} C^*(\mathbb{F}) \longrightarrow B(\ell^2) \otimes_{\min} \mathcal{A} \longrightarrow 0$$

is exact. By the Effros-Haagerup Lifting Theorem (Theorem 2.4.15), the identity map on  $\mathcal{A}$  is locally liftable with respect to the surjective  $*$ -homomorphism  $\pi: C^*(\mathbb{F}) \rightarrow \mathcal{A}$ . By Proposition 3.3.4, this shows that  $\mathcal{A}$  has the local lifting property.

Now, suppose instead that  $\mathcal{A}$  has the local lifting property. By Proposition 3.3.4 there exist a free group  $\mathbb{F}$  and a surjective  $*$ -homomorphism  $\rho: C^*(\mathbb{F}) \rightarrow \mathcal{A}$ , such that the identity map on  $\mathcal{A}$  is locally liftable. By the Effros-Haagerup Lifting Theorem we get that the sequence

$$0 \longrightarrow B(\mathcal{H}) \otimes_{\min} \ker \rho \longrightarrow B(\mathcal{H}) \otimes_{\min} C^*(\mathbb{F}) \longrightarrow B(\mathcal{H}) \otimes_{\min} \mathcal{A} \longrightarrow 0$$



is exact. We also know from Proposition 1.5.7 that the sequence

$$0 \longrightarrow B(\mathcal{H}) \otimes_{\max} \ker \rho \longrightarrow B(\mathcal{H}) \otimes_{\max} C^*(\mathbb{F}) \longrightarrow B(\mathcal{H}) \otimes_{\max} \mathcal{A} \longrightarrow 0$$

is exact. Since  $B(\mathcal{H}) \otimes_{\max} C^*(\mathbb{F}) = B(\mathcal{H}) \otimes_{\min} C^*(\mathbb{F})$  by Theorem 3.2.6, we deduce, that  $B(\mathcal{H}) \otimes_{\max} \ker \rho = B(\mathcal{H}) \otimes_{\min} \ker \rho$ , and further that  $B(\mathcal{H}) \otimes_{\max} \mathcal{A} = B(\mathcal{H}) \otimes_{\min} \mathcal{A}$ . Hence the proposition is proved.  $\square$

Before we end this section, let us combine these two tensorial characterizations, giving a sufficient criteria for when there is a unique  $C^*$ -algebra norm on the algebraic tensor product of two  $C^*$ -algebras.

**Proposition 3.3.6.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be  $C^*$ -algebras. Then  $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$  if  $\mathcal{A}$  has the WEP and  $\mathcal{B}$  has the LLP.*

*Proof.* We may assume that  $\mathcal{A} \subseteq B(\mathcal{H})$  for an infinite dimensional Hilbert space  $\mathcal{H}$ . Since  $\mathcal{A}$  has the WEP, we get that  $\mathcal{A}$  is relatively weakly injective in  $B(\mathcal{H})$ , so by Theorem 3.3.1, the canonical map  $\mathcal{A} \otimes_{\max} \mathcal{B} \rightarrow B(\mathcal{H}) \otimes_{\max} \mathcal{B}$  is isometric. The canonical map  $\mathcal{A} \otimes_{\min} \mathcal{B} \rightarrow B(\mathcal{H}) \otimes_{\min} \mathcal{B}$  is also isometric, and  $B(\mathcal{H}) \otimes_{\max} \mathcal{B} = B(\mathcal{H}) \otimes_{\min} \mathcal{B}$ , since  $\mathcal{B}$  has the LLP. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{A} \otimes_{\max} \mathcal{B} & \longrightarrow & B(\mathcal{H}) \otimes_{\max} \mathcal{B} , \\ \downarrow & & \parallel \\ \mathcal{A} \otimes_{\min} \mathcal{B} & \longrightarrow & B(\mathcal{H}) \otimes_{\min} \mathcal{B} \end{array}$$

where all the maps are the obvious ones. Since we argued above that the horizontal maps are isometric, the map on the left must be isometric, as well. Hence  $\mathcal{A} \otimes_{\max} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$ .  $\square$

Let us make an easy remark on these tensorial characterizations. We saw earlier the Choi-Effros Lifting Theorem, which had the consequence that all separable nuclear  $C^*$ -algebras have the lifting property. Now, from Proposition 3.3.5 it follows that, in fact, all nuclear  $C^*$ -algebras—separable or not—has the local lifting property.<sup>3</sup> Also in the same manner, it follows from Proposition 3.3.2 that all nuclear  $C^*$ -algebras has the weak expectation property.

### 3.4 QWEP and The Connes Embedding Problem

In this section we establish that the QWEP conjecture is equivalent to an affirmative answer to the Connes Embedding Problem, as well as various other characterizations of the QWEP conjecture.

We have already done most of the work in the previous sections, so let us start with a small lemma, and then begin proving equivalences.

**Lemma 3.4.1.** *Suppose that  $\pi: \mathcal{A} \rightarrow B(\mathcal{K})$  is a faithful representation of a  $C^*$ -algebra  $\mathcal{A}$  on a Hilbert space  $\mathcal{K}$ . Let  $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$  be contractive completely positive, with  $\mathcal{H}$  a Hilbert space. Then there exists a contractive completely positive map  $\psi: B(\mathcal{K}) \rightarrow B(\mathcal{H})$  such that  $\psi \circ \pi = \phi$ .*

<sup>3</sup>This is using the tensorial characterization of nuclearity, as stated below Definition 2.4.11.

*Proof.* Since the representation  $\pi$  is faithful, we can define a contractive completely positive map  $\psi': \pi(\mathcal{A}) \rightarrow B(\mathcal{K})$  by  $\pi(x) \mapsto \phi(x)$ ,  $x \in \mathcal{A}$ . By Corollary B.3.7, the map  $\psi'$  extends to a contractive completely positive map  $\psi: B(\mathcal{H}) \rightarrow B(\mathcal{K})$ . By construction, this map  $\psi$  satisfies  $\psi \circ \pi = \phi$ .  $\square$

**Proposition 3.4.2.** *If a  $C^*$ -algebra is QWEP and has the LP, then it has the WEP.*

*Proof.* Figure 3.1 contains a commutative diagram illustrating the proof. Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra which is QWEP and has the LP. Since  $\mathcal{A}$  is QWEP there exist a  $C^*$ -algebra  $\mathcal{B}$  which has the WEP and a surjective  $*$ -homomorphism  $\pi: \mathcal{B} \rightarrow \mathcal{A}$ . Since  $\mathcal{B}$  has the WEP there exist a faithful representation  $\rho: \mathcal{B} \rightarrow B(\mathcal{H})$  on a Hilbert space  $\mathcal{H}$ , and a contractive completely positive map  $\Phi: B(\mathcal{H}) \rightarrow \mathcal{B}^{**}$  such that  $\Phi \circ \pi(b) = b$ , for all  $b \in \mathcal{B}$ . Let  $\hat{\rho}: \mathcal{A} \rightarrow B(\mathcal{K})$  be a faithful representation of  $\mathcal{A}$  on a Hilbert space  $\mathcal{K}$ . Now, because  $\mathcal{A}$  has the lifting property we can lift the identity map  $\mathcal{A} \rightarrow \mathcal{A}$  to a contractive completely positive map  $\phi: \mathcal{A} \rightarrow \mathcal{B}$  such that  $\pi \circ \phi(a) = a$ , for all  $a \in \mathcal{A}$ . The map  $\rho \circ \phi$  is contractive completely positive, so by Lemma 3.4.1 we can find a contractive completely positive map  $\psi: B(\mathcal{K}) \rightarrow B(\mathcal{H})$  such that  $\psi \circ \hat{\rho} = \rho \circ \phi$ . Now, the map  $\Psi: B(\mathcal{K}) \rightarrow \mathcal{A}^{**}$  given by  $\Psi = \pi^{**} \circ \Phi \circ \psi$  is clearly contractive completely positive, since it is the composition of such maps. It is straightforward to check that  $(\pi^{**} \circ \Phi \circ \psi \circ \hat{\rho})(a) = a$ , for all  $a \in \mathcal{A}$ , by construction. Thus  $\mathcal{A}$  has the WEP.  $\square$

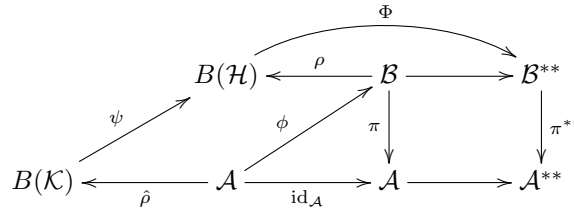


Figure 3.1: Commutative diagram illustrating the proof of Proposition 3.4.2

**Remark 3.4.3.** A thing definitely worth noticing about Proposition 3.4.2, is that in the proof we did not fully use the fact that  $\mathcal{A}$  had the LP. In fact, the only thing we used was that the identity map on  $\mathcal{A}$  is liftable with respect to some surjection onto  $\mathcal{A}$  of a  $C^*$ -algebra with the weak expectation property.

**Theorem 3.4.4.** *The following conjectures are equivalent:*

- (i) all  $C^*$ -algebras are QWEP;
- (ii) all separable  $C^*$ -algebras are QWEP;
- (iii) the  $C^*$ -algebra  $C^*(\mathbb{F}_\infty)$  is QWEP;
- (iv) LP implies WEP;
- (v) the  $C^*$ -algebra  $C^*(\mathbb{F}_\infty)$  is WEP;
- (vi)  $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$ .

*Proof.* Clearly (i)  $\implies$  (ii). We have (ii)  $\iff$  (iii) from Corollary 3.1.4. The implication (i)  $\implies$  (iv) follows from Proposition 3.4.2, and so does the implication (iii)  $\implies$  (v), since  $C^*(\mathbb{F}_\infty)$  has the lifting property by Theorem 3.1.7. The implication (v)  $\implies$  (iii) is clear, and the lifting property of  $C^*(\mathbb{F}_\infty)$  also shows that (iv)  $\implies$  (v). The equivalence (v)  $\iff$  (vi) is just Proposition 3.3.2, and adding up, one realizes that it now suffices to prove (vi)  $\implies$  (i). So suppose that (vi) holds, and let  $\mathcal{A}$  be any  $C^*$ -algebra. Choose a free group  $\mathbb{F}$  such that there exists a surjective  $*$ -homomorphism  $\pi: C^*(\mathbb{F}) \rightarrow \mathcal{A}$ . Then  $\mathcal{A}$  is a quotient of  $C^*(\mathbb{F})$ , and so it suffices to prove that  $C^*(\mathbb{F})$  has the WEP. By Theorem 3.1.9 we get that  $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F})$ , since  $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$  by assumption. According to Proposition 3.3.2, this precisely shows that  $C^*(\mathbb{F})$  has the WEP, and so we have proved (vi)  $\implies$  (i).  $\square$

**Proposition 3.4.5.** *The hyperfinite  $\text{II}_1$ -factor  $\mathcal{R}$  is QWEP, and so are all tracial ultrapowers of it.*

*Proof.* It follows from Proposition 2.3.9 that  $\mathcal{R}$  is QWEP, since it contains an increasing sequence of finite dimensional factors, whose union is weak operator dense in  $\mathcal{M}$  by definition being approximately finite. By Proposition 2.3.5, all ultrapowers of  $\mathcal{R}$  are QWEP.  $\square$

Next thing, we state a deep theorem of Tomita-Takesaki theory (see [BO08, Theorem 9.3.5 & Lemma 9.3.6] and references therein). We state without a reference to Tomita-Takesaki theory, but formulate it as the exact statement we need.

**Theorem 3.4.6.** *Every countably decomposable von Neumann algebra  $\mathcal{N}$  embeds into a semi-finite and countably decomposable von Neumann algebra  $\mathcal{M}$ , in such a way that there exists a conditional expectation from  $\mathcal{M}$  onto  $\mathcal{N}$ .*

Now we prove the theorem, which relates the QWEP Conjecture to the Connes Embedding Problem.

**Theorem 3.4.7.** *The following conjectures are equivalent:*

- (i) *every von Neumann algebra  $\text{II}_1$ -factor with separable predual embed into  $\mathcal{R}^\omega$ , for some choice of free ultrafilter  $\omega$  on  $\mathbb{N}$ ;*
- (ii) *all  $C^*$ -algebras are QWEP.*

*Proof.* First, note that from Proposition 3.4.5 we know that ultrapowers of  $\mathcal{R}^\omega$  are QWEP, and by Corollary 2.3.3 we get that finite von Neumann algebras that embeds into ultrapowers of  $\mathcal{R}$  are QWEP, as well.

Suppose first that every von Neumann algebra  $\text{II}_1$ -factor with separable predual embeds into  $\mathcal{R}^\omega$ , for some choice of a free ultrafilter  $\omega$  on  $\mathbb{N}$ . We start by proving that all von Neumann algebras are QWEP.

Assume that  $\mathcal{M}$  is a finite von Neumann algebra with separable predual. By Theorem A.1.5 we know that  $\mathcal{M}$  embeds into a  $\text{II}_1$ -factor with separable predual. The latter embeds into an ultrapower of  $\mathcal{R}$  by assumption, and is therefore QWEP, by the first part of the proof. Hence all finite von Neumann algebras with separable predual are QWEP.

Assume now that  $\mathcal{M}$  is a finite and countably decomposable von Neumann algebra. Let  $(\mathcal{M}_\alpha)_{\alpha \in A}$  be the net of finitely generated von Neumann subalgebras of  $\mathcal{M}$ , directed by inclusion. Let  $\alpha \in A$ . Clearly  $\mathcal{M}_\alpha$  is finite and countably decomposable,

since  $\mathcal{M}$  is finite and countably decomposable, and since  $\mathcal{M}_\alpha$  is also finitely generated, we get by Theorem 1.3.11 that  $\mathcal{M}_\alpha$  has separable predual. Hence, we know from above, that  $\mathcal{M}_\alpha$  is QWEP. Clearly  $\mathcal{M} = \bigcup_{\alpha \in A} \mathcal{M}_\alpha$ , since each  $x \in \mathcal{M}$  generates a finitely generated von Neumann subalgebra of  $\mathcal{M}$ , so by Proposition 2.3.9 we get that  $(\bigcup_{\alpha \in A} \mathcal{M}_\alpha)'' = \mathcal{M}'' = \mathcal{M}$  is QWEP.

Suppose that  $\mathcal{M}$  is a semi-finite and countably decomposable von Neumann algebra. We know that all von Neumann subalgebras of  $\mathcal{M}$  are countably decomposable, since  $\mathcal{M}$  is. From Proposition 1.2.6, we know that  $\mathcal{M}$  is the strong operator closure of an increasing union of finite von Neumann algebras. Since the latter are QWEP, we get again by Proposition 2.3.9, that  $\mathcal{M}$  is QWEP.

Suppose that  $\mathcal{M}$  is a countably decomposable von Neumann algebra. By Theorem 3.4.6 we know that  $\mathcal{M}$  embeds into a semi-finite and countably decomposable von Neumann algebra  $\mathcal{N}$ , in such a way that there exists a conditional expectation from  $\mathcal{N}$  onto  $\mathcal{M}$ . In particular,  $\mathcal{M}$  is relatively weakly injective in  $\mathcal{N}$ . We know from above that  $\mathcal{N}$  is QWEP since it is semi-finite, and so by Proposition 2.3.2 we conclude that  $\mathcal{M}$  is QWEP.

Last, assume that  $\mathcal{M}$  is just any von Neumann algebra. By Proposition 1.3.7  $\mathcal{M}$  is the strong operator closure of an increasing union of countably decomposable von Neumann subalgebras, so since we by now know that these are QWEP, we get by Proposition 2.3.9 that  $\mathcal{M}$  is QWEP.

Now we have proved that all von Neumann algebras are QWEP, but this also shows that all  $C^*$ -algebras are QWEP, since a  $C^*$ -algebra is QWEP if and only if its double dual is QWEP, by Corollary 2.3.10.

This proves one implication, so let us prove the reverse one. Assume therefore that all  $C^*$ -algebras are QWEP. In particular,  $C^*(\mathbb{F}_\infty)$  is QWEP, and by Proposition 3.4.2 it also has the WEP. Let  $\mathcal{M}$  be a  $\text{II}_1$ -factor with separable predual, and let  $\tau$  denote the tracial state on  $\mathcal{M}$ . By Theorem 1.3.11 and Proposition 3.1.3, we may choose some  $*$ -homomorphism  $\pi: C^*(\mathbb{F}_\infty) \rightarrow \mathcal{M}$ , whose range is weak operator dense in  $\mathcal{M}$ . The linear functional  $\tau \circ \pi$  is a trace on  $C^*(\mathbb{F}_\infty)$ . Choose some faithful representation  $\rho: C^*(\mathbb{F}_\infty) \rightarrow B(\mathcal{H})$  on a Hilbert space  $\mathcal{H}$ , and consider the map  $\phi: \rho(C^*(\mathbb{F}_\infty)) \rightarrow \mathcal{M}$  defined by  $\phi(x) = \pi(x)$ ,  $x \in C^*(\mathbb{F}_\infty)$ , which by the way is well-defined since  $\rho$  is faithful. If we think of  $\phi$  as a representation of  $\rho(C^*(\mathbb{F}_\infty))$  on a Hilbert space, then, since  $C^*(\mathbb{F}_\infty)$  has the WEP, this map extends to a contractive completely positive map  $\Phi: B(\mathcal{H}) \rightarrow \mathcal{M}'' = \mathcal{M}$ , by Theorem 3.3.1. Now, by construction  $\tau \circ \Phi$  is a state on  $B(\mathcal{H})$  satisfying  $\tau \circ \Phi \circ \rho = \tau \circ \pi$ , so by Corollary A.2.3 we get a trace-preserving  $*$ -homomorphism from  $C^*(\mathbb{F}_\infty)$  into  $\mathcal{R}^\omega$ , for some free ultrafilter  $\omega$  on  $\mathbb{N}$ . Let  $\psi$  denote this map, and let  $\tau_\omega$  denote the trace on  $\mathcal{R}^\omega$ . Now, define a map  $\psi': \pi(C^*(\mathbb{F}_\infty)) \rightarrow \mathcal{R}^\omega$  by  $\psi'(x) = \psi(x)$ , and let us argue why this is well-defined. We know that  $\tau_\omega \circ \psi = \tau \circ \pi$ , and therefore  $\psi'$  is well-defined, since it preserves a faithful trace. In particular, it is also injective, and it extends to an embedding  $\tilde{\psi}: \mathcal{M} \rightarrow \mathcal{R}^\omega$ , since  $\pi(C^*(\mathbb{F}_\infty))$  is weak operator dense in  $\mathcal{M}$ . This proves the reverse implication.  $\square$

At this point we have proved that an affirmative answer to the Connes Embedding Problem is equivalent to the QWEP Conjecture. Thus concluding this chapter.

## Chapter 4

# Ultraproducts

There are several different notions of ultraproducts. In this thesis we will only consider the metric ultraproduct of groups and the tracial ultraproduct of  $C^*$ -algebras. There are two other notions of ultraproducts that are natural to mention, namely the algebraic ultraproduct of groups and the metric ultraproduct of  $C^*$ -algebras. The construction of these last two notions of ultraproducts are explained quickly as remarks the end of section 4.1 and section 4.2, respectively.

### 4.1 Metric ultraproduct of groups

The metric ultraproduct of groups only makes sense for metric groups with bi-invariant metrics, so we start by introducing these concepts.

**Definition 4.1.1.** A **metric group** is a group  $G$  together with a metric  $d$  on  $G$ . The metric  $d$  is called **bi-invariant** if

$$d(gh, gk) = d(h, k) = d(hg, kg)$$

for all  $g, h, k \in G$ . ◀

All discrete groups are of course metric groups with the discrete metric, which is also bi-invariant. Besides this, there are more interesting examples. One example that will be of interest in this thesis is the finite rank unitary group  $\mathcal{U}(n)$ ,  $n \in \mathbb{N}$ , that is, the unitary group of  $\mathbb{M}_n$ . We equip  $\mathcal{U}(n)$  with the **Hilbert-Schmidt distance**, which is defined by

$$d_{HS}(u, v) = \|u - v\|_2 = \text{tr}_n((u - v)^*(u - v)).$$

Clearly the Hilbert-Schmidt distance is bi-invariant on  $\mathcal{U}(n)$ .

Now, let us introduce the metric ultraproduct of groups. Suppose that  $(G_i)_{i \in I}$  is a family of metric groups with bi-invariant metrics. Let  $d_i$  and  $\mathbf{1}_i$  denote the metric on  $G_i$  and the neutral element in  $G_i$ , respectively. Fix some ultrafilter  $\omega$  on  $I$ . Consider the set

$$N_\omega = \left\{ (g_i)_{i \in I} \in \ell_\infty(I; G_i) : \lim_{i \rightarrow \omega} d_i(g_i, \mathbf{1}_i) = 0 \right\}.$$

Let us show that  $N_\omega$  is a normal subgroup of  $\ell_\infty(I; G_i)$ . If  $(g_i)_{i \in I}, (h_i)_{i \in I} \in N_\omega$ , then using that  $d_i$  is bi-invariant for all  $i \in I$ , we get

$$\begin{aligned} \lim_{i \rightarrow \omega} d_i(g_i h_i^{-1}, \mathbf{1}_i) &\leq \lim_{i \rightarrow \omega} d_i(g_i h_i^{-1}, h_i^{-1}) + d_i(h_i^{-1}, \mathbf{1}_i) \\ &= \lim_{i \rightarrow \omega} d_i(g_i, \mathbf{1}_i) + d_i(\mathbf{1}_i, h_i) \\ &= 0. \end{aligned}$$

Thus  $gh^{-1} \in H_\omega$ , which shows that  $N_\omega$  is a subgroup. Now if  $(g_i)_{i \in I} \in N_\omega$  and  $(h_i)_{i \in I} \in \ell_\infty(I; G_i)$ , then

$$\lim_{i \rightarrow \omega} d_i(h_i g_i h_i^{-1}, \mathbf{1}_i) = \lim_{i \rightarrow \omega} d_i(g_i, h_i^{-1} h_i) = \lim_{i \rightarrow \omega} d_i(g_i, \mathbf{1}_i) = 0.$$

which shows that  $hgh^{-1} \in N_\omega$ , so  $N_\omega$  is a normal subgroup.

**Definition 4.1.2.** With the notation above, the quotient of  $\ell_\infty(I; G_i)$  by the normal subgroup  $N_\omega$  is called the **metric ultraproduct** of the groups  $(G_i)_{i \in I}$ , and it is denoted by  $\prod_{i \in I}^\omega G_i$ . In the case when  $G_i = G$  for all  $i \in I$  and some fixed metric group  $G$ , the ultraproduct  $\prod_{i \in I}^\omega G_i$  is called the **ultrapower**, and it is denoted by  $G^\omega$ . ◀

**Proposition 4.1.3.** *The metric ultraproduct  $\prod_{i \in I}^\omega G_i$ , of a family of metric groups  $(G_i)_{i \in I}$  with bi-invariant metrics is a metric group equipped with the bi-invariant metric  $d_\omega$  defined by*

$$d_\omega([(g_i)_{i \in I}], [(h_i)_{i \in I}]) = \lim_{i \rightarrow \omega} d_i(g_i, h_i),$$

for all  $[(g_i)_{i \in I}], [(h_i)_{i \in I}] \in \prod_{i \in I}^\omega G_i$ , where  $d_i$  denotes the metric on  $G_i$ ,  $i \in I$ .

*Proof.* First let us show that  $d_\omega$  is well-defined. Suppose that  $(g_i)_{i \in I}, (h_i)_{i \in I} \in \ell_\infty(I; G_i)$  and  $(n_i)_{i \in I} \in N_\omega$ , then we need to show that

$$\lim_{i \rightarrow \omega} d_i(g_i, h_i) = \lim_{i \rightarrow \omega} d_i(g_i, h_i n_i) = \lim_{i \rightarrow \omega} d_i(g_i n_i, h_i).$$

It suffices to prove the first equality, since the second follows from the first by replacing  $(n_i)_{i \in I}$  with  $(n_i^{-1})_{i \in I}$  and using that  $d_i$  is bi-invariant for all  $i \in I$ . Since  $(n_i)_{i \in I} \in N_\omega$  we have  $\lim_{i \rightarrow \omega} d_i(n_i, \mathbf{1}_i) = 0$  and therefore

$$\begin{aligned} \lim_{i \rightarrow \omega} d_i(g_i, h_i n_i) &\leq \lim_{i \rightarrow \omega} d_i(g_i, h_i) + d_i(h_i, h_i n_i) \\ &= \lim_{i \rightarrow \omega} d_i(g_i, h_i) + d_i(n_i, \mathbf{1}_i) = \lim_{i \rightarrow \omega} d_i(g_i, h_i). \end{aligned}$$

Now, since also  $(n_i^{-1})_{i \in I}$  is in  $N_\omega$ , the same argument shows that

$$\lim_{i \rightarrow \omega} d_i(g_i, h_i) = \lim_{i \rightarrow \omega} d_i(g_i, h_i n_i n_i^{-1}) \leq \lim_{i \rightarrow \omega} d_i(g_i, h_i n_i).$$

Thus  $d_\omega$  is well-defined. The fact that  $d_\omega$  is a bi-invariant metric follows simply from the fact that  $d_i$  is a bi-invariant metric for each  $i \in I$ . ◻

Unless otherwise specified, when  $\prod_{i \in I}^\omega G_i$  is referred to as a metric group,  $d_\omega$  is the metric in question.

It is straightforward to check that if  $\omega$  is a principal ultrafilter, based at  $i$  say, then  $\prod_{i \in I}^{\omega} G_i$  is isometrically isomorphic to  $G_i$ . Thus, the case when  $\omega$  is principal is not that interesting.

If we are given two families  $(G_i)_{i \in I}$  and  $(H_i)_{i \in I}$  of metric groups with bi-invariant metrics such that  $G_i$  is isometrically isomorphic to  $H_i$  for all  $i \in I$ —in fact, it only need to be the case for all  $i$  in some set  $F \in \omega$ —then it is not difficult to see that  $\prod_{i \in I}^{\omega} G_i$  is isometrically isomorphic to  $\prod_{i \in I}^{\omega} H_i$ . More generally, under certain conditions one can obtain a group homomorphism between metric ultraproducts, if one has a family of group homomorphisms between the groups. Some sufficient conditions are specified in the following proposition, which is straightforward to check:

**Proposition 4.1.4.** *Let  $I$  be an index set and  $\omega$  an ultrafilter on  $I$ . Suppose for each  $i \in I$  that  $(G_i, d_{G_i})$  and  $(H_i, d_{H_i})$  are metric groups and  $\pi_i: G_i \rightarrow H_i$  a group homomorphism. If there exists a bounded set  $(C_i)_{i \in I}$  of non-negative real numbers such that  $d_{H_i}(\pi_i(g_i), \pi_i(g'_i)) \leq C_i \cdot d_{G_i}(g_i, g'_i)$  for each  $g_i, g'_i \in G_i$  and  $i \in I$  then the map  $\prod_{i \in I}^{\omega} G_i \rightarrow \prod_{i \in I}^{\omega} H_i$  defined by  $[(g_i)_{i \in I}] \mapsto [(\pi_i(g_i))_{i \in I}]$ , is a well-defined homomorphism. If, in addition, there exists a bounded set  $(c_i)_{i \in I}$  of non-negative real numbers such that  $d_{G_i}(g_i, g'_i) \leq c_i \cdot d_{H_i}(\pi_i(g_i), \pi_i(g'_i))$  for each  $i \in I$  and  $g_i, g'_i \in G_i$ , then the given map is injective.*

*In fact the maps  $(\pi_i)_{i \in I}$  need not be homomorphisms, but just asymptotically multiplicative and unital in the sense that  $\lim_{i \rightarrow \omega} d_{H_i}(\pi_i(g_i g'_i), \pi_i(g_i) \pi_i(g'_i)) = 0$  and  $\lim_{i \rightarrow \omega} d_{H_i}(\pi_i(\mathbf{1}_{G_i}), \mathbf{1}_{H_i}) = 0$  for all  $[(g_i)_{i \in I}], [(g'_i)_{i \in I}] \in \prod_{i \in I}^{\omega} G_i$ .*

**Remark 4.1.5.** So this was the basics on the metric ultraproduct of groups. There is also a construction called the algebraic ultraproduct of groups. Let us just quickly go through how this construction is done.

Suppose that we are given a family of groups  $(G_i)_{i \in I}$  and an ultrafilter  $\omega$  on  $I$ . Let  $G$  denote the direct sum  $\prod_{i \in I} G_i$ , and let  $N$  denote the subset of  $G$  given by

$$N = \left\{ (g_i)_{i \in I} \in \prod_{i \in I} G_i : \{i \in I : g_i = \mathbf{1}_{G_i}\} \in \omega \right\}.$$

Clearly  $N$  is a normal subgroup of  $G$ , and the quotient  $G/N$  is called the algebraic ultraproduct of the groups  $(G_i)_{i \in I}$ . It is not hard to realize that the **algebraic ultraproduct** of the groups  $(G_i)_{i \in I}$  is just the metric ultraproduct  $\prod_{i \in I}^{\omega} G_i$  when each  $G_i$  is equipped with the discrete metric. ◀

## 4.2 Tracial ultraproduct of $C^*$ -algebras

Suppose that we are given an index set  $I$  and an ultrafilter  $\omega$  on  $I$ . Suppose also that we are given a family of  $C^*$ -algebras  $(\mathcal{A}_i)_{i \in I}$  and for each  $i \in I$  a trace  $\tau_i$  is on  $\mathcal{A}_i$ . Further, for each  $i \in I$  we let  $\|\cdot\|_{\tau_i}$  denote the trace semi-norm on  $\mathcal{A}_i$  associated with  $\tau_i$ . It is straightforward to check that the set  $\tau((x_i)_{i \in I}) = \lim_{i \rightarrow \omega} \tau_i(x_i)$  defines a trace on  $\mathcal{A}$ , which is well-defined since  $\omega$  is an ultrafilter. Moreover

$$\mathcal{I} = \left\{ (x_i)_{i \in I} \in \mathcal{A} : \lim_{i \rightarrow \omega} \|x_i\|_{\tau_i} = 0 \right\}$$

is a closed two-sided ideal in  $\mathcal{A}$ . The quotient  $\mathcal{A}/\mathcal{I}$  is of course a  $C^*$ -algebra, and  $\tau$  induces a faithful trace  $\tau_{\omega}$  on  $\mathcal{A}/\mathcal{I}$  given by  $\tau_{\omega}([(x_i)_{i \in I}]) = \tau((x_i)_{i \in I})$ . Let  $(\pi, \mathcal{H}, \xi)$  denote the GNS representation of  $\mathcal{A}/\mathcal{I}$  corresponding to  $\tau_{\omega}$

**Definition 4.2.1.** With the notation above, the image  $C^*$ -algebra  $\pi(\mathcal{A}/\mathcal{I})$  is called the **tracial ultraproduct** of the  $C^*$ -algebras, and it is denoted by  $\prod_{i \in I}^{\omega} \mathcal{A}_i$ . In the case when  $\mathcal{A}_i = \mathcal{A}$ , for all  $i \in I$ , and some fixed  $C^*$ -algebra  $\mathcal{A}$ , the ultraproduct  $\prod_{i \in I}^{\omega} \mathcal{A}$  is called the **tracial ultrapower** of  $\mathcal{A}$ , and it is denoted by  $\mathcal{A}^{\omega}$ . ◀

In the following, when considering tracial ultraproducts, we will not distinguish between  $\mathcal{A}/\mathcal{I}$  and  $\pi(\mathcal{A}/\mathcal{I})$ , and therefore omit the representation  $\pi$ .

As in the case of metric ultraproduct of groups, the construction of the tracial ultraproduct is quite simple, but in contrast to the metric ultraproduct of groups, there is a lot of work left. Namely, because the main reason for using the tracial ultraproduct, as opposed to the metric ultraproduct of  $C^*$ -algebras explained in Remark 4.2.9, is that in many cases the tracial ultraproduct turns out to be a von Neumann algebra. Before we are ready to prove this, we need an intermediate result characterizing von Neumann algebras, see Corollary 4.2.3.

The proof of the following theorem is taken from [Haa91, Proposition 3.10] in which Uffe Haagerup proves a more general result in the setting of  $C^*$ -algebras with quasi-traces.

**Theorem 4.2.2.** *Suppose that  $\mathcal{M}$  is a von Neumann algebra with a faithful normal tracial state  $\tau$ . Then the closed unit ball  $\mathcal{M}_1$  of  $\mathcal{M}$  is complete in the norm induced by  $\tau$ .*

*Proof.* Let us start by showing that the unitary group  $\mathcal{U}(\mathcal{M})$  is complete in the trace norm, which we denote by  $\|\cdot\|_2$ . So let  $(u_n)_{n \in \mathbb{N}}$  be a Cauchy sequence of unitaries (in the trace norm). It suffices to prove that this sequence has a convergent subsequence, so by passing to a subsequence we can ensure that  $\|u_n - u_{n+1}\|_2 < 4^{-n}$ . Let  $e_n$  denote the projection  $1_{[0, 2^{-n}]}(|u_n - u_{n+1}|)$ , then  $\|(u_n - u_{n+1})e_n\| \leq 2^{-n}$  since  $\sup\{|z|1_{[0, 2^{-n}]}(|z|)| : z \in \mathbb{C}\} = 2^{-n}$ . Likewise  $\|\mathbf{1} - e_n\| \leq 2^n \|u_n - u_{n+1}\|$  since  $1 - 1_{[0, 2^{-n}]}(|z|) \leq 2^n |z|$  for all  $z \in \mathbb{C}$ . In particular

$$\tau(\mathbf{1} - e_n) \leq 2^n \tau(|u_n - u_{n+1}|) = 2^n \|\mathbf{1}\|_2 \|u_n - u_{n+1}\|_2 < 2^{-n}$$

for all  $n \in \mathbb{N}$ . Now fix  $n \in \mathbb{N}$ , and let  $f_n$  denote the greatest lower bound of the projections  $e_n, e_{n+1}, e_{n+2}, \dots$ , that is,  $f_n = \bigwedge_{k \geq n} e_k$ . Then  $\mathbf{1} - f_n$  is the least upper bound of the projections  $\mathbf{1} - e_n, \mathbf{1} - e_{n+1}, \mathbf{1} - e_{n+2}, \dots$ , so since  $\tau$  is normal we get

$$\tau(\mathbf{1} - f_n) = \sup_{k \geq n} \tau(\mathbf{1} - e_k) \leq \sum_{k \geq n} \tau(\mathbf{1} - e_k) < \sum_{k \geq n} 2^{-k} = 2^{1-n}.$$

Now for all  $k \geq n$  we have that  $f_n \leq e_k$ , so

$$(u_k - u_{k+1})f_n(u_k - u_{k+1})^* \leq (u_k - u_{k+1})e_k(u_k - u_{k+1})^*.$$

This shows that  $\|(u_k - u_{k+1})f_n\| \leq \|(u_k - u_{k+1})e_k\| < 2^{-k}$ . Hence the sum  $\sum_{k=n}^{\infty} (u_k - u_{k+1})f_n$  is absolutely convergent. In particular it is convergent, and so the sequence  $u_k f_n$  has a limit as  $k \rightarrow \infty$  in the norm topology. Let  $v_n = \lim_{k \rightarrow \infty} u_k f_n$ . Since  $v_n^* v_n = \lim_{k \rightarrow \infty} f_n u_k^* u_k f_n = f_n$  we know that  $v_n$  is a partial isometry. We denote the projection  $v_n v_n^*$  by  $q_n$ . Let  $v_0 = f_0 = q_0 = 0$ . Clearly  $f_0 \leq f_1 \leq f_2 \leq \dots$ , so for  $n, m \in \mathbb{Z}$  with  $n \geq m \geq 0$  we have

$$v_n f_m = \lim_{k \rightarrow \infty} u_k f_n f_m = \lim_{k \rightarrow \infty} u_k f_m = v_m.$$



From this it follows that  $q_1 \leq q_2 \leq q_3 \leq \dots$ , since  $q_m = v_m v_m^* = v_n f_m v_n^* \leq v_n f_n v_n^* = q_n$  when  $m \leq n$ . We also have for  $n, m \geq 0$  that

$$v_n^* v_m = \lim_{k \rightarrow \infty} f_n u_k^* u_k f_m = f_n f_m = f_{\min\{n, m\}}$$

and in the same way that  $v_n v_m^* = q_{\min\{n, m\}}$ . From this it follows that with  $w_n = v_n - v_{n-1}$  we have  $w_n^* w_n = f_n - f_{n-1}$  and  $w_n w_n^* = q_n - q_{n-1}$  for all  $n \geq 1$ . Thus the projections  $f_n - f_{n-1}$  and  $q_n - q_{n-1}$  are Murray-von Neumann equivalent for all  $n \in \mathbb{N}$ . By [KR86, Proposition 6.2.2] and proof hereof the projections  $\sum_{n=1}^{\infty} (f_n - f_{n-1})$  and  $\sum_{n=1}^{\infty} (q_n - q_{n-1})$  are Murray-von Neumann equivalent, and the sum  $\sum_{n=1}^{\infty} w_n$  is strong operator convergent to a partial isometry  $w$  with  $w^* w = \sum_{n=1}^{\infty} (f_n - f_{n-1})$  and  $w w^* = \sum_{n=1}^{\infty} (q_n - q_{n-1})$ . Then  $\mathbf{1} - w^* w$  is the strong operator limit as  $n \rightarrow \infty$  of the sequence  $\mathbf{1} - \sum_{n=1}^k (f_n - f_{n-1}) = \mathbf{1} - f_n$ . Since  $\tau$  is normal, it follows that

$$\tau(\mathbf{1} - w^* w) = \lim_{n \rightarrow \infty} \tau(\mathbf{1} - f_n) \leq \lim_{n \rightarrow \infty} 2^{1-n} = 0.$$

Thus  $w^* w = \mathbf{1}$  since  $\tau$  is faithful, and likewise,  $w w^* = \mathbf{1}$  since  $\tau(\mathbf{1} - w w^*) = \tau(\mathbf{1} - w^* w) = 0$ , so  $w$  is a unitary. Because the product is strong operator continuous in each variable separately it follows that  $w(f_n - f_{n-1}) = w_n$ , for each  $n \in \mathbb{N}$ , and since  $v_n = \sum_{k=1}^n w_k$  we get that  $v_n = \sum_{k=1}^n w(f_k - f_{k-1}) = w f_n$ . Hence by definition of  $v_n$  we get  $\lim_{k \rightarrow \infty} \|(u_k - w) f_n\| = 0$ , and in particular  $\lim_{k \rightarrow \infty} \|(u_k - w) f_n\|_2 = 0$ . Now it follows that

$$\begin{aligned} \|u_k - w\|_2 &\leq \|(u_k - w) f_n\|_2 + \|(u_k - w)(\mathbf{1} - f_n)\|_2 \\ &\leq \|(u_k - w) f_n\|_2 + 2\|\mathbf{1} - f_n\|_2 \\ &< \|(u_k - w) f_n\|_2 + 2^{2-n}, \end{aligned}$$

for all  $n \in \mathbb{N}$ , and therefore that  $\lim_{k \rightarrow \infty} \|u_k - w\|_2 \leq 2^{2-n}$ , for all  $n \in \mathbb{N}$ . Thus  $\lim_{k \rightarrow \infty} \|u_k - w\|_2 = 0$ , which shows that  $\mathcal{U}(\mathcal{M})$  is complete in the trace norm.

Now let us prove that the set  $(\mathcal{M}_{sa})_1$  of self-adjoint elements in  $\mathcal{M}$  of norm less than or equal to 1 is also complete in the trace norm. So suppose that  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $(\mathcal{M}_{sa})_1$  with respect to the trace norm. For  $n \in \mathbb{N}$  let  $u_n$  denote the Cayley transform of  $x_n$ , that is,  $u_n = (x_n + i\mathbf{1})(x_n - i\mathbf{1})^{-1}$ . Since  $x_n$  is self-adjoint, the continuous functional calculus tells us that  $\|(x_n - i\mathbf{1})^{-1}\| \leq 1$  for each  $n \in \mathbb{N}$ , and in particular  $\|(x_n - i\mathbf{1})^{-1}\|_2 \leq 1$  for each  $n \in \mathbb{N}$ . Thus  $\|u_n - u_m\|_2 \leq 2\|x_m - x_n\|_2$  for each  $n, m \in \mathbb{N}$  since

$$u_n - u_m = 2(x_n - i\mathbf{1})^{-1}(x_m - x_n)(x_m - i\mathbf{1})^{-1}.$$

This shows that  $(u_n)_{n \in \mathbb{N}}$  is a Cauchy sequence as well, with respect to the trace norm, that is. Let  $u$  denote the limit of this sequence, which exists by the previous part of the proof. By the spectral mapping theorem we know that  $\sigma(u_n) \subseteq \{z \in \mathbb{T} : \operatorname{Re} z \leq 0\}$ , since  $\sigma(x_n) \subseteq [-1, 1]$ . Because

$$\{z \in \mathbb{T} : \operatorname{Re} z \leq 0\} = \{z \in \mathbb{T} : |1 + z| \leq \sqrt{2}\}$$

we get by the spectral mapping theorem that  $\|\mathbf{1} + u_n\| \leq \sqrt{2}$  for all  $n \in \mathbb{N}$ , so since the closed balls is also closed in the trace norm, we get  $\|\mathbf{1} + u\| \leq \sqrt{2}$ . Thus it follows from the equality above that  $\sigma(u) \subseteq \{z \in \mathbb{T} : \operatorname{Re} z \leq 0\}$ , and so  $\|(u -$

$\mathbf{1})^{-1}\| \leq (\sqrt{2})^{-1}$ . Now if we let  $x$  denote the inverse Cayley transform of  $u$ , that is  $x = i(u + \mathbf{1})(u - \mathbf{1})^{-1}$ , then since

$$x_n - x = 2i(u_n - \mathbf{1})^{-1}(u - u_n)(u - \mathbf{1})^{-1}.$$

we get that  $\|x_n - x\|_2 \leq \|u - u_n\|_2$ . Thus  $x_n \rightarrow x$  as  $n \rightarrow \infty$  in the trace norm. Clearly  $\|x\|_2 \leq 1$  since the closed unit ball of  $\mathcal{M}$  is closed and  $\|x_n\| \leq 1$  for all  $n \in \mathbb{N}$ . Thus we have proved the completeness of  $(\mathcal{M}_{sa})_1$ .

Now suppose that  $(y_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $(\mathcal{M})_1$  with respect to the trace norm. Then it is straight forward to check that  $\operatorname{Re} y_n$  and  $\operatorname{Im} y_n$  are both Cauchy sequences in  $(\mathcal{M})_1$  with respect to the trace norm, and hence convergent to some elements  $a$  and  $b$  in  $(\mathcal{M}_{sa})_1$ , respectively. Since  $y_n = \operatorname{Re} y_n + i \operatorname{Im} y_n$  we get that  $(y_n)_{n \in \mathbb{N}}$  is convergent with limit  $y = a + ib$ . Clearly  $y \in (\mathcal{M})_1$ , and so this proves completeness of the closed unit ball.  $\square$

It is worth pointing out, that in the process of proving the completeness of the closed unit ball in the above result, we actually proved completeness of both the set unitary elements and the set of self-adjoint elements of norm less than or equal to one. The converse of this theorem is also true, as shown by the corollary:

**Corollary 4.2.3.** *Suppose that  $\mathcal{H}$  is a Hilbert space and  $\mathcal{A} \subseteq B(\mathcal{H})$  a  $C^*$ -algebra containing the unit and equipped with a normal tracial state  $\tau$ . Then  $\mathcal{A}$  is a von Neumann algebra if and only if the closed unit ball of  $\mathcal{A}$  is complete in the norm induced by  $\tau$ .*

*Proof.* Assume that the closed unit ball of  $\mathcal{A}$  is complete in the norm induced by  $\tau$ . Let  $\mathcal{M}$  denote the weak operator closure of  $\mathcal{A}$  in  $B(\mathcal{H})$ , and extend  $\tau$  to a faithful normal tracial state  $\tilde{\tau}$  on  $\mathcal{M}$ . Suppose that  $(x_\alpha)_{\alpha \in A}$  is a net in the closed unit ball of  $\mathcal{A}$ , which is strong operator convergent to some  $x \in B(\mathcal{H})$ . Then  $x \in \mathcal{M}$ , and since  $\tau$  is strong operator continuous, we get that  $\|x - x_\alpha\|_{\tilde{\tau}}$  converges to zero as  $\alpha$  runs through  $A$ . In particular  $(x_\alpha)_{\alpha \in A}$  is a Cauchy net in the closed unit ball of  $\mathcal{A}$  with respect to the norm induced by  $\tau$ . Thus by assumption the net is convergent in the closed unit ball of  $\mathcal{A}$ , and of course the limit must necessarily be  $x$ . This shows that the closed unit ball of  $\mathcal{A}$  is strong operator closed, so  $\mathcal{A}$  must be a von Neumann algebra (see [Zhu93, Corollary 19.6]).  $\square$

With this characterization of von Neumann algebras we are now ready to prove that the ultraproduct of von Neumann algebras with faithful normal tracial states is a von Neumann algebra. The proof is again taken from [Haa91], wherein a more general result in the setting of  $C^*$ -algebras with quasi-traces is established.

**Theorem 4.2.4.** *Suppose that  $(\mathcal{M}_i)_{i \in I}$  is a family of von Neumann algebras with normal faithful tracial states  $(\tau_i)_{i \in I}$  and that  $\omega$  is a free ultrafilter on  $I$ . Then the tracial ultraproduct  $\prod_{i \in I}^{\omega} \mathcal{M}_i$  is a von Neumann algebra.*

*Proof.* By Corollary 4.2.3 it suffices to show that the closed unit ball of the ultraproduct is complete in the trace norm. Suppose that  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in the closed unit ball of  $\prod_{i \in I}^{\omega} \mathcal{M}_i$  with respect to the trace norm. Since we want to show that the sequence is convergent, it suffices to show that it has a convergent subsequence. Thus we may assume that  $\|x_n - x_{n+1}\|_2 < 2^{-n}$ , for all  $n \geq 1$ . Choose  $x_n^{(i)} \in \mathcal{M}_i$  with  $\|x_n^{(i)}\| \leq 2$  for each  $i \in I$  and  $n \in \mathbb{N}$  so that  $x_n = [(x_n^{(i)})_{i \in I}]$  for all  $n \in \mathbb{N}$ . This is possible since the quotient map  $\ell_\infty(I; \mathcal{M}_i) \rightarrow \prod_{i \in I}^{\omega} \mathcal{M}_i$  maps the

open ball with center 0 and radius 2 of  $\ell_\infty(I; \mathcal{M}_i)$  onto the open ball with center 0 and radius 2 of  $\prod_{i \in I}^\omega \mathcal{M}_i$  by Proposition 1.4.3. By assumption, the set

$$F_n = \{i \in I : \|x_k^{(i)} - x_{k+1}^{(i)}\|_2 < 2^{-k}, k = 1, 2, \dots, n\}$$

is in  $\omega$  for each  $n \in \mathbb{N}$ . Let  $F_0 = I$  and  $F = \bigcap_{k=0}^\infty F_k$ . Then clearly

$$F_0 \supseteq F_1 \supseteq \dots \supseteq F_n \supseteq \dots \supseteq F.$$

It follows that  $I$  is the disjoint union of the sets  $F_n \setminus F_{n+1}$ ,  $n \in \mathbb{N}$  and  $F$ . Suppose that  $i \in I \setminus F$ , then there exists some  $k \geq 0$  such that  $i \in F_k \setminus F_{k+1}$ . Set  $x^{(i)} = x_n^{(i)}$ . For  $i \in F$  we have  $\|x_n^{(i)} - x_{n+1}^{(i)}\|_2 < 2^{-n}$ , for each  $n \in \mathbb{N}$ , so  $(x_n^{(i)})_{n \in \mathbb{N}}$  is a Cauchy sequence in the closed unit ball of  $\mathcal{M}_i$  with respect to the trace norm on  $\mathcal{M}_i$ . Thus, by Theorem 4.2.2, the sequence is convergent. Let  $x^{(i)}$  denote the limit, and notice that that

$$\|x_n^{(i)} - x^{(i)}\|_2 = \left\| \sum_{k=n}^\infty x_k^{(i)} - x_{k+1}^{(i)} \right\|_2 \leq \sum_{k=n}^\infty \|x_k^{(i)} - x_{k+1}^{(i)}\|_2 < \sum_{k=n}^\infty 2^{-k} = 2^{1-n}$$

for each  $n \in \mathbb{N}$ .

Let  $x = [(x^{(i)})_{i \in I}]$ , then  $x \in \prod_{i \in I}^\omega \mathcal{M}_i$  since  $\sup_{i \in I} \|x^{(i)}\| \leq 2$ . Now fix some  $n \in \mathbb{N}$ . Let  $i \in F_n$ . If  $i \in F$ , then from above we know that  $\|x_n^{(i)} - x^{(i)}\|_2 < 2^{1-n}$ . If  $i \notin F$ , then there exists some  $k \geq n$  with  $i \in F_k \setminus F_{k+1}$ . If  $k = n$ , then  $\|x_n^{(i)} - x^{(i)}\|_2 = \|x_n^{(i)} - x_n^{(i)}\|_2 = 0$ . If  $k > n$ , then

$$\|x_n^{(i)} - x_k^{(i)}\|_2 = \left\| \sum_{m=n}^{k-1} x_m^{(i)} - x_{m+1}^{(i)} \right\|_2 \leq \sum_{m=n}^{k-1} \|x_m^{(i)} - x_{m+1}^{(i)}\|_2 < \sum_{m=n}^{k-1} \frac{1}{2^m} < 2^{1-n}$$

So in any case  $\|x_n^{(i)} - x^{(i)}\|_2 < 2^{1-n}$  for all  $i \in F_n$ . Since  $F_n \in \omega$  this shows that  $\|x_n - x\|_2 \leq 2^{1-n}$ , and so we conclude that  $x_n \rightarrow x$  as  $n \rightarrow \infty$  in the trace norm. Now the completeness of the closed unit ball of the tracial ultraproduct is proved, and therefore it must be a von Neumann algebra by Corollary 4.2.3.  $\square$

**Remark 4.2.5.** A more careful analysis of the proof above shows that, essentially with the same arguments, one can obtain a more general result *provided that* one puts some restrictions on the ultrafilter  $\omega$ . More precisely, one can prove that the ultraproduct of tracial  $C^*$ -algebras is a von Neumann algebra, by putting some extra restriction on the ultrafilter. Indeed, note that the only time we used the fact that  $\mathcal{M}_i$  is a von Neumann algebra and that the trace  $\tau_i$  is faithful and normal,  $i \in I$ , was when applying Theorem 4.2.2 in the proof. An appropriate restriction on the ultrafilter can make this step superfluous. Namely, consider a free ultrafilter  $\omega$  on  $I$ , which contains a countable family of sets  $(E_k)_{k \in \mathbb{N}}$ , such that  $\bigcap_{n=1}^\infty E_n = \emptyset$ . We may assume that this sequence of sets is decreasing, by replacing  $E_n$  with the intersection of  $E_1, \dots, E_n$ , for each  $n \in \mathbb{N}$ , and we may also assume that  $E_1 = I$ . By replacing the set  $F_k$  in the proof by the set  $F'_k = F_k \cap E_k$  we get a new sequence of sets, with the same properties as the original sequence, but now with  $\bigcap_{k=1}^\infty F'_k$  empty. This makes the part of the proof, which deals with indices in  $F$ , and therefore also the use of Theorem 4.2.2, superfluous. This idea of putting a restriction on the ultrafilter is taken from the paper [HL09] of Don Hadwin and Weihua Li.

Note that if the index set  $I$  is countable, say  $I = \mathbb{N}$ , then all free ultrafilters  $\omega$  on  $I$  satisfy this condition. More precisely,  $\omega$  must contain the sets  $E_k = \{n \in \mathbb{N} : n \geq k\}$ ,  $k \in \mathbb{N}$ , since they have finite complement.  $\blacktriangleleft$

Naturally, the next thing we are interested in is determining under what conditions the tracial ultraproduct of von Neumann algebras is a factor, and in such cases, which type of factor it actually is. Before we get to this we need a few intermediate result.

**Proposition 4.2.6.** *Suppose that  $(\mathcal{M}_i)_{i \in I}$  is a family of von Neumann algebras with normal faithful tracial states  $(\tau_i)_{i \in I}$  and that  $\omega$  is a free ultrafilter on  $I$ . Then each projection in  $\prod_{i \in I}^{\omega} \mathcal{M}_i$  lifts to a projection in  $\ell_{\infty}(I; \mathcal{M}_i)$ .*

*Proof.* Suppose that  $p$  is a projection in  $\prod_{i \in I}^{\omega} \mathcal{M}_i$ , and let  $(x_i)_{i \in I}$  be a positive lift of  $p$ . For each  $i \in I$  let  $q_i = 1_{[\frac{1}{2}, \infty)}(x_i)$ , and let  $q = [(q_i)_{i \in I}]$ . Our goal is to prove that  $p = [q]$ . Since  $2^{-1}q_i \leq x_i$  and  $\|(1 - q_i)x_i\|_i \leq 2^{-1}$ , for each  $i \in I$ , it follows that

$$\frac{1}{2}[q] \leq p \quad \text{and} \quad \|(1 - [q])p\| \leq \frac{1}{2}.$$

Since the square root of positive elements is order preserving, see [KR83, Proposition 4.2.8], we obtain that  $2^{-1/2}[q] \leq p$ , and iterating this, we obtain  $2^{-1/2^n}[q] \leq p$  for all  $n \in \mathbb{N}$ . Hence  $[q] \leq p$ . This shows that  $p - [q] = (1 - [q])p$  is a projection, so since we know that it has norm less than  $2^{-1}$ , we conclude that  $p = [q]$ .  $\square$

**Lemma 4.2.7.** *A von Neumann algebra  $\mathcal{M}$  with a faithful tracial state  $\tau$  is a factor if and only if, for each non-zero projection  $p$  in  $\mathcal{M}$  with  $\tau(p) \leq 2^{-1}$ , there exists a projection  $q$  in  $\mathcal{M}$  equivalent to  $p$  such that  $q \leq 1 - p$ .*

*Proof.* Suppose that  $\mathcal{M}$  is a factor, and  $p$  a non-zero projection. Since  $\mathcal{M}$  is a factor, we deduce that either  $p$  is equivalent to a projection below  $1 - p$ , or  $1 - p$  is equivalent to a projection below  $p$ , see [Zhu93, Corollary 25.5]. If the trace of  $p$  is strictly less than  $2^{-1}$ , then the latter cannot be the case, and we conclude that there exists some projection  $q \leq 1 - p$  which is equivalent to  $p$ . If the trace of  $p$  is equal to  $2^{-1}$ , then  $p$  and  $1 - p$  are equivalent, so we may choose  $q = 1 - p$ .

Now suppose that  $\mathcal{M}$  is not a factor, and let  $p$  be a non-trivial central projection in  $\mathcal{M}$ . By interchanging  $p$  and  $1 - p$  we can assume that  $p$  has trace less than or equal to  $2^{-1}$ . Now suppose that  $q$  is a projection in  $\mathcal{M}$  which is equivalent to  $p$ . Let  $v \in \mathcal{M}$  be a partial isometry such that  $v^*v = p$  and  $vv^* = q$ . Now since  $p$  is the support projection for  $v$  and it is central, we see that  $q = vv^* = vp^2v^* = pvv^*p \leq p$ . This shows that  $p$  is not equivalent to a projection below  $1 - p$ .  $\square$

We already know that if the tracial ultraproduct is a von Neumann algebra, then it is a finite von Neumann algebra, since it possesses a faithful trace. The next theorem states that if sufficiently many of the terms in the ultraproduct are von Neumann algebra factors, then the ultraproduct is a factor.

**Theorem 4.2.8.** *Suppose that  $(\mathcal{M}_i)_{i \in I}$  is a family of von Neumann algebras with faithful normal tracial states  $(\tau_i)_{i \in I}$  and that  $\omega$  is a free ultrafilter on  $I$ . If the set  $\{i \in I : \mathcal{M}_i \cap \mathcal{M}'_i = \mathbb{C}\mathbf{1}_i\}$  is in  $\omega$ , then the tracial ultraproduct  $\prod_{i \in I}^{\omega} \mathcal{M}_i$  is a finite factor. Moreover,*

- (i) *the tracial ultraproduct  $\prod_{i \in I}^{\omega} \mathcal{M}_i$  is a  $\text{II}_1$ -factor if and only if the set  $\{i \in I : \dim \mathcal{M}_i \geq k^2\}$  is in  $\omega$  for all  $k \in \mathbb{N}$ ;*
- (ii) *the tracial ultraproduct  $\prod_{i \in I}^{\omega} \mathcal{M}_i$  is a  $\text{I}_n$ -factor if and only if the set  $\{i \in I : \dim \mathcal{M}_i = n^2\}$  is in  $\omega$ .*

*Proof.* Let  $F$  denote the set  $\{i \in I : \mathcal{M}_i \cap \mathcal{M}'_i = \mathbb{C}\mathbf{1}_i\}$ , and assume that  $F \in \omega$ . Let  $p$  be a non-zero projection in  $\mathcal{M}$  with  $\tau_\omega(p) \leq 2^{-1}$ . By Proposition 4.2.6 we can find lift  $p$  to a projection  $(p_i)_{i \in I}$  in  $\ell_\infty(I; \mathcal{M}_i)$ . By interchanging  $p$  with  $\mathbf{1} - p$  and  $p_i$  with  $\mathbf{1}_i - p_i$ , for each  $i \in I$ , we may assume that the set  $A = \{i \in I : \tau_i(p_i) \leq 2^{-1}\}$  is in  $\omega$ . For each  $i \in A \cap F$  we can choose a projection  $q_i \in \mathcal{M}_i$  and a partial isometry  $v_i \in \mathcal{M}_i$  satisfying

$$v_i^* v_i = p_i, \quad v_i v_i^* = q_i \quad \text{and} \quad q_i \leq \mathbf{1} - p_i,$$

by Lemma 4.2.7. For each  $i \in I \setminus (A \cap F)$ , let  $q_i = v_i = 0$ . Now, since  $A \cap F \in \omega$ , we obtain that  $q = [(q_i)_{i \in I}]$  is a projection in  $\prod_{i \in I}^\omega \mathcal{M}_i$ . By setting  $v = [(v_i)_{i \in I}]$  we get

$$v^* v = p, \quad v v^* = q \quad \text{and} \quad q \leq \mathbf{1} - p.$$

Thus, by Lemma 4.2.7 we conclude that  $\prod_{i \in I}^\omega \mathcal{M}_i$  is a factor.

Suppose that  $n \in \mathbb{N}$ . Let  $D_n$  denote the set  $\{i \in I : \dim \mathcal{M}_i \geq n^2\}$ , and assume that  $D_n \in \omega$ . Let  $i \in F \cap D_n$ . By dimension considerations, we see that  $\mathcal{M}_i$  is a  $\text{II}_1$ -factor or a  $\text{I}_k$ -factor, for some  $k \geq n$ . In either case we can choose orthogonal projections  $p_1^{(i)}, p_2^{(i)}, \dots, p_k^{(i)}$  in  $\mathcal{M}_i$  with trace greater than or equal to  $\frac{1}{2n}$ . For each  $i \in I \setminus (F \cap D_n)$ , let  $p_1^{(i)} = p_2^{(i)} = \dots = p_k^{(i)} = 0$ . Let  $p_j = [(p_j^{(i)})_{i \in I}]$ , for  $j = 1, 2, \dots, n$ . Since  $F \cap D_n \in \omega$ , we get that  $p_1, p_2, \dots, p_n$  are orthogonal projections with trace greater than or equal to  $(2n)^{-1}$ . Thus  $\prod_{i \in I}^\omega \mathcal{M}_i$  cannot be of type  $\text{I}_k$  for  $k < n$ .

If  $D_k$  is in  $\omega$  for all  $k \in \mathbb{N}$ , then by the previous part  $\prod_{i \in I}^\omega \mathcal{M}_i$  cannot be of type  $\text{I}_n$  for  $n$  all  $n \in \mathbb{N}$ . Hence  $\prod_{i \in I}^\omega \mathcal{M}_i$  is of type  $\text{II}_1$ . Now, if this is not the case, then since

$$I = D_1 \subseteq D_2 \subseteq D_3 \subseteq \dots \subseteq D_k \subseteq D_{k+1} \subseteq \dots$$

and  $\omega$  is a filter, this means that there exist some  $k \in \mathbb{N}$  such that  $D_k \in \omega$ , but  $D_{k+1} \notin \omega$ . Since  $\omega$  is an ultrafilter, we conclude that

$$\{i \in I : \dim \mathcal{M}_i = k^2\} = D_k \cap (I \setminus D_{k+1})$$

is in  $\omega$ . Now, if  $p$  is a non-zero projection in  $\prod_{i \in I}^\omega \mathcal{M}_i$ , then by Proposition 4.2.6 we can lift  $p$  to a projection  $(p_i)_{i \in I}$  in  $\ell_\infty(I; \mathcal{M}_i)$ . For each  $i \in D_k \cap (I \setminus D_{k+1})$  the trace of a projection in  $\mathcal{M}_i$  is a multiple of  $\frac{1}{k}$ . Since the set

$$\{i \in I : \dim \mathcal{M}_i = k^2, p_i \neq 0\}$$

is in  $\omega$  as  $p$  is non-zero, we conclude that  $\tau_\omega(p)$  must be greater than or equal to  $\frac{1}{k}$ . Thus every non-zero projection in  $\prod_{i \in I}^\omega \mathcal{M}_i$  has trace greater than or equal to  $\frac{1}{k}$ . In particular  $\prod_{i \in I}^\omega \mathcal{M}_i$  cannot be either a  $\text{II}_1$ -factor or a  $\text{I}_n$ -factor for  $n > k$ , since these all contain non-zero projections of trace strictly less than  $\frac{1}{k}$ . On the other hand, we know that  $D_k \in \omega$  so by the previous part,  $\prod_{i \in I}^\omega \mathcal{M}_i$  cannot be a  $\text{I}_n$ -factor for  $n < k$ . Thus we conclude that  $\prod_{i \in I}^\omega \mathcal{M}_i$  must be a  $\text{I}_k$ -factor.

To summarize, we have proved that if the set  $\{i \in I : \dim \mathcal{M} \geq k^2\}$  is not in  $\omega$  for all  $k \in \mathbb{N}$ , then there exists some  $n \in \mathbb{N}$  such that the set  $\{i \in I : \dim \mathcal{M} = n^2\}$  is in  $\omega$ . Clearly this goes both ways, that is, if there exists some  $n \in \mathbb{N}$  such that the set  $\{i \in I : \dim \mathcal{M} = n^2\}$  is in  $\omega$ , then the set  $\{i \in I : \dim \mathcal{M} \geq k^2\}$  is not in  $\omega$ , for

all  $k \in \mathbb{N}$ . Hence the conditions that the set  $\{i \in I : \dim \mathcal{M} \geq k^2\}$  is not in  $\omega$ , for all  $k \in \mathbb{N}$  and the condition that  $\{i \in I : \dim \mathcal{M} = n^2\}$  is in  $\omega$ , for some  $n \in \mathbb{N}$ , are mutually exclusive and represent all possibilities. In the former case we found that the tracial ultraproduct  $\prod_{i \in I}^{\omega} \mathcal{M}_i$  is a  $\text{II}_1$ -factor, and in the latter case we found that the tracial ultraproduct  $\prod_{i \in I}^{\omega} \mathcal{M}_i$  is a  $I_n$ -factor. Thus the proof is complete.  $\square$

**Remark 4.2.9.** As mentioned in the beginning of this chapter, there is also another notion of ultraproduct for  $C^*$ -algebras, the so-called metric ultraproduct. We briefly explain its construction.

Suppose that  $(\mathcal{A}_i)_{i \in I}$  is a family of  $C^*$ -algebras. Let  $\|\cdot\|_i$  denote the norm on  $\mathcal{A}_i$  for each  $i \in I$ . The set

$$\mathcal{J} = \{(x_i)_{i \in I} \in \ell_{\infty}(I; \mathcal{A}_i) : \lim_{i \rightarrow \omega} \|x_i\|_i = 0\}$$

is a closed two-sided ideal in the  $C^*$ -algebra  $\ell_{\infty}(I; \mathcal{A}_i)$ . The quotient of  $\ell_{\infty}(I; \mathcal{A}_i)$  by the ideal  $\mathcal{J}$  is therefore a  $C^*$ -algebra and it is called the **metric ultraproduct** of the  $C^*$ -algebras. This ultraproduct will not be used in this thesis, so every reference to ultraproducts of  $C^*$ -algebra will be to the tracial ultraproduct.  $\blacktriangleleft$

### 4.3 Tensor product ultrafilter—ultraproducts of ultraproducts

**Definition 4.3.1.** Suppose that  $\omega$  and  $\nu$  are filters on the sets  $I$  and  $J$ , respectively. Let  $\omega \otimes \nu$  denote the set of subsets  $U \subseteq I \times J$  such that

$$\{i \in I : \{j \in J : (i, j) \in U\} \in \nu\} \in \omega.$$

This set  $\omega \otimes \nu$  is called the **tensor product** of the filters  $\omega$  and  $\nu$ .

**Proposition 4.3.2.** *Suppose that  $\omega$  and  $\nu$  are filters on  $I$  and  $J$ , respectively. Then their tensor product  $\omega \otimes \nu$  is a filter. Moreover, if  $\omega$  and  $\nu$  are both ultrafilters, then  $\omega \otimes \nu$  is again an ultrafilter, and  $\omega \otimes \nu$  is free if and only if either  $\omega$  or  $\nu$  is free.*

*Proof.* Let us start by proving that the tensor product is a filter. Suppose that  $U, V \in \omega \otimes \nu$ . For each  $i \in I$  we have  $\{j \in J : (i, j) \in U \cap V\} = \{j \in J : (i, j) \in U\} \cap \{j \in J : (i, j) \in V\}$ , so  $\{j \in J : (i, j) \in U \cap V\} \in \nu$  if and only if both  $\{j \in J : (i, j) \in U\} \in \nu$  and  $\{j \in J : (i, j) \in V\} \in \nu$ . From this it follows that

$$\begin{aligned} & \{i \in I : \{j \in J : (i, j) \in U \cap V\} \in \nu\} \\ &= \{i \in I : \{j \in J : (i, j) \in U\} \in \nu\} \cap \{i \in I : \{j \in J : (i, j) \in V\} \in \nu\} \end{aligned}$$

Since by assumption the two sets on the right hand side both belong to  $\omega$ , the set on the left hand side belongs to  $\omega$ , as well. This shows that  $U \cap V \in \omega \otimes \nu$ . Suppose now that  $U \in \omega \otimes \nu$ , and  $V \subseteq I \times J$  with  $U \subseteq V$ . Since

$$\{i \in I : \{j \in J : (i, j) \in U\} \in \nu\} \subseteq \{i \in I : \{j \in J : (i, j) \in V\} \in \nu\},$$

and  $\omega$  is a filter, we deduce that the set on the right hand side is in  $\omega$ , so  $V \in \omega \otimes \nu$ . Last, we observe that the empty set is not in  $\omega \otimes \nu$  and that  $\omega \otimes \nu$  is non-empty, which is clear since  $I \times J \in \omega \otimes \nu$ .

Now suppose that  $\omega$  and  $\nu$  are both ultrafilters, and let us show that  $\omega \otimes \nu$  is also an ultrafilter. Suppose that  $A \subseteq I \times J$  with  $A \notin \omega \otimes \nu$ . Let  $B = (I \times J) \setminus A$ . Since  $\omega$  is an ultrafilter and  $A \notin \omega \otimes \nu$ , we know that

$$C := \{i \in I : \{j \in J : (i, j) \in A\} \notin \nu\} \in \omega$$

since  $C$  is the complement of  $\{i \in I : \{j \in J : (i, j) \in A\} \in \nu\}$  in  $I \times J$ . Now if  $i \in C$  then  $\{j \in J : (i, j) \in A\} \notin \nu$ , so since  $\nu$  is an ultrafilter we conclude that  $\{j \in J : (i, j) \in B\} \in \nu$ . Thus

$$\{i \in I : \{j \in J : (i, j) \in B\} \in \nu\} \supseteq C \in \omega.$$

This proves that  $B \in \omega \otimes \nu$ , and therefore  $\omega \otimes \nu$  is an ultrafilter.

Let us show that  $\omega \otimes \nu$  is free if and only if either  $\omega$  or  $\nu$  is free. Assume say, that  $\omega$  is free. For each  $U \in \omega$ , we have  $U \times J \in \omega \otimes \nu$ , so

$$\bigcap_{F \in \omega \otimes \nu} F \subseteq \bigcap_{U \in \omega} (U \times J) = \left( \bigcap_{U \in \omega} U \right) \times J = \emptyset \times J = \emptyset.$$

Hence  $\omega \otimes \nu$  is free. A similar argument applies when  $\nu$  is free. Suppose instead that both  $\omega$  and  $\nu$  are principal, based at  $i_0 \in I$  and  $j_0 \in J$ , respectively. The claim is then that  $\omega \otimes \nu$  is principal, based at  $(i_0, j_0)$ . Suppose that  $A \in \omega \otimes \nu$ . Since  $\omega$  is principal based at  $i_0$ , we get that  $i_0 \in \{i \in I : \{j \in J : (i, j) \in A\} \in \nu\}$ . So, in particular,  $\{j \in J : (i_0, j) \in A\} \in \nu$ . Thus  $(i_0, j_0) \in A$ , and we have proved that  $\omega \otimes \nu$  is principal.  $\square$

**Proposition 4.3.3.** *Suppose that  $\mathfrak{X}$  is a Hausdorff topological space and let  $\omega$  and  $\nu$  filters on  $I$  and  $J$ , respectively. Suppose that we are given  $x_{i,j} \in \mathfrak{X}$  for each  $i \in I$  and  $j \in J$  so that  $\lim_{i \rightarrow \omega} (\lim_{j \rightarrow \nu} x_{i,j})$  exists. Then  $\lim_{(i,j) \rightarrow \omega \otimes \nu} x_{i,j}$  exists and*

$$\lim_{(i,j) \rightarrow \omega \otimes \nu} x_{i,j} = \lim_{i \rightarrow \omega} \lim_{j \rightarrow \nu} x_{i,j}$$

*Proof.* For each  $i \in I$ , let  $x^i$  denote the limit of  $(x_{i,j})_{j \in J}$  along  $\nu$ , and let  $x$  denote the limit of  $(x^i)_{i \in I}$  along  $\omega$ . Suppose that  $U \subset \mathfrak{X}$  is an open neighbourhood of  $x$  in  $\mathfrak{X}$ . Since  $\lim_{i \rightarrow \omega} x^i = x$ , we deduce that  $A = \{i \in I : x^i \in U\} \in \omega$ . Now, for each  $i \in A$ , we know that  $x^i \in U$  and  $\lim_{j \rightarrow \nu} x_{i,j} = x^i$ , so  $B_i = \{j \in J : x_{i,j} \in U\} \in \nu$ . Set  $F = \bigcup_{i \in A} \{i\} \times B_i$ . If we show that  $F \in \omega \otimes \nu$ , then since  $U$  was an arbitrary open neighbourhood of  $x$  in  $\mathfrak{X}$  we have proved that  $\lim_{(i,j) \rightarrow \omega \otimes \nu} x_{i,j} = x$ . Note that, for  $i \in I \setminus A$ ,  $\{j \in J : (i, j) \in F\} = \emptyset \notin \nu$ , while for  $i \in A$ ,  $\{j \in J : (i, j) \in F\} = B_i \in \nu$ . Thus  $\{j \in J : (i, j) \in F\} \in \nu$  if and only if  $i \in A$ . It now follows that

$$\{i \in I : \{j \in J : (i, j) \in F\} \in \nu\} = A \in \omega,$$

which by the definition of the tensor product ultrafilter means that  $F \in \omega \otimes \nu$ .  $\square$

**Proposition 4.3.4.** *Suppose that  $\omega$  and  $\nu$  are ultrafilters on  $I$  and  $J$ , respectively. For each  $i \in I$  and  $j \in J$  let  $(G_{i,j}, d_{i,j})$  be a metric group. Then there is a canonical isometric isomorphism of metric groups*

$$\prod_{i \in I}^{\omega} \left( \prod_{j \in J}^{\nu} G_{i,j} \right) \cong \prod_{(i,j) \in I \times J}^{\omega \otimes \nu} G_{i,j}$$

*Proof.* Let  $\varphi: \prod_{i \in I}^{\omega} \left( \prod_{j \in J}^{\nu} G_{i,j} \right) \rightarrow \prod_{(i,j) \in I \times J}^{\omega \otimes \nu} G_{i,j}$  denote the map given by  $[[[(g_{i,j})_{j \in J}]_{i \in I}]] \mapsto [(g_{i,j})_{(i,j) \in I \times J}]$ . For each  $i \in I$ , let  $d_{i,\nu}$  denote the metric on  $\prod_{j \in J}^{\nu} G_{i,j}$ , let  $d_{\omega,\nu}$  denote the metric on  $\prod_{i \in I}^{\omega} \left( \prod_{j \in J}^{\nu} G_{i,j} \right)$  and let  $d_{\omega \otimes \nu}$  denote the metric on  $\prod_{(i,j) \in I \times J}^{\omega \otimes \nu} G_{i,j}$ . Suppose that  $g, h \in \prod_{i \in I}^{\omega} \left( \prod_{j \in J}^{\nu} G_{i,j} \right)$  and write  $g = [[[(g_{i,j})_{j \in J}]_{i \in I}]]$  and  $h = [[[(h_{i,j})_{j \in J}]_{i \in I}]]$  for some  $g_{i,j}, h_{i,j} \in G_{i,j}$ ,  $(i, j) \in I \times J$ . Now, by Proposition 4.3.3 we get

$$\begin{aligned} d_{\omega,\nu}(g, h) &= \lim_{i \rightarrow \omega} d_{i,\nu}([(g_{i,j})_{j \in J}], [(h_{i,j})_{j \in J}]) \\ &= \lim_{i \rightarrow \omega} \lim_{j \rightarrow \nu} d_{i,j}(g_{i,j}, h_{i,j}) \\ &= \lim_{(i,j) \rightarrow \omega \otimes \nu} d_{i,j}(g_{i,j}, h_{i,j}) \\ &= d_{\omega \otimes \nu}([(g_{i,j})_{(i,j) \in I \times J}], [(h_{i,j})_{(i,j) \in I \times J}]) \end{aligned}$$

which precisely shows that  $\varphi$  is isometric. This, in particular, implies that  $\varphi$  is well-defined and isometric. Clearly  $\varphi$  is also surjective, hence  $\varphi$  is an isometric isomorphism of metric groups.  $\square$

**Proposition 4.3.5.** *Suppose that  $\omega$  and  $\nu$  are ultrafilters on  $I$  and  $J$ , respectively. For each  $i \in I$  and  $j \in J$ , let  $(\mathcal{A}_{i,j}, \tau_{i,j})$  be a tracial  $C^*$ -algebra. Then there is a canonical  $*$ -isomorphism*

$$\prod_{i \in I}^{\omega} \left( \prod_{j \in J}^{\nu} \mathcal{A}_{i,j} \right) \cong \prod_{(i,j) \in I \times J}^{\omega \otimes \nu} \mathcal{A}_{i,j}$$

*Proof.* The argument is almost identical to the one in the proof of Proposition 4.3.4. One shows that the map  $\varphi$  is well defined by proving that it is trace-preserving, which also makes it injective, so since it is clearly surjective it becomes a  $*$ -homomorphism.  $\square$



## Chapter 5

# Hyperlinear groups

In this thesis hyperlinear groups are of interest because of their connection to the Connes Embedding Problem. We shall see later, that countable discrete groups with infinite conjugacy classes are exactly the ones that satisfy the Connes Embedding Problem for Groups. Naturally, the chapter starts with the definition of hyperlinear groups, and from this we move towards proving that hyperlinear groups have the explained connection to the Connes Embedding problem.

### 5.1 The definition of a hyperlinear group

The term *hyperlinear group* was first introduced by Florin Rădulescu in his paper [Răd08] of 2008. Originally, Rădulescu defined a countable group with infinite conjugacy classes to be hyperlinear if it embeds into the unitary group  $\mathcal{U}(\mathcal{R}^\omega)$  of the ultrapower  $\mathcal{R}^\omega$  for some choice of free ultrafilter  $\omega$  on  $\mathbb{N}$ . In this thesis we consider a different definition (see Definition 5.1.1), which we will eventually show to be equivalent to the definition of Rădulescu (see Proposition 5.3.5), with some slight changes. More precisely, we do not require the group to be countable and have infinite conjugacy classes, nor do we require  $\omega$  to be a free ultrafilter on  $\mathbb{N}$ , but rather an ultrafilter on some index set  $I$ , thus allowing more groups to be hyperlinear.

At a glance, the definition of a hyperlinear group does not seem to have much to do with the Connes Embedding Problem, whilst the original definition of Rădulescu seems to have a more explicit connection. The current definition below is chosen for the sake of exposition, and the connection to the Connes Embedding Problem will become apparent later in this chapter.

**Definition 5.1.1.** A group  $G$  is called **hyperlinear** if for every finite subset  $F$  of  $G$  and every  $\varepsilon > 0$ , there exist some  $n \in \mathbb{N}$  and a map  $\varphi: G \rightarrow \mathcal{U}(n)$  satisfying:

- (i)  $\|\varphi(gh) - \varphi(g)\varphi(h)\|_2 \leq \varepsilon$ , for all  $g, h \in F$ ;
- (ii)  $\|\varphi(g) - \varphi(h)\|_2 \geq \sqrt{2} - \varepsilon$ , for all  $g, h \in F$  with  $g \neq h$ . ◀

A curious thing one might notice about this definition is that even though the map  $\varphi$  above is defined on all of  $G$ , there is only a restriction of its values on the set  $F \cup F^2$ , and so the map may be changed freely outside this set.

An easy observation about hyperlinear groups is the following:

**Proposition 5.1.2.** *For a group  $G$  the following are equivalent:*

- (i) The group  $G$  is hyperlinear;
- (ii) Every subgroup of  $G$  is hyperlinear;
- (iii) Every countable subgroup of  $G$  is hyperlinear;
- (iv) Every finitely generated subgroup of  $G$  is hyperlinear.

*Proof.* Clearly (ii)  $\Rightarrow$  (iii) and (iii)  $\Rightarrow$  (vi), and the two implications (i)  $\Rightarrow$  (ii) and (vi)  $\Rightarrow$  (i) are also easy.  $\square$

The following two results are of use when working with hyperlinear groups in terms of Definition 5.1.1.

**Lemma 5.1.3.** *For each  $n, m \in \mathbb{N}$  with  $m \geq n$ , there is an injective homomorphism  $\rho_{n,m}: \mathcal{U}(n) \rightarrow \mathcal{U}(m)$  so that  $\frac{1}{\sqrt{2}}\|u - v\|_2 \leq \|\rho_{n,m}(u) - \rho_{n,m}(v)\|_2 \leq \|u - v\|_2$ .*

*Proof.* Choose  $k \in \mathbb{N}$  and  $r \in \mathbb{Z}$  with  $0 \leq r < n$  so that  $m = kn + r$ . Let  $\rho_{n,m}: \mathcal{U}(n) \rightarrow \mathcal{U}(m)$  be defined by  $\rho_{n,m}(u) = u \oplus u \oplus \dots \oplus u \oplus \mathbf{1}_r$  ( $k$  copies of  $u$ ), that is,

$$\rho_{n,m}(u) = \begin{bmatrix} u & 0 & \dots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & u & 0 \\ 0 & \dots & \dots & 0 & \mathbf{1}_r \end{bmatrix}.$$

Clearly  $\rho_{n,m}$  is an injective homomorphism with  $\text{tr}_m(\rho_{n,m}(u)) = \frac{m-r}{m}\text{tr}_n(u) + \frac{r}{m}$ . Let  $\alpha = \frac{m-r}{m}$ , then  $\frac{1}{2} < \alpha \leq 1$ . It is straightforward to check that

$$\|\rho_{n,m}(u) - \rho_{n,m}(v)\|_2 = \sqrt{\alpha}\|u - v\|_2.$$

From this the desired inequality follows.  $\square$

**Lemma 5.1.4.** *For each  $n, k \in \mathbb{N}$  there exists a group homomorphism  $\pi_{n,k}: \mathcal{U}(n) \rightarrow \mathcal{U}((2n)^k)$  with the property that if  $u, v \in \mathcal{U}(n)$ , then*

$$\|\pi_{n,k}(u) - \pi_{n,k}(v)\|_2^2 = 2 - 2^{1-k} \text{Re}(1 + \text{tr}_n(u^*v))^k.$$

*Proof.* Fix  $n, k \in \mathbb{N}$ . There is a natural isomorphism  $f_{n,k}: \mathcal{U}(2n)^{\otimes k} \rightarrow \mathcal{U}((2n)^k)$  of the  $k$ -fold tensor product of  $\mathcal{U}(2n)$  onto  $\mathcal{U}((2n)^k)$ . Furthermore, let  $\pi_{n,k}: \mathcal{U}(n) \rightarrow \mathcal{U}((2n)^k)$  denote the map

$$\pi_{n,k}(u) = f_{n,k}((u \oplus \mathbf{1}_n) \otimes (u \oplus \mathbf{1}_n) \otimes \dots \otimes (u \oplus \mathbf{1}_n)).$$

Then  $\pi_{n,k}$  is a group homomorphism. It is straightforward to check that

$$\text{tr}_{(2n)^k}(\pi_{n,k}(u)) = 2^{-k}(\text{tr}_n(u) + 1)^k,$$

since  $f_{n,k}$  must necessarily preserve the trace. Now,

$$\begin{aligned} \|\pi_{n,k}(u) - \pi_{n,k}(v)\|_2^2 &= \text{tr}_{(2n)^k}((\pi_{n,k}(u) - \pi_{n,k}(v))^*(\pi_{n,k}(u) - \pi_{n,k}(v))) \\ &= 2 - 2 \text{Re} \text{tr}_{(2n)^k}(\pi_{n,k}(u^*v)) \\ &= 2 - 2^{1-k} \text{Re}(\text{tr}_n(u^*v) + 1)^k, \end{aligned}$$

which is what we needed to prove.  $\square$

There are a number of different variations of Definition 5.1.1 in the literature, and of course in some cases one is more handy than the other. The following proposition proves the equivalence of some of these.

**Proposition 5.1.5.** *For a group  $G$  the following set of conditions are equivalent:*

- (1) *For each finite subset  $F \subseteq G$  and every  $\varepsilon > 0$ , there exist some  $n \in \mathbb{N}$  and a map  $\varphi: G \rightarrow \mathcal{U}(n)$  with the properties:
 
  - (1.i)  $\|\varphi(gh) - \varphi(g)\varphi(h)\|_2 \leq \varepsilon$ , for each  $g, h \in F$ ;
  - (1.ii)  $\|\varphi(\mathbf{1}_G) - \mathbf{1}_{\mathcal{U}(n)}\|_2 \leq \varepsilon$ ;
  - (1.iii)  $\|\varphi(g) - \varphi(h)\|_2 \geq \sqrt{2} - \varepsilon$ , for each  $g, h \in F$  with  $g \neq h$ .*
- (2) *For each finite subset  $F \subseteq G$  and every  $\varepsilon > 0$ , there exist some  $n \in \mathbb{N}$  and a map  $\varphi: G \rightarrow \mathcal{U}(n)$  with the properties:
 
  - (2.i)  $\|\varphi(gh) - \varphi(g)\varphi(h)\|_2 \leq \varepsilon$  for each  $g, h \in F$ ;
  - (2.ii)  $\|\varphi(g) - \varphi(h)\|_2 \geq \sqrt{2} - \varepsilon$ , for each  $g, h \in F$  with  $g \neq h$ .*
- (3) *For every constant  $\delta \in (0, \sqrt{2})$ , each finite subset  $F \subseteq G$  and every  $\varepsilon > 0$ , there exist some  $n \in \mathbb{N}$  and a map  $\varphi: G \rightarrow \mathcal{U}(n)$  with the properties:
 
  - (3.i)  $\|\varphi(gh) - \varphi(g)\varphi(h)\|_2 \leq \varepsilon$  for each  $g, h \in F$ ;
  - (3.ii)  $\|\varphi(g) - \varphi(h)\|_2 \geq \delta$ , for each  $g, h \in F$  with  $g \neq h$ .*
- (4) *There exists some  $\delta > 0$  such that for each finite subset  $F \subseteq G$  and every  $\varepsilon > 0$ , there exist some  $n \in \mathbb{N}$  and a map  $\varphi: G \rightarrow \mathcal{U}(n)$  with the properties:
 
  - (4.i)  $\|\varphi(gh) - \varphi(g)\varphi(h)\|_2 \leq \varepsilon$ , for each  $g, h \in F$ ;
  - (4.ii)  $\|\varphi(g) - \varphi(h)\|_2 \geq \delta$ , for each  $g, h \in F$  with  $g \neq h$ .*
- (5) *For each finite subset  $F \subseteq G$  there exists some  $\delta_F > 0$  such that for every  $\varepsilon > 0$  there exists some  $n \in \mathbb{N}$  and a map  $\varphi: G \rightarrow \mathcal{U}(n)$  with the properties:
 
  - (5.i)  $\|\varphi(gh) - \varphi(g)\varphi(h)\|_2 \leq \varepsilon$ , for each  $g, h \in F$ ;
  - (5.ii)  $\|\varphi(g) - \varphi(h)\|_2 \geq \delta_F$ , for each  $g, h \in F$  with  $g \neq h$ .*

*Proof.* One can see right-away, that the implications (1)  $\Rightarrow$  (2), (3)  $\Rightarrow$  (4) and (4)  $\Rightarrow$  (5) are trivial, and the implication (2)  $\Rightarrow$  (3) follows by replacing  $\varepsilon$  in (2) with  $\min\{\varepsilon, \delta\}$ . Thus the only implication we need to prove is (5)  $\Rightarrow$  (1). Now assume that  $G$  satisfies (5), and let  $F \subseteq G$  be finite and  $\varepsilon > 0$ . We may assume that  $\varepsilon \leq \sqrt{2}$  and that  $F$  has at least two elements. To prove (1) we use the amplification trick from Proposition 5.1.4. Suppose that  $z \in \mathbb{C}$  with  $|z| \leq 1$ , then

$$|z + 1|^2 = 1 + 2 \operatorname{Re} z + |z|^2 \leq 2 + 2 \operatorname{Re} z.$$

Thus, if we let  $k, m \in \mathbb{N}$  and  $u, v \in \mathcal{U}(m)$ , then

$$\begin{aligned} \operatorname{Re} (1 + \operatorname{tr}_m(u^*v))^k &\leq |1 + \operatorname{tr}_m(u^*v)|^k \\ &\leq (2 + 2 \operatorname{Re} \operatorname{tr}_m(u^*v))^{k/2} \\ &= (4 - \|u - v\|_2^2)^{k/2}. \end{aligned}$$

Let  $K = F \cup \{1_G\}$ , and choose the constant  $\delta_K > 0$  as in condition (5). Then

$$\|\pi_{m,k}(u) - \pi_{m,k}(v)\|_2^2 = 2 - 2^{1-k} \operatorname{Re}(1 + \operatorname{tr}_m(u^*v))^k \geq 2 - 2^{1-k} (4 - \delta_K^2)^{k/2}.$$

Since  $K$  has at least two element, the condition (5.ii) ensures that  $\delta_K \leq 2$ , so in particular  $0 \leq 4 - \delta_K^2 < 4$  and we can choose  $k \in \mathbb{N}$  so that

$$2 - 2^{1-k} (4 - \delta_K^2)^{k/2} \geq (\sqrt{2} - \varepsilon)^2,$$

but then  $\|\pi_{m,k}(u) - \pi_{m,k}(v)\|_2 \geq \sqrt{2} - \varepsilon$  for all  $m \in \mathbb{N}$  and  $u, v \in \mathcal{U}(m)$  with  $\|u - v\|_2 \geq \delta_K$ . Now, since  $2 - 2^{1-k} \operatorname{Re}(1 + z)^k \rightarrow 0$  when  $z \rightarrow 1$ , we can choose some  $\varepsilon' > 0$  so that  $2 - 2^{1-k} \operatorname{Re}(1 + z)^k \leq \varepsilon^2$  when  $|z - 1| < \varepsilon'$ . Let  $u, v \in \mathcal{U}(m)$ . Since

$$\begin{aligned} |\operatorname{tr}_m(u^*v) - 1|^2 &= 1 + |\operatorname{tr}_m(u^*v)| - 2 \operatorname{Re} \operatorname{tr}_m(u^*v) \\ &\leq 2 - 2 \operatorname{Re} \operatorname{tr}_m(u^*v) \\ &= \|u - v\|_2^2, \end{aligned}$$

we see that

$$\|\pi_{m,k}(u) - \pi_{m,k}(v)\|_2^2 = 2 - 2^{1-k} \operatorname{Re}(1 + \operatorname{tr}_m(u^*v))^k \leq \varepsilon^2,$$

whenever  $\|u - v\|_2 \leq \varepsilon'$ . Thus, if we choose some  $m \in \mathbb{N}$  and a map  $\psi: G \rightarrow \mathcal{U}(m)$  with the properties (5.i) and (5.ii) corresponding to  $K$  and  $\varepsilon'$ , then the map

$$\varphi = \pi_{m,k} \circ \psi: G \rightarrow \mathcal{U}((2m)^k)$$

will satisfy (1.i) and (1.iii). The property (1.ii) follows from the fact that  $\mathbf{1}_G \in K$ , since then

$$\|\varphi(\mathbf{1}_G) - \mathbf{1}_{\mathcal{U}((2m)^k)}\|_2 = \|\varphi(\mathbf{1}_G)^2 - \varphi(\mathbf{1}_G^2)\|_2 \leq \varepsilon.$$

Thus the group  $G$  satisfies the condition (1).  $\square$

**Remark 5.1.6.** The fact that condition (4) and condition (3) of Proposition 5.1.5 are equivalent implies that  $G$  is hyperlinear if and only if satisfies:

(6) For each finite subset  $F \subseteq G$  and every  $\varepsilon > 0$  there exists some  $n \in \mathbb{N}$  and a map  $\varphi: G \rightarrow \mathcal{U}(n)$  with the properties

$$(6.i) \quad \|\varphi(gh) - \varphi(g)\varphi(h)\|_2 \leq \varepsilon \text{ for each } g, h \in F;$$

$$(6.ii) \quad \|\varphi(g) - \varphi(h)\|_2 \geq \frac{1}{10^5} \text{ for each } g, h \in F \text{ with } g \neq h.$$

Here the constant  $\frac{1}{10^5}$  is just an arbitrary constant in the interval  $(0, \sqrt{2})$ . This equivalent definition—maybe with another constant—is also sometimes used as the definition of a hyperlinear group.  $\blacktriangleleft$

## 5.2 Hyperlinear groups in terms of ultraproducts

The following result from 2005 is due to Gábor Elek and Endre Szabó (see [ES05]). More precisely, Elek and Szabó proved a similar result for sofic groups (see Proposition 6.5.2 below), but the argument is almost identical.

**Proposition 5.2.1.** *A group  $G$  is hyperlinear if and only if it embeds into a metric ultraproduct of finite rank unitary groups, that is, there exist an embedding  $\varphi: G \rightarrow \prod_{i \in I}^{\omega} \mathcal{U}(n_i)$  for some index set  $I$ , some ultrafilter  $\omega$  on  $I$  and some family of natural numbers  $(n_i)_{i \in I}$ .*

*Proof.* Suppose that  $G$  satisfies condition (1) from proposition 5.1.5. Let  $\mathbb{F}$  denote the set of finite subsets of  $G$  and let  $I = \mathbb{F} \times \mathbb{N}$ . For each  $(F, n) \in I$  choose some natural number  $k(F, n)$  and a map  $\varphi_{(F, n)}: G \rightarrow \mathcal{U}(k(F, n))$  so that  $\|\varphi_{(F, n)}(gh) - \varphi_{(F, n)}(g)\varphi_{(F, n)}(h)\|_2 < \frac{1}{n}$  for each  $g, h \in F$ ;  $\|\varphi_{(F, n)}(g) - \varphi_{(F, n)}(h)\|_2 > \sqrt{2} - \frac{1}{n}$  for each  $g, h \in F$  with  $g \neq h$ ; and  $\|\varphi_{(F, n)}(\mathbf{1}_G) - \mathbf{1}_{k(F, n)}\|_2 < \frac{1}{n}$ . Let  $\omega$  be the ultrafilter on  $I$  containing all the sets  $\{(F', n') \in I \mid F \subseteq F', n \leq n'\}$ . This ultrafilter exists by Proposition 1.6.3. Now define a map  $\varphi: G \rightarrow \prod_{i \in I}^{\omega} \mathcal{U}(k(i))$  by  $\varphi(g) = [(\varphi_i(g))_{i \in I}]$ . Let us show that this map is in fact an injective groups homomorphism. Suppose that  $g, h \in G$ . By construction,

$$\lim_{i \rightarrow \omega} \|\varphi_i(gh) - \varphi_i(g)\varphi_i(h)\|_2 = 0, \quad \text{and if } g \neq h, \quad \lim_{i \rightarrow \omega} \|\varphi_i(g) - \varphi_i(h)\|_2 = \sqrt{2}$$

so  $\varphi$  is multiplicative and injective. Likewise,  $\lim_{i \rightarrow \omega} \|\varphi_i(\mathbf{1}_G) - \mathbf{1}_{k(i)}\|_2 = 0$ , and so  $\varphi$  is an embedding of  $G$  into the metric ultraproduct  $\prod_{i \in I}^{\omega} \mathcal{U}(k(i))$ , which was what we needed to find.

Now suppose instead that  $\varphi: G \rightarrow \prod_{i \in I}^{\omega} \mathcal{U}(n_i)$  is an embedding, for some index set  $I$ , an ultrafilter  $\omega$  on  $I$  and a set  $\{n_i : i \in I\}$  of natural numbers. Denote the metric on  $\prod_{i \in I}^{\omega} \mathcal{U}(n_i)$  by  $d_{\omega}$ . Let us prove that  $G$  satisfies conditions (5) from Proposition 5.1.5. Let  $F \subseteq G$  be a finite subset and  $\varepsilon > 0$ , and let  $\delta = \frac{1}{2} \min\{d_{\omega}(\varphi(g), \varphi(h)) : g, h \in F, g \neq h\}$ . Since  $\varphi$  is an embedding  $\delta > 0$ . Let  $(\theta_i)_{i \in I}: G \rightarrow \ell_{\infty}(I; G_i)$  be any lift of  $\varphi$ . For each  $g, h \in F$ , we let

$$A_{g, h} = \{i \in I : \|\theta_i(gh) - \theta_i(g)\theta_i(h)\|_2 \leq \varepsilon\}$$

Then  $A_{g, h} \in \omega$ , since  $\theta$  is a homomorphism. Also, if  $g \neq h$  we let

$$B_{g, h} = \{i \in I : \|\theta_i(g) - \theta_i(h)\|_2 \geq \delta\},$$

which is also in  $\omega$  since  $d_{\omega}(\theta(g), \theta(h)) \geq 2\delta$ . Pick an element

$$j \in \left( \bigcap_{g, h \in F} A_{g, h} \right) \cap \left( \bigcap_{\substack{g, h \in F \\ g \neq h}} B_{g, h} \right).$$

This is possible since the above set is in  $\omega$  and is therefore non-empty. Now  $\theta_j: G \rightarrow \mathcal{U}(n_j)$  satisfies the conditions (5.i) and (5.ii) by choice of  $j$ . Thus by Proposition 5.1.5 the group  $G$  is hyperlinear.  $\square$

**Remark 5.2.2.** There are a few things in the proof of Proposition 5.2.1 worth noticing. For one, the embedding  $\varphi$  constructed actually satisfies  $d_{\omega}(\varphi(g), \varphi(h)) = \sqrt{2}$  when  $g \neq h$ , so the Proposition is still true if this is added as a requirement on  $\varphi$ . Besides this, if the group  $G$  is countable, then the set of finite subsets of  $G$  is also countable. Thus the index set  $I$  will be countable as well. This means that if the group is countable, then we can assume that the index set is  $\mathbb{N}$ .  $\blacktriangleleft$

In fact, one can exchange the finite rank unitary groups with a fixed group, if this fixed group contains an increasing sequence of finite rank unitary groups—or at least isomorphic copies of such—whose union is dense in this fixed group. This statement is made precise in the following proposition.

**Proposition 5.2.3.** *Suppose that  $\Gamma$  is a metric group, with a bi-invariant metric  $d$ , containing an increasing sequence of subgroups  $(H_k)_{k \in \mathbb{N}}$ , where  $H_n$  is isometrically isomorphic to  $\mathcal{U}(n_k)$ , for some unbounded sequence  $(n_1, n_2, n_3, \dots)$  of natural numbers. Moreover assume that the union  $\bigcup_{k=1}^{\infty} H_n$  is dense in  $\Gamma$ . Then a group  $G$  is hyperlinear if and only if  $G$  embeds into an ultrapower of  $\Gamma$ .*

*Proof.* Suppose that  $G$  is hyperlinear. By Theorem 5.2.1 we may choose an ultrafilter  $\omega$  on an index set  $I$  and a family of natural numbers  $(m_i)_{i \in I}$ , such that  $G$  embeds into  $\prod_{i \in I}^{\omega} \mathcal{U}(m_i)$ . Let  $\varphi$  be such an embedding. For each  $k \in \mathbb{N}$  let  $\pi_k: \mathcal{U}(n_k) \rightarrow \Gamma$  be an isometric isomorphism of  $\mathcal{U}(n_k)$  onto  $H_k$ . For each  $i \in I$ , choose  $k(i) \in \mathbb{N}$  so that  $m_i \leq n_{k(i)}$ . With the notation of Lemma 5.1.3 we define

$$\psi: \prod_{i \in I}^{\omega} \mathcal{U}(m_i) \longrightarrow \prod_{i \in I}^{\omega} \Gamma \quad \text{by} \quad \varphi([(u_i)_{i \in I}]) = [(\pi_{k(i)} \circ \rho_{m_i, k(i)}(u_i))_{i \in I}]$$

This map is a well-defined injective homomorphism by Proposition 4.1.4 since we can choose the constants  $C_i = 1$  and  $c_i = \frac{1}{\sqrt{2}}$ , for all  $i \in I$ , due to Lemma 5.1.3, and the fact that  $\pi_k$  is isometric, for all  $k \in \mathbb{N}$ . In particular  $\psi \circ \varphi: G \rightarrow \prod_{i \in I}^{\omega} \Gamma$  is an embedding of  $G$  into  $\prod_{i \in I}^{\omega} \Gamma$ .

Now, suppose on the other hand that  $\omega$  is an ultrafilter on  $I$  and that  $G$  embeds into  $\prod_{i \in I}^{\omega} \Gamma$ . Let  $\psi$  be such an embedding. Choose some map  $(\psi_i)_{i \in I}: G \rightarrow \ell_{\infty}(I; G_i)$  such that  $\psi(g) = [(\psi_i(g))_{i \in I}]$ , for all  $g \in G$ . We want to show that  $G$  satisfies condition (5) of Proposition 5.1.5, because then it is hyperlinear. So, let  $F \subseteq G$  be a finite subset. Let

$$\delta_F = \frac{1}{4} \min\{d_{\omega}(\psi(g), \psi(h)) : g, h \in G, g \neq h\}.$$

Since  $F$  is finite and  $\psi$  is injective, we have  $\delta_F > 0$ . For  $\varepsilon > 0$ . For  $g, h \in F^2 \cup F$  define

$$A_{g,h} = \{i \in I : \|\psi_i(gh) - \psi_i(g)\psi_i(h)\|_2 \leq \frac{1}{3}\varepsilon\},$$

which is an element of  $\omega$ , since  $\psi$  is a homomorphism. For  $g, h \in F$  with  $g \neq h$ , let

$$B_{g,h} = \{i \in I : \|\psi_i(g) - \psi_i(h)\|_2 \geq 3\delta_F\},$$

which is an element of  $\omega$  by the choice of  $\delta_F$ . Now, since all these sets are in  $\omega$ , the set

$$\left( \bigcap_{g,h \in F^2 \cup F} A_{g,h} \right) \cap \left( \bigcap_{\substack{g,h \in F \\ g \neq h}} B_{g,h} \right)$$

is also in  $\omega$ , and therefore non-empty. Let  $i$  be an element in this set. Since  $\bigcup_{k=1}^{\infty} H_k$  is dense in  $\Gamma$ , we can, for each  $g \in F^2 \cup F$ , choose  $n_g \in \mathbb{N}$  and  $u_g \in H_{n_g}$ , so that  $d(\psi_i(g), u_g) \leq \min\{\delta_F, \frac{1}{3}\varepsilon\}$ . Further, for each  $g \in G \setminus (F^2 \cup F)$  let  $u_g = \mathbf{1}_{\Gamma}$ . Let  $m = \max\{n_g : g \in F^2 \cup F\}$ , then  $u_g \in H_m$  for all  $g \in G$ , since  $(H_k)_{k \in \mathbb{N}}$  is an increasing sequence of subgroups. Define  $\varphi: G \rightarrow \mathcal{U}(n_m)$  by  $\varphi(g) = \pi_m^{-1}(u_g)$ . What we need to show is that this map satisfies the conditions (5.i) and (5.ii) from Proposition 5.1.5. First, if  $g, h \in F$ , then  $gh, g, h \in F^2 \cup F$ , and so

$$\begin{aligned} \|\pi_k^{-1}(u_{gh}) - \pi_k^{-1}(u_g u_h)\|_2 &= d(u_{gh}, u_g u_h) \\ &\leq d(u_{gh}, \psi_i(gh)) + d(\psi_i(gh), \psi_i(g)\psi_i(h)) + d(\psi_i(g)\psi_i(h), u_g u_h) < \varepsilon, \end{aligned}$$

which shows that  $\varphi$  satisfies (5.i). Now if  $g, h \in G$  with  $g \neq h$ , then

$$\begin{aligned} \|\pi_k^{-1}(u_g) - \pi_k^{-1}(u_h)\|_2 &= d(u_g, u_h) \\ &\geq d(\psi_i(g), \psi_i(h)) - d(u_g, \psi_i(g)) - d(\psi_i(h), u_h) \\ &\geq 3\delta_F - \delta_F - \delta_F = \delta_F, \end{aligned}$$

which shows that  $\varphi$  satisfies (5.ii). Thus by Theorem 5.2.1, the group  $G$  embeds into an ultraproduct of finite rank unitary groups.  $\square$

**Remark 5.2.4.** If the group  $G$  in Proposition 5.2.3 is countable, then the index set  $I$  constructed in the proof can be assumed to be countable by Remark 5.2.2. Thus, a countable group is hyperlinear if and only if it embeds into  $\Gamma^\omega$ , for some choice of ultrafilter on  $\mathbb{N}$ .  $\blacktriangleleft$

### 5.3 Embedding in ultrapowers of the hyperfinite $\text{II}_1$ -factor

The reason hyperlinear groups are of interest—at least in this thesis—is their connection to the Connes Embedding Problem. In this section we discuss the direct connection between hyperlinear groups and the Connes Embedding Problem. We also relate the present Definition 5.1.1 of a hyperlinear group to the original definition of Rădulescu. But first we need a couple of intermediate results.

**Lemma 5.3.1.** *Let  $\mathcal{M}$  be a finite factor and  $v \in \mathcal{M}$  a partial isometry. Then there exist a unitary  $u \in \mathcal{M}$  such that  $u$  and  $v$  agree on the support of  $v$ . In particular, if  $x \in \mathcal{M}$  then there exist a unitary  $u$  such that  $x = u|x|$ .*

*Proof.* Let  $p = v^*v$  and  $q = vv^*$ . Since  $p \sim q$  and  $\mathcal{M}$  is a finite factor we know  $\mathbf{1} - p \sim \mathbf{1} - q$ . Choose  $\tilde{v} \in \mathcal{M}$  with  $\tilde{v}^*\tilde{v} = \mathbf{1} - p$  and  $\tilde{v}\tilde{v}^* = \mathbf{1} - q$ . Let  $u = v + \tilde{v}$ . Since  $v^*\tilde{v} = 0$  and  $\tilde{v}^*v = 0$  we get  $u^*u = uu^* = \mathbf{1}$ , so  $u$  is unitary. The support of  $v$  and the kernel of  $\tilde{v}$  agrees, so  $u$  agree with  $v$  on the support of  $v$ .  $\square$

**Proposition 5.3.2.** *Suppose that  $(\mathcal{M}_i)_{i \in I}$  is a family of finite factors and  $\omega$  is an ultrafilter on  $I$ . If  $u \in \prod_{i \in I}^\omega \mathcal{M}_i$  is unitary, then there exist unitaries  $u_i \in \mathcal{M}_i$  ( $i \in I$ ) so that  $u = [(u_i)_{i \in I}]$ .*

*Proof.* Choose  $v_i \in \mathcal{M}_i$  for each  $i \in I$  so that  $u = [(v_i)_{i \in I}]$ . For each  $i \in I$  choose a unitary  $u_i \in \mathcal{M}_i$  so that  $v_i = u_i|v_i|$ . This is possible by Lemma 5.3.1. Now we see that  $u = [(u_i|v_i|)_{i \in I}] = [(u_i)_{i \in I}][(|v_i|)_{i \in I}]$ , and so we only have to argue that  $[(|v_i|)_{i \in I}] = \mathbf{1}$ . Since  $\varphi(y)^{1/2} = \varphi(y^2/2)$  for every positive element  $y$  in a  $C^*$ -algebra and every  $*$ -homomorphism  $\varphi$ , we get that

$$[(|v_i|)_{i \in I}] = [((v_i^*v_i)^{1/2})_{i \in I}] = [((v_i^*v_i))_{i \in I}]^{1/2} = (u^*u)^{1/2} = \mathbf{1}.$$

Thus we have proved the proposition.  $\square$

**Corollary 5.3.3.** *The natural map  $\varphi: \prod_{i \in I}^\omega \mathcal{U}(\mathcal{M}_i) \rightarrow \mathcal{U}(\prod_{i \in I}^\omega \mathcal{M}_i)$  is an isomorphism of groups.*

*Proof.* First, note that  $\varphi$  is the map

$$(\ell_\infty(I; \mathcal{U}(\mathcal{M}_i))) / (\mathcal{I} \cap \ell_\infty(I; \mathcal{U}(\mathcal{M}_i))) \hookrightarrow (\ell_\infty(I; \mathcal{M}_i)) / \mathcal{I}$$

induced by the inclusion  $\ell_\infty(I; \mathcal{U}(\mathcal{M}_i)) \hookrightarrow \mathcal{U}(\ell_\infty(I; \mathcal{M}_i))$ . Thus,  $\varphi$  is clearly injective, and in Proposition 5.3.2 we have proved that it is also surjective. The fact that it is a group homomorphism is straightforward to check.  $\square$

**Lemma 5.3.4.** *Suppose that  $\mathcal{M}$  is a von Neumann algebra with a faithful normal trace  $\tau$ , and  $\mathcal{A}$  is a  $*$ -subalgebra. If the unitary group of  $\mathcal{A}$  is strong operator dense in the unitary group of  $\mathcal{M}$ , then it is also dense with respect to the norm induced by  $\tau$ .*

*Proof.* Suppose that  $u$  is a unitary element in  $\mathcal{M}$ , and let  $(u_\alpha)_{\alpha \in A}$  be a net of unitary elements in  $\mathcal{A}$  converging to  $u$  in strong operator topology. Then  $(u - u_\alpha)^*(u - u_\alpha)$  converges to zero in the weak operator topology, so since  $\tau$  is weak operator continuous,  $\|u - u_\alpha\|_\tau = \tau((u - u_\alpha)^*(u - u_\alpha))$  converges to zero as well. Hence the net  $(u_\alpha)_{\alpha \in A}$  converges to  $u$  in the norm induced by  $\tau$ , which shows that the unitary group of  $\mathcal{A}$  is dense in the unitary group of  $\mathcal{M}$  with respect to the norm induced by  $\tau$ .  $\square$

Now we are ready to prove that our definition of a hyperlinear group is indeed equivalent to—a slightly extended version of—Rădulescu’s original definition of hyperlinear groups (see Remark 5.3.6 for details on this).

**Proposition 5.3.5.** *A group is hyperlinear if and only if it embeds into  $\mathcal{U}(\mathcal{R}^\omega)$  for some set  $I$  (not necessarily countable) and some ultrafilter  $\omega$  on  $I$ .*

*Proof.* The strategy of the proof is to prove that  $\mathcal{U}(\mathcal{R})$  satisfy the conditions of the group  $\Gamma$  in Proposition 5.2.3, and then use Corollary 5.3.3.

Since  $\mathcal{R}$  is approximately finite there exists an ascending sequence of von Neumann subalgebras of  $\mathcal{R}$ , say  $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots$ , such that  $\mathcal{M}_k$  is a factor of type  $I_{n_k}$ , for each  $k \in \mathbb{N}$ , for some strictly increasing sequence  $n_1, n_2, n_3, \dots$  of natural numbers. Since  $\mathcal{M}_k$  is of type  $I_{n_k}$ , and thus isomorphic to  $\mathbb{M}_{n_k}$  (see [KR86, Theorem 6.6.1]), we get that  $\mathcal{U}(\mathcal{M}_k)$  is isomorphic to  $\mathcal{U}(n_k)$ . Since the normalized trace on a finite factor is unique, the normalized trace on  $\mathcal{R}$  restricts to the normalized trace on  $\mathcal{M}_k$ , for all  $k \geq 1$ . Thus the Hilbert-Schmidt distance on  $\mathcal{U}(\mathcal{R})$  restricts to the Hilbert-Schmidt distance on  $\mathcal{U}(\mathcal{M}_k)$ , for all  $k \geq 1$ . Now all we need to show is that  $\bigcup_{k=1}^{\infty} \mathcal{U}(\mathcal{M}_k)$  is dense in  $\mathcal{U}(\mathcal{R})$  with respect to the Hilbert-Schmidt distance. The set  $\mathcal{M} = \bigcup_{k=1}^{\infty} \mathcal{M}_k$  is a  $*$ -subalgebra of  $\mathcal{R}$ , with the property that when  $x \in \mathcal{M}$  is normal and  $f \in C(\sigma(x); \mathbb{C})$ , then  $f(x) \in \mathcal{M}$ . By Kaplansky’s Density Theorem the unitary group of  $\mathcal{M}$ , that is,  $\bigcup_{k=1}^{\infty} \mathcal{U}(\mathcal{M}_k)$ , is strong operator dense in  $\mathcal{U}(\mathcal{R})$  (see [KR86, Corollary 5.3.7], and the proof thereof). By Lemma 5.3.4 we get that  $\bigcup_{k=1}^{\infty} \mathcal{U}(\mathcal{M}_k)$  is dense in  $\mathcal{U}(\mathcal{R})$  with respect to the topology induced by the Hilbert-Schmidt distance. Now, by Proposition 5.2.3, a group  $G$  is hyperlinear if and only if  $G$  embeds into the ultrapower  $\prod_{i \in I}^{\omega} \mathcal{U}(\mathcal{R})$  for some index set  $I$  and some choice of an ultrafilter  $\omega$  on  $I$ . By Corollary 5.3.3 we know that  $\prod_{i \in I}^{\omega} \mathcal{U}(\mathcal{R})$  is isomorphic to  $\mathcal{U}(\mathcal{R}^\omega)$ . Thus a group is hyperlinear if and only if it embeds into  $\mathcal{U}(\mathcal{R}^\omega)$  for some index set  $I$  and some ultrafilter  $\omega$  on  $I$ .  $\square$

**Remark 5.3.6.** Suppose that a group  $G$  is countable. Then, in view of Remark 5.2.4, Proposition 5.3.5 reads that the group  $G$  is hyperlinear if and only if it embeds into  $\mathcal{U}(\mathcal{R}^\omega)$  for some choice of an ultrafilter  $\omega$  on  $\mathbb{N}$ . In particular, a group is hyperlinear in the sense of Definition 5.1.1 if and only if the group is hyperlinear in the sense of the definition originally introduced by Rădulescu [Răd08, Definition 2.6].  $\blacktriangleleft$

Now we are ready to establish the connection between hyperlinear groups and the Connes Embedding Conjecture. First we need an intermediate result.



**Proposition 5.3.7.** *Suppose that  $\varphi: G \rightarrow \mathcal{U}(\mathcal{M})$  is an embedding of a group  $G$  with infinite conjugacy classes into the unitary group of a von Neumann algebra  $\text{II}_1$ -factor  $\mathcal{M}$ . Then  $\varphi$  extends to an embedding  $\tilde{\varphi}: \text{LG} \rightarrow \mathcal{M}$  if and only if  $\text{tr}(\varphi(g)) = 0$ , for all  $g \neq \mathbf{1}_G$ .*

*Proof.* Since  $\text{LG}$  is a  $\text{II}_1$ -factor, a potential extension to an embedding  $\tilde{\varphi}: \text{LG} \rightarrow \mathcal{M}$  must necessarily be trace-preserving (when  $\text{LG}$  and  $\mathcal{M}$  are equipped with their corresponding tracial states). Hence, the fact that  $\text{tr}(\varphi(g)) = 0$  for all  $g \neq \mathbf{1}_G$  is a necessary requirement. Let us show that is also sufficient. Clearly  $\varphi$  extends to a trace-preserving  $*$ -homomorphism from  $\text{span}\{\delta_g : g \in G\}$  to  $\mathcal{M}$ . Since  $\text{span}\{\delta_g : g \in G\}$  is a weak operator dense  $*$ -subalgebra of  $\text{LG}$ , we can extend the given map to a trace-preserving  $*$ -homomorphism  $\tilde{\varphi}: \text{LG} \rightarrow \mathcal{M}$ . This map is clearly an embedding, since it preserves the faithful trace.  $\square$

**Lemma 5.3.8.** *Suppose that  $\mathcal{M}$  is a von Neumann algebra with faithful trace  $\tau$ . If  $u \in \mathcal{U}(\mathcal{M})$  with  $\tau(u) = 1$ , then  $u = \mathbf{1}$ . In particular, if  $u \neq \mathbf{1}$  then we must have  $0 < |\tau(u) + 1| < 2$ .*

*Proof.* It is straightforward to check that if  $\tau(u) = 1$  then  $\|u - \mathbf{1}\|_2 = 0$ , so by faithfulness of  $\tau$  we conclude that  $u = \mathbf{1}$ .  $\square$

The following proposition is due to Rădulescu:

**Proposition 5.3.9.** *A group  $G$  is hyperlinear if and only if  $\text{LG}$  embeds into  $\mathcal{R}^\omega$  for some choice of an index set  $I$  and of an ultrafilter  $\omega$  on  $I$ .*

*Proof.* One direction is easy. If  $\varphi$  is an embedding of  $\text{LG}$  into  $\mathcal{R}^\omega$  then the map  $g \mapsto \varphi(\lambda_g)$  will be an embedding of  $G$  into  $\mathcal{U}(\mathcal{R}^\omega)$ .

Suppose conversely that  $\varphi$  is an embedding of  $G$  into  $\mathcal{U}(\mathcal{R}^\omega)$ . By successive application of Proposition 1.2.8 we obtain an isomorphism  $\pi_k: \text{M}_2(\mathcal{R} \otimes \mathcal{R})^{\otimes k} \rightarrow \mathcal{R}$ , for each  $k \in \mathbb{N}$ . These isomorphisms  $\pi_k$  must necessarily be trace-preserving, for all  $k \in \mathbb{N}$ , since both  $\mathcal{R}$  and  $\text{M}_2(\mathcal{R} \otimes \mathcal{R})^{\otimes k}$  are  $\text{II}_1$ -factors. Now let  $\rho_k: \mathcal{R} \rightarrow \text{M}_2(\mathcal{R} \otimes \mathcal{R})^{\otimes k}$  denote the map

$$\rho_k(x) = \begin{bmatrix} x \otimes \mathbf{1}_{\mathcal{R}} & \\ & x \otimes x \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} x \otimes \mathbf{1}_{\mathcal{R}} & \\ & x \otimes x \end{bmatrix} \quad (k \text{ copies})$$

This map is unital, multiplicative and preserves the  $*$ -operation. Thus  $\pi_k \circ \rho_k$  is a unital and multiplicative map of  $\mathcal{R}$  into  $\mathcal{R}$  which preserves the  $*$ -operation. In particular, it restricts to a group homomorphism from  $\mathcal{U}(\mathcal{R})$  to itself. Now since  $\pi_k$  is trace-preserving and  $\text{tr}(\rho_k(x)) = (\frac{1}{2}\text{tr}(x) + \frac{1}{2}\text{tr}(x)^2)^k$ , we get that

$$\text{tr}(\pi_k \circ \rho_k(x)) = \left(\frac{1}{2}\text{tr}(x) + \frac{1}{2}\text{tr}(x)^2\right)^k,$$

for all  $k \in \mathbb{N}$ . By Proposition 5.3.1 we can choose a lift  $(\theta_i)_{i \in I}: G \rightarrow \ell_\infty(I; \mathcal{R})$  of  $\varphi$  so that  $\theta_i(g) \in \mathcal{U}(\mathcal{R})$ , for all  $g \in G$  and  $i \in I$ . Let  $\nu$  be any free ultrafilter in  $\mathbb{N}$  and define a map

$$\psi: G \rightarrow \mathcal{U}((\mathcal{R}^\omega)^\nu) \quad \text{by} \quad \psi(g) = \left[ \left( \left[ (\pi_k \circ \rho_k(\theta_i(g)))_{i \in I} \right]_{k \in \mathbb{N}} \right) \right]$$

Since  $\pi_k \circ \rho_k(\theta_i(g))$  is unitary for all  $k \in \mathbb{N}$ ,  $i \in I$  and  $g \in G$ , this map is well-defined. Note further that if  $g, h \in G$ , then

$$\begin{aligned} d(\psi(g), \psi(h))^2 &= \lim_{k \rightarrow \nu} \lim_{i \rightarrow \omega} \|\pi_k \rho_k(\theta_i(g)) - \pi_k \rho_k(\theta_i(h))\|_2^2 \\ &= \lim_{k \rightarrow \nu} \lim_{i \rightarrow \omega} \left( 2 - 2 \operatorname{Re} \operatorname{tr}(\pi_k \rho_k(\theta_i(g)^* \theta_i(h))) \right) \\ &= 2 - 2 \lim_{k \rightarrow \nu} \lim_{i \rightarrow \omega} \operatorname{Re} \frac{1}{2^k} \left( \operatorname{tr}(\theta_i(g)^* \theta_i(h)) + \operatorname{tr}(\theta_i(g) \theta_i(h))^2 \right)^k. \end{aligned}$$

In particular, since  $\varphi$  is a homomorphism,  $\lim_{i \rightarrow \omega} \operatorname{tr}(\theta_i(gh)^* \theta_i(g) \theta_i(h)) = 1$ , and hence it follows that

$$\begin{aligned} d(\psi(gh), \psi(g)\psi(h))^2 &= 2 - 2 \lim_{k \rightarrow \nu} \lim_{i \rightarrow \omega} \operatorname{Re} \frac{1}{2^k} \left( \operatorname{tr}(\theta_i(gh)^* \theta_i(g) \theta_i(h)) + \operatorname{tr}(\theta_i(gh)^* \theta_i(g) \theta_i(h))^2 \right)^k \\ &= 2 - 2 \lim_{k \rightarrow \nu} \operatorname{Re} \frac{1}{2^k} 2^k = 0. \end{aligned}$$

Thus  $\psi$  is a homomorphism, and it follows that, if  $g \neq h$ , then  $[(\theta_i(g)^* \theta_i(h))_{i \in I}] \neq \mathbf{1}$ . Let  $z = \lim_{i \rightarrow \omega} \operatorname{tr}(\theta_i(g)^* \theta_i(h))$ , then by Lemma 5.3.8 we get that  $z \neq 1$ , and since we know that  $|z| \leq 1$  we conclude that  $\frac{1}{2^k} (z + z^2)^k \rightarrow 0$  as  $k \rightarrow \infty$ . Now, since  $\nu$  is a free ultrafilter

$$\begin{aligned} d(\psi(g), \psi(h))^2 &= 2 - 2 \lim_{k \rightarrow \nu} \lim_{i \rightarrow \omega} \operatorname{Re} \frac{1}{2^k} \left( \operatorname{tr}(\theta_i(gh)^* \theta_i(g) \theta_i(h)) + \operatorname{tr}(\theta_i(gh)^* \theta_i(g) \theta_i(h))^2 \right)^k \\ &= 2 - 2 \lim_{k \rightarrow \nu} \operatorname{Re} \frac{1}{2^k} (z + z^2)^k = 2. \end{aligned}$$

This shows that  $d(\psi(g), \psi(h)) = \sqrt{2}$ , for all  $g, h \in G$  with  $g \neq h$ , or equivalently, that  $\operatorname{tr}(\psi(g)) = 0$  for all  $g \neq \mathbf{1}_G$ . By Proposition 5.3.7 the map  $\psi$  extends to an embedding  $\psi: LG \rightarrow (\mathcal{R}^\omega)^\nu$ , and by Proposition 4.3.5 we know that  $(\mathcal{R}^\omega)^\nu$  is isomorphic to  $\mathcal{R}^{\nu \otimes \omega}$ . Hence by composing the right maps we obtain an embedding of  $LG$  into  $\mathcal{R}^{\nu \otimes \omega}$ , which concludes the proof.  $\square$

**Corollary 5.3.10.** *A countable group is hyperlinear if and only if its group von Neumann algebra embeds into  $\mathcal{R}^\omega$  for some choice of an ultrafilter  $\omega$  on  $\mathbb{N}$ . This ultrafilter can be chosen to be free.*

*Proof.* Let us recall the proof of Proposition 5.3.9 when we restrict our attention to countable groups. Therein we took an embedding  $\varphi$  of  $G$  into  $\mathcal{R}^\omega$  for some choice of an ultrafilter  $\omega$  on some index set  $I$ . Afterwards, we constructed an embedding into  $\mathcal{R}^{\nu \otimes \omega}$ , where  $\nu \otimes \omega$  is an ultrafilter on  $\mathbb{N} \times I$ . Thus, if the group is countable, then by Remark 5.3.6 the set  $I$  can be chosen to be countable, and the set  $\mathbb{N} \times I$  will therefore also be countable. Since  $\nu$  was chosen to be a free ultrafilter,  $\nu \otimes \omega$  is also free by Proposition 4.3.2.  $\square$

## Chapter 6

# Sofic groups

In this chapter we introduce the concept of sofic groups, which is a relatively new one—introduced around 1999. Sofic groups are of interest for several reasons, but in this thesis they are of interest particularly because of their connection to the Connes Embedding Problem. More precisely, we shall see that sofic groups are hyperlinear and thus satisfy the Connes Embedding Problem for Groups.

To mention a few other applications of sofic groups, then the following two conjectures are solved in the positive for sofic groups.

**Gottschalk’s Surjectivity Conjecture.** Every countable group  $G$  is surjunctive, that is, every continuous  $G$ -equivariant map  $f: \{1, 2, \dots, n\}^G \rightarrow \{1, 2, \dots, n\}^G$ , which is injective is necessarily surjective.<sup>1</sup> By continuous we mean with respect to the product topology on  $\{1, 2, \dots, n\}^G$ , when  $\{1, 2, \dots, n\}$  has the discrete topology.

Let us explain what a  $G$ -equivariant map means. The group  $G$  acts naturally on the space  $\{1, 2, \dots, n\}^G$  by shifting the index, that is,  $g \cdot (n_h)_{h \in G} = (n_{g^{-1}h})_{h \in G}$ . A map  $f: \{1, 2, \dots, n\}^G \rightarrow \{1, 2, \dots, n\}^G$  is called  $G$ -equivariant if  $f(g \cdot x) = g \cdot f(x)$  for all  $x \in \{1, 2, \dots, n\}^G$ .

**Kaplansky’s Direct Finiteness Conjecture.** For any group  $G$  and commutative field  $\mathbb{K}$ , the group algebra  $\mathbb{K}G$  is directly finite, that is,  $ab = 1$  in  $\mathbb{K}G$  implies  $ba = 1$ .

### 6.1 The definition of a sofic group

Sofic groups—at least the finitely generated ones—were first introduced by Mikhail Gromov his article [Gro99] of 1999. Gromov defined a notion for graphs he called *initially subamenable*, and sofic groups were introduced as finitely generated groups whose Cayley graph is initially subamenable. The name *sofic* is due to Benjamin Weiss—introduced in his article [Wei00] of 2000. In our presentation, we choose a different definition than the original one of Gromov (see Definition 6.1.3), but first we need to introduce the Hamming metric.

**Definition 6.1.1.** For a finite set  $F$  we denote by  $\text{Sym}(F)$  the symmetric group on  $F$ , that is, the set of all permutations of the elements of  $F$ , and for  $\alpha \in \text{Sym}(F)$  we let

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<sup>1</sup>Here  $X^Y$  denote the set of maps  $Y \rightarrow X$ .

$\text{fix}(\alpha)$  denote the fixed points of  $\alpha$ . The **Hamming metric** on  $\text{Sym}(F)$  is the metric  $d_F$  defined by

$$d_F(\alpha, \beta) = \frac{1}{|F|} |\{f \in F : \alpha(f) \neq \beta(f)\}| = \frac{1}{|F|} (|F| - |\text{fix}(\alpha^{-1}\beta)|),$$

for all  $\alpha, \beta \in \text{Sym}(F)$ . ◀

Let us just for precaution check that the Hamming metric is indeed a metric. Clearly  $d_F(\alpha, \beta)$  is greater than or equal to zero, and equal to zero if and only if  $\alpha = \beta$ , so what we need to show is that the triangle inequality holds. Suppose that  $\alpha, \beta, \gamma \in \text{Sym}(F)$ . If  $f_0 \in \{f \in F : \alpha(f) \neq \gamma(f)\}$  then

$$\text{either } f_0 \in \{f \in F : \alpha(f) \neq \beta(f)\}, \quad \text{or } f_0 \in \{f \in F : \beta(f) \neq \gamma(f)\}.$$

In particular it follows that  $d_F(\alpha, \gamma) \leq d_F(\alpha, \beta) + d_F(\beta, \gamma)$ , and so we have proved the triangle inequality.

**Proposition 6.1.2.** *The Hamming metric  $d_F$  on  $\text{Sym}(F)$  for a finite set  $F$  is a bi-invariant metric.*

*Proof.* This follows directly from the fact that for  $f \in F$ ,  $\alpha(f) \neq \beta(f)$  if and only if  $\gamma\alpha(f) \neq \gamma\beta(f)$ , and likewise, that  $\alpha(f) \neq \beta(f)$  if and only if  $\alpha\gamma(f) \neq \beta\gamma(f)$ .  $\square$

We now define the concept of a sofic group. The present definition is due to Elek and Szabó (see [ES06, Definition 1.1 & Lemma 2.1]), who proved that it was equivalent to the original definition of Gromov in the case of finitely generated groups (see [ES04, Proposition 4.4]).

**Definition 6.1.3.** A group  $G$  is **Sofic** if for every finite set  $K \subseteq G$  and every  $\varepsilon > 0$ , there exist a non-empty finite set  $F$  and a map  $\varphi: G \rightarrow \text{Sym}(F)$  satisfying:

- (i)  $d_F(\varphi(gh), \varphi(g)\varphi(h)) \leq \varepsilon$ , for each  $g, h \in K$ ;
- (ii)  $d_F(\varphi(g), \varphi(h)) \geq 1 - \varepsilon$ , for each  $g, h \in K$  with  $g \neq h$ . ◀

As in the definition of hyperlinear groups, we observe that there are no restrictions on the map  $\varphi$  outside the set  $K \cup K^2$ . Besides this, the requirement on  $\varphi$  is vacuous in the case where  $\varepsilon \geq 1$ . As we will often need to speak about maps  $\varphi$  satisfying the conditions of the above definition, we will call such a map  $\varphi$  a  $(K, \varepsilon)$ -**almost homomorphism**.

Surprisingly enough, it is not known whether all groups are in fact sofic. Since being sofic is a local property, to ask whether all groups are sofic is the same as asking whether all countable groups, or even all finitely generated groups, are sofic.

Suppose that we are given  $n \in \mathbb{N}$  and finite sets  $F_1, \dots, F_n$ . There is a natural map  $\Phi: \prod_{k=1}^n \text{Sym}(F_k) \rightarrow \text{Sym}(F_1 \times \dots \times F_n)$ , namely the one defined as follows: for  $\alpha_i \in \text{Sym}(F_i)$  and  $f_i \in F_i$  ( $i = 1, \dots, n$ ), we let

$$\Phi(\alpha_1, \dots, \alpha_n)(f_1, \dots, f_n) = (\alpha_1(f_1), \dots, \alpha_n(f_n)).$$

The following proposition expresses the distance between elements in the image of  $\Phi$  in terms of the metrics  $d_{F_1}, d_{F_2}, \dots, d_{F_n}$ .

**Proposition 6.1.4.** *Given  $n \in \mathbb{N}$  and finite sets  $F_1, \dots, F_n$ , let  $F$  denote the set  $F_1 \times \dots \times F_n$ . Let  $\Phi : \prod_{i=1}^n \text{Sym}(F_i) \rightarrow \text{Sym}(F)$  be the natural map described above. If  $\alpha = (\alpha_1, \dots, \alpha_n)$  and  $\beta = (\beta_1, \dots, \beta_n)$  are in  $\prod_{i=1}^n \text{Sym}(F_i)$ , then*

$$d_F(\Phi(\alpha), \Phi(\beta)) = 1 - \prod_{i=1}^n (1 - d_{F_i}(\alpha_i, \beta_i))$$

*Proof.* Since  $|F| = \prod_{i=1}^n |F_i|$ , the statement that we want to prove is equivalent to showing that

$$|F|(1 - d_F(\Phi(\alpha), \Phi(\beta))) = \prod_{i=1}^n |F_i|(1 - d_{F_i}(\alpha_i, \beta_i)).$$

The number on the left hand side is equal to  $|\text{fix}(\alpha^{-1}\beta)|$ , whilst the number on the right hand side is equal to  $\prod_{i=1}^n |\text{fix}(\alpha_i^{-1}\beta_i)|$ . Thus we only need to show that these two numbers agree. It follows easily by inspection that

$$\text{fix}(\alpha^{-1}\beta) = \text{fix}(\alpha_1^{-1}\beta_1) \times \dots \times \text{fix}(\alpha_n^{-1}\beta_n),$$

and therefore the two numbers must coincide as desired.  $\square$

The above proposition is an analogue of Lemma 5.1.4 for hyperlinear groups, and the next proposition is an analogue of Proposition 5.1.5.

**Proposition 6.1.5.** *For a group  $G$  the following set of conditions are equivalent:*

- (1) *For each finite subset  $K \subseteq G$  and every  $\varepsilon > 0$ , there exist a non-empty finite set  $F$  and a map  $\varphi: G \rightarrow \text{Sym}(F)$  with the properties:*
  - (1.i)  $d_F(\varphi(gh), \varphi(g)\varphi(h)) \leq \varepsilon$  for each  $g, h \in K$ ;
  - (1.ii)  $d_F(\varphi(\mathbf{1}_G), \mathbf{1}_{\text{Sym}(F)}) \leq \varepsilon$ ;
  - (1.iii)  $d_F(\varphi(g), \varphi(h)) \geq 1 - \varepsilon$  for each  $g, h \in K$  with  $g \neq h$ .
- (2) *For each finite subset  $K \subseteq G$  and every  $\varepsilon > 0$ , there exist a non-empty finite set  $F$  and a map  $\varphi: G \rightarrow \text{Sym}(F)$  with the properties:*
  - (2.i)  $d_F(\varphi(gh), \varphi(g)\varphi(h)) \leq \varepsilon$  for each  $g, h \in K$ ;
  - (2.ii)  $d_F(\varphi(g), \varphi(h)) \geq 1 - \varepsilon$  for each  $g, h \in K$  with  $g \neq h$ .
- (3) *For every constant  $\delta \in (0, 1)$ , each finite subset  $K \subseteq G$  and every  $\varepsilon > 0$ , there exist a non-empty finite set  $F$  and a map  $\varphi: G \rightarrow \text{Sym}(F)$  with the properties:*
  - (3.i)  $d_F(\varphi(gh), \varphi(g)\varphi(h)) \leq \varepsilon$  for each  $g, h \in K$ ;
  - (3.ii)  $d_F(\varphi(g), \varphi(h)) \geq \delta$  for each  $g, h \in K$  with  $g \neq h$ .
- (4) *There exist some  $\delta > 0$  such that for each finite subset  $K \subseteq G$  and every  $\varepsilon > 0$ , there exist a non-empty finite set  $F$  and a map  $\varphi: G \rightarrow \text{Sym}(F)$  with the properties:*
  - (4.i)  $d_F(\varphi(gh), \varphi(g)\varphi(h)) \leq \varepsilon$  for each  $g, h \in K$ ;
  - (4.ii)  $d_F(\varphi(g), \varphi(h)) \geq \delta$  for each  $g, h \in K$  with  $g \neq h$ .

(5) For each finite subset  $K \subseteq G$  there exists some  $\delta_K > 0$  such that for every  $\varepsilon > 0$ , there exist a non-empty finite set  $F$  and a map  $\varphi: G \rightarrow \text{Sym}(F)$  with the properties:

$$(5.i) \quad d_F(\varphi(gh), \varphi(g)\varphi(h)) \leq \varepsilon \text{ for each } g, h \in K;$$

$$(5.ii) \quad d_F(\varphi(g), \varphi(h)) \geq \delta_K \text{ for each } g, h \in K \text{ with } g \neq h.$$

*Proof.* The implications (1)  $\Rightarrow$  (2), (3)  $\Rightarrow$  (4) and (4)  $\Rightarrow$  (5) are trivial, and the implication (2)  $\Rightarrow$  (3) follows by replacing  $\varepsilon$  in (2) with  $\min\{\varepsilon, \delta\}$ . Thus the only implication we need to prove is (5)  $\Rightarrow$  (1). Now assume that  $G$  satisfies (5). Let  $K \subseteq G$  be finite and  $\varepsilon > 0$ . We may assume that  $K$  has at least two elements, or else we may just choose  $F$  to be a one-point set, in which case there is only one choice of the map  $\varphi$ , and this choice is a homomorphism. We will prove that  $G$  satisfies (1) by using Proposition 6.1.4. Since  $K$  has at least two elements it follows from (5.ii) that  $\delta_K < 1$ , and we may therefore choose  $m \in \mathbb{N}$  such that  $1 - (1 - \delta_K)^m \geq 1 - \varepsilon$ . Likewise, we may choose  $\tilde{\varepsilon} > 0$  so that  $1 - (1 - \tilde{\varepsilon})^m \leq \varepsilon$ . Now choose a finite set  $\tilde{F}$  and a map  $\tilde{\varphi}: G \rightarrow \text{Sym}(\tilde{F})$  satisfying conditions (5.i) and (5.ii) for the finite set  $\{\mathbf{1}_G\} \cup K$  and  $\varepsilon'$ . Let  $\Phi: \prod_{n=1}^m \text{Sym}(\tilde{F}) \rightarrow \text{Sym}(\tilde{F}^m)$  be the map from Proposition 6.1.4. Let  $F = \tilde{F}^m$  and define  $\varphi: G \rightarrow \text{Sym}(F)$  by

$$\varphi(g) = \Phi(\tilde{\varphi}(g), \tilde{\varphi}'(g), \dots, \tilde{\varphi}'(g)), \quad g \in G.$$

By using Proposition 6.1.4, it is straightforward to check that  $F$  and  $\varphi$  satisfy the conditions (1.i), (1.ii) and (1.iii) corresponding to  $K$  and  $\varepsilon$ .  $\square$

**Remark 6.1.6.** Combining condition (4) and condition (3) of Proposition 6.1.5 we infer that a group  $G$  is sofic if and only if satisfies:

(6) For each finite subset  $K \subseteq G$  and every  $\varepsilon > 0$ , there exist a non-empty finite set  $F$  and a map  $\varphi: G \rightarrow \text{Sym}(F)$  with the properties:

$$(6.i) \quad d_F(\varphi(gh), \varphi(g)\varphi(h)) \leq \varepsilon \text{ for each } g, h \in K;$$

$$(6.ii) \quad d_F(\varphi(g), \varphi(h)) \geq \frac{1}{10^8} \text{ for each } g, h \in K \text{ with } g \neq h.$$

The constant  $\frac{1}{10^8}$  is just an arbitrary constant in the interval  $(0, 1)$ . This equivalent formulation is also sometimes used as the definition of a sofic group—possibly with a different constant.  $\blacktriangleleft$

## 6.2 Connection to the Connes Embedding Problem

It is well-known that the symmetric group of order  $n$  can be embedded into the unitary group  $\mathcal{U}(n)$  in a natural way—which is illustrated in Theorem 6.2.1—but it is not the case that the Hilbert-Schmidt distance on  $\mathcal{U}(n)$  restricts to the Hamming distance on this image of the symmetric group. There is though a connection between the two, and this allows us to prove the following proposition.

**Theorem 6.2.1.** *If a group is sofic, then it is hyperlinear.*

*Proof.* Suppose that  $G$  is a sofic group, and let us prove that  $G$  is hyperlinear. So suppose that  $K \subseteq G$  is a finite set and  $\varepsilon > 0$ . We need to find  $n \in \mathbb{N}$  and a map  $\varphi: G \rightarrow \mathcal{U}(n)$  satisfying  $\|\varphi(gh) - \varphi(g)\varphi(h)\|_2 \leq \varepsilon$ , for all  $g, h \in K$  and  $\|\varphi(g) - \varphi(h)\|_2 \geq 1 - \varepsilon$ , for all  $g, h \in K$  with  $g \neq h$ .

Let  $K \subset G$  be finite and  $\varepsilon > 0$ . We may assume that  $\varepsilon < 2\sqrt{2}$ . Let  $\varepsilon' = \min\{\frac{\varepsilon^2}{2}, 1 - \frac{1}{2}(\sqrt{2} - \varepsilon)^2\}$ , and note that then  $\sqrt{2\varepsilon'} \leq \varepsilon$  and  $\sqrt{2(1 - \varepsilon')} \geq \sqrt{2} - \varepsilon$ . Since  $G$  is sofic we can choose a finite set  $F$  and a  $(K, \varepsilon')$ -almost homomorphism  $\varphi': G \rightarrow \text{Sym}(F)$ . We may further assume that  $F = \{1, 2, \dots, n\}$ , for some  $n \in \mathbb{N}$ , so that  $\text{Sym}(F) = \text{Sym}(n)$ . Now let  $\pi: \text{Sym}(n) \rightarrow \mathcal{U}(n)$  denote the natural unitary representation, that is,  $\pi(\sigma)e_i = e_{\sigma(i)}$ , for all  $\sigma \in \text{Sym}(n)$  and  $i \in \{1, 2, \dots, n\}$ , when  $e_1, e_2, \dots, e_n$  denotes the standard orthogonal basis for  $\mathbb{C}$ . It is straightforward to check that

$$d_{HS}(\pi(\sigma), \pi(\tau))^2 = 2d_F(\sigma, \tau).$$

So with  $\varphi = \pi \circ \varphi': G \rightarrow \mathcal{U}(n)$  we have that for all  $g, h \in K$ ,

$$d_{HS}(\varphi(gh), \varphi(g)\varphi(h)) = \sqrt{2d_F(\varphi'(gh), \varphi'(g)\varphi'(h))} \leq \sqrt{2\varepsilon'} \leq \varepsilon,$$

and if  $g \neq h$  then

$$d_{HS}(\varphi(g), \varphi(h)) = \sqrt{2d_F(\varphi'(g), \varphi'(h))} \geq \sqrt{2(1 - \varepsilon')} \geq \sqrt{2} - \varepsilon.$$

Thus  $G$  is hyperlinear by definition, since  $F$  and  $\varepsilon$  was arbitrary.  $\square$

Now we are ready to state the result, which captures our main interest in the Connes Embedding Problem, namely that sofic groups satisfy Connes Embedding Problem for Groups.

**Corollary 6.2.2.** *If  $G$  is a sofic group, then  $\text{LG}$  embeds into  $\mathcal{R}^\omega$  for some choice of an ultrafilter  $\omega$  on an index set  $I$ . This ultrafilter may be chosen to be free, and if the group is countable, then  $I$  may be chosen to be  $\mathbb{N}$ .*

*Proof.* This is an immediate consequence of Theorem 5.3.9, Corollary 5.3.10 and Theorem 6.2.1.  $\square$

### 6.3 Intermezzo—Local embeddability

In this section we introduce the notion of local embeddability and a few related concepts. As mentioned earlier, the point of this is, to introduce a terminology which is well-suited for describing some large classes of sofic groups, and prove some permanence properties of such. This section is mostly based on [CSC10, Section 7.1].

**Definition 6.3.1.** A **class of groups**  $\mathcal{C}$  is a collection of groups, so that if  $G \in \mathcal{C}$  and  $H$  is a group with  $H \cong G$ , then  $H \in \mathcal{C}$ .  $\blacktriangleleft$

It is easy to come up with examples of classes of groups: the class of finite groups; the class of countable groups; the class of finitely generated groups; and the class of abelian groups. Also the amenable groups and the torsion-free groups form classes of groups. One could also mix these examples and obtain classes of groups such as the class of finitely generated abelian groups, or the class of countable torsion-free groups, and so on.

**Definition 6.3.2.** Suppose that  $G$  and  $H$  are groups. For a finite subset  $K \subseteq G$ , a map  $\varphi: G \rightarrow H$  is called a  **$K$ -almost homomorphism** if  $\varphi|_K$  is injective and  $\varphi(gh) = \varphi(g)\varphi(h)$ , for all  $g, h \in K$ .  $\blacktriangleleft$

**Definition 6.3.3.** Let  $\mathcal{C}$  be a class of groups. A group  $G$  is called **locally embeddable** in the class  $\mathcal{C}$  if for each  $K \subseteq G$  finite, there exist a group  $H \in \mathcal{C}$  and a  $K$ -almost homomorphism from  $G$  to  $H$ .

An example of local embeddability could be the following: let  $\mathcal{C}$  denote the class of finite groups, then  $\mathbb{Z}$  is locally embeddable into  $\mathcal{C}$ . In fact, if  $K \subseteq \mathbb{Z}$  is finite, then with  $m = \max\{|k| : k \in K\}$  we have a surjective  $*$ -homomorphism  $\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ . Thus  $\mathbb{Z}$  is locally embeddable into the class of amenable groups.

**Proposition 6.3.4.** *Suppose that  $\mathcal{C}$  is a class of groups. The groups which are locally embeddable into  $\mathcal{C}$  form a class  $\overline{\mathcal{C}}$ . If a group is locally embeddable into  $\overline{\mathcal{C}}$  then the group is itself in  $\overline{\mathcal{C}}$ .*

*Proof.* Now suppose that  $G$  is a group in  $\overline{\mathcal{C}}$ ,  $H$  is another group and  $\varphi: H \rightarrow G$  is an isomorphism. If  $K \subseteq H$  is finite, then since  $\varphi(K)$  is finite we can find a group  $C$  in  $\mathcal{C}$  and a  $\varphi(K)$ -almost homomorphism  $\psi: G \rightarrow C$ . Clearly,  $\psi \circ \varphi$  is a  $K$ -almost homomorphism  $H \rightarrow C$ . Hence  $H$  is locally embeddable into  $\mathcal{C}$ .

The latter part of the statement is proved similarly, with the exception that  $\varphi$  is now a  $K$ -almost homomorphism. Their composition will still be a  $K$ -almost homomorphism.  $\square$

Next, let us state a few easy observations.

**Proposition 6.3.5.** *Suppose that  $\mathcal{C}$  is a class of groups which is closed under taking subgroups. Then if a finite group is locally embeddable into  $\mathcal{C}$ , the group is itself in  $\mathcal{C}$ .*

**Proposition 6.3.6.** *Suppose that  $\mathcal{C}$  is a class of groups and  $G$  a group which is locally embeddable into  $\mathcal{C}$ . Then any subgroup of  $G$  is locally embeddable in  $\mathcal{C}$ .*

**Proposition 6.3.7.** *Suppose that  $\mathcal{C}$  is a class of groups. A group is locally embeddable in  $\mathcal{C}$  if and only if each finitely generated subgroups of the group is locally embeddable into  $\mathcal{C}$ .*

It is easy to imagine that one would prefer working with classes of groups which are closed under passing to subgroups. It turns out that another very nice property for a class of groups  $\mathcal{C}$  to have—at least from the point of view of local embeddability—is to be closed under taking finite direct products. The next proposition and the following corollaries illustrates this point.

**Proposition 6.3.8.** *Suppose that  $\mathcal{C}$  is a class of groups which is closed under taking finite direct products. If  $(G_i)_{i \in I}$  is a family of groups which is locally embeddable into  $\mathcal{C}$ , then their direct product  $\prod_{i \in I} G_i$  is locally embeddable into  $\mathcal{C}$ .*

*Proof.* Let  $G$  denote the group  $\prod_{i \in I} G_i$ , let  $K \subseteq G$  be finite and set  $L = \{k_1 k_2^{-1} : k_1, k_2 \in K, k_1 \neq k_2\}$ . Clearly  $L$  is also finite and  $\mathbf{1}_G \notin L$ . For each  $l \in L$  we may choose  $j \in I$  with  $\pi_i(l) \neq \mathbf{1}_{G_i}$ , where  $\pi_i : G \rightarrow G_i$  denotes the natural projection. Let  $J$  denote the set of all such  $j$ . Let  $G_J$  denote the product group  $\prod_{j \in J} G_j$  and let  $\pi_J : G \rightarrow G_J$  denote the natural projection given by  $\pi_J((g_i)_{i \in I}) = (g_j)_{j \in J}$ . By assumption,  $G_J$  is in  $\mathcal{C}$ , and since  $\pi_J$  is a homomorphism it would suffice to show that  $\pi_J|_K$  is injective in order to prove that  $\pi_J$  is a  $K$ -almost homomorphism. Now, by construction  $\pi_J(l) \neq \mathbf{1}_{G_J}$  for all  $l \in L$ . Thus  $\pi_J(k_1)\pi_J(k_2)^{-1} \neq \mathbf{1}_{G_J}$  for all  $k_1, k_2 \in K$  with  $k_1 \neq k_2$ , or, in other words,  $\pi_J|_K$  is injective. Next, for each  $j \in J$  choose some  $H_j$  in  $\mathcal{C}$  and a  $\pi_j(K)$ -almost homomorphism  $\varphi_j : G_j \rightarrow H_j$ . Let



$H_J = \prod_{j \in J} H_j$  and let  $\varphi_J: G_J \rightarrow H$  be the product map. Let the map  $\varphi: G \rightarrow H_J$  be the composition  $\varphi_J \circ \pi_J$ . Then it is straightforward to check that  $\varphi$  is a  $K$ -almost homomorphism, since  $\pi_J(K) \subseteq \prod_{j \in J} \pi_j(K)$ . Hence  $G$  is locally embeddable into  $\mathcal{C}$ , since  $H_J$  is in  $\mathcal{C}$  by assumption.  $\square$

Since both the direct sum of groups and the projective limit of groups are subgroups of the direct product (see [CSC10, Appendix E] for construction of the projective limit), we immediately get the following two corollaries:

**Corollary 6.3.9.** *Suppose that  $\mathcal{C}$  is a class of groups which is closed under taking finite direct products. If  $(G_i)_{i \in I}$  is a family of groups which is locally embeddable into  $\mathcal{C}$ , then their direct sum  $\bigoplus_{i \in I} G_i$  is locally embeddable into  $\mathcal{C}$ .*

**Corollary 6.3.10.** *Suppose that  $\mathcal{C}$  is a class of groups which is closed under taking finite direct products. If  $(G_i)_{i \in I}$  is a projective system of groups which is locally embeddable into  $\mathcal{C}$ , then their projective limit  $\varprojlim G_i$  is locally embeddable into  $\mathcal{C}$ .*

**Definition 6.3.11.** Let  $\mathcal{C}$  be a class of group. A group  $G$  is said to be **residually in  $\mathcal{C}$**  (or **residually  $\mathcal{C}$** ) if for each  $g \in G$ , there exist a group  $H$  in  $\mathcal{C}$  and a surjective homomorphism  $\varphi: G \rightarrow H$  with  $\varphi(g) \neq \mathbf{1}_H$ . A group  $G$  is called **locally residually in  $\mathcal{C}$**  (or **locally residually  $\mathcal{C}$** ) if every finitely generated subgroup of  $G$  is residually in  $\mathcal{C}$ .  $\blacktriangleleft$

One might at first suspect that residually  $\mathcal{C}$  implies locally residually  $\mathcal{C}$ , but this is not always the case. The problem one encounters is that one can not necessarily satisfy the surjectivity condition when passing to subgroups. If the class  $\mathcal{C}$  in question is closed under taking subgroups though, then a group which is residually in  $\mathcal{C}$  is also locally residually in  $\mathcal{C}$ . An example where it goes wrong could be the following: let  $\mathcal{C}$  denote the class of infinite torsion groups, that is, infinite groups where all elements have finite order. The complex unit circle  $\mathbb{T}$  with multiplication is an example of such a group. Now if  $G$  is a group in  $\mathcal{C}$ , then clearly  $G$  is residually in  $\mathcal{C}$ , but since every subgroup of  $G$  is finite,  $G$  cannot be locally residually in  $\mathcal{C}$ .

Another thing one could note, is that if a group is locally embeddable in  $\mathcal{C}$  then it is also locally residually in  $\mathcal{C}$ .

**Proposition 6.3.12.** *Suppose that  $\mathcal{C}$  is a class of groups which is closed under taking finite direct products. If  $G$  is a group which is either residually in  $\mathcal{C}$  or locally residually in  $\mathcal{C}$ , then  $G$  is locally embeddable into  $\mathcal{C}$ .*

*Proof.* Suppose that  $K \subseteq G$  is finite. We may assume that  $G$  is residually in  $\mathcal{C}$ . If  $G$  were only locally residually in  $\mathcal{C}$ , then we use the same argument, but just with the group generated by  $K$  instead.

Let  $L = \{k_1 k_2^{-1} : k_1, k_2 \in K, k_1 \neq k_2\}$ . Clearly  $L$  is finite and does not contain the neutral element. Since  $G$  is residually in  $\mathcal{C}$ , then for each  $l \in L$  there exist a group  $H_l$  in  $\mathcal{C}$  and a surjective homomorphism  $\varphi_l: G \rightarrow H_l$  such that  $\varphi_l(l) \neq \mathbf{1}_{H_l}$ . Let  $H_L$  denote the group  $\prod_{l \in L} H_l$ , and let  $\varphi_L: G \rightarrow H_L$  denote the map  $\varphi(g) = (\varphi_l(g))_{l \in L}$ . Clearly  $\varphi_L$  is a homomorphism, and the group  $H_L$  is in  $\mathcal{C}$  by assumption, so if we can prove that  $\varphi|_K$  is injective, then we are done. Suppose that  $k_1, k_2 \in K$  with  $k_1 \neq k_2$ . Then  $k_1 k_2^{-1} \in L$  and  $\varphi_{k_1 k_2^{-1}}(k_1) \neq \varphi_{k_1 k_2^{-1}}(k_2)$ . In particular  $\varphi_L(k_1) \neq \varphi_L(k_2)$ , which shows that  $\varphi_L$  is injective.  $\square$

## 6.4 Examples and permanence properties

We are now ready to begin proving some permanence properties for sofic groups and give examples of sofic groups, including the residually finite groups and the amenable groups. Most of the proofs in this section follows the ones presented in [CSC10, Chapter 7].

**Proposition 6.4.1.** *The sofic groups constitute a class of groups, and this class is closed under passing to subgroups and taking finite direct products. Moreover if a group is locally embeddable into the class of sofic groups, then it is itself sofic.*

*Proof.* The fact that the sofic groups constitute a class of groups which is, moreover closed under parsing to subgroups is straightforward to check.

Suppose that  $G_1$  and  $G_2$  are sofic. Let  $K \subseteq G_1 \times G_2$  be finite and let  $\varepsilon > 0$ . Choose  $\varepsilon' \in (0, \varepsilon]$  so that  $2\varepsilon' - \varepsilon'^2 < \varepsilon$  and  $K_1 \subseteq G_1$  and  $K_2 \subseteq G_2$  both finite so that  $K \subseteq K_1 \times K_2$ . Choose also finite subsets  $F_i \subseteq G_i$  and  $(K_i, \varepsilon')$ -almost homomorphisms  $\varphi_i : G_i \rightarrow \text{Sym}(F_i)$ , for  $i = 1, 2$ . Now let  $F = F_1 \times F_2$  and define a map  $\varphi : G_1 \times G_2 \rightarrow \text{Sym}(F)$  by

$$\varphi(g_1, g_2)(f_1, f_2) = (\varphi_1(g_1)(f_1), \varphi_2(g_2)(f_2)), \quad (g_1, g_2) \in G_1 \times G_2$$

We aim at showing that  $\varphi$  is a  $(K, \varepsilon)$ -almost homomorphism. In fact, we will show that it is a  $(K_1 \times K_2, \varepsilon)$ -almost homomorphism. Suppose that  $k_i, k'_i \in K_i$ , for  $i = 1, 2$ . Then by Proposition 6.1.4 we know that

$$\begin{aligned} & d_F(\varphi(k'_1 k_1, k'_2 k_2), \varphi(k'_1, k'_2)\varphi(k_1, k_2)) \\ &= 1 - (1 - d_{F_1}(\varphi_1(k'_1 k_1), \varphi_1(k'_1)\varphi_1(k_1))) \times \\ & \quad (1 - d_{F_2}(\varphi_2(k'_2 k_2), \varphi_2(k'_2)\varphi_2(k_2))) \end{aligned}$$

Since  $\varphi_i$  is a  $(K_i, \varepsilon')$ -almost homomorphism, for  $i = 1, 2$ , we know that each of the two terms in the product on the right hand side is greater than or equal to  $1 - \varepsilon'$ , and so it follows that

$$d_F(\varphi(k'_1 k_1, k'_2 k_2), \varphi(k'_1, k'_2)\varphi(k_1, k_2)) \leq 1 - (1 - \varepsilon')^2 = 2\varepsilon' - \varepsilon'^2 \leq \varepsilon$$

Now suppose that  $(k'_1, k'_2) \neq (k_1, k_2)$ . Again by Proposition 6.1.4 we get

$$\begin{aligned} & d_F(\varphi(k'_1, k'_2), \varphi(k_1, k_2)) \\ &= 1 - (1 - d_{F_1}(\varphi_1(k'_1), \varphi_1(k_1))) \times (1 - d_{F_2}(\varphi_2(k'_2), \varphi_2(k_2))) \end{aligned}$$

Since the Hamming metric is always less than or equal to 1, we know that both terms in the product on the right hand side are less than or equal to 1. Moreover, since  $\varphi_i$  is a  $(K_i, \varepsilon')$ -almost homomorphism, for  $i = 1, 2$  and  $(k'_1, k'_2) \neq (k_1, k_2)$ , at least one of the terms in the product is less than or equal to  $1 - (1 - \varepsilon') = \varepsilon'$ . From this it follows that

$$d_F(\varphi(k'_1, k'_2), \varphi(k_1, k_2)) \geq 1 - \varepsilon' \geq 1 - \varepsilon.$$

This proves that  $\varphi$  is a  $(K_1 \times K_2, \varepsilon)$ -almost homomorphism, and in particular a  $(K, \varepsilon)$ -almost homomorphism. Thus  $G_1 \times G_2$  is sofic, which proves that the class of sofic groups is closed under taking finite direct products.

Last, suppose that  $G$  is a group which is locally embeddable into the class of sofic groups. Let  $K \subseteq G$  be a finite subset and  $\varepsilon > 0$ . Since  $G$  is locally embeddable into the class of sofic groups we can find some sofic group  $H$  and a  $K$ -almost homomorphism  $\varphi: G \rightarrow H$ . Since  $H$  is sofic and  $\varphi(K)$  is finite, we can choose some finite non-empty set  $F$  and a map  $\psi: H \rightarrow \text{Sym}(F)$  as in the definition of a sofic group. But then  $\psi \circ \varphi$  satisfy the requirement in the definition of a sofic group. Hence  $G$  is sofic.  $\square$

By combining the above proposition with Proposition 6.3.8, Corollary 6.3.9 and Corollary 6.3.10 we obtain the following two corollaries.

**Corollary 6.4.2.** *Suppose that  $(G_i)_{i \in I}$  is a family of sofic groups. Then both  $\prod_{i \in I} G_i$  and  $\bigoplus_{i \in I} G_i$  are sofic.*

**Corollary 6.4.3.** *Suppose that  $(G_i)_{i \in I}$  is a projective system of sofic groups. Then the projective limit  $\varprojlim G_i$  is also sofic.*

Let us recall the following theorem characterizing amenable groups as being those groups that satisfy the Følner condition—see for example [BO08, Theorem 2.6.8] or [CSC10, Proposition 4.1.7 & Theorem 4.9.1].

**Theorem 6.4.4.** *A group  $G$  is amenable if and only if it satisfies the following condition known as the Følner condition: for every finite subset  $S \subseteq G$  and every  $\varepsilon > 0$ , there exists a non-empty finite subset  $F \subseteq G$ , so that  $|F \setminus gF| < \varepsilon|F|$ , for all  $g \in S$ .*

This theorem will be used to establish the following technical lemma, which, in term, will enable use to prove Proposition 6.4.6 and Proposition 6.4.7. It essentially states that for amenable groups one can choose the desired  $(K, \varepsilon)$ -almost homomorphism to be a bit more nice.

**Lemma 6.4.5.** *Suppose that  $G$  is an amenable group,  $K \subseteq G$  a finite subset and  $\varepsilon > 0$ . Then there exist  $F \subseteq G$  finite,  $E \subseteq F$  with  $|E| \geq (1 - \varepsilon)|F|$  and a map  $\varphi: G \rightarrow \text{Sym}(F)$  so that*

$$\varphi(k)(f) = kf, \quad \varphi(h)(kf) = hkf \quad \text{and} \quad \varphi(hk)(f) = hkf,$$

for all  $h, k \in K$  and  $f \in F$ . In particular,  $\varphi(hk)(f) = (\varphi(h)\varphi(k))(f)$ , for all  $f \in E$  and  $h, k \in K$ , and  $\varphi(h)(f) \neq \varphi(k)(f)$ , for all  $f \in E$  whenever  $h, k \in K$  with  $h \neq k$ .

*Proof.* Let  $S = (\{1_G\} \cup K \cup K^{-1})^2$ , then  $S$  is finite with  $S = S^{-1}$  and  $K \subseteq S$ . Since  $G$  is amenable, we can choose a non-empty finite subset  $F \subseteq G$  so that

$$\frac{1}{|F|} |F \setminus gF| < \frac{\varepsilon}{|S|},$$

for all  $g \in S$ , by Theorem 6.4.4. Let  $E = \bigcap_{g \in S} gF$ , and note that since  $S^{-1} = S$  and  $1_G \in S$ , we have  $E \subseteq F$  and  $gE \subseteq g(g^{-1}F) = F$ . Furthermore, observe that

$$|F \setminus E| = \left| F \setminus \bigcap_{g \in S} gF \right| = \left| \bigcup_{g \in S} F \setminus gF \right| \leq \sum_{g \in S} |F \setminus gF| \leq |F|\varepsilon,$$

by the choice of  $F$ , so we get

$$|E| = |F| - |F \setminus E| \geq |F| - \varepsilon|F| = (1 - \varepsilon)|F|.$$

Now that we have found  $E$  and  $F$ , and so we only need to construct the map  $\varphi$ . Let  $g \in G$ . Since  $|gF| = |F|$ , we see that

$$|F \setminus gF| = |F| - |F \cap gF| = |gF| - |F \cap gF| = |gF \setminus F|,$$

and because of this we may now choose a bijection  $\alpha_g : gF \setminus F \rightarrow F \setminus gF$ . Next, let us define the map  $\varphi : G \rightarrow \text{Sym}(F)$  as follows: for  $g \in G$  and  $f \in F$ , let

$$\varphi(g)(f) = \begin{cases} gf & \text{if } gf \in F \\ \alpha_g(gf) & \text{if } gf \notin F \end{cases}.$$

It is straightforward to check that  $\varphi(g)$  is injective, and hence bijective, for all  $g \in G$ , which means that the map is well-defined. The properties we want for  $f$  follow directly from the fact that  $kf \in F$  and  $hkf \in F$ , for all  $h, k \in K$  and  $f \in E$ , since  $K \subseteq S$ ,  $K^2 \subseteq S$  and  $gE \subseteq F$ , for all  $g \in S$ .  $\square$

Now, with the above lemma we can prove the following two proposition, the first of which was originally proved by Weiss in [Wei00], and the second of which was originally proved by Elek and Szabó in [ES06]. The proof—including the part in Lemma 6.4.5—is from [CSC10], which is the same proof as the original one by Elek and Szabó, but with different terminology.

**Proposition 6.4.6.** *Every amenable group is sofic.*

*Proof.* Suppose that  $G$  is an amenable group,  $K \subseteq G$  finite and  $\varepsilon > 0$ . Then it is straightforward to check that the map  $\varphi$  from Lemma 6.4.5 is a  $(K, \varepsilon)$ -almost homomorphism.  $\square$

**Proposition 6.4.7.** *Extensions of sofic groups by amenable groups are sofic, that is, if  $G$  is a group with a normal subgroup  $N \subseteq G$ , so that  $N$  is sofic and  $G/N$  is amenable, then  $G$  is sofic.*

*Proof.* Let  $K \subseteq G$  be finite and  $\varepsilon > 0$ . We may assume that  $\varepsilon < 1$  and let  $\varepsilon' = 1 - \sqrt{1 - \varepsilon}$ . If  $\pi : G \rightarrow G/N$  denote the quotient map, then since  $\pi(K)$  is finite we can choose a finite set  $F_1 \subseteq G/N$ , a subset  $E_1 \subseteq F_1$  and a map  $\varphi_1$  as in Lemma 6.4.5. Let  $\sigma : G/N \rightarrow G$  be a right inverse of  $\pi$ , that is,  $\pi\sigma = \text{id}_G$ . Consider the set

$$M = \{\sigma(f_1)^{-1}k\sigma(f_2) : f_1, f_2 \in F_1, k \in K\} \cap N.$$

This set is clearly finite since  $K$  and  $F_1$  are finite, so we may choose some finite set  $F_2$  and a  $(M, \varepsilon')$ -almost homomorphism  $\varphi_2 : N \rightarrow \text{Sym}(F_2)$ . Let  $F = F_1 \times F_2$  and define  $\Phi : G \rightarrow \text{Sym}(F)$  as follows

$$\Phi(g)(f_1, f_2) = \begin{pmatrix} \varphi_1(\pi(g))(f_1) \\ \varphi_2(\sigma(\pi(g)f_1)^{-1}g\sigma(f_1))(f_2) \end{pmatrix}, \quad g \in G, (f_1, f_2) \in F.$$

The values of the map are written in column form purely for convenience. We will show that this map is a  $(K, \varepsilon)$ -almost homomorphism. Let us start by showing that it is well-defined, that is, for  $g \in G$  and  $f_1 \in F_1$  we have  $\sigma(\pi(g)f_1)^{-1}g\sigma(f_1) \in N$ . This though is easy, since it follows directly from the fact that

$$\begin{aligned} \pi\left(\sigma(\pi(g)f_1)^{-1}g\sigma(f_1)\right) &= \pi(\sigma(\pi(g)f_1))^{-1}\pi(g)\pi(\sigma(f_1)) \\ &= (\pi(g)f_1)^{-1}\pi(g)f_1 \\ &= 1_{G/N}. \end{aligned}$$

Fix  $h, k \in K$  and let  $f_1 \in F_1$ . Let

$$m = \sigma(\pi(k)f_1)^{-1}k\sigma(f_1) \quad \text{and} \quad m' = \sigma(\pi(hk)f_1)^{-1}h\sigma(\pi(k)f_1). \quad (6.1)$$

Clearly  $m, m' \in M$ , so since  $\varphi_2$  is a  $(M, \varepsilon')$ -almost homomorphism, the set

$$E_{f_1} = \{f \in F_2 : \varphi_2(m'm)(f) = (\varphi_2(m')\varphi_2(m))(f)\}$$

has cardinality larger than  $(1 - \varepsilon')|F_2|$ . In other words,  $|E_f| \geq (1 - \varepsilon')|F_2|$ . Now if  $f_2 \in E_{f_1}$ , then (using the notation  $m$  and  $m'$  in (6.1) again—just to shorten notation),

$$\begin{aligned} (\Phi(h)\Phi(k))(f_1, f_2) &= \Phi(h) \left( \begin{array}{c} \varphi_1(\pi(k))(f_1) \\ \varphi_2(\sigma(\pi(k)f_1)^{-1}k\sigma(f_1))(f_2) \end{array} \right) \\ &= \Phi(h) \left( \begin{array}{c} \pi(k)f_1 \\ \varphi_2(m)(f_2) \end{array} \right) \\ &= \left( \begin{array}{c} \varphi_1(\pi(h))(\pi(k)f_1) \\ \varphi_2(\sigma(\pi(h)\pi(k)f_1)^{-1}h\sigma(\pi(k)f_1))(\varphi_2(m)(f_2)) \end{array} \right) \\ &= \left( \begin{array}{c} \pi(h)\pi(k)f_1 \\ \varphi_2(m')(\varphi_2(m)(f_2)) \end{array} \right), \end{aligned}$$

where the second and fourth equality follow from the fact that  $f_1 \in E_1$ . Since  $f_2 \in E_{f_1}$ , we know that  $\varphi_2(m')(\varphi_2(m)(f_2)) = \varphi_2(m'm)(f_2)$ , and so it follows that

$$(\Phi(h)\Phi(k))(f_1, f_2) = \left( \begin{array}{c} \pi(hk)f_1 \\ \varphi_2(m'm)(f_2) \end{array} \right).$$

If we calculate  $m'm$ , we see that

$$\begin{aligned} m'm &= \sigma(\pi(hk)f_1)^{-1}h\sigma(\pi(k)f_1)\sigma(\pi(k)f_1)^{-1}k\sigma(f_1) \\ &= \sigma(\pi(hk)f_1)^{-1}hk\sigma(f_1), \end{aligned}$$

and so we conclude that

$$(\Phi(h)\Phi(k))(f_1, f_2) = \left( \begin{array}{c} \pi(hk)f_1 \\ \varphi_2(m'm)(f_2) \end{array} \right) = \Phi(hk)(f_1, f_2).$$

So far we know that  $(\Phi(h)\Phi(k))(f_1, f_2) = \Phi(hk)(f_1, f_2)$ , whenever  $f_1 \in E_1$  and  $f_2 \in E_{f_1}$ , or, in other words,  $(\Phi(h)\Phi(k))(f) = \Phi(hk)(f)$ , for all  $f \in E$ , where

$$E = \bigcup_{f_1 \in E_1} \{f_1\} \times E_{f_1}$$

Clearly this is a disjoint union, and so, if we recall that  $|E_1| \geq (1 - \varepsilon')|F_1|$  and  $|E_{f_1}| \geq (1 - \varepsilon')|F_2|$ , we then get

$$\begin{aligned} |E| &= \sum_{f_1 \in E_1} |\{f_1\} \times E_{f_1}| \\ &\geq \sum_{f_1 \in E_1} (1 - \varepsilon')|F_2| \\ &\geq (1 - \varepsilon')^2|F_1||F_2| \\ &= (1 - \varepsilon)|F|. \end{aligned}$$

From this it follows that

$$d_F(\Phi(hk), \Phi(h)\Phi(k)) \leq \frac{1}{|F|}(|F| - |E|) \leq \varepsilon.$$

Next suppose that  $h \neq k$ . We consider two cases. Suppose first that  $\pi(h) \neq \pi(k)$ . Then for  $f \in E_1$ , we know that  $\varphi_1(\pi(h))(f) = \pi(h)f \neq \pi(k)f = \varphi_1(k)(f)$ . In particular,

$$\Phi(h)(f_1, f_2) \neq \Phi(k)(f_1, f_2),$$

for all  $(f_1, f_2) \in E_1 \times F_2$ , since the first coordinates are different. Now because

$$|E_1 \times F_2| = |E_1||F_2| \geq (1 - \varepsilon')|F_1||F_2| = (1 - \varepsilon')|F|,$$

we get that

$$d_F(\Phi(h), \Phi(k)) \geq \frac{1}{|F|}|E_1 \times F_2| \geq (1 - \varepsilon') \geq (1 - \varepsilon).$$

Suppose instead that  $\pi(h) = \pi(k)$ , then for  $f_1 \in F_1$  we have

$$\sigma(\pi(h)f_1)^{-1}h\sigma(f_1) = \sigma(\pi(k)f_1)^{-1}h\sigma(f_1) \neq \sigma(\pi(k)f_1)^{-1}k\sigma(f_1).$$

Let us denote the left hand side by  $g_h$  and the right hand side by  $g_k$ . Since  $\varphi_2$  is an  $(M, \varepsilon')$ -almost homomorphism, there exists a set  $B \subseteq F_2$  so that  $\varphi_2(g_h)(f) \neq \varphi_2(g_k)(f)$ , for all  $f \in B$  and with  $|B| \geq (1 - \varepsilon')|F_2|$ . In particular  $\Phi(h)(f_1, f_2) \neq \Phi(k)(f_1, f_2)$ , for all  $(f_1, f_2) \in F_1 \times B$  since the second coordinates do not agree. From this it follows that

$$d_F(\Phi(h), \Phi(k)) \geq \frac{1}{|F|}|F_1 \times B| \geq \frac{1}{|F|}(1 - \varepsilon')|F_1||F_2| \geq (1 - \varepsilon).$$

Thus, since  $h, k \in K$  where arbitrary, we conclude that  $\Phi$  is a  $(K, \varepsilon)$ -almost homomorphism, and that  $G$  must then be sofic.  $\square$

## 6.5 Embedding in ultraproducts

In the setting of sofic groups there is an analogue of Proposition 5.2.1, namely Proposition 6.5.2. In fact, this is the original result that Elek and Szabó proved in [ES05].

Clearly, the group  $\text{Sym}(F)$  is sofic when  $F$  is a finite non-empty set, and the following proposition shows that metric ultraproducts of such finite symmetric groups is sofic.

**Proposition 6.5.1.** *Suppose that  $I$  is an index set,  $\omega$  an ultrafilter on  $I$  and  $(F_i)_{i \in I}$  a family of finite sets. Then the metric ultraproduct  $\prod_{i \in I}^{\omega} \text{Sym}(F_i)$  is sofic.*

*Proof.* Suppose that  $K \subseteq G_\omega$  is finite and  $\varepsilon > 0$  (we may also assume that  $\varepsilon < 1$ ). Let  $G_i$  denote the group  $\text{Sym}(F_i)$  and for each  $g \in G_\omega$  choose a representative  $\tilde{g} = (\tilde{g}_i)_{i \in I} \in G_I$ , that is,  $g = \tilde{g}N_\omega$ . If  $k_1, k_2 \in K$  with  $k_1 \neq k_2$  then  $d_\omega(\tilde{k}_1, \tilde{k}_2) > 0$ , and so if we let

$$t = 2^{-1} \min\{d_\omega(\tilde{k}_1, \tilde{k}_2) : k_1, k_2 \in K, k_1 \neq k_2\},$$

then  $0 < t \leq \frac{1}{2}$ . Choose an integer  $m$  so that  $(1-t)^m \leq \varepsilon$ . This is possible since  $0 < (1-t) < 1$ . From this we get that  $1 - (1-t)^m \geq 1 - \varepsilon$ . Choose also  $s \in (0, 1)$  with  $1 - (1-s)^m \leq \varepsilon$ , which is possible since  $1 - (1-x)^m \rightarrow 0$  as  $x \rightarrow 0$ . To summarize, we have chosen  $t, s$  and  $m$  so that, for all  $h, k \in K$ ,

$$d_\omega(h, k) \geq 2t, \quad 1 - (1-t)^m \geq 1 - \varepsilon \quad \text{and} \quad 1 - (1-s)^m \leq \varepsilon.$$

For  $h, k \in K$  we have  $\widetilde{hk}N_\omega = hk = \widetilde{h}N_\omega\widetilde{k}N_\omega = \widetilde{h}\widetilde{k}N_\omega$ , where the last equality follows from the fact that  $N_\omega$  is normal. But because  $\widetilde{hk}N_\omega = \widetilde{h}\widetilde{k}N_\omega$ , we have  $d_\omega(\widetilde{hk}, \widetilde{h}\widetilde{k}) = 0$ . In particular, if we let

$$A(h, k) = \{i \in I : d_{F_i}((\widetilde{hk})_i, \widetilde{h}_i\widetilde{k}_i) \leq s\},$$

then  $A(h, k) \in \omega$ . If  $h \neq k$  then  $d_\omega(\widetilde{h}, \widetilde{k}) \geq 2t$ , and so since  $t > 0$  the set

$$B(h, k) = \{i \in I : d_{F_i}(\widetilde{h}_i, \widetilde{k}_i) \geq t\}$$

belongs to  $\omega$ . From this we deduce that the set

$$S = \left( \bigcap_{h, k \in K} A(h, k) \right) \cap \left( \bigcap_{\substack{h, k \in K \\ h \neq k}} B(h, k) \right)$$

is in  $\omega$ , since it is a finite intersection of sets of  $\omega$ . In particular, it is non-empty. Choose  $j \in S$ . Then, for  $h, k \in K$  we have  $d_{F_j}((\widetilde{hk})_j, \widetilde{h}_j\widetilde{k}_j) \leq s$ , and if  $h \neq k$  then  $d_{F_j}(\widetilde{h}_j, \widetilde{k}_j) \geq t$ . Now, if we define a map  $\psi : G_\omega \rightarrow \text{Sym}(F_j)$  by  $\psi(g) = \widetilde{g}_j$ , then by the choice of  $j$  we know that  $d_{F_j}(\psi(hk), \psi(h)\psi(k)) \leq s$  and if  $h \neq k$  then  $d_{F_j}(\psi(h), \psi(k)) \geq t$ . Now consider the Cartesian product  $F = F_j \times F_j \times \cdots \times F_j$  ( $m$  factors) and define a map  $\varphi : G_\omega \rightarrow \text{Sym}(F)$  by

$$\varphi(g)(f_1, f_2, \dots, f_m) = (\psi(g)(f_1), \psi(g)(f_2), \dots, \psi(g)(f_m)),$$

for all  $g \in G_\omega$  and  $(f_1, f_2, \dots, f_m) \in F$ . Suppose that  $h, k \in K$ , then it follows from Lemma 6.1.4 and the choice of  $s$  that

$$\begin{aligned} d_F(\varphi(hk), \varphi(h)\varphi(k)) &= 1 - (1 - d_{F_j}(\psi(hk), \psi(h)\psi(k)))^m \\ &\leq 1 - (1 - s)^m \leq \varepsilon. \end{aligned}$$

If, moreover we assume that  $h \neq k$ , then by the choice of  $t$

$$\begin{aligned} d_F(\varphi(h), \varphi(k)) &= 1 - (1 - d_{F_j}(\psi(h), \psi(k)))^m \\ &\geq 1 - (1 - t)^m \geq 1 - \varepsilon. \end{aligned}$$

This shows that  $\varphi$  is a  $(K, \varepsilon)$ -almost homomorphism, and thus  $G_\omega$  is sofic.  $\square$

**Proposition 6.5.2.** *A group is sofic if and only if there exist an index set  $I$ , an ultrafilter  $\omega$  on  $I$  and a family of finite sets  $(F_i)_{i \in I}$  such that the group embeds into the metric ultraproduct of the groups  $\prod_{i \in I}^\omega \text{Sym}(F_i)$ .*

*Proof.* If the group embeds into such a metric ultraproduct, then the group is sofic since it is isomorphic to a sofic group, by Proposition 6.5.1.

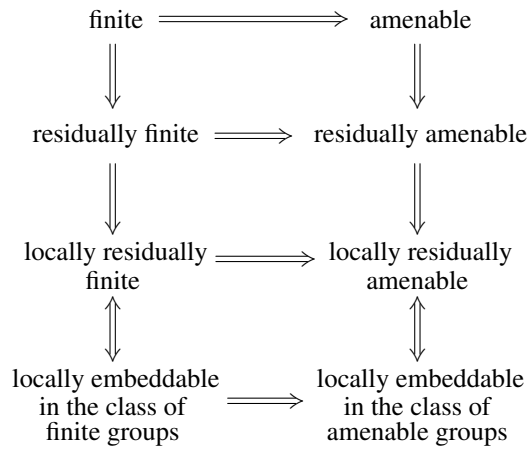
Conversely, suppose that  $G$  is a sofic group. Let  $\mathbb{F}$  denote the set of finite subsets of  $G$  and let  $I = \mathbb{F} \times \mathbb{N}$ . For each  $K \in \mathbb{F}$  and  $n \in \mathbb{N}$ , choose a finite set  $K_{(F, n)}$

and a  $(K_{F,n}, \frac{1}{n})$ -almost homomorphism  $\varphi_{(F,n)}: G \rightarrow \text{Sym}(K_{(F,n)})$ . Let  $\omega$  be an ultrafilter on  $I$  containing all the sets  $\{(K', n') \in I \mid K \subseteq F', n \leq n'\}$ . Such an ultrafilter exists by Proposition 1.6.3. Now define a map  $\varphi: G \rightarrow \prod_{i \in I}^{\omega} \text{Sym}(K_{(F,n)})$  by  $\varphi(g) = [(\varphi_i(g))_{i \in I}]$ . It is straightforward to show that  $\varphi$  is in fact a group homomorphism with  $d_{\omega}(g, h) = 1$  for  $g, h \in G$  with  $g \neq h$ . In particular,  $\varphi$  is an embedding of  $G$  into a metric ultraproduct of the desired type.  $\square$

### 6.6 Summary

Let us make a summary of what we have proved in this chapter. Most importantly, we have proved that sofic groups are hyperlinear, and thus—if countable—satisfy the Connes Embedding Conjecture for Groups. Besides this, we have characterized sofic groups in terms of their embeddings in ultraproduct of finite symmetric groups, we have proved some permanence properties of sofic groups, and have given some examples of classes of sofic groups.

Let us start with the examples of sofic groups. We have introduced various properties for groups. First of all we have introduced the concepts of residually finite, residually amenable, locally residually finite, locally residually amenable, locally embeddable in the class of finite groups and locally embeddable in the class of amenable groups, and other such. There are some obvious implications between these properties and there are some which we have proved, but there are also some not so obvious one. Some of this can be captured in the following diagram



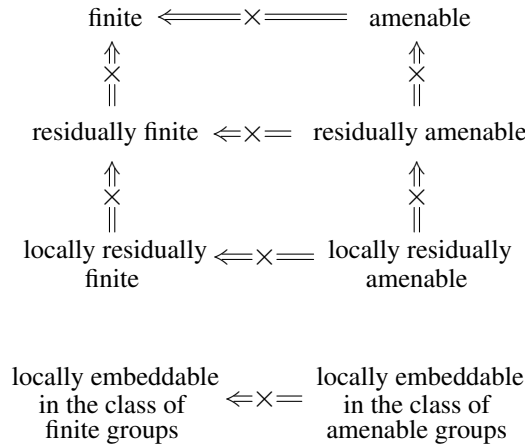
Let us discuss why the different implications are true, and then why those left out are *not*. Clearly finite implies residually finite and amenable implies residually amenable. The rest of the vertical implications follow from Proposition 6.3.12 and the discussion just above the same, since the class of finite groups and the class of amenable groups are both closed under passing to subgroups and taking finite direct products. We know that finite groups are amenable, and the horizontal implications follows just from this fact and the definition of the involved concepts.

Let us now discuss why the implications left out in the above diagram are not true. Clearly, neither amenable nor residually finite implies finite, a counterexample could be  $\mathbb{Z}$ . The free group  $\mathbb{F}_2$  on two generators is non-amenable, but residually finite and hence residually amenable, so residually amenable does not imply amenable.

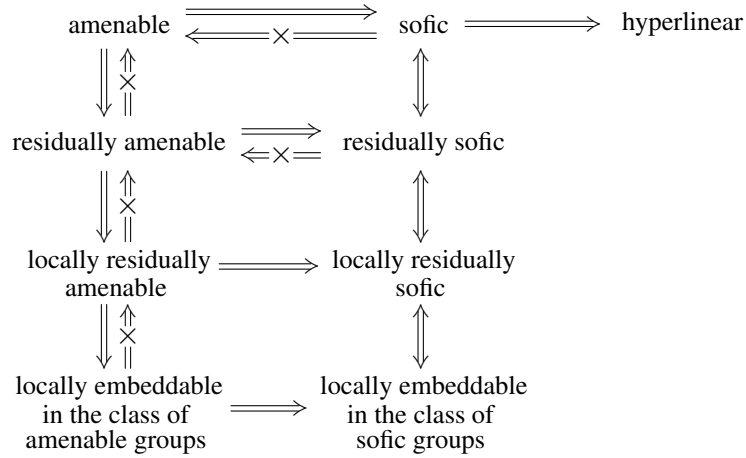


Next, consider the subgroup  $G$  of the set  $\text{Sym}(\mathbb{Z})$  of permutations of  $\mathbb{Z}$  generated by the bijection  $n \mapsto n + 1$ ,  $n \in \mathbb{N}$ , and the transposition  $(0\ 1)$ . By [CSC10, Proposition 7.3.9] the group  $G$  is amenable and locally embeddable into the class of finite groups, but not residually finite. From this we deduce that locally residually finite does not imply residually finite. In fact, this example also show that residually amenable does not imply residually finite, given that amenable imply locally residually amenable.

The Baumslag-Solitar groups, denoted  $BS(n, m)$ ,  $n, m \in \mathbb{Z}$  are two generator one relation groups. The group  $BS(n, m)$  is given by  $\langle a, b \mid a^{-1}b^na = b^m \rangle$ , that is, the group on two generators,  $a$  and  $b$ , with relation  $a^{-1}b^na = b^m$ . In the article [BS62] of 1962 Gilbert Baumslag and Donald Solitar proved that, in many cases—the cases where  $n$  and  $m$  are what is called meshed—the group  $BS(n, m)$  is non-Hopfian (see [CSC10, Definition 2.4.1] or [BS62] for the definition). One of these cases is  $BS(2, 3)$ . In particular, by [CSC10, Theorem 2.4.3] the group  $BS(2, 3)$  is not residually finite. It was proved in [Kro90] by Peter Kropholler that the Baumslag-Solitar group  $BS(2, 3)$  is residually solvable. In particular, it is residually amenable and locally embeddable into the class of amenable groups. This was also the case with the group  $G$  above, but in this case the group is not locally embeddable into the class of finite groups, since this is equivalent to being residually finite for finitely presented groups (see [CSC10, Proposition 7.3.8]). Thus locally embeddable in the class of amenable groups does not imply locally embeddable into the class of finite groups. Given the equivalences we already know, we also get that locally residually amenable does not imply locally residually finite. To summarize, we have the following inverted diagram, where  $\Leftarrow \times \Rightarrow$  means that the implication is false



Now, let us connect all this to the class of sofic groups. By Proposition 6.3.12 and Proposition 6.4.1 we know that sofic, residually sofic, locally residually sofic and locally embeddable into the class of sofic groups are equivalent. By Proposition 6.4.6 amenable groups are sofic. Hence the following diagram



Let us discuss some permanence properties of sofic groups. More precisely, recall that we have proved the following:

- ◇ subgroups of sofic groups are sofic;
- ◇ direct products of sofic groups are sofic;
- ◇ direct sums of sofic groups are sofic;
- ◇ projective limits of sofic groups are sofic;
- ◇ extension of sofic groups by amenable groups are sofic.

Besides these, there are some other permanence properties of sofic groups known to be true:

- ◇ injective limits of sofic groups are sofic;
- ◇ free products of sofic groups with amalgamation over amenable groups are again sofic.

The proof that injective limits of sofic groups are sofic is due to Elek and Szabó, proved in [ES06]. In this article they also proved that the free product of sofic groups are sofic. This was later generalized to free product of sofic groups with amalgamation over amenable groups, independently by Păunescu in [Pău11], respectively, Elek and Szabó in [ES11].

# Appendix A

## Various results needed

This chapter contains a few results which we prove, but with the help of references, which to do not instead to prove. We will, as much as possible, explain what these result say, and how they are used in this thesis. More explicitly, the results in this chapter which are used in this thesis are: Theorem A.1.5 and Corollary A.2.3. These results are used in Section 3.4, and only in this section.

### A.1 Free products of von Neumann algebras

The purpose of this section is to prove that every finite von Neumann algebra with separable predual embeds into a von Neumann algebra  $\text{II}_1$ -factor with separable predual, in a trace-preserving way. We start by introducing the free product of Hilbert spaces, and then the free product of von Neumann algebras. The mentioned result relies on a result on factoriality of free product of von Neumann algebras, that we state without a proof.

In general, the theory of free products is part of the theory of *free probability*, which was introduced by Dan-Virgil Voiculescu in the 1980's. Voiculescu introduced free probability with the purpose of investigating the free group factor problem, that is, the problem of determining whether there are isomorphisms between the free group factors. For more material on free probability, including some of the constructions considered here, the reader may consult [VDN92].

Suppose that  $I$  is an index set, and, for each  $i \in I$ , that  $\mathcal{H}_i$  is a Hilbert space with some distinguished unit vector  $\xi_i$ —which just mean a predetermined unit vector. For each  $i \in I$  let  $\mathcal{H}_i^o$  denote the orthogonal complement of  $\xi_i$  in  $\mathcal{H}_i$ . The **free product** of this family of Hilbert spaces, is defined to be the Hilbert space

$$\mathcal{H} = \mathbb{C}\xi \oplus \bigoplus_{n \in \mathbb{N}} \left( \bigoplus_{i_1 \neq i_2 \neq \dots \neq i_n} \mathcal{H}_{i_1}^o \otimes \mathcal{H}_{i_2}^o \otimes \dots \otimes \mathcal{H}_{i_n}^o \right),$$

where is implicit that the indexes  $i_1, i_2, \dots, i_n$  are all in  $I$ . We denote this free product by  $\star_{i \in I} (\mathcal{H}_i, \xi_i)$ . Note that  $\xi$  is just a symbol we use for what will denote the distinguished unit vector of  $\mathcal{H}$ . Suppose now that, for each  $i \in I$ , we are given a von Neumann algebra  $\mathcal{M}_i$  acting on  $\mathcal{H}_i$ . For each  $j \in J$  let  $\mathcal{K}_j$  denote the Hilbert space given by

$$\mathcal{K}_j = \mathbb{C}\xi \oplus \bigoplus_{n \in \mathbb{N}} \left( \bigoplus_{\substack{i_1 \neq i_2 \neq \dots \neq i_n \\ j \neq i_1}} \mathcal{H}_{i_1}^o \otimes \mathcal{H}_{i_2}^o \otimes \dots \otimes \mathcal{H}_{i_n}^o \right),$$

and let  $\mathcal{K}_j^o$  denote the orthogonal complement of  $\xi$  in  $\mathcal{K}_j$ . Fix some  $j \in I$ . With a little thought, one can convince one self, that we get, by rewriting and using the distributive law, that

$$\mathcal{H} = \mathbb{C}\xi \oplus \mathcal{K}_j^o \oplus \mathcal{H}_j^o \oplus (\mathcal{H}_j^o \otimes \mathcal{K}_j^o).$$

Obviously the maps  $\eta \mapsto \xi \otimes \eta$  and  $\eta' \mapsto \eta' \otimes \xi$  are isomorphisms Hilbert spaces  $\mathcal{K}_j^o \cong \mathbb{C}\xi \otimes \mathcal{K}_j^o$  and  $\mathcal{H}_j^o \cong \mathcal{H}_j^o \otimes \mathbb{C}\xi$ , respectively. Using these we get that

$$\mathcal{H} \cong \mathbb{C}\xi \oplus (\mathbb{C}\xi \otimes \mathcal{K}_j^o) \oplus (\mathcal{H}_j^o \otimes \mathbb{C}\xi) \oplus (\mathcal{H}_j^o \otimes \mathcal{K}_j^o) = (\mathbb{C}\xi \oplus \mathcal{H}_j^o) \otimes (\mathbb{C}\xi \oplus \mathcal{K}_j^o)$$

Since  $\mathcal{H}_j \cong \mathbb{C}\xi \oplus \mathcal{H}_j^o$ , we see that

$$\mathcal{H} \cong (\mathbb{C}\xi \oplus \mathcal{H}_j^o) \otimes \mathcal{K}_j \cong \mathcal{H}_j \otimes \mathcal{K}_j.$$

To summarize, we have a unitary operator  $U_j: \mathcal{H}_j \otimes \mathcal{K}_j \rightarrow \mathcal{H}$ , so that

$$\begin{aligned} U_j(\xi_j \otimes \xi) &= \xi, & U_j(\eta \otimes \xi) &= \eta, \\ U_j(\xi_j \otimes \eta') &= \eta' & \text{and} & & U_j(\eta \otimes \eta') &= \eta \otimes \eta', \end{aligned}$$

for all  $\eta \in \mathcal{H}_j^o$  and  $\eta' \in \mathcal{K}_j^o$ . Now, define a \*-homomorphism  $\rho_j: \mathcal{M}_j \rightarrow B(\mathcal{H})$  by

$$\rho_j(x) = U_j(x \otimes \mathbf{1}_{\mathcal{K}_j})U_j^*, \quad x \in \mathcal{M}_j.$$

Since  $U_j$  is unitary, this map is clearly unital and injective. Before we go on to define the free product of von Neumann algebras. Let us just prove the following proposition about these representations:

**Proposition A.1.1.** *In the setup above, we have for each  $j \in I$  that:*

- (i) *the map  $\rho_j$  is a faithful unital representation of  $\mathcal{M}_j$ , which is ultraweak operator-to-weak operator continuous;*
- (ii) *the image of  $\mathcal{M}_j$  by  $\rho_j$  is a von Neumann subalgebra of  $B(\mathcal{H})$ ;*
- (iii) *for each  $x \in \mathcal{M}_j$  we have that  $\langle x\xi_j \mid \xi_j \rangle = \langle \rho_j(x)\xi \mid \xi \rangle$ ;*
- (iv) *if  $\{x_\beta : \beta \in B\}$  is a generating set of  $\mathcal{M}_j$ , that is, if  $\{x_\beta : \beta \in A\}'' = \mathcal{M}_j$ , then  $\{\rho_j(x_\beta) : \beta \in B\}$  is a generating set for  $\rho_j(\mathcal{M}_j)$ ;*
- (v) *if  $\{x_\alpha^j : \alpha \in A_j\}$  is a generating set of  $\mathcal{M}_j$ , for each  $j \in I$ , then  $\{\rho_j(x_\alpha^j) : j \in I, \alpha \in A_j\}$  is a generating set for  $\star_{i \in I} \mathcal{M}_i$ .*

*Proof.* Let us start from the top. The first assertion follows from Proposition 1.2.12. The second assertion follows from the first point, because if we know that the map is ultraweak operator-to-weak operator continuous, then it is in particular weak operator-to-weak operator continuous, and so the image of the unit ball must be weak operator compact. Thus, since the map is isometric, the unit ball of  $\rho_j(\mathcal{M}_j)$  is weak operator compact, and the image is therefore a von Neumann algebra. Now, for  $x \in \mathcal{M}_j$  we see that

$$\langle \rho_j(x)\xi \mid \xi \rangle = \langle (x \otimes \mathbf{1}_{\mathcal{K}_j})\xi_j \otimes \xi \mid \xi_j \otimes \xi \rangle = \langle x\xi_j \mid \xi_j \rangle \|\xi\|^2 = \langle x\xi_j \mid \xi_j \rangle.$$

The last assertions follows directly from Proposition 1.3.9 together with the first two assertions.  $\square$

Now, we come to the definition. The **free product** of the von Neumann algebras  $(\mathcal{M}_i)_{i \in I}$  is defined to be the von Neumann algebra  $(\bigcup_{i \in I} \rho_i(\mathcal{M}_i))''$ , and is denoted by  $\star_{i \in I} \mathcal{M}_i$ . This free product of course depends on the choice of distinguished unit vector for each of the Hilbert spaces.

There is also another notion of free product of von Neumann algebras. Suppose that, for each  $i \in I$ , the von Neumann algebra  $\mathcal{M}_i$  is equipped with a normal faithful state  $\phi_i$ , then we may form the GNS construction  $(\pi_{\phi_i}, \mathcal{H}_{\phi_i}, \xi_{\phi_i})$  corresponding to  $\phi_i$ . In this case we denote by  $\star_{i \in I}(\mathcal{M}_i, \phi_i)$  the free product  $\star_{i \in I} \pi_{\phi_i}(\mathcal{M}_i)$  of the family of von Neumann algebras  $(\pi_{\phi_i}(\mathcal{M}_i))_{i \in I}$  acting on the free product Hilbert space  $\star_{i \in I}(\mathcal{H}_{\phi_i}, \xi_{\phi_i})$ . Let us make some observations about this free product—which we do not prove.

**Proposition A.1.2.** *In the setup above, and with  $\phi$  denoting the vector state corresponding to  $\xi$ , it holds that:*

- (i) *for each  $j \in I$  and  $x \in \mathcal{M}_j$ , we have that  $\phi(\rho_j(x)) = \phi_j(x)$ ;*
- (ii) *if  $\phi_j$  is a trace, for each  $j \in I$ , then  $\xi$  is a cyclic and separating trace vector for  $\star_{i \in I}(\mathcal{M}_i, \phi_i)$ . In particular,  $\phi$  is a faithful trace. In particular,  $\star_{i \in I}(\mathcal{M}_i, \phi_i)$  is countably decomposable.*

The first assertion is straight forward to check, just by calculating. For the last assertion, see [VDN92, Remark 1.6.6], and note that  $\xi_j$  is a cyclic and separating trace vector for  $\mathcal{M}_j$  if  $\phi_j$  is a trace, for each  $j \in I$ .

**Remark A.1.3.** It follows from the above proposition, that if  $(\mathcal{M}_i)_{i \in I}$  is a family of finite von Neumann algebras, and  $\tau_i$  is a faithful normal trace on  $\mathcal{M}_i$ , for each  $i \in I$ , then  $\star_{i \in I}(\mathcal{M}_i, \tau_i)$  is a finite von Neumann algebra, since it possesses a faithful trace. Let us show that in addition  $I$  has more than two elements, then the free product  $\star_{i \in I}(\mathcal{M}_i, \tau_i)$  is infinite dimensional. So let  $i, j \in I$  be distinct. First, note that the Hilbert space  $\mathcal{H}$  must be infinite dimensional, since it contains the sequence

$$\mathcal{H}_i, \quad \mathcal{H}_i \otimes \mathcal{H}_j, \quad \mathcal{H}_i \otimes \mathcal{H}_j \otimes \mathcal{H}_i, \quad \mathcal{H}_i \otimes \mathcal{H}_j \otimes \mathcal{H}_i \otimes \mathcal{H}_j, \quad \dots$$

of orthogonal subspaces of  $\mathcal{H}$ . Second, let us show that, if  $\mathcal{K}$  is an infinite dimensional Hilbert space, and  $\mathcal{A} \subseteq B(\mathcal{K})$  a  $C^*$ -algebra with a cyclic vector  $\eta$ , then  $\mathcal{A}$  is infinite dimensional. This will clearly imply that  $\star_{i \in I}(\mathcal{M}_i, \tau_i)$  is infinite dimensional, since  $\mathcal{H}$  is infinite dimensional and  $\xi$  is a cyclic vector. So suppose that  $n \in \mathbb{N}$  and let us show that  $\mathcal{A}$  has dimension larger than  $n$ . Since  $\mathcal{A}\mathcal{K}$  is a dense subspace of an infinite dimensional Hilbert space, it must itself be infinite dimensional. Let  $\eta_1, \eta_2, \dots, \eta_n$  be a set of non-trivial linearly independent vectors in  $\mathcal{A}\eta$ . For each  $k \in \{1, 2, \dots, n\}$  choose  $x_k \in \mathcal{A}$  so that  $\eta_k = x_k \eta$ . The claim is now, that the vectors  $x_1, x_2, \dots, x_n$  are linearly independent. Suppose that  $\lambda_1, \lambda_2, \dots, \lambda_n$  are complex numbers, such that  $\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = 0$ . By applying this operator to  $\eta$  we get that  $\lambda_1 \eta_1 + \lambda_2 \eta_2 + \dots + \lambda_n \eta_n = 0$ , and so since these vectors were linearly independent, we conclude that  $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ . Thus the set  $x_1, x_2, \dots, x_n$  are linearly independent, and it follows that the dimension of  $\mathcal{A}$  is greater than or equal to  $n$ , so since  $n$  was arbitrary, we conclude that  $\mathcal{A}$  is infinite dimensional. ◀

We are now interested in when the free product of von Neumann algebras is a factor. The following result—of which we omit the proof—gives a sufficient criteria for when this is the case. The result is due to Kenneth J. Dykema. More precisely, the theorem below is a combination of Theorem 2.5 and Lemma 3.2 in [Dyk94].

**Theorem A.1.4.** *Suppose that  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are von Neumann algebras with normal faithful states  $\phi_1$  and  $\phi_2$ , respectively, such that both  $\mathcal{M}_1$  and  $\mathcal{M}_2$  have linear dimension greater than or equal to two, and at least one has linear dimension greater than or equal to three. If neither  $\mathcal{M}_1$  nor  $\mathcal{M}_2$  contains a minimal projection  $p$ , such that  $\phi_1(p) > \frac{1}{2}$  or  $\phi_2(p) > \frac{1}{2}$ , respectively, then the free product  $(\mathcal{M}_1, \phi_1) \star (\mathcal{M}_2, \phi_2)$  is a factor.*

Now, we are ready to prove the following theorem, which is the purpose of this section.

**Theorem A.1.5.** *Every finite and countably decomposable von Neumann algebra embeds into a  $\text{II}_1$ -factor in a trace preserving way. Also, if the finite von Neumann algebra has separable predual, then it embeds into a von Neumann algebra  $\text{II}_1$ -factor with separable predual.*

*Proof.* Let  $\mathcal{M}$  be a finite von Neumann algebra, let  $\mathcal{M}_1$  denote the direct sum  $\mathcal{M} \oplus \mathcal{M}$  and let  $\mathcal{M}_2$  denote the von Neumann algebra  $\ell_\infty^3$  acting on  $\mathbb{C}^3$  as diagonal operators. Choose a faithful normal tracial state  $\tau$  on  $\mathcal{M}$ . This is possible by Theorem 1.3.6. Let  $\phi_1$  and  $\phi_2$  denote the faithful normal tracial states on  $\mathcal{M}_1$  and  $\mathcal{M}_2$  given by

$$\phi_1(x \oplus y) = \frac{1}{2}(\tau(x) + \tau(y)) \quad \text{and} \quad \phi_2(\xi_1, \xi_2, \xi_3) = \frac{1}{3}(\xi_1 + \xi_2 + \xi_3),$$

for  $x, y \in \mathcal{M}$  and  $(\xi_1, \xi_2, \xi_3) \in \mathcal{M}_2$ , respectively. Clearly  $\mathcal{M}_1$  has linear dimension greater than or equal to two, and  $\mathcal{M}_2$  linear dimension equal to three. Also, every minimal projection in  $\mathcal{M}$  has trace less than or equal to  $\frac{1}{2}$ , since all minimal projections are of the form  $p \oplus 0$  or  $0 \oplus p$  for some minimal projection  $p \in \mathcal{M}$ , and all the minimal projections in  $\mathcal{M}_2$  has trace equal to  $\frac{1}{3}$ , so by Theorem A.1.4 we get that the free product  $\mathcal{M}_1 \star \mathcal{M}_2$  is a factor. By Remark A.1.3 the free product is a finite von Neumann algebra since it has a faithful normal trace  $\tau_*$ , which restricts the  $\phi_1$ , when  $\mathcal{M}_1$  is embedded into  $\mathcal{M}_1 \star \mathcal{M}_2$ , and infinite dimensional since we are taking free product with more than one factor. Thus we conclude that  $\mathcal{M}_1 \star \mathcal{M}_2$  must be a  $\text{II}_1$ -factor.

Suppose that  $\mathcal{M}$  has separable predual, and let us show that so does  $\mathcal{M}_1 \star \mathcal{M}_2$ . Since  $\mathcal{M}$  has separable predual, it has a finite generating set  $\mathfrak{X}$  by Theorem 1.3.11. The set  $\{x \oplus y : x, y \in \mathfrak{X}\}$  is clearly also countable, and it generates  $\mathcal{M}_1$ . The von Neumann algebra  $\mathcal{M}_2$  is clearly finitely generated, so both  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are countably generated, and it follows from Proposition A.1.1 that  $\mathcal{M}_1 \star \mathcal{M}_2$  is also countably generated. Since we know that  $\mathcal{M}_1 \star \mathcal{M}_2$  is a finite von Neumann algebra with a faithful normal trace  $\tau_*$ , it is countably decomposable. Thus  $\mathcal{M}_1 \star \mathcal{M}_2$  has separable predual by Theorem 1.3.11, since it is countably generated and countably decomposable.  $\square$

With this last result we conclude this section. All in all, it seemed straight forward and easy to prove. But the bulk of proving this theorem, lies in the result on factoriality of free products, which we skipped. So to complete this proof without a reference, one would need to work a bit harder than this.

## A.2 Amenable traces

In this section we state a theorem about amenable traces, and prove a corollary of this theorem. This corollary gives a sufficient condition for when a unital separable  $C^*$ -algebra embeds into an ultrapower of the hyperfinite  $\text{II}_1$ -factor.

Let us start by defining what it means for a trace to be amenable.

**Definition A.2.1.** Let  $\mathcal{H}$  be a Hilbert space, and  $\mathcal{A} \subseteq B(\mathcal{H})$  a  $C^*$ -algebra. A tracial state  $\tau$  on  $\mathcal{A}$  is called **amenable**, if it extends to a state  $\phi$  on  $B(\mathcal{H})$ , such that  $\phi(xy) = \phi(yx)$ , whenever  $x \in \mathcal{A}$  and  $y \in B(\mathcal{H})$ . ◀

The next theorem is part of an important theorem on amenable traces. This theorem will allow us to obtain the result we need. We will not prove the theorem, and a proof can be found in [BO08, Theorem 6.2.7 & Remark 6.2.8].

**Theorem A.2.2.** *If  $\mathcal{A}$  is a unital separable  $C^*$ -algebra, and  $\tau$  is an amenable trace on  $\mathcal{A}$ , then there exist a sequence of natural numbers  $(k_1, k_2, k_3, \dots)$ , and a sequence of unital completely positive maps  $\phi_n: \mathcal{A} \rightarrow \mathbb{M}_{k_n}$ , such that*

- (i)  $\tau(x) = \lim_{n \rightarrow \infty} \text{tr}_{k_n} \phi_n(x)$ , for all  $x \in \mathcal{A}$ ;
- (ii)  $\lim_{n \rightarrow \infty} \|\phi_n(xy) - \phi_n(x)\phi_n(y)\|_2 = 0$ , for all  $x, y \in \mathcal{A}$ .

**Corollary A.2.3.** *Suppose that  $\mathcal{A}$  is a unital separable  $C^*$ -algebra, with a trace  $\tau$ . If there exist a faithful representation  $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$  of  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$ , such that the induced trace on  $\pi(\mathcal{A})$ , given by  $\pi(x) \mapsto \tau(x)$ ,  $x \in \mathcal{A}$ , is amenable, then there is a trace-preserving  $*$ -homomorphism from  $\mathcal{A}$  into  $\mathcal{R}^\omega$ , for any choice of free ultrafilter  $\omega$  on  $\mathbb{N}$ .*

*Proof.* Let  $\hat{\tau}$  denote the tracial state on  $\mathcal{R}$ , and let  $\tau'$  denote the trace  $\pi(x) \mapsto \tau(x)$  on  $\pi(\mathcal{A})$ . Choose a sequence of natural numbers  $(k_1, k_2, k_3, \dots)$  and a sequence of unital completely positive maps  $\phi_n: \pi(\mathcal{A}) \rightarrow \mathbb{M}_{k_n}$ , with the properties that  $\tau'(x) = \lim_{n \rightarrow \infty} \text{tr}_{k_n} \phi_n(x)$  and  $\lim_{n \rightarrow \infty} \|\phi_n(xy) - \phi_n(x)\phi_n(y)\|_2 = 0$ , for all  $x, y \in \mathcal{A}$ . Since  $\mathcal{R}$  contains a von Neumann algebra subfactor of type  $I_{k_n}$ , for all  $n \in \mathbb{N}$ , we may assume that  $\phi_n$  is a unital completely positive map from  $\pi(\mathcal{A})$  to  $\mathcal{R}$ , for each  $n \in \mathbb{N}$ , with the properties that  $\tau'(x) = \lim_{n \rightarrow \infty} \hat{\tau} \circ \phi_n(x)$  and  $\lim_{n \rightarrow \infty} \|\phi_n(xy) - \phi_n(x)\phi_n(y)\|_2 = 0$ , for all  $x, y \in \mathcal{A}$ . Now, let  $\omega$  be a free ultrafilter on  $\mathbb{N}$ , and define  $\rho: \mathcal{A} \rightarrow \mathcal{R}^\omega$  by  $\rho(x) = [(\phi_n(\pi(x)))_{n \in \mathbb{N}}]$  for  $x \in \mathcal{A}$ . Since

$$\lim_{n \rightarrow \infty} \|\phi_n(\pi(xy)) - \phi_n(\pi(x))\phi_n(\pi(y))\|_2 = 0$$

and  $\omega$  is a free ultrafilter, we conclude that  $\rho$  is multiplicative. Also, since

$$\tau(x) = \tau'(\pi(x)) = \lim_{n \rightarrow \infty} \hat{\tau}(\phi_n(\pi(x))),$$

for all  $x \in \mathcal{A}$ , we conclude that  $\hat{\tau}_\omega(\rho(x)) = \tau(x)$ , that is,  $\rho$  is trace-preserving. Because the maps  $(\phi_n)_{n \in \mathbb{N}}$  are all linear and Hermitian, it follows that also  $\rho(x^*) = \rho(x)^*$ , for all  $x \in \mathcal{A}$ . Thus  $\rho$  is a trace preserving  $*$ -homomorphism as desired. ◻





## Appendix B

# Operator spaces and operator systems

This appendix is about operator spaces, operator systems, completely bounded maps and completely positive maps. We will mostly deal with operator systems and completely positive maps, since these are used the most throughout the thesis. Also for this reason we introduce operator systems before introducing operator spaces, even though the latter is a generalization of the former. This causes some redundancy, but not much.

In this thesis operator spaces and systems will always be subsets of  $C^*$ -algebras. There is also an abstract definition of operator spaces and operator systems which does not refer to any ambient  $C^*$ -algebra, but since all operator spaces and systems are occurring in this thesis is, in fact, subspaces of  $C^*$ -algebras, this seems natural.

We start by introducing the notation connected to operator spaces and systems. For a linear space  $V$  and  $n, m \in \mathbb{N}$ , we let  $M_{n,m}(V)$  denote the  $n \times m$  matrices over  $V$ , that is, the set of  $n \times m$  matrices whose entries are elements of  $V$ . This is again a linear space, with entry-wise operations, and when  $m = n$  we will denote  $M_{n,n}(V)$  by  $M_n(V)$ . Elements of  $M_{n,m}(V)$  are usually denoted by  $[v_{i,j}]$  indicating that  $v_{i,j} \in V$  for all  $i = 1, \dots, n$  and  $j = 1, \dots, m$ , and that the entry in the  $i$ 'th row and  $j$ 'th column is  $v_{i,j}$ . In some cases, where it is less clear which index corresponds to the row and which corresponds to the column, we will use the notation  $[v_{i,j}]_{i,j}$  to indicate that  $i$  denotes the row index and  $j$  denotes the column index. If we are given  $v \in M_{n,m}(V)$  and  $w \in M_{p,q}(V)$ , then we define  $v \oplus w$  to be the element

$$\begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{n+p,m+q}(V).$$

In the special case where  $V = \mathbb{C}$  we will denote  $M_{n,m}(\mathbb{C})$  by  $\mathbb{M}_{n,m}$ , respectively,  $M_n(\mathbb{C})$  by  $\mathbb{M}_n$ . We will denote the standard matrix units in  $\mathbb{M}_{n,m}$  by  $E_{i,j}$ ,  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, m$ , meaning that  $E_{i,j}$  is the  $n \times m$  complex matrix in  $\mathbb{M}_{n,m}$  having 1 in the  $i, j$ 'th entry and zeroes elsewhere.

Given two linear spaces  $V$  and  $W$ , a linear map  $\phi : V \rightarrow W$  and  $n, m \in \mathbb{N}$ , we will let  $\phi_{n,m}$  denote the linear map

$$\phi_{n,m} : M_{n,m}(V) \rightarrow M_{n,m}(W) \quad \text{defined by} \quad [v_{i,j}] \mapsto [\phi(v_{i,j})],$$

and  $\phi_n$  the linear map  $\phi_{n,n}$ . Clearly  $\phi_{n+p,m+q}(v \oplus w) = \phi_{n,m}(v) \oplus \phi_{p,q}(w)$ , whenever  $v \in M_{n,m}(V)$  and  $w \in M_{p,q}(V)$ . Also, if we are given another linear map  $\psi$ , for which the composition  $\psi \circ \phi$  makes sense, then  $(\psi \circ \phi)_n = \psi_n \circ \phi_n$ , for all  $n \in \mathbb{N}$ .

There is a canonical identification  $M_{n,m}(V)$  with the tensor product  $V \otimes M_{n,m}$ , namely via the map

$$[v_{i,j}] \mapsto \sum_{i,j=1}^n v_{i,j} \otimes E_{i,j}; \quad M_n(V) \longrightarrow V \otimes M_{n,m}.$$

This identification will sometimes be used without being mentioned. An advantage of this, is that the notation sometimes becomes easier when expressing it in terms of this tensor product, for example, the matrix in  $M_n(V)$  having  $a$  in the  $i, j$ 'th entry and zeroes elsewhere corresponds to the element  $a \otimes E_{i,j}$ . With this notation we see that, for a linear map  $\phi : V \rightarrow W$ , the map  $\phi_{n,m}$  just corresponds to  $\phi \otimes \mathbf{1}_{n,m}$ .

## B.1 Operator systems

We start with the definition of an operator system. Afterwards we will explain what structure operator systems possess that makes them so interesting.

**Definition B.1.1.** A linear subspace  $S$  of a unital  $C^*$ -algebra is called an **operator system** if it is self-adjoint and contains the unit.  $\blacktriangleleft$

Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra, and consider the set  $M_n(\mathcal{A})$  of  $n \times n$  matrices over  $\mathcal{A}$ . This linear space can be given the structure of a  $*$ -algebra in the canonical way. More precisely, we already know the linear structure, the multiplication is defined as the  $i, j$ 'th entry of a product

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,n} \end{bmatrix}$$

being the sum  $\sum_{k=1}^n a_{i,k} b_{k,j}$  and the involution given by

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}^* = \begin{bmatrix} a_{1,1}^* & \cdots & a_{n,1}^* \\ \vdots & \ddots & \vdots \\ a_{1,n}^* & \cdots & a_{n,n}^* \end{bmatrix}.$$

These rules just follow the usual rules for multiplication and involution in the matrices  $M_n$ , and indeed, if  $\mathcal{A} = \mathbb{C}$ , then the structure described above is the usual structure of  $M_n$ . If  $\mathcal{A}$  is unital, then  $M_n(\mathcal{A})$  is unital as well, with unit being the diagonal matrix having  $\mathbf{1}$  in the diagonal, that is, the matrix  $\mathbf{1} \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}$  ( $n$  copies).

There is also a canonical choice of norm on  $M_n(\mathcal{A})$ . To describe this norm suppose first that  $\mathcal{A}$  is a concrete  $C^*$ -algebra, that is, if  $\mathcal{A} = B(\mathcal{H})$ , for some Hilbert space  $\mathcal{H}$ . For  $n \in \mathbb{N}$ , there is a natural identification of  $M_n(B(\mathcal{H}))$  with the space  $B(\mathcal{H}^{\oplus n})$ . Namely, by letting a matrix  $[x_{i,j}]$  in  $M_n(B(\mathcal{H}))$  correspond to the operator on  $\mathcal{H}^{\oplus n}$  given by

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} x_{1,1}\xi_1 + \cdots + x_{1,n}\xi_n \\ \vdots \\ x_{n,1}\xi_1 + \cdots + x_{n,n}\xi_n \end{bmatrix},$$

for each  $(\xi_1, \dots, \xi_n) \in \mathcal{H}^{\oplus n}$ . In this way we get an induced norm on  $M_n(B(\mathcal{H}))$ , which obviously makes  $M_n(B(\mathcal{H}))$  a  $C^*$ -algebra. It is straightforward to check, that for an element  $[x_{i,j}] \in M_n(B(\mathcal{H}))$ , this norm satisfies

$$\max_{i,j} \|x_{i,j}\| \leq \left\| \begin{bmatrix} x_{1,1} & \dots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \dots & x_{n,n} \end{bmatrix} \right\| \leq \sum_{i,j=1}^n \|x_{i,j}\|.$$

These inequalities will be referred to as **the standard matrix estimates**, since they are immensely useful, and we will be referring to them a couple of times. In fact, the latter inequality is actually just the triangle inequality.

Now, for general  $\mathcal{A}$ , we may choose some faithful representation  $\pi$  on a Hilbert space  $\mathcal{K}$ . The map  $\pi_n: M_n(\mathcal{A}) \rightarrow M_n(B(\mathcal{K}))$  then becomes an injective  $*$ -homomorphism. The fact that it is injective follows from the standard matrix estimates. In this way we obtain a norm on  $M_n(\mathcal{A})$ , and it also follows from the standard matrix estimates, that this norm makes  $M_n(\mathcal{A})$  into a  $C^*$ -algebra. In particular, this norm is independent of the choice of faithful representation, by uniqueness of the norm on a  $C^*$ -algebra.

Another way of describing the norm on  $M_n(\mathcal{A})$  for a  $C^*$ -algebra  $\mathcal{A}$  is the following. As mentioned in the beginning of this appendix, we may identify  $M_n(\mathcal{A})$  with the space  $\mathcal{A} \otimes \mathbb{M}_n$  in a natural way. It is easy to check that the structure on  $M_n(\mathcal{A})$  described above is just the usual  $*$ -algebra structure on  $\mathcal{A} \otimes \mathbb{M}_n$ , which makes this space into a  $C^*$ -algebra. In particular, with the chosen structure, the identification of  $M_n(\mathcal{A})$  with  $\mathcal{A} \otimes \mathbb{M}_n$  is a  $*$ -isomorphism.<sup>1</sup>

Now, returning to the operator systems, we see that if  $\mathcal{S}$  is an operator system, say with ambient  $C^*$ -algebra  $\mathcal{A}$ , then for each  $n \in \mathbb{N}$ , we get an induced norm on  $M_n(\mathcal{S})$  from the inclusion  $M_n(\mathcal{S}) \subseteq M_n(\mathcal{A})$ . We also see that  $M_n(\mathcal{S})$  becomes an operator system in a natural way. Concerning the matrix norm on an operator system  $\mathcal{S}$ , it holds that

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \right\| = \max\{\|x\|, \|y\|\},$$

whenever  $x, y \in \mathcal{S}$ . These equalities are easily obtained by proving that they hold for operators in Hilbert spaces. This easy observation frequently comes in handy, when applying  $2 \times 2$  matrix tricks, and often in the case where  $y = x$  or  $y = x^*$ . These equalities will be used without mentioning.

With this structure on the matrix spaces over an operator system, a natural thing to consider is maps that preserve this structure. For a bounded linear map  $\phi: \mathcal{S} \rightarrow \mathcal{S}'$  from one operator system  $\mathcal{S}$  to another  $\mathcal{S}'$ , we get from the standard matrix estimates, that for  $[x_{i,j}] \in M_n(\mathcal{S})$  we have

$$\|\phi_n([x_{i,j}])\| \leq \sum_{i,j=1}^n \|\phi(x_{i,j})\| \leq \|\phi\| \sum_{i,j=1}^n \|x_{i,j}\| \leq n^2 \|\phi\| \| [x_{i,j}] \|.$$

In particular, boundedness of  $\phi$  implies boundedness of  $\phi_n$ , and with  $\|\phi_n\| \leq n^2 \|\phi\|$ . This though seems like a pretty bad estimate for the norm of  $\phi_n$ , and indeed in many cases one can do better, but not always. For example, it is not true that if  $\phi$  is bounded,

<sup>1</sup>The same conclusion naturally holds with  $\mathcal{A} \otimes \mathbb{M}_n$  replaced by  $\mathbb{M}_n \otimes \mathcal{A}$ .

then  $(\|\phi_n\|)_{n \in \mathbb{N}}$  is uniformly bounded. For this reason the following quantity is of interest:

$$\|\phi\|_{\text{cb}} = \sup\{\|\phi_n\| : n \in \mathbb{N}\}.$$

This obviously defines a norm for those linear maps for which it is finite, and it is called the **completely bounded norm**.

This leads us to the following definition:

**Definition B.1.2.** Suppose that  $\mathcal{S}$  and  $\mathcal{S}'$  are operator systems, and let  $\phi: \mathcal{S} \rightarrow \mathcal{S}'$  be a linear map. The map  $\phi$  is called **completely bounded** if  $\|\phi\|_{\text{cb}} < \infty$  and **completely contractive** if in addition  $\|\phi\|_{\text{cb}} \leq 1$ . In the case where  $\phi_n$  is an isometry for all  $n \in \mathbb{N}$ , we say that  $\phi$  is **completely isometric**, or that  $\phi$  is a **complete isometry**. ◀

Particularly nice are the complete isometries, since they carry all the information of the different matrix norms. But there are more information to be preserved. Indeed, in operator systems there is a notion of positivity, and therefore also a notion of order. Hence, an extension of the notion of positive maps would be in order.

**Definition B.1.3.** Suppose that  $\mathcal{S}$  is an operator system and  $\mathcal{B}$  a  $C^*$ -algebra. Let  $\phi: \mathcal{S} \rightarrow \mathcal{B}$  be a linear map. If  $\phi_n$  is positive then  $\phi$  is said to be  **$n$ -positive**, and if  $\phi$  is  $n$ -positive for all  $n \in \mathbb{N}$ , then then we say that  $\phi$  is **completely positive**.

It of course goes with out saying, that a **unital completely positive** maps, is a completely positive map from an operator system to a unital  $C^*$ -algebra, which carries the unit to the unit.

So far we have not addressed the problem, whether the described structure on an operator system depends on the choice of ambient  $C^*$ -algebra. Of course it does not, but there are a subtlety. Suppose that  $\mathcal{S}$  is an operator system, and  $\mathcal{A}, \mathcal{B}$  are two  $C^*$ -algebras containing  $\mathcal{S}$ . If  $\mathcal{S}$  generates the same  $C^*$ -algebra, say  $\mathcal{C}$ , in both  $\mathcal{A}$  and  $\mathcal{B}$ , same algebraic structure and all, then the inclusion of  $\mathcal{C}$  into  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, is a completely positive map and a complete isometry.<sup>2</sup> In particular, we see that the operator system structure on  $\mathcal{S}$  does not depend on this choice of ambient  $C^*$ -algebra. Because of this free choice of ambient  $C^*$ -algebra it may always be arranged that the ambient  $C^*$ -algebra is unital, even  $B(\mathcal{H})$ , for some Hilbert space  $\mathcal{H}$ , without altering the matrix structure.

Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are unital  $C^*$ -algebras, and that  $\pi: \mathcal{A} \rightarrow \mathcal{B}$  is a  $*$ -homomorphism. It is straightforward to check that  $\pi$  is completely positive and completely contractive. If in addition  $\pi$  is injective, then  $\pi$  is a complete isometry. All this follows just from the fact that, if  $\pi$  is a homomorphism, then  $\pi_n$  is a homomorphism, and, if  $\pi$  is injective, the  $\pi_n$  is injective. Now, given an operator system  $\mathcal{S}$ , we can represent the ambient  $C^*$ -algebra faithfully and non-degenerately on a Hilbert space  $\mathcal{H}$ , and in this way obtain a unital completely positive map  $\mathcal{S} \rightarrow B(\mathcal{H})$ , which is also a complete isometry. Because of this we are often allowed to assume that our operator system consists of bounded operators on some Hilbert space.

If a map is  $n$ -positive, for some  $n \in \mathbb{N}$ , then it is also  $k$ -positive, for all  $k = 1, 2, \dots, n$ . This follows directly from the fact that an element of the form

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}$$

<sup>2</sup>Here “the same”  $C^*$ -algebra is a bit vague, but what we mean is, that the identity on  $\mathcal{S}$  extends to  $*$ -isomorphism from the  $C^*$ -algebra generated by  $\mathcal{S}$  in  $\mathcal{A}$  to the  $C^*$ -algebra generated by  $\mathcal{S}$  in  $\mathcal{B}$ .

is positive if and only if  $x$  is positive. Besides this observation, we also have the following remark, which states that all the concepts introduced above behave nicely with respect to compositions:

**Remark B.1.4.** If  $\phi$  and  $\psi$  are two maps linear maps, for which the composition  $\phi \circ \psi$  makes sense, then we know that  $(\phi \circ \psi)_n = \psi_n \circ \phi_n$ , for all  $n \in \mathbb{N}$ . Thus, it easily follows that:

- ◇ the composition of completely bounded maps is again completely bounded;
- ◇ the composition of completely contractive maps is again completely contractive;
- ◇ the composition of completely isometric maps is again completely isometric;
- ◇ the composition of  $n$ -positive maps is again  $n$ -positive;
- ◇ the composition of completely positive maps is again completely positive. ◀

The rest of this section will mostly be concerned with positive maps between operator systems. First, let us prove the following proposition, which frequently is of use:

**Proposition B.1.5.** *Let  $\mathcal{S}$  be an operator system and let  $x \in \mathcal{S}$ . Then  $\|x\| \leq 1$  if and only if the following element is positive in  $M_2(\mathcal{S})$ :*

$$\begin{bmatrix} \mathbf{1} & x \\ x^* & \mathbf{1} \end{bmatrix}$$

*Proof.* We may assume that  $\mathcal{S} \subseteq B(\mathcal{H})$  for some Hilbert space, by representing the ambient  $C^*$ -algebra faithfully on a Hilbert space. First note that for each  $\xi, \eta \in \mathcal{H}$  we have

$$\left\langle \begin{bmatrix} \mathbf{1} & x \\ x^* & \mathbf{1} \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \middle| \begin{bmatrix} \xi \\ \eta \end{bmatrix} \right\rangle = \|\xi\|^2 + \|\eta\|^2 + 2 \operatorname{Re}(x\eta \mid \xi). \quad (\text{B.1})$$

Now, assume that  $\|x\| \leq 1$ , and let us show that the specified matrix is positive. For each  $\xi, \eta \in \mathcal{H}$ , we see that

$$\begin{aligned} \|\xi\|^2 + \|\eta\|^2 + 2 \operatorname{Re}(x\eta \mid \xi) &\geq \|\xi\|^2 + \|\eta\|^2 - 2\|\xi\|\|x\eta\| \\ &\geq \|\xi\|^2 + \|\eta\|^2 - 2\|\xi\|\|\eta\| \\ &= (\|\xi\| - \|\eta\|)^2 \geq 0 \end{aligned}$$

which, by (B.1), shows that the matrix in question is indeed positive.

Now for the converse implication, suppose that  $\|x\| > 1$ . Then we want to show that the inner product (B.1) is not always non-negative. Since  $\|x\| > 1$  we can find some unit vectors  $\xi, \eta \in \mathcal{H}$ , with  $(x\xi \mid \eta) < -1$ . But then

$$\|\xi\|^2 + \|\eta\|^2 + 2 \operatorname{Re}(x\eta \mid \xi) < 0$$

which shows that the inner product (B.1) is not always non-negative, for every  $\xi, \eta \in \mathcal{H}$ , and the matrix in question can therefore not be positive.  $\square$

As with linear functionals on  $C^*$ -algebras, positive maps between operator systems are automatically bounded. The following proposition gives an explicit bound, on the norm:

**Proposition B.1.6.** *Let  $\mathcal{S}$  be an operator system, and let  $\mathcal{B}$  be a  $C^*$ -algebra. If  $\phi : \mathcal{S} \rightarrow \mathcal{B}$  is a positive map, then  $\|\phi(x)\| \leq \|x\|\|\phi(\mathbf{1})\|$ , for all self-adjoint  $x \in \mathcal{S}$ . In particular,  $\phi$  is bounded with  $\|\phi\| \leq 2\|\phi(\mathbf{1})\|$ .*

*Proof.* Suppose that  $x \in \mathcal{S}$  is self-adjoint. Then  $-\|x\|\mathbf{1} \leq x \leq \|x\|\mathbf{1}$ , and since  $\phi$  is positive  $-\|x\|\phi(\mathbf{1}) \leq \phi(x) \leq \|x\|\phi(\mathbf{1})$ . This shows that  $\|\phi(x)\| \leq \|x\|\|\phi(\mathbf{1})\|$ .

Now, if we do not assume that  $x$  is self-adjoint, then by writing  $x = \operatorname{Re} x + i \operatorname{Im} x$  and remembering that  $\|\operatorname{Re} x\| \leq \|x\|$  and  $\|\operatorname{Im} x\| \leq \|x\|$ , we see that

$$\begin{aligned} \|\phi(x)\| &\leq \|\phi(\operatorname{Re} x)\| + \|\phi(\operatorname{Im} x)\| \\ &\leq \|\operatorname{Re} x\|\|\phi(\mathbf{1})\| + \|\operatorname{Im} x\|\|\phi(\mathbf{1})\| \\ &\leq 2\|x\|\|\phi(\mathbf{1})\|. \end{aligned}$$

Thus  $\|\phi\| \leq 2\|\phi(\mathbf{1})\|$ , as desired.  $\square$

As a fact, one cannot do better than 2 in the above proposition, unless some extra assumptions are made (see, for example, [Pau02, Example 2.2]). But in many cases one can do better. Already if the map is 2-positive, then the estimate gets better. Before we prove this, note that if a map  $\phi : \mathcal{S} \rightarrow \mathcal{S}'$  between operator systems, is positive, then it is also Hermitian. This follows from the fact that if  $x \in \mathcal{S}$ , then also  $\operatorname{Re} x$  and  $\operatorname{Im} x$  belong to  $\mathcal{S}$ .

**Proposition B.1.7.** *Let  $\mathcal{S}$  be an operator system and let  $\mathcal{B}$  be a  $C^*$ -algebra. If  $\phi : \mathcal{S} \rightarrow \mathcal{B}$  is positive and 2-positive, then  $\|\phi\| = \|\phi(\mathbf{1})\|$ .*

*Proof.* Let  $x \in \mathcal{S}$ , and let us show that  $\|\phi(x)\| \leq \|x\|$ . Since  $\phi$  is positive we get by Proposition B.1.6 that

$$\left\| \begin{bmatrix} 0 & \phi(x) \\ \phi(x)^* & 0 \end{bmatrix} \right\| = \left\| \phi_2 \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix} \right\| \|\phi_2(\mathbf{1})\|.$$

But the left hand side equals  $\|\phi(x)\|$ , and the right hand side equals  $\|x\|\|\phi(\mathbf{1})\|$ , and so we have the inequality  $\|\phi(x)\| \leq \|x\|$ . Thus  $\|\phi\| \leq \|\phi(\mathbf{1})\|$ , while the other inequality is trivial.  $\square$

**Corollary B.1.8.** *Let  $\mathcal{S}$  be an operator system and  $\mathcal{B}$  a  $C^*$ -algebra. If  $\phi : \mathcal{S} \rightarrow \mathcal{B}$  is completely positive, then  $\phi$  is completely bounded with  $\|\phi\| = \|\phi\|_{\text{cb}} = \|\phi(\mathbf{1})\|$ .*

*Proof.* Let  $n \in \mathbb{N}$ . Since  $\phi$  is completely positive,  $\phi_n$  is, in particular, positive and 2-positive, hence  $\|\phi_n\| = \|\phi_n(\mathbf{1})\| = \|\phi(\mathbf{1})\|$ .  $\square$

Also in the case of linear functionals one can do better. The following proposition is a generalization of a well-known result for unital  $C^*$ -algebras, which also holds for operator systems:

**Proposition B.1.9.** *A linear functional  $\psi$  on an operator system is positive if and only if  $\|\psi\| = \psi(\mathbf{1})$ . Moreover each such linear functional extends a positive linear functional on an ambient  $C^*$ -algebra.*

*Proof.* Suppose that  $\mathcal{A}$  is a unital  $C^*$ -algebra,  $\mathcal{S} \subseteq \mathcal{A}$  an operator system and  $\psi$  a linear functional on  $\mathcal{S}$ .

Suppose first that  $\|\psi\| = \psi(\mathbf{1})$ . By the Hahn-Banach Theorem we can extend  $\psi$  to a linear functional  $\tilde{\psi}$  on  $\mathcal{A}$  of the same norm. Since  $\|\tilde{\psi}\| = \|\psi\| = \psi(\mathbf{1}) = \tilde{\psi}(\mathbf{1})$  we get that  $\tilde{\psi}$  is positive. In particular  $\psi$  is positive, and we have seen that it extends to a positive linear functional on  $\mathcal{A}$ .

Suppose now instead, that  $\psi$  is positive. Fix  $x \in \mathcal{S}$  with  $\|x\| \leq 1$ , and choose  $\lambda \in \mathbb{C}$ , so that  $\psi(\lambda x) \geq 0$ . Then, since  $\psi$  is Hermitian,

$$\psi(\operatorname{Re} x) = \frac{1}{2}(\psi(\lambda x) + \overline{\psi(\lambda x)}) = \psi(x).$$

We know from Proposition B.1.6  $|\psi(\operatorname{Re} x)| \leq \psi(\mathbf{1})\|x\|$ , since  $\operatorname{Re} x$  is self-adjoint, but then  $|\psi(x)| \leq \psi(\mathbf{1})\|x\|$ , and since  $x$  was arbitrary, we conclude that  $\|\psi\| = \psi(\mathbf{1})$ .  $\square$

The next proposition is a generalization of the fact that a contractive linear functionals which sends the unit to one (in other words, a state), is actually positive.

**Proposition B.1.10.** *Let  $\mathcal{S}$  be an operator system and  $\mathcal{B}$  a  $C^*$ -algebra. If  $\phi : \mathcal{S} \rightarrow \mathcal{B}$  is a unital contraction, then  $\phi$  is positive.*

*Proof.* We may assume that  $\mathcal{B} = B(\mathcal{H})$ , for some Hilbert space  $\mathcal{H}$ , and let  $\mathcal{A}$  denote the ambient Hilbert space of  $\mathcal{S}$ . Let  $\xi$  be a unit vector in  $\mathcal{H}$ . Define a linear functional  $f_\xi$  on  $\mathcal{S}$  by  $f_\xi(x) = \langle \phi(x)\xi \mid \xi \rangle$ ,  $x \in \mathcal{S}$ . Since  $\phi$  is unital and contractive, so is  $f_\xi$ . Extend  $f_\xi$  to a linear functional  $\hat{f}_\xi$  on  $\mathcal{A}$  of the same norm. Since  $\hat{f}_\xi$  is unital of norm one, it is a state on  $\mathcal{A}$ . In particular, it is positive, so  $\langle \phi(x)\xi \mid \xi \rangle$  is non-negative for all  $x \in \mathcal{S}$ . Fix  $x \in \mathcal{S}$ . Since  $\xi$  was arbitrary of norm one, we know that  $\langle \phi(x)\xi \mid \xi \rangle$  is non-negative, for all  $\xi \in \mathcal{H}$ , that is  $\phi(x)$  is positive. Since  $x \in \mathcal{S}$  was arbitrary, this shows that  $\phi$  is positive.  $\square$

**Corollary B.1.11.** *Unital completely contractive linear map is automatically unital completely positive.*

## B.2 Completely positive maps

The main result in this section is the result known as Stinespring's Dilation Theorem, which, in a manner of speaking, says that a contractive completely positive map is the corner of a  $*$ -representation. This celebrated result has a vast amount of applications.

After we have proved Stinespring's Dilation Theorem, we look at the space of completely positive maps.

Let us start by considering some ways in which completely positive maps occur. Suppose that we are given a unital  $C^*$ -algebra  $\mathcal{A}$  and an operator system  $\mathcal{S} \subseteq \mathcal{A}$ .

First, for  $x \in \mathcal{A}$ , the map  $\phi_x : \mathcal{A} \rightarrow \mathcal{A}$  defined by  $\phi_x(y) = x^*yx$  is completely positive. To see this, one just notices that, for  $n \in \mathbb{N}$  and  $[y_{i,j}] \in M_n(\mathcal{A})$ , the matrix  $(\phi_x)_n([y_{i,j}])$  is given by the product

$$\begin{bmatrix} x & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & x \end{bmatrix}^* \begin{bmatrix} y_{1,1} & \dots & y_{1,n} \\ \vdots & \ddots & \vdots \\ y_{n,1} & \dots & y_{n,n} \end{bmatrix} \begin{bmatrix} x & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & x \end{bmatrix}^* .$$

So if  $[y_{i,j}] \in M_n(\mathcal{A})$  is positive, then  $(\phi_x)_n([y_{i,j}])$  is positive. Hence  $\phi_x$  is completely positive.

Second, suppose that  $\psi: \mathcal{S} \rightarrow B(\mathcal{H})$  a bounded linear map, with  $\mathcal{H}$  a Hilbert space. If  $\mathcal{K}$  is another Hilbert space and  $V: \mathcal{K} \rightarrow \mathcal{H}$  a bounded linear operator, then we may define a bounded linear map

$$\phi: \mathcal{S} \rightarrow B(\mathcal{K}) \quad \text{by} \quad \phi(x) = V^* \psi(x) V, \quad \text{for} \quad x \in \mathcal{S}.$$

It is not necessarily the case that  $\phi$  is completely positive. This, of course, depends on  $\psi$ . More precisely,  $\psi$  is  $n$ -positive, then  $\phi$  is also  $n$ -positive. In particular, if  $\psi$  is completely positive, then  $\phi$  is also completely positive. Let us see why this is the case. If  $n \in \mathbb{N}$  and  $[x_{i,j}] \in M_n(\mathcal{S})$ , then

$$\begin{aligned} & \left\langle \begin{bmatrix} V^* \psi(x_{1,1}) V & \dots & V^* \psi(x_{1,n}) V \\ \vdots & \ddots & \vdots \\ V^* \psi(x_{n,1}) V & \dots & V^* \psi(x_{n,n}) V \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \mid \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_n \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} \psi(x_{1,1}) & \dots & \psi(x_{1,n}) \\ \vdots & \ddots & \vdots \\ \psi(x_{n,1}) & \dots & \psi(x_{n,n}) \end{bmatrix} \begin{bmatrix} V \xi_1 \\ \vdots \\ V \xi_n \end{bmatrix} \mid \begin{bmatrix} V \eta_1 \\ \vdots \\ V \eta_n \end{bmatrix} \right\rangle \end{aligned}$$

for all  $(\xi_1, \dots, \xi_n)$  and  $(\eta_1, \dots, \eta_n)$  in  $\mathcal{K}^{\oplus n}$ . Hence  $\phi_n$  is positive if  $\psi_n$  is positive.

A special case of the last example is when  $\mathcal{S} = \mathcal{A}$  is a unital  $C^*$ -algebra and the map  $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$  is a  $*$ -homomorphism. This case is of special interest, because it turns out that all completely positive maps have this form in a sense. This is the contents of the next theorem, which will be referred to as Stinespring's Dilation Theorem.

**Theorem B.2.1.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and  $\mathcal{H}$  be a Hilbert space. Then, for each completely positive map  $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$ , there exist a Hilbert space  $\mathcal{K}$ , a unital  $*$ -homomorphism  $\pi: \mathcal{A} \rightarrow B(\mathcal{K})$  and a bounded operator  $V: \mathcal{H} \rightarrow \mathcal{K}$ , with  $\|\phi(\mathbf{1})\| = \|V\|^2$ , such that*

$$\phi(x) = V^* \pi(x) V, \quad x \in \mathcal{A}.$$

*Proof.* Let  $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$  be a completely positive map. First we want to define a sesquilinear form  $\langle \cdot \mid \cdot \rangle_\phi$  on the algebraic tensor product  $\mathcal{A} \odot \mathcal{H}$ . This is done by the formula

$$\langle a \otimes \xi \mid b \otimes \eta \rangle_\phi = \langle \phi(b^* a) \xi \mid \eta \rangle_{\mathcal{H}},$$

for  $a, b \in \mathcal{A}$  and  $\xi, \eta \in \mathcal{H}$ . Here  $\langle \cdot \mid \cdot \rangle_{\mathcal{H}}$  denotes the inner product on  $\mathcal{H}$ . It is straightforward to check that this is well-defined, and that

$$\left\langle \sum_{i=1}^n a_i \otimes \xi_i \mid \sum_{i=1}^n a_i \otimes \xi_i \right\rangle_\phi = \langle \phi_n([a_j^* a_i]) \xi \mid \xi \rangle_{\mathcal{H}^{\oplus n}},$$

when  $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{H}^{\oplus n}$  and  $a_i \in \mathcal{A}$ , for  $i = 1, \dots, n$ . Because  $\phi_n$  is positive by assumption, and because

$$\begin{bmatrix} a_1^* a_1 & \dots & a_1^* a_n \\ \vdots & \ddots & \vdots \\ a_n^* a_1 & \dots & a_n^* a_n \end{bmatrix} = \begin{bmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}^* \begin{bmatrix} a_1 & \dots & a_n \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix},$$



it follows that  $\langle \cdot | \cdot \rangle_\phi$  is positive semi-definite, that is,  $\langle v | v \rangle \geq 0$  for all  $v \in \mathcal{A} \odot \mathcal{H}$ . Now, if we take the quotient of  $\mathcal{A} \odot \mathcal{H}$  by the closed subspace

$$\mathcal{N} = \{v \in \mathcal{A} \odot \mathcal{H} : \langle v | v \rangle_\phi = 0\},$$

then we get a pre-Hilbert space. Let  $\mathcal{K}$  denote its completion. Fix  $a \in \mathcal{A}$ , and consider the linear map

$$\tilde{\pi}(a) : \mathcal{A} \odot \mathcal{H} \rightarrow \mathcal{A} \odot \mathcal{H} \quad \text{defined by} \quad \tilde{\pi}(a) \left( \sum_{i=1}^n b_i \otimes \xi_i \right) = \sum_{i=1}^n ab_i \otimes \xi_i,$$

for  $b_1, \dots, b_n$  in  $\mathcal{A}$  and  $\xi_1, \dots, \xi_n$  in  $\mathcal{H}$ . It is straightforward to check that, for  $b_1, \dots, b_n$  in  $\mathcal{A}$  and  $\xi_1, \dots, \xi_n$  in  $\mathcal{H}$ , we have  $[b_j^* a^* ab_i] \leq \|a^* a\| [b_j^* b_i]$ , and so

$$\begin{aligned} \left\| \tilde{\pi}(a) \sum_{i=1}^n b_i \otimes \xi_i \right\|_\phi^2 &= \langle \phi_n([b_j^* a^* ab_i]) \xi | \xi \rangle_{\mathcal{H}^{\oplus n}} \\ &\leq \|a^* a\| \langle \phi_n([b_j^* b_i]) \xi | \xi \rangle_{\mathcal{H}^{\oplus n}} \\ &= \|a^* a\| \left\| \sum_{i=1}^n b_i \otimes \xi_i \right\|_\phi^2, \end{aligned}$$

where  $\xi = (\xi_1, \dots, \xi_n)$ . This shows that  $\tilde{\pi}(a)$  leaves  $\mathcal{N}$  invariant, and therefore induces a well-defined bounded linear operator on the quotient  $(\mathcal{A} \odot \mathcal{H})/\mathcal{N}$ . We will denote this operator by  $\pi'(a)$ . The above calculation also shows that  $\|\pi'(a)\| \leq \|a\|$ , so  $\pi'(a)$  extends to a linear operator on  $\mathcal{K}$  by continuity. We denote this extension by  $\pi(a)$ . We know have a map  $\pi : \mathcal{A} \rightarrow B(\mathcal{K})$ , and it is easy to see that this map is a unital  $*$ -homomorphism. Define a map

$$V : \mathcal{H} \rightarrow \mathcal{K} \quad \text{by} \quad V\xi = \mathbf{1} \otimes \xi + \mathcal{N}, \quad \text{for } \xi \in \mathcal{H}.$$

Clearly this is a linear operator, and we see that

$$\|V\xi\|^2 = \langle \mathbf{1} \otimes \xi | \mathbf{1} \otimes \xi \rangle_\phi = \langle \phi(\mathbf{1})\xi | \xi \rangle_{\mathcal{H}} \leq \|\phi(\mathbf{1})\| \|\xi\|^2.$$

In particular,  $V$  is bounded. Since  $\phi(\mathbf{1})$  is a positive operator we get that  $\|\phi(\mathbf{1})\| = \sup\{\langle \phi(\mathbf{1})\xi | \xi \rangle_{\mathcal{H}} : x \in \mathcal{H}, \|x\| \leq 1\}$ , and so it actually follows from the equality  $\|V\xi\|^2 = \langle \phi(\mathbf{1})\xi | \xi \rangle_{\mathcal{H}}$  that  $\|V\|^2 = \|\phi(\mathbf{1})\|$ . For each  $a \in \mathcal{A}$  we see that

$$\langle V^* \pi(a) V \xi | \eta \rangle_{\mathcal{H}} = \langle \pi(a) \mathbf{1} \otimes \xi | \mathbf{1} \otimes \eta \rangle_\phi = \langle \phi(a)\xi | \eta \rangle_{\mathcal{H}}$$

for all  $\xi, \eta \in \mathcal{H}$ . Hence  $V^* \pi(a) V = \phi(a)$  for all  $a \in \mathcal{A}$ , and the proof is complete.  $\square$

There are several comments that can be made concerning Stinespring's Dilation Theorem, and its proof. Most importantly, if  $\phi$  is unital, then  $V^* V = V^* \pi(\mathbf{1}) V = \phi(\mathbf{1}) = \mathbf{1}$ , which shows that  $V$  is an isometry. This we will often use, and it will be implicit that this is a consequence of Stinespring's Dilation Theorem. Besides this, we can observe in the proof that  $\pi(\mathcal{A})V\mathcal{H}$  spans a dense subset of  $\mathcal{K}$ . Last, if both  $\mathcal{A}$  and  $\mathcal{H}$  are separable, then  $\mathcal{A} \odot \mathcal{H}$  is separable. Hence  $\mathcal{K}$  is also separable.

Let us before we move on give the following corollary to Stinespring's Dilation Theorem, which is just a non-unital version:

**Corollary B.2.2.** *Suppose that  $\mathcal{A}$  is a non-unital  $C^*$ -algebra,  $\mathcal{H}$  a Hilbert space and  $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$  a contractive completely positive map. Then there exist a Hilbert space  $\mathcal{K}$ , a representation  $\pi: \mathcal{A} \rightarrow B(\mathcal{K})$  and an isometry  $V: \mathcal{H} \rightarrow \mathcal{K}$ , so that  $\phi(a) = V^*\pi(a)V$ , for all  $a \in \mathcal{A}$ .*

*Proof.* By [BO08, Proposition 2.2.1] the map  $\phi$  extends to a unital completely positive map  $\tilde{\phi}$  from  $\tilde{\mathcal{A}}$  to  $B(\mathcal{H})$ . Let  $(\pi, V, \mathcal{H})$  be a Stinespring representation for  $\tilde{\phi}$ . Since  $\tilde{\phi}$  is unital, then operator  $V$  is an isometry. Now  $(\pi|_{\mathcal{A}}, V, \mathcal{H})$  is a Stinespring representation for  $\phi$  with  $V$  an isometry.  $\square$

It is of course natural to believe that there could be several Hilbert spaces and several  $*$ -homomorphisms with the properties listed in Stinespring's Dilation Theorem. This is, in fact, the case, but by adding some sort of non-degeneracy condition, the triple  $(\pi, V, \mathcal{H})$  essentially becomes unique.

**Definition B.2.3.** A triple  $(\pi, V, \mathcal{K})$  associated to a completely positive map  $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$  as in Stinespring's Dilation Theorem is called a **Stinespring representation** for  $\phi$ . If in addition  $\pi(\mathcal{A})V\mathcal{H}$  spans a dense subset of  $\mathcal{K}$ , then  $(\pi, V, \mathcal{K})$  is called a **minimal Stinespring representation** for  $\phi$ .  $\blacktriangleleft$

Let us prove the minimal Stinespring representations are essentially unique.

**Proposition B.2.4.** *Suppose that  $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$  is completely positive, with  $\mathcal{A}$  a unital  $C^*$ -algebra and  $\mathcal{H}$  a Hilbert space. The map  $\phi$  has a minimal Stinespring representation  $(\pi_1, V_1, \mathcal{K}_1)$ . If  $(\pi_2, V_2, \mathcal{K}_2)$  is another Stinespring representation of  $\phi$ , then there exist an isometry  $U: \mathcal{K}_1 \rightarrow \mathcal{K}_2$  such that  $V_2 = UV_1$  and  $\pi_2 = U^*\pi_1U$ . Moreover, if  $(\pi_2, V_2, \mathcal{K}_2)$  is also minimal, then  $U$  is unitary.*

*Proof.* The existence is already taken care of—see the note above Definition B.2.3. We start by defining  $U: \mathcal{K}_1 \rightarrow \mathcal{K}_2$  by the letting

$$U\left(\sum_{i=1}^n \pi_1(a_i)V_1\xi_i\right) = \sum_{i=1}^n \pi_2(a_i)V_2\xi_i$$

for all  $a_i \in \mathcal{A}$ ,  $\xi_i \in \mathcal{H}$  ( $i = 1, \dots, n$ ). Let us show that this describes a well-defined operator  $\mathcal{K}_1 \rightarrow \mathcal{K}_2$ . First note that the expression above only specifies the operator on the span of  $\pi_1(\mathcal{A})V_1\mathcal{H}$ , but since the span of this is dense in  $\mathcal{K}_1$ , we just have to show that the operator is well-defined and bounded on the span of  $\pi_1(\mathcal{A})V_1\mathcal{H}$ . Therefore it suffices to show that the above prescription is isometric, which follows from the fact that

$$\left\|\sum_{i=1}^n \pi_k(a_i)V_k\xi_i\right\|^2 = \sum_{i,j=1}^n (V_k\pi_k(a_i^*a_j)V_k\xi_j \mid \xi_i) = \sum_{i,j=1}^n (\phi(a_i^*a_j)\xi_j \mid \xi_i)$$

for  $k = 1, 2$ . Thus  $U$  is a well-defined isometry, and by construction it satisfies the specified relations. If we also assume that  $(\pi_2, V_2, \mathcal{K}_2)$  is minimal, then the image of  $U$  must be dense since it contains the span of  $\pi_2(\mathcal{A})V_2\mathcal{H}$ . Hence it must be surjective, which means that  $U$  is a unitary.  $\square$

Now we turn away from completely positive maps a bit, to prove that certain spaces of bounded linear operators, can be thought of as dual Banach spaces.

In the rest of this section, we will use  $\langle \cdot, \cdot \rangle$  to denote the duality between a normed space  $\mathfrak{X}$  and its dual Banach space  $\mathfrak{X}^*$ . In other words, for a bounded linear functional  $\phi$  on  $\mathfrak{X}$  and an element  $x \in \mathfrak{X}$ , we let  $\langle \phi, x \rangle$  denote the evaluation of  $\phi$  in  $x$ .

For normed spaces  $\mathfrak{X}$  and  $\mathfrak{Y}$  we can identify the algebraic tensor product  $\mathfrak{X} \odot \mathfrak{Y}$  with a subset of  $B(\mathfrak{X}, \mathfrak{Y}^*)^*$ . Namely, by letting a basic tensor  $x \otimes y$  act as a linear functional on  $B(\mathfrak{X}, \mathfrak{Y}^*)$ , according to the rule:  $\langle x \otimes y, F \rangle = \langle F(x), y \rangle$ , when  $F \in B(\mathfrak{X}, \mathfrak{Y}^*)$ . It is easy to show that identification is well-defined, that is, that this prescription of a linear functional to a pair  $(x, y)$  is bilinear and injective. In this way we obtain a norm on  $\mathfrak{X} \odot \mathfrak{Y}$ .

**Proposition B.2.5.** *If  $\mathfrak{X}$  and  $\mathfrak{Y}$  are normed spaces, then  $B(\mathfrak{X}, \mathfrak{Y}^*)$  is isometrically isomorphic to the dual space  $(\mathfrak{X} \odot \mathfrak{Y})^*$  of the algebraic tensor product  $\mathfrak{X} \odot \mathfrak{Y}$ , with the norm described above, via the duality given by*

$$\langle L, x \otimes y \rangle = \langle L(x), y \rangle,$$

for all  $L \in B(\mathfrak{X}, \mathfrak{Y}^*)$ ,  $x \in \mathfrak{X}$  and  $y \in \mathfrak{Y}$ .

*Proof.* Let  $\iota: \mathfrak{X} \odot \mathfrak{Y} \rightarrow B(\mathfrak{X}, \mathfrak{Y}^*)^*$  denote the identification described above. By definition this map is isometric, and so we get an isometric inclusion  $\iota^*: B(\mathfrak{X}, \mathfrak{Y}^*)^{**} \rightarrow (\mathfrak{X} \odot \mathfrak{Y})^*$ . By restricting  $\iota^*$  to  $B(\mathfrak{X}, \mathfrak{Y}^*)$  inside  $B(\mathfrak{X}, \mathfrak{Y}^*)^{**}$  we get an isometric inclusion of  $B(\mathfrak{X}, \mathfrak{Y}^*)$  into  $(\mathfrak{X} \odot \mathfrak{Y})^*$ , and it is straightforward to check that the duality described in the statement of the proposition, is the duality one obtains in this way.

What we are left to prove, is that this inclusion is surjective. So fix  $f \in (\mathfrak{X} \odot \mathfrak{Y})^*$ . For each  $x \in \mathfrak{X}$  and  $y \in \mathfrak{Y}$ , we know that  $|f(x, y)| \leq \|f\| \|x\| \|y\|$ , so the map

$$f_x: \mathfrak{Y} \rightarrow \mathbb{C} \quad \text{given by} \quad f_x(y) = f(x \otimes y), \quad \text{for} \quad y \in \mathfrak{Y},$$

is a bounded linear functional, bounded by  $\|x\| \|f\|$ . Hence, the linear map  $L: \mathfrak{X} \rightarrow \mathfrak{Y}^*$  give by  $L(x) = f_x$  is well-defined, and bounded, bounded by  $\|f\|$ . Clearly  $\iota^*(L) = f$ , which shows that  $\iota^*$  is surjective, and therefore that  $B(\mathfrak{X}, \mathfrak{Y}^*)$  can be identified with the dual space of  $\mathfrak{X} \odot \mathfrak{Y}$  in the described way.  $\square$

**Remark B.2.6.** Proposition B.2.5 allows us to put a weak\*-topology on  $B(\mathfrak{X}, \mathfrak{Y}^*)$ , namely, the topology on  $B(\mathfrak{X}, \mathfrak{Y}^*)$  induced by the completion of  $\mathfrak{X} \odot \mathfrak{Y}$ .<sup>3</sup> This weak\*-topology is given, on bounded sets, as follows: a bounded net  $(L_\alpha)_{\alpha \in A}$  in  $B(\mathfrak{X}, \mathfrak{Y}^*)$  converges to some  $L \in B(\mathfrak{X}, \mathfrak{Y}^*)$  if and only if  $\lim_{\alpha \in A} \langle L_\alpha(x), y \rangle = \langle L(x), y \rangle$ , for all  $x \in \mathfrak{X}$  and  $y \in \mathfrak{Y}$ . The convergence on unbounded sets is a bit more tricky to handle, but for most cases we are only interested in the bounded case.

A special case of Proposition B.2.5, is when  $\mathfrak{X} = \mathfrak{Y} = \mathcal{H}$  is a Hilbert space. In this case  $\mathcal{H}^* = \mathcal{H}$ , and so  $B(\mathcal{H}, \mathcal{H}^*) = B(\mathcal{H})$ . The conclusion of the proposition is then that  $B(\mathcal{H})$  is a dual space  $\mathcal{H} \odot \mathcal{H}$ , with the specified duality. In this duality  $\mathcal{H} \odot \mathcal{H}$  exactly corresponds to the weak operator continuous linear functionals, and recalling that the closure of these are the ultraweakly continuous linear functionals, we once more obtain this result.

Another special case of Proposition B.2.5, is the case where  $\mathfrak{X}$  is just some normed space and  $\mathfrak{Y}^* = B(\mathcal{H})$ .

<sup>3</sup>There is a subtle point here, namely, that  $\mathfrak{X} \odot \mathfrak{Y}$ , by all probability, is not a Banach space. In general, if  $\mathfrak{X}$  is a Banach space and  $\mathfrak{Y} \subseteq \mathfrak{X}$  a dense subspace, then the topologies on  $\mathfrak{X}^*$  induced by  $\mathfrak{X}$  and  $\mathfrak{Y}$ , respectively, do not necessarily agree. It is though the case, that they agree on bounded sets.

**Proposition B.2.7.** *Given a normed space  $\mathfrak{X}$  and a Hilbert space  $\mathcal{H}$ , the weak\*-topology<sup>4</sup> on  $B(\mathfrak{X}, B(\mathcal{H}))$  is described on bounded sets as follows: a bounded net  $(\phi_\alpha)_{\alpha \in A}$  in  $B(\mathfrak{X}, B(\mathcal{H}))$  converges to some  $\phi$  if and only if*

$$\lim_{\alpha \in A} \langle \phi_\alpha(x)\xi \mid \eta \rangle = \langle \phi(x)\xi \mid \eta \rangle,$$

for all  $\xi, \eta \in \mathcal{H}$  and  $x \in \mathfrak{X}$ .

*Proof.* As mentioned above, we may realize  $B(\mathcal{H})$  as the dual space of the algebraic tensor product  $\mathcal{H} \odot \mathcal{H}$ , and therefore  $B(\mathfrak{X}, B(\mathcal{H}))$  as the dual space of the algebraic tensor product  $\mathfrak{X} \odot \mathcal{H} \odot \mathcal{H}$ . The duality, which identifies  $B(\mathfrak{X}, B(\mathcal{H}))$  with the dual space of  $\mathfrak{X} \odot \mathcal{H} \odot \mathcal{H}$  is the duality described above. Hence, the weak\*-topology on  $B(\mathfrak{X}, B(\mathcal{H}))$  is determined by  $\mathfrak{X} \odot \mathcal{H} \odot \mathcal{H}$  as described.  $\square$

Now, let us return to the completely positive maps. Given two operator systems  $\mathcal{S}$  and  $\mathcal{S}'$ , we denote the set of completely positive maps from  $\mathcal{S}$  to  $\mathcal{S}'$  by  $\text{CP}(\mathcal{S}, \mathcal{S}')$ . This set is a cone in  $B(\mathcal{S}, \mathcal{S}')$ , and if  $\mathcal{S}' = B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ , then we get a weak\*-topology on  $\text{CP}(\mathcal{S}, B(\mathcal{H}))$ . The following proposition is the reason why we introduced this weak\*-topology:

**Proposition B.2.8.** *Given an operator system  $\mathcal{S}$  and a Hilbert space  $\mathcal{H}$ , the sets*

$$\{\phi \in \text{CP}(\mathcal{S}, B(\mathcal{H})) : \|\phi\|_{\text{cb}} \leq r\} \quad \text{and} \quad \{\phi \in \text{CP}(\mathcal{S}, B(\mathcal{H})) : \phi(\mathbf{1}) = \mathbf{1}\}$$

are compact in the weak\*-topology, for all  $r > 0$ .

*Proof.* Let us start with the space to the left. By the Banach-Alaoglu Theorem the ball of radius  $r$  in  $B(\mathcal{S}, B(\mathcal{H}))$  is compact in the weak\*-topology.<sup>5</sup> Hence it suffices to show that  $\text{CP}_r(\mathcal{S}, B(\mathcal{H}))$  is closed in the weak\*-topology, relative to  $B(\mathcal{S}, B(\mathcal{H}))$ . So suppose that  $(\phi_\alpha)_{\alpha \in A}$  is a net in  $\text{CP}_r(\mathcal{S}, B(\mathcal{H}))$  converging to some  $\phi \in B(\mathcal{S}, B(\mathcal{H}))$  in the weak\*-topology. By Proposition B.2.7 this means that

$$\lim_{\alpha \in A} \langle \phi_\alpha(x)\xi \mid \eta \rangle = \langle \phi(x)\xi \mid \eta \rangle,$$

for all  $\xi, \eta \in \mathcal{H}$  and  $x \in \mathcal{S}$ . Let us show that  $\phi$  is completely positive. Let  $n \in \mathbb{N}$  and let  $[x_{i,j}] \in M_n(\mathcal{S})$  be positive. For each  $\xi = (\xi_1, \dots, \xi_n) \in \mathcal{H}^{\oplus n}$ , we have

$$\langle \phi_n([x_{i,j}])\xi \mid \xi \rangle = \lim_{\alpha \in A} \sum_{i,j=1}^n \langle \phi_\alpha(x_{i,j})\xi_j \mid \xi_i \rangle = \lim_{\alpha \in A} \langle (\phi_\alpha)_n([x_{i,j}])\xi \mid \xi \rangle$$

Since  $(\phi_\alpha)_n$  is positive, for each  $\alpha \in A$ , the right hand side is non-negative, and since  $\xi$  was arbitrary, this shows that  $\phi_n([x_{i,j}])$  is positive. Hence  $\phi$  is completely positive. Clearly  $\|\phi\| \leq r$  since  $\|\phi_\alpha\| \leq \|\phi_\alpha\|_{\text{cb}} \leq r$ , for all  $\alpha \in A$ . By Corollary B.1.8 we get that  $\|\phi\|_{\text{cb}} = \|\phi\|$ , since  $\phi$  is completely positive, and therefore  $\|\phi\|_{\text{cb}} \leq r$ . This proves compactness of the given set  $\{\phi \in \text{CP}(\mathcal{S}, B(\mathcal{H})) : \|\phi\|_{\text{cb}} \leq r\}$ .

Clearly the weak\* limit of unital maps is again unital, so since

$$\{\phi \in \text{CP}(\mathcal{S}, B(\mathcal{H})) : \phi(\mathbf{1}) = \mathbf{1}\} \subseteq \{\phi \in \text{CP}(\mathcal{S}, B(\mathcal{H})) : \|\phi\|_{\text{cb}} \leq 1\},$$

it follows that the set  $\{\phi \in \text{CP}(\mathcal{S}, B(\mathcal{H})) : \phi(\mathbf{1}) = \mathbf{1}\}$  is also weak\*-compact.  $\square$

<sup>4</sup>This refers to the weak\*-topology of Proposition B.2.5, which one obtains by considering  $B(\mathcal{H})$  a dual space, as explained just above this proposition.

<sup>5</sup>Here ball refers to the operator norm on  $B(\mathcal{S}, B(\mathcal{H}))$ , and not the completely bounded norm.

We end this section with a result on contractive completely positive maps and unitizations.

**Proposition B.2.9.** *Suppose that  $\mathcal{A}$  and  $\mathcal{B}$  are  $C^*$ -algebras which both are unital. Let  $\phi: \mathcal{A} \rightarrow \mathcal{B}$  be a contractive completely positive map. Then  $\phi$  extends to a contractive completely positive map  $\tilde{\mathcal{A}} \rightarrow \mathcal{B}$  with the same norm. Moreover if  $\phi$  is unital then the extension can be chosen unital.*

*Proof.* We may assume that  $\mathcal{B} \subseteq B(\mathcal{H})$ , but we do not assume that  $\mathcal{B}$  contains the unit of  $B(\mathcal{H})$ . Let  $(\pi, V, \mathcal{K})$  be a minimal Stinespring triple for  $\phi$ . By Proposition 1.1.8 the representation  $\pi$  extends to a unital representation  $\tilde{\pi}: \tilde{\mathcal{A}} \rightarrow B(\mathcal{K})$ . Now define  $\tilde{\phi}: \tilde{\mathcal{A}} \rightarrow \mathcal{B}$  by  $\tilde{\phi}(x) = V^* \tilde{\pi}(x) V$  for all  $x \in \tilde{\mathcal{A}}$ . Clearly  $\tilde{\phi}$  is a completely positive map extending  $\phi$ , so we only need to make sure that  $\tilde{\phi}$  is contractive and that its image is contained in  $\mathcal{B}$ . Since  $\pi$  is non-degenerate it is unital, so it follows that  $\tilde{\phi}(\mathbf{1}_{\tilde{\mathcal{A}}}) = \phi(\mathbf{1}_{\mathcal{A}})$ . In particular by Corollary B.1.8

$$\|\tilde{\phi}\| = \|\tilde{\phi}(\mathbf{1}_{\tilde{\mathcal{A}}})\| = \|\phi(\mathbf{1}_{\mathcal{A}})\| = \|\phi\|.$$

From this we also deduce that  $\tilde{\phi}$  is contractive completely positive. Now that  $\tilde{\phi}$  is unital if  $\phi$  is unital follows from the identity  $\tilde{\phi}(\mathbf{1}_{\tilde{\mathcal{A}}}) = \phi(\mathbf{1}_{\mathcal{A}})$ .  $\square$

### B.3 Positive maps and matrices

In this section we discuss two theorems, which characterize completely positive maps, from or to  $\mathbb{M}_n$ , and afterwards we prove Arveson's Extension Theorem.

**Theorem B.3.1.** *Suppose that  $\phi: \mathbb{M}_n \rightarrow \mathcal{A}$  is a linear map. Then  $\phi$  is completely positive if and only if  $[\phi(E_{i,j})]_{i,j}$  is positive in  $M_n(\mathcal{A})$ . Moreover, there is a bijective correspondence given by:*

$$\text{CP}(\mathbb{M}_n, \mathcal{A}) \ni \phi \longleftrightarrow [\phi(E_{i,j})]_{i,j} \in M_n(\mathcal{A})_+.$$

*Proof.* Clearly a linear map from  $\mathbb{M}_n$  to  $\mathcal{A}$  is uniquely determined by its values on the standard matrix units, and each choice of values defines a linear map by linear independence of the matrix units. In other words, we have a bijective correspondence

$$B(\mathbb{M}_n, \mathcal{A}) \ni \phi \longleftrightarrow [\phi(E_{i,j})]_{i,j} \in M_n(\mathcal{A}).$$

Let  $\Phi: B(\mathbb{M}_n, \mathcal{A}) \rightarrow M_n(\mathcal{A})$  denote the map  $\phi \mapsto [\phi(E_{i,j})]_{i,j}$ ,  $\phi \in B(\mathbb{M}_n, \mathcal{A})$ . Our aim is to prove that  $\phi$  is completely positive if and only if  $\Phi(\phi)$  is positive. First notice that for  $\phi \in B(\mathbb{M}_n, \mathcal{A})$  we have

$$\Phi(\phi) = \phi_n([\mathbf{E}_{i,j}]_{i,j}).$$

So if we can show that the matrix  $[\mathbf{E}_{i,j}]_{i,j}$  is positive in  $M_n(\mathbb{M}_n)$ , the  $\Phi(\phi)$  is positive by complete positivity of  $\phi$ . This is easy, since

$$\begin{bmatrix} E_{1,1} & \dots & E_{1,n} \\ \vdots & \ddots & \vdots \\ E_{n,1} & \dots & E_{n,n} \end{bmatrix} = \begin{bmatrix} E_{1,1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_{n,1} & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} E_{1,1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ E_{n,1} & 0 & \dots & 0 \end{bmatrix}^*$$

by the usual multiplication rules for the standard matrix units. Thus all we need to show is that if  $\phi \in B(\mathbb{M}_n, \mathcal{A})$  with  $\Phi(\phi)$  positive, then  $\phi$  is completely positive. So assume that  $[\phi(E_{i,j})]$  is positive, and choose some  $x = [x_{i,j}] \in M_n(\mathcal{A})$ , such that  $x^*x = [\phi(E_{i,j})]$ . In other words,

$$\phi(E_{i,j}) = \sum_{k=1}^n b_{k,i}^* b_{k,j},$$

for each  $i, j \in \{1, 2, \dots, n\}$ . Let  $\zeta_1, \zeta_2, \dots, \zeta_n$  denotes the standard orthonormal basis for  $\mathbb{C}^n$ , and let  $\pi: \mathcal{A} \rightarrow B(\mathcal{H})$  be a faithful representation of  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$ . Define a linear map  $V: \mathcal{H} \rightarrow \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{H}$ , by

$$V\xi = \sum_{j,k=1}^n \zeta_j \otimes \zeta_k \otimes \pi(b_{k,j})\xi, \quad \xi \in \mathcal{H}.$$

Now, suppose that  $[\lambda_{i,j}] \in \mathbb{M}_n$ , and denote this matrix by  $\Lambda$ . For  $\xi, \eta \in \mathcal{H}$ , we see that

$$\begin{aligned} \langle V^*(\Lambda \otimes \mathbf{1}_n \otimes \mathbf{1}_{\mathcal{H}})V\xi \mid \eta \rangle &= \sum_{i,j,k,l=1}^n \langle \Lambda \zeta_j \mid \zeta_i \rangle \langle \zeta_k \mid \zeta_l \rangle \langle \pi(b_{k,j})\xi \mid \pi(b_{l,i})\eta \rangle \\ &= \sum_{i,j,k=1}^n \langle \Lambda \zeta_j \mid \zeta_i \rangle \langle \pi(b_{k,i}^* b_{k,j})\xi \mid \eta \rangle \\ &= \sum_{i,j=1}^n \lambda_{i,j} \langle \pi(\phi(E_{i,j}))\xi \mid \eta \rangle \\ &= \langle \pi(\phi(\Lambda))\xi \mid \eta \rangle. \end{aligned}$$

Thus we see that  $\phi(\Lambda) = V^*(\Lambda \otimes \mathbf{1}_n \otimes \mathbf{1}_{\mathcal{H}})V$ , for all  $\Lambda \in \mathbb{M}_n$ . The map from  $\mathbb{M}_n$  to  $B(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathcal{H})$  given by  $\Lambda \mapsto \Lambda \otimes \mathbf{1}_n \otimes \mathbf{1}_{\mathcal{H}}$ ,  $\Lambda \in \mathbb{M}_n$ , is a  $*$ -homomorphism, so the map  $\Lambda \mapsto V^*(\Lambda \otimes \mathbf{1}_n \otimes \mathbf{1}_{\mathcal{H}})V$  is completely positive. Hence  $\phi$  is completely positive.  $\square$

**Corollary B.3.2.** *Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra and  $n \in \mathbb{N}$ . Then a linear map  $\phi: \mathbb{M}_n \rightarrow \mathcal{A}$  is completely positive if and only if it is  $n$ -positive.*

*Proof.* Clearly the linear map  $\phi$  is  $n$ -positive if it is completely positive. So suppose that  $\phi$  is  $n$ -positive, then we need to show that  $[\phi(E_{i,j})]_{i,j}$  is positive in  $M_n(\mathcal{A})$ . As noted in the proof above, the matrix  $[E_{i,j}]_{i,j}$  is positive in  $M_n(\mathbb{M}_n)$ , so since  $[\phi(E_{i,j})]_{i,j} = \phi_n([E_{i,j}]_{i,j})$ , it follows that  $[\phi(E_{i,j})]_{i,j}$  is positive.  $\square$

**Remark B.3.3.** Let us write down the inverse of the map  $\Phi: B(\mathbb{M}_n, \mathcal{A}) \rightarrow M_n(\mathcal{A})$  from the theorem, given by  $\phi \mapsto [\phi(E_{i,j})]_{i,j}$ ,  $\phi \in B(\mathbb{M}_n, \mathcal{A})$ . This was the map we proved that gave a bijective correspondence. The inverse is given as follows: suppose that  $x = [x_{i,j}] \in M_n(\mathcal{A})_+$ , then  $\Phi^{-1}(x)$  is is map  $\phi_x: \mathbb{M}_n \rightarrow \mathcal{A}$ , defined, for  $[\lambda_{i,j}] \in \mathbb{M}_n$ , by

$$\phi_x([\lambda_{i,j}]) = \sum_{i,j=1}^n \lambda_{i,j} x_{i,j}.$$

To see this, one just has to notice that  $\phi_x(E_{i,j}) = x_{i,j}$ . Another think to notice, is that  $\Phi$  is actually linear.  $\blacktriangleleft$

Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra. For a linear map  $\phi: \mathcal{A} \rightarrow \mathbb{M}_n$ , let  $\hat{\phi}$  denote the linear functional on  $M_n(\mathcal{A})$  given by

$$\hat{\phi}([x_{i,j}]) = \sum_{i,j}^n \phi(x_{i,j})_{i,j}, \quad [x_{i,j}] \in M_n(\mathcal{A}),$$

where  $\phi(x_{i,j})_{i,j}$  denotes the  $i, j$ 'th entry in the  $n \times n$  scalar matrix  $\phi(x_{i,j})$ . Said in another way, if  $\zeta_1, \zeta_2, \dots, \zeta_n$  denotes the standard orthonormal basis for  $\mathbb{C}^n$ , and  $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n)$ , then

$$\hat{\phi}([x_{i,j}]) = \langle \phi_n([x_{i,j}])\zeta \mid \zeta \rangle, \quad [x_{i,j}] \in M_n(\mathcal{A}).$$

This last way of writing  $\hat{\phi}$  also shows that  $\hat{\phi}$  is indeed a bounded linear functional if  $\phi$  is bounded.

**Theorem B.3.4.** *Suppose that  $\mathcal{A}$  is a  $C^*$ -algebra. With the notation above, a linear map  $\phi: \mathcal{A} \rightarrow \mathbb{M}_n$  is completely positive if and only if  $\hat{\phi}$  is positive. Moreover, there is a bijective correspondence given by:*

$$\text{CP}(\mathcal{A}, \mathbb{M}_n) \ni \phi \longleftrightarrow \hat{\phi} \in (M_n(\mathcal{A})^*)_+.$$

*Proof.* Let  $\Psi: B(\mathcal{A}, \mathbb{M}_n) \rightarrow M_n(\mathcal{A})^*$  denote the map  $\phi \mapsto \hat{\phi}$ ,  $\phi \in B(\mathcal{A}, \mathbb{M}_n)$ , with the notation above. Let us start by proving that this map is a bijection, with no regard to any positivity. Suppose that  $\psi \in M_n(\mathcal{A})^*$ , and define a linear map  $\phi_\psi: \mathcal{A} \rightarrow \mathbb{M}_n$ , as follows: for  $x \in \mathcal{A}$ , the  $i, j$ 'th entry of  $\phi_\psi(x)$  should be  $\psi(x \otimes E_{i,j})$ . Said in another way,  $\langle \phi_\psi(x)\zeta_j \mid \zeta_i \rangle = \psi(x \otimes E_{i,j})$ , and for an explicit expression for  $\phi_\psi$ , we write

$$\phi_\psi(x) = \sum_{i,j=1}^n \psi(x \otimes E_{i,j}) \otimes E_{i,j}, \quad x \in \mathcal{A}.$$

It is straightforward to check that  $\phi_\psi$  is bounded using the standard matrix estimates. In fact  $\phi_\psi$  is bounded by  $n^2 \|\psi\|$ . It is also easy to check that, if  $\phi \in B(\mathcal{A}, \mathbb{M}_n)$ , then  $\hat{\phi}_\psi = \phi$ . Hence we have a bijective correspondence

$$B(\mathcal{A}, \mathbb{M}_n) \ni \phi \longleftrightarrow \hat{\phi} \in M_n(\mathcal{A})^*.$$

Our aim is to prove that  $\phi$  is completely positive if and only if  $\Psi(\phi)$  is positive. That  $\Psi(\phi)$  is positive if  $\phi$  is completely positive is apparent from the expression

$$\hat{\phi}([x_{i,j}]) = \langle \phi_n([x_{i,j}])\zeta \mid \zeta \rangle, \quad [x_{i,j}] \in M_n(\mathcal{A}),$$

so let us prove the converse implication. Suppose that  $\phi \in B(\mathcal{A}, \mathbb{M}_n)$ , with  $\hat{\phi}$  positive. Consider the GNS-construction  $(\pi_\phi, \mathcal{H}_\phi, \xi_\phi)$  corresponding to  $\phi$ . Define an operator

$$V: \mathbb{C}^n \rightarrow \mathcal{H}_\phi \quad \text{by} \quad \zeta_j \mapsto \pi_\phi(E_{1,j})\xi_\phi \quad \text{for} \quad j = 1, \dots, n,$$

and define a representation  $\tilde{\pi}: \mathcal{A} \rightarrow B(\mathcal{H}_\phi)$  of  $\mathcal{A}$  on  $\mathcal{H}_\phi$ , by

$$\tilde{\pi}(x) = \pi \left( \sum_{i=1}^n x \otimes E_{i,i} \right), \quad x \in \mathcal{A}.$$

Now we want to show that  $\psi(x) = V^*\tilde{\pi}(x)V$  for all  $x \in \mathcal{A}$ , because this will clearly imply that  $\psi$  is completely positive. A computation shows that

$$(V^*\pi([x_{i,j}])V\zeta_l \mid \zeta_k) = (\pi(E_{k,1}[x_{i,j}]E_{1,l})\xi \mid \xi) = \hat{\psi}(E_{k,1}[x_{i,j}]E_{1,l}).$$

Since  $E_{k,1}[x_{i,j}]E_{1,l} = x_{1,1} \otimes E_{k,l}$  we see that

$$(V^*\tilde{\pi}(x)V\zeta_l \mid \zeta_k) = \hat{\psi}(x \otimes E_{k,l}) = \psi(x)_{k,l}$$

and thus  $\psi(x) = V^*\tilde{\pi}(x)V$ , which was what we needed to prove.  $\square$

**Corollary B.3.5.** *Suppose that  $\mathcal{A}$  is a unital  $C^*$ -algebra and  $\mathcal{S} \subseteq \mathcal{A}$  an operator system. A linear map  $\hat{\phi}: \mathcal{S} \rightarrow \mathbb{M}_n$  is completely positive if and only if it is an  $n$ -positive map, and in this case there exists a completely positive map  $\psi: \mathcal{A} \rightarrow \mathbb{M}_n$  extending  $\hat{\phi}$ .*

*Proof.* It suffices to show that if  $\hat{\phi}: \mathcal{S} \rightarrow \mathbb{M}_n$  is  $n$ -positive, then there exists a completely positive map  $\psi: \mathcal{A} \rightarrow \mathbb{M}_n$  extending  $\hat{\phi}$ . So suppose that  $\hat{\phi}: \mathcal{S} \rightarrow \mathbb{M}_n$  is  $n$ -positive. Define a linear functional  $\hat{\phi}$  on  $M_n(\mathcal{S})$  as above Theorem B.3.4. Even though we only did the construction for  $C^*$ -algebras it makes sense for general operator systems. It still holds that

$$\hat{\phi}([x_{i,j}]) = \langle \phi_n([x_{i,j}])\zeta \mid \zeta \rangle, \quad [x_{i,j}] \in M_n(\mathcal{S}),$$

by the same calculation. So since  $\hat{\phi}$  is  $n$ -positive by assumption,  $\hat{\phi}$  is a positive linear functional. By Proposition B.1.9 we can extend  $\hat{\phi}$  to a positive linear functional on  $M_n(\mathcal{A})$ . Combining this with the bijective correspondence, we get that there exist some completely positive map  $\psi: \mathbb{M}_n \rightarrow \mathcal{A}$ , such that  $\hat{\psi}|_{M_n(\mathcal{S})} = \hat{\phi}$ . It follows from the formula

$$\psi(x) = \sum_{i,j=1}^n \hat{\psi}(x \otimes E_{i,j}) \otimes E_{i,j}, \quad x \in \mathcal{A},$$

that  $\psi$  extends  $\hat{\phi}$ . Thus we have extended  $\hat{\phi}$  to a completely positive map on  $\mathcal{A}$ , and the proof is complete.  $\square$

Now we prove a famous theorem known as Arveson's Extension Theorem.

**Theorem B.3.6.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra,  $\mathcal{S} \subseteq \mathcal{A}$  an operator system and  $\mathcal{H}$  a Hilbert space. Then every completely positive map  $\hat{\phi}: \mathcal{S} \rightarrow B(\mathcal{H})$ , then  $\hat{\phi}$  extends to a completely positive map  $\mathcal{A} \rightarrow B(\mathcal{H})$ , with the same completely bounded norm.*

*Proof.* Suppose that  $\hat{\phi}: \mathcal{A} \rightarrow B(\mathcal{H})$  is completely positive, and let  $\mathcal{F}$  denote the set of finite dimensional subspaces of  $\mathcal{H}$ . For each  $F \in \mathcal{F}$ , let  $\phi_F$  denote the compression of  $\hat{\phi}$  to  $F$ , that is,  $\phi_F(a) = P_F\hat{\phi}(a)P_F$ , where  $P_F$  denotes the orthogonal projection of  $\mathcal{H}$  onto  $F$ . Clearly  $\phi_F$  is completely positive since  $\hat{\phi}$  is completely positive. The range of  $\phi_F$  is contained in  $P_FB(\mathcal{H})P_F$ , and this space is naturally isomorphic to  $B(F)$ , which, in turn, is isomorphic to  $M_k(\mathbb{C})$ , for  $k = \dim F$ . Thus by Corollary B.3.5 we can extend  $\phi_F$  to a completely positive map  $\psi_F: \mathcal{A} \rightarrow B(\mathcal{H})$ , whose range is also contained in  $P_FB(\mathcal{H})P_F$ .

By considering  $\mathcal{F}$  as a directed set under inclusion, we obtain a net  $(\psi_F)_{F \in \mathcal{F}}$  in  $CP(\mathcal{A}, B(\mathcal{H}))$ . From Corollary B.1.8 we know that

$$\|\psi_F\|_{cb} = \|\psi_F(\mathbf{1})\| = \|\phi_F(\mathbf{1})\| \leq \|\hat{\phi}(\mathbf{1})\| = \|\hat{\phi}\|_{cb}.$$



So in fact  $\psi_F (F \in \mathcal{F})$  is a net in  $\text{CP}_r(\mathcal{A}, B(\mathcal{H}))$  with  $r = \|\phi\|$ . Since  $\text{CP}_r(\mathcal{A}, B(\mathcal{H}))$  is compact by Proposition B.2.8, we may choose a convergent subnet  $\psi_{F_\alpha}$  ( $\alpha \in A$ ). Let  $\psi \in \text{CP}_r(\mathcal{A}, B(\mathcal{H}))$  denote the limit of this convergent net. Now the claim is that  $\psi$  is actually an extension of  $\phi$ , or, equivalently, that  $\langle \phi(a)\xi \mid \eta \rangle = \langle \psi(a)\xi \mid \eta \rangle$  for all  $\xi, \eta \in \mathcal{H}$  and  $a \in \mathcal{S}$ . Suppose that  $\xi, \eta \in \mathcal{H}$  and  $a \in \mathcal{S}$ . Let  $F'$  denote the span of  $\xi$  and  $\eta$ , then

$$\langle \phi(a)\xi \mid \eta \rangle = \langle \phi(a)P_{F_\alpha}\xi \mid P_{F_\alpha}\eta \rangle = \langle \psi_{F_\alpha}(a)\xi \mid \eta \rangle$$

for all  $\alpha \in A$  with  $F_\alpha \supseteq F'$ . Now by Proposition B.2.7 we know that the limit over  $\alpha$  on the right hand side equals  $\langle \psi(a)\xi \mid \eta \rangle$ , that is,

$$\lim_{\alpha \in A} \langle \psi_{F_\alpha}(a)\xi \mid \eta \rangle = \langle \psi(a)\xi \mid \eta \rangle$$

But since this net is constant  $\langle \phi(a)\xi \mid \eta \rangle$  from a certain point on, we conclude that  $\langle \phi(a)\xi \mid \eta \rangle = \langle \psi(a)\xi \mid \eta \rangle$ . Thus  $\psi$  extends  $\phi$ , and we have proved the theorem.  $\square$

We end this section with the following non-unital version of Arveson's Extension Theorem:

**Corollary B.3.7.** *Suppose that  $\mathcal{B}$  is a  $C^*$ -algebra and  $\mathcal{A}$  is a  $C^*$ -subalgebra of  $\mathcal{B}$ —none of which have to be unital. Let  $\mathcal{H}$  be a Hilbert space. Then every contractive completely positive map  $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$  extends to a contractive completely positive map  $\mathcal{B} \rightarrow B(\mathcal{H})$ . This map can be chosen unital unless  $\mathcal{B}$  is unital,  $\mathbf{1}_{\mathcal{B}} \in \mathcal{A}$  and  $\phi(\mathbf{1}_{\mathcal{B}}) \neq \mathbf{1}_{B(\mathcal{H})}$ .*

*Proof.* We may assume that  $\mathcal{B}$  is unital, for if not, then we just extend the map to a map from the unitization of  $\mathcal{B}$  to  $B(\mathcal{H})$ , and then restrict it to a map  $\mathcal{B} \rightarrow B(\mathcal{H})$ . If  $\mathcal{A}$  contains the unit of  $\mathcal{B}$ , then Arveson's Extension Theorem gives the desired conclusion. So assume that this is not the case, and let  $\mathcal{A}'$  denote the  $C^*$ -algebra generated by  $\mathcal{A}$  and  $\mathbf{1}_{\mathcal{B}}$ , that is,  $\mathcal{A}' = \mathcal{A} + \mathbb{C}\mathbf{1}_{\mathcal{B}}$ . Let  $\phi: \mathcal{A} \rightarrow B(\mathcal{H})$  be a contractive completely positive map. By [BO08, Proposition 2.2.1] the map  $\phi$  extends to a unital completely positive map  $\phi': \mathcal{A}' \rightarrow B(\mathcal{H})$  given by  $\phi'(x + \lambda\mathbf{1}_{\mathcal{B}}) = \phi(x) + \lambda\mathbf{1}_{B(\mathcal{H})}$ . Now by Arveson's Extension Theorem the map  $\phi'$  extends to a unital completely positive map  $\mathcal{B} \rightarrow B(\mathcal{H})$ , which was what we needed to prove.  $\square$

## B.4 Operator spaces

Operator spaces are a generalization of operator systems.

**Definition B.4.1.** A linear subspace (not necessarily closed) of a  $C^*$ -algebra is called an **operator space**.

Since operator systems are subsets of  $C^*$ -algebras by definition, they carry a structure on their matrix algebras. Indeed, if  $\mathcal{M}$  is an operator spaces in a  $C^*$ -algebra  $\mathcal{A}$ , then  $M_n(\mathcal{M}) \subseteq M_n(\mathcal{A})$ , for all  $n \in \mathbb{N}$ . In particular we have a natural choice of norm on  $M_n(\mathcal{M})$ , for all  $n \in \mathbb{N}$ . Because of this the concept of completely boundedness makes sense for operator spaces as well.

**Definition B.4.2.** Suppose that  $\mathcal{M}$  and  $\mathcal{M}'$  are operator spaces, and  $\phi: \mathcal{M} \rightarrow \mathcal{M}'$  be a linear map. The map  $\phi$  is called **completely bounded** if  $\|\phi\|_{\text{cb}} < \infty$  and **completely contractive** if in addition  $\|\phi\|_{\text{cb}} \leq 1$ . In the case where  $\phi_n$  is an isometry for all  $n \in \mathbb{N}$ , we say that  $\phi$  is **completely isometric**, or that  $\phi$  is a **complete isometry**.  $\blacktriangleleft$

As with operator systems, the structure of an operator space does not depend on the choice of ambient  $C^*$ -algebra. Again, as with operator systems, there is a subtlety in this statement, namely the following: if  $\mathcal{M}$  is an operator space, and  $\mathcal{A}, \mathcal{B}$  are two ambient  $C^*$ -algebras, then they induce the same operator space structure on  $\mathcal{M}$ , if the  $C^*$ -algebras generated by  $\mathcal{M}$  in  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, are the same, algebraic structure and all. Indeed, if we denote this  $C^*$ -algebra by  $\mathcal{C}$ , then the inclusion of  $\mathcal{C}$  into  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, is a complete isometry. If it is not the case that the generated  $C^*$ -algebras are the same, then they do not induce it is not the case. By ambient  $C^*$ -algebra we will always understand a  $C^*$ -algebra inducing the correct operator space structure.

Because of this free choice of ambient  $C^*$ -algebra it may always be arranged that the ambient  $C^*$ -algebra is unital, even  $B(\mathcal{H})$ , for some Hilbert space  $\mathcal{H}$ , without altering the matrix structure.

For operator spaces  $\mathcal{M}$  and  $\mathcal{M}'$  we denote by  $\text{CB}(\mathcal{M}, \mathcal{M}')$  the set of completely bounded maps from  $\mathcal{M}$  to  $\mathcal{M}'$ . This is a linear subspace of  $B(\mathcal{M}, \mathcal{M}')$ , and a normed space with  $\|\cdot\|_{\text{cb}}$ .

Given  $n, m \in \mathbb{N}$  and a  $C^*$ -algebra  $\mathcal{A}$ , there is a canonical isomorphism from  $M_n(M_m(\mathcal{A}))$  to  $M_m(M_n(\mathcal{A}))$ , namely, for  $x_{i,j}^{k,l} \in \mathcal{A}$ , for  $i, j = 1, 2, \dots, n$  and  $k, l = 1, 2, \dots, m$ , the map is given by

$$\begin{bmatrix} [x_{i,j}^{1,1}]_{i,j} & \cdots & [x_{i,j}^{1,m}]_{i,j} \\ \vdots & \ddots & \vdots \\ [x_{i,j}^{m,1}]_{i,j} & \cdots & [x_{i,j}^{m,m}]_{i,j} \end{bmatrix} \mapsto \begin{bmatrix} [x_{1,1}^{k,l}]_{k,l} & \cdots & [x_{1,n}^{k,l}]_{k,l} \\ \vdots & \ddots & \vdots \\ [x_{n,1}^{k,l}]_{k,l} & \cdots & [x_{n,n}^{k,l}]_{k,l} \end{bmatrix}$$

One can check that this map is a  $*$ -isomorphism. Indeed, it is just a permutations of the rows and columns. This map goes under the name of **the canonical shuffle**.

The following lemma provides a way of passing between operator spaces and operator systems, and it is known as Paulsen's trick (or Paulsen's "off-diagonal" trick):

**Lemma B.4.3.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be a unital  $C^*$ -algebras and let  $\mathcal{M} \subseteq \mathcal{A}$  be an operator space. Let  $\phi : \mathcal{M} \rightarrow \mathcal{B}$  be a linear map and let  $\mathcal{S}_{\mathcal{M}}$  denote the operator system*

$$\mathcal{S}_{\mathcal{M}} = \left\{ \begin{bmatrix} \lambda \mathbf{1} & a \\ b^* & \mu \mathbf{1} \end{bmatrix} : \lambda, \mu \in \mathbb{C} \text{ and } a, b \in \mathcal{M} \right\}.$$

Define the map  $\Phi : \mathcal{S}_{\mathcal{M}} \rightarrow M_2(\mathcal{B})$  by

$$\Phi \left( \begin{bmatrix} \lambda \mathbf{1} & a \\ b^* & \mu \mathbf{1} \end{bmatrix} \right) = \begin{bmatrix} \lambda \mathbf{1} & \phi(a) \\ \phi(b)^* & \mu \mathbf{1} \end{bmatrix}.$$

Then if  $\phi$  is completely contractive then  $\Phi$  is completely positive.

*Proof.* Suppose that  $\phi$  is completely contractive. Let  $[S_{i,j}] \in M_n(\mathcal{M})$  be positive, and write

$$S_{i,j} = \begin{bmatrix} \lambda_{i,j} \mathbf{1} & a_{i,j} \\ b_{i,j}^* & \mu_{i,j} \mathbf{1} \end{bmatrix}$$

for some  $\lambda_{i,j}, \mu_{i,j} \in \mathbb{C}$  and  $a_{i,j}, b_{i,j} \in \mathcal{M}$ , for  $i, j \in \{1, 2, \dots, n\}$ . Since  $M_n(\mathcal{S}_{\mathcal{M}})$  is a subset  $M_n(M_2(\mathcal{A}))$  we may, by doing the canonical shuffle, regard  $M_n(\mathcal{S}_{\mathcal{M}})$  as a subset of  $M_2(M_n(\mathcal{A}))$ , and  $\Phi_n$  as a map from  $M_2(M_n(\mathcal{S}_{\mathcal{M}}))$  to  $M_2(M_n(\mathcal{B}))$ .

More precisely, by doing the canonical shuffle, and letting  $H = [\lambda_{i,j}\mathbf{1}]$ ,  $K = [\mu_{i,j}\mathbf{1}]$ ,  $A = [a_{i,j}]$  and  $B = [b_{j,i}]$ , the matrix  $[S_{i,j}]$  becomes the matrix

$$\begin{bmatrix} H & A \\ B^* & K \end{bmatrix} \quad (\text{B.2})$$

and the map  $\Phi$  is given by

$$\Phi\left(\begin{bmatrix} H & A \\ B^* & K \end{bmatrix}\right) = \begin{bmatrix} H & \phi_n(A) \\ \phi_n(B)^* & K \end{bmatrix}. \quad (\text{B.3})$$

So what we need to show is that if (B.2) is positive then (B.3) is positive. Now if (B.2) is positive, then  $H$  and  $K$  must be positive and  $A = B$ . Now, if, for  $\varepsilon > 0$ , we let  $H_\varepsilon = H + \varepsilon\mathbf{1}$  and  $K_\varepsilon = K + \varepsilon\mathbf{1}$ , then  $H_\varepsilon$  and  $K_\varepsilon$  are positive and invertible, and it is straightforward to check that

$$\begin{aligned} & \begin{bmatrix} H_\varepsilon^{-1/2} & 0 \\ 0 & K_\varepsilon^{-1/2} \end{bmatrix} \begin{bmatrix} H_\varepsilon & A \\ A^* & K_\varepsilon \end{bmatrix} \begin{bmatrix} H_\varepsilon^{-1/2} & 0 \\ 0 & K_\varepsilon^{-1/2} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1} & H_\varepsilon^{-1/2}AK_\varepsilon^{-1/2} \\ (H_\varepsilon^{-1/2}AK_\varepsilon^{-1/2})^* & \mathbf{1} \end{bmatrix} \end{aligned}$$

which shows that the matrix on the right hand side is positive, and by Proposition B.1.5 we obtain that  $\|H_\varepsilon^{-1/2}AK_\varepsilon^{-1/2}\| \leq 1$ . Now since  $\phi$  is completely contractive also  $\|\phi_n(H_\varepsilon^{-1/2}AK_\varepsilon^{-1/2})\| \leq 1$ . It is straight forward to check that since  $\phi$  is linear and  $H_\varepsilon$  and  $K_\varepsilon$  are almost just scalar matrices it easy to see that

$$\phi_n(H_\varepsilon^{-1/2}AK_\varepsilon^{-1/2}) = H_\varepsilon^{-1/2}\phi_n(A)K_\varepsilon^{-1/2},$$

and so it follows that

$$\begin{aligned} & \begin{bmatrix} H_\varepsilon & \phi_n(A) \\ \phi_n(A)^* & K_\varepsilon \end{bmatrix} = \begin{bmatrix} H_\varepsilon^{1/2} & 0 \\ 0 & K_\varepsilon^{1/2} \end{bmatrix} \\ & \times \begin{bmatrix} \mathbf{1} & \phi_n(H_\varepsilon^{-1/2}AK_\varepsilon^{-1/2}) \\ \phi_n(H_\varepsilon^{-1/2}AK_\varepsilon^{-1/2})^* & \mathbf{1} \end{bmatrix} \begin{bmatrix} H_\varepsilon^{1/2} & 0 \\ 0 & K_\varepsilon^{1/2} \end{bmatrix} \end{aligned}$$

The matrix in the middle on the right hand side is positive by Proposition B.1.5 since  $\|\phi_n(H_\varepsilon^{-1/2}AK_\varepsilon^{-1/2})\| \leq 1$ , so the matrix on the right hand side must be positive as well. Now the matrix on the right hand side is

$$\Phi\left(\begin{bmatrix} H & A \\ B^* & K \end{bmatrix}\right) + \varepsilon \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{bmatrix}.$$

so since this was positive for all  $\varepsilon > 0$  we conclude that (B.3) must be positive. Thus  $\Phi$  is completely positive.  $\square$

With Paulsen's trick we can now prove Wittstock's Extension Theorem, which can be thought of as Arveson's Extension Theorem for completely bounded maps instead of completely positive maps.

**Theorem B.4.4.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and  $\mathcal{M} \subseteq \mathcal{A}$  an operator space. Let  $\phi : \mathcal{M} \rightarrow B(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$  be a completely bounded map. Then there exist a completely bounded map  $\psi : \mathcal{A} \rightarrow B(\mathcal{H})$  extending  $\phi$ , with  $\|\phi\|_{\text{cb}} = \|\psi\|_{\text{cb}}$ .*

*Proof.* We may assume that  $\mathcal{A} \subseteq B(\mathcal{K})$  for some Hilbert space  $\mathcal{K}$  and that  $\|\phi\|_{\text{cb}} = 1$ . Let  $\mathcal{S}_{\mathcal{M}}$  and  $\Phi$  be as in Lemma B.4.3. Since  $\Phi$  is completely positive, by Arveson's Extension Theorem (that is, Theorem B.3.6)  $\Phi$  extends to a completely positive map  $\Psi : M_2(\mathcal{A}) \rightarrow M_2(B(\mathcal{H}))$ . Now we define a map  $\psi : \mathcal{A} \rightarrow B(\mathcal{H})$  by

$$\psi(a) = P_2 \Psi \left( \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \right) P_1^*$$

where  $P_i : \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H}$  denotes the projection onto the  $i$ 'th coordinate ( $i = 1, 2$ ). Said more intuitively we define  $\psi$  so that

$$\Psi \left( \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} * & \psi(a) \\ * & * \end{bmatrix}$$

It should be clear that  $\psi$  extends  $\phi$  since  $\Psi$  extends  $\Phi$ , and what we need to show is that  $\psi$  is completely contractive. Suppose that  $A = (a_{i,j}) \in M_n(\mathcal{A})$ , and let  $B$  denote the matrix with  $i, j$ 'th entry given by

$$\begin{bmatrix} 0 & a_{i,j} \\ 0 & 0 \end{bmatrix}$$

Now by a canonical shuffle the above matrix becomes the matrix

$$\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$$

so we see that  $\|B\| = \|A\|$ . Since

$$\psi_n(A) = \begin{bmatrix} P_2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & P_2 \end{bmatrix} \Psi_n(B) \begin{bmatrix} P_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & P_1 \end{bmatrix}^*$$

we see that  $\|\psi_n(A)\| \leq \|\Psi_n(B)\|$ , and since  $\Psi_n$  is unital and completely positive we have  $\|\Psi\| = 1$  by Lemma B.1.8, and it follows that

$$\|\psi_n(A)\| \leq \|\Psi_n(B)\| \leq \|B\| = \|A\|.$$

Thus we conclude that  $\psi$  is a complete contraction, and since  $\psi$  extends  $\phi$  with  $\|\phi\|_{\text{cb}} = 1$  we must have  $\|\psi\|_{\text{cb}} = 1$ .  $\square$

Like Wittstock's Extension Theorem can be thought of as Arveson's Extension Theorem for completely bounded maps, there is also a variant of Stinespring's Dilation Theorem for completely bounded maps. We will not prove this result, but just state it here:

**Theorem B.4.5.** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra,  $\mathcal{H}$  a Hilbert space and  $\phi : \mathcal{A} \rightarrow B(\mathcal{H})$  be a completely bounded map. Then there exist a Hilbert space  $\mathcal{K}$ , a  $*$ -homomorphism  $\pi : \mathcal{A} \rightarrow B(\mathcal{K})$  and bounded operators  $V_i : \mathcal{H} \rightarrow \mathcal{K}$ ,  $i = 1, 2$ , with  $\|\phi\|_{\text{cb}} = \|V_1\| \|V_2\|$ , such that*

$$\phi(x) = V_1^* \pi(x) V_2 \quad \text{for all } x \in \mathcal{A}.$$

*Moreover, if  $\|\phi\|_{\text{cb}} = 1$ , then  $V_1$  and  $V_2$  can be chosen to be isometries.*

For a proof of this theorem see [Pau02, Theorem 8.4] or [BO08, Theorem B.7].

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