Supremum and infimum in $[-\infty, \infty]$

Axiom + Observation: For all sets $A \subset [-\infty, \infty]$ there is a smallest number larger than all numbers in A called $\sup A$ – the supremum of A. Likewise there is a largest number smaller than all numbers in A called $\inf A$ – the infimum of A.

Observation: For all increasing sequences $x_1 \leq x_2 \leq \ldots$ in $[-\infty, \infty]$ we have

$$x_n \nearrow \sup\{x_n \mid n \in \mathbb{N}\}.$$

Likewise for a decreasing sequence $x_1 \geq x_2 \geq \ldots$ in $[-\infty, \infty]$ we have

$$x_n \searrow \inf\{x_n \mid n \in \mathbb{N}\}.$$

Limes superior and limes inferior

If $x_1, x_2, \ldots$ is a sequence in $[-\infty, \infty]$ we define the running supremum

$$y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}$$

Observe that $y_1 \geq y_2 \geq y_3 \geq \ldots$ and the y_n-sequence converges. The limit is called limes superior.

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{k\geq n} x_k\right) = \inf_{n\in\mathbb{N}}\left(\sup_{k\geq n} x_k\right)$$

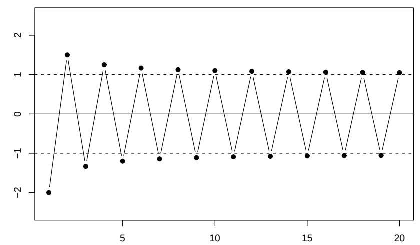
Likewise limes inferior is

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{k\geq n} x_k\right) = \sup_{n\in\mathbb{N}}\left(\inf_{k\geq n} x_k\right)$$

Example

Consider

$$x_n = (-1)^n\left(1 + \frac{1}{n}\right)$$



We observe that

$$\liminf_{n\to\infty} x_n = -1\ , \qquad \limsup_{n\to\infty} x_n = 1\ .$$

Simple rules

1) If $a \leq x_n \leq b$ for all n, $a, b \in [-\infty, \infty]$, then

$$a \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq b$$

2) If $x_n \leq y_n$ for all $n \in \mathbb{N}$ then

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n \quad \text{and} \quad \limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$$

3) We always have

$$\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n$$

4) For sequences in $[0, \infty]$ it holds that

$$\limsup_{n\to\infty} x_n + y_n \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

Liminf, limsup and limes

Lemma: If $x_1, x_2, \ldots$ is a sequence in $[-\infty, \infty]$ then $x_n \to x$ for $n \to \infty$ if and only if $\liminf_{n\to\infty} x_n = x = \limsup_{n\to\infty} x_n$.

Proof: Note that for $n \geq N$

$$\inf_{k\geq N} x_k \leq x_n \leq \sup_{k\geq N} x_k.$$

The left hand side converges to $\liminf_{n\to\infty} x_n$ for $n \to \infty$ and the right hand side to $\limsup_{n\to\infty} x_n$ for $n \to \infty$. This implies that if the two limits are equal then x_n converges toward that common limit.

And if the limits are different there will arbitrarily far out in the sequence be x_n's that are close to the $\limsup$ and x_n's that are close to the $\liminf$, hence x_n can not converge.

The function class $\mathcal{M}$

Lemma: If $f_1, f_2, \ldots$ is a sequence of functions with $f_n \in \mathcal{M}(\mathcal{X}, \mathbb{E})$ for all $n \in \mathbb{N}$ then

$$x \mapsto \sup_{n\in\mathbb{N}} f_n(x) \quad \text{and} \quad x \mapsto \inf_{n\in\mathbb{N}} f_n(x)$$

belongs to $\mathcal{M}(\mathcal{X}, \mathbb{E})$ provided that $\sup_{n\in\mathbb{N}} f_n(x) < \infty$ or $\inf_{n\in\mathbb{N}} f_n(x) > -\infty$, respectively, for all $x \in \mathcal{X}$. Also

$$x \mapsto \liminf_{n\to\infty} f_n(x), \quad \text{and} \quad x \mapsto \limsup_{n\to\infty} f_n(x)$$

are in $\mathcal{M}(\mathcal{X}, \mathbb{E})$ – provided that the limits are finite, and if there is a function $f : \mathcal{Y} \to \mathbb{R}$ such that

$$\lim_{n\to\infty} f_n(x) = f(x)$$

for all $x \in \mathcal{X}$ then $f \in \mathcal{M}(\mathcal{X}, \mathbb{E})$.

Further properties of pre-integrals

The following result replaces Lemma 6.3.

Corollary 6.5: If $s \in \mathcal{S}^+$ can be written $s = \sum_{i=1}^n c_i 1_{A_i}$ with $A_1, \ldots, A_n \in \mathbb{E}$ and $c_1, \ldots, c_n \in [0, \infty)$ then

$$I(s) = \sum_{i=1}^n c_i \mu(A_i).$$

Corollary 6.6: If $s, t \in \mathcal{S}^+$ and

$$s(x) \leq t(x) \quad \text{for all } x \in \mathcal{X}$$

then

$$I(s) \leq I(t).$$

Approximations

Theorem 6.7: Let $s \in \mathcal{S}^+$ and $E_1 \subset E_2 \subset \ldots \in \mathbb{E}$ such that $\bigcup_{n=1}^\infty E_n = \mathcal{X}$. Then

$$I(1_{E_n} s) \to I(s) \quad \text{for } n \to \infty\,.$$

Proof: Write s on the form $s(x) = \sum_{i=1}^m c_i\, 1_{A_i}(x)$. Then

$$1_{E_n}(x)\, s(x) = \sum_{i=1}^m c_i\, 1_{A_i}(x)\, 1_{E_n}(x) = \sum_{i=1}^m c_i\, 1_{A_i \cap E_n}(x)$$

Upward continuity for the measure implies that

$$I(1_{E_n} s) = \sum_{i=1}^m c_i\, \mu(A_i \cap E_n) \to \sum_{i=1}^m c_i\, \mu(A_i) \qquad \text{for } n \to \infty$$

The Lebesgue integral

$(\mathcal{X}, \mathbb{E}, \mu)$ is a measure space.

Definition: The integral of $f \in \mathcal{M}^+$ is defined as

$$\int f \mathrm{d}\mu = \sup\{I(s) \mid s \leq f,\ s \in \mathcal{S}^+\}.$$

If $s_1 \leq s_2 \leq \ldots \in \mathcal{S}^+$ with $s_n(x) \nearrow f(x)$ for $n \to \infty$ for all $x \in \mathcal{X}$ then by monotone convergence

$$I(s_n) = \int s_n \mathrm{d}\mu \nearrow \int f \mathrm{d}\mu$$

for $n \to \infty$.

Definition: The integral of $f \in \mathcal{M}^+$ over the set $A \in \mathbb{E}$ is defined as

$$\int_A f \mathrm{d}\mu = \int 1_A f \mathrm{d}\mu.$$

Properties of the Lebesgue integral

Theorem: Let $f, g \in \mathcal{M}^+$ and $c \in [0, \infty]$ then

- If $f \leq g$ then $\int f \mathrm{d}\mu \leq \int g \mathrm{d}\mu$.
- $\int c f \mathrm{d}\mu = c \int f \mathrm{d}\mu$.
- $\int f + g \mathrm{d}\mu = \int f \mathrm{d}\mu + \int g \mathrm{d}\mu$.
- In fact, if $f_1, f_2, \ldots \in \mathcal{M}^+$ then

$$\int \sum_{n=1}^{\infty} f_n \mathrm{d}\mu = \sum_{n=1}^{\infty} \int f_n \mathrm{d}\mu.$$

Proof: Points 2+3 by the rules for simple functions and monotone convergence of simple function approximations to f and g.

Point 4 follows by using Point 3 and monotone convergence.

Convergence Theorems

$(\mathcal{X}, \mathbb{E}, \mu)$ is a measure space.

The monotone convergence theorem: If $f_1 \leq f_2 \leq \ldots \in \mathcal{M}^+$, if

$$f_n(x) \nearrow f(x)$$

for $n \to \infty$ for all $x \in \mathcal{X}$ then $\int f_n \mathrm{d}\mu \nearrow \int f \mathrm{d}\mu$ for $n \to \infty$.

The dominated convergence theorem: If $f_1, f_2, \ldots \in \mathcal{M}^+$, if

$$f_n(x) \to f(x)$$

for $n \to \infty$ for all $x \in \mathcal{X}$ then if there is a $g \in \mathcal{M}^+$ with $\int g \mathrm{d}\mu < \infty$ such that $f_n \leq g$ then $\int f_n \mathrm{d}\mu \to \int f \mathrm{d}\mu$ for $n \to \infty$.

Integration over $\mathbb{N}$

Consider the measure space $\big(\mathbb{N}, \mathbb{P}(\mathbb{N}), \tau\big)$ – where τ is the counting measure.

An $\mathcal{M}$-function $f : \mathbb{N} \to \mathbb{R}$ is the same as a sequence:

$$f \quad \leftrightarrow \quad (f(1), f(2), \ldots)$$

Likewise, an $\mathcal{M}^+(\mathbb{N})$-function is that same as a sequence with elements in $[0, \infty]$.

Lemma If $f : \mathbb{N} \to [0, \infty]$ then

$$\int f \, d\tau = \sum_{n=1}^{\infty} f(n).$$

Integration w.r.t. the counting measure

To

$$\boxed{\text{integrate w.r.t. the counting measure on } \mathbb{N}}$$

is the same as computing

$$\boxed{\text{infinite sums}}$$

We have in particular all the results from integration theory available for manipulating with infinite sums.

Example

It holds that

$$\boxed{\int e^{-x^2/2}\, dm < \infty}$$

This follows from

$$e^{-x^2/2} \le e^{-n^2/2} \quad \text{when} \quad x \in [n, n+1) \cup (-(n+1), -n]$$

and hence

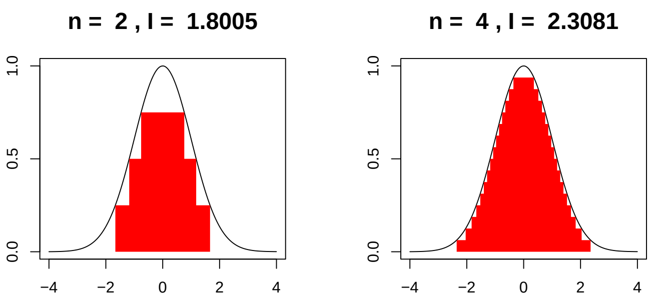
$$\int e^{-x^2/2}\, dm \le 2 \sum_{n=0}^{\infty} e^{-n^2/2} \le 2 \sum_{n=0}^{\infty} e^{-n/2} = \frac{2}{1 - e^{-1/2}}$$

Example

What is the value of

$$\int e^{-x^2/2}\, dm = 2.5066\ldots (= \sqrt{2\pi})$$

The simple approximations $s_n = \phi_n \circ f$



Example

What is

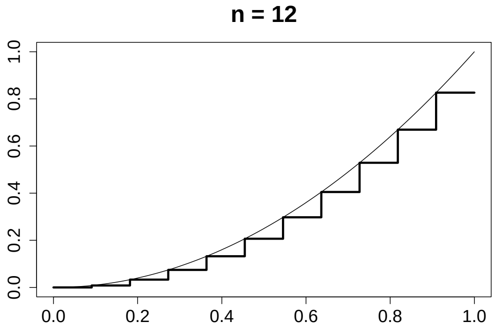
$$\int_{(0,1)} x^2\, dm$$

Put

$$s_n(x) = \begin{cases} 0 & \text{for } x \in \left(0, \frac{1}{n}\right), \\ \left(\frac{1}{n}\right)^2 & \text{for } x \in \left[\frac{1}{n}, \frac{2}{n}\right), \\ \left(\frac{2}{n}\right)^2 & \text{for } x \in \left[\frac{2}{n}, \frac{3}{n}\right), \\ \ldots & \ldots \\ \left(\frac{n-1}{n}\right)^2 & \text{for } x \in \left[\frac{n-1}{n}, 1\right), \end{cases}$$

and $s_n(x) = 0$ for $x \notin (0, 1$.

Example



$$I(s_n) = \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{k=1}^{n-1} k^2$$

Convergence is dominated by the integrable function $1_{(0,1)}$, say – or one can take advantage of the monotone convergence along the subsequence $n_k = 2^k$.

Example

Consider the measure space $(\mathbb{R}, \mathbb{B}, m)$ and the function

$$f_n(x) = \frac{n\sqrt{x}}{1+n^2x^2} \quad x > 0.$$

Then $f_n(x) \to 0$ for $n \to \infty$ for all $x \in (0, \infty)$ and

$$f_n(x) \leq \frac{1}{2\sqrt{x}}.$$

Since $x \mapsto \frac{1}{2\sqrt{x}}$ is integrable over $(0,1)$ it follows from dominated convergence that

$$\int_{(0,1)} f_n \mathrm{d}\mu \to 0$$

for $n \to \infty$.