The Long and Short of Static Hedging With Frictions

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Abstract
We investigate static hedging of barrier options with plain vanilla in the presence of bid/ask spreads. Spreads for options are much larger than for the underlying, and this levels the playing field in relation to traditional dynamic hedging. However, hedging with options is still the best way to reduce the risk of large losses.

Keywords
static hedging, barrier option, bid/ask-spread

1 Introduction
The literature on static hedging of barrier options is large, but surprisingly quiet on anything that has to do with market frictions. In this paper we remedy that by extending the risk-minimizing static hedges from Siven & Poulsen (2008) to incorporate bid/ask-spreads. The method relies on simulation and numerical optimization and is very flexible with regard to the type of contract to be hedged, the choice of hedge instruments, the model for the underlying, and the choice of risk-measure. To incorporate bid/ask-spreads in the modelling, long and short positions in the hedge instruments have to be treated separately, but the convexity of the optimization problem is retained so the numerical solution is still straightforward.

For dynamic delta hedging, it is well-known that frictions must be considered as an integral part of the hedging problem. It is too expensive to use the no frictions strategy, smile politely and pay the costs, see, e.g., Whalley & Wilmott (1997). We demonstrate that when hedging barrier options with plain vanilla this is “even more true.”

At a first glance, bid/ask-spreads may seem less important when hedging statically with vanilla puts and calls rather than dynamically with the underlying. This is naive though, since the bid/ask-spreads are much larger for options than for the underlying — typically two orders of magnitude. We compare dynamic and static hedging of an up-and-out call option under realistic assumptions on the bid/ask-spreads in simulation experiments, and find that whether or not to statically hedge depends non-trivially both on the size of the spreads for the different contracts, and on the choice of risk-measure. The static hedges do a better job of eliminating large losses, even for large vanilla option spreads.

The paper is structured as follows. In Section 2 we review the risk-minimizing approach to static hedging from Siven & Poulsen (2008) and show how to extend it to incorporate flexible modelling of bid/ask-spreads. Section 3 describes how the bid/ask-spreads typically look in the currency market where barrier options are most heavily traded. Sections 4 and 5 contain simulation experiments and Section 6 concludes.

2 Static Hedging with Bid/Ask-spreads
We now review the risk-minimizing static hedges from Siven & Poulsen (2008) and show how to extend them to take bid/ask-spreads into account.

Static hedging. Let \((S_t)_{t\in[0,T]}\) be a process modelling the mid-price of a traded asset; the underlying. Let \(P\) denote the time-\(\tau\) payoff of a contingent claim, an exotic, illiquid, or path dependent option on the underlying, where \(\tau\) is a stopping time bounded by \(T\). Furthermore, let \(H = (H_1, \ldots, H_M)\) denote the time-\(\tau\) payoff of \(M\) hedging instruments: typically the market value of plain vanilla options, but more generally the cash flows from any assets that can be exercised or liquidated at time \(\tau\). A static hedge strategy is some \(c = (c_1, \ldots, c_M) \in \mathbb{R}^M\) where \(c_k\) repre-
sents the position in the $k$th hedging instrument ($c_k > 0$ is a long position), so the time-$\tau$ value of the corresponding hedge portfolio is $H \cdot c = \sum_{k=1}^{M} c_k H_k$. We let $P(0)$ and $H(0) = (H_1(0), \ldots, H_M(0))$ denote the time-0 prices of the target option and the hedging instruments.

**Risk minimization.** Generally, it is not possible to build a perfect static hedge of the payoff $P$, so it is natural to view hedging as an approximation problem: choose a strategy $c$ that makes the hedge error small in some sense. The quantity

$$e^{-\tau} (P - H \cdot c)$$

is the discounted P&L for an investor hedging a short position in the target option with the strategy $c$. We now take a convex function $u$ (e.g., $u(x) = x^2$) and define a risk-minimizing hedge as a solution to

$$\min_{c \in D} E[u(e^{-\tau} (P - H \cdot c))],$$

where the set $D$ of admissible hedging strategies is typically defined by linear constraints: limits on the hedge weights and a budget restriction, (time-0 cost of hedging portfolio) = $H(0) \cdot c \leq \text{const}$. The stochastic programming problem (1) can be solved with the so called sample average approximation method: simulate $N$ samples $(P^n, H^n, \tau^n)$ of the triplet $(P, H, \tau)$, approximate the goal function $g(c) = E[u(e^{-\tau} (P - H \cdot c))]$ with

$$\hat{g}(c) = \frac{1}{N} \sum_{n=1}^{N} u(e^{-\tau} (P^n - H^n \cdot c)),$$

and minimize this deterministic and convex function over $D$ with some numerical method. Simulating $(P, H, \tau)$ amounts to simulating paths of the underlying asset (this requires assumptions about the real-world dynamics) and evaluating the payoff from the target options and the hedge instruments (this requires assumptions about the pricing measure). The optimization problem is convex and thus readily solved, most off-the-shelf software packages have efficient solution algorithms. Not only is the sample average approximation method intuitively appealing, it can be shown, see Shapiro (2008) and the references therein, that under mild conditions the minimizer of (2) converges to a solution of (1) as $N \to \infty$.

**Incorporating bid/ask-spreads.** The risk-minimizing approach to static hedging can easily be modified to incorporate flexible modelling of bid/ask-spreads. Assume that the $M$ liquid contracts that are used as hedging instruments have time-$\tau$ mid-prices $H^n_m$, $\ldots$, $H^M_m$ (superscript $m$ for “mid”). Denote the spread for the $k$th instrument by $s_k$ and write $H^n_k = H^n_m + \frac{1}{2} s_k$ for the time-$\tau$ bid and ask prices, where the superscripts $l$ and $s$ stand for “long” and “short” (a hedger long the $k$th instrument faces the market price $H^n_k$ when liquidating, and vice versa).

Treating long and short positions separately, we consider the vector

$$H = (H^1, \ldots, H^M_m, H^l, \ldots, H^M_m),$$

and correspondingly denote the (observable) vector of initial bid and ask prices by $H(0)$. A static hedge portfolio is now a vector

$$c = (c^1, \ldots, c^M_m, c^l, \ldots, c^M_m),$$

where $c^l_k \geq 0$ and $c^s_k \leq 0$ respectively correspond to long and short positions in the $k$th hedging instrument. To compute the risk-minimizing hedge, we just solve the minimization problem (1) with the extra constraints that $c^l_k \geq 0$ and $c^s_k \leq 0$ for all $k$.

In the modelling, the spreads $s_k$ may in principle depend on anything (the time $\tau$, the whole mid-price history, ...) as long as they can be simulated together with $P, H^n_1, \ldots, H^n_M$ and $\tau$. However, in the experimental setting below, further simulations (not reported) indicate that refined modelling of the time-$\tau$ spreads is of minor importance, it is the time-0 spreads that matter.

If the $k$th hedging instrument is not liquidated but expires and is settled in cash, we have $s_k = 0$.

### 3 Bid/Ask-spreads in Practice

**Bid/Ask-spreads in the spot.** For the underlying, bid/ask-spreads are approximately proportional to the spot level. So, if the time-$\tau$ mid-price is $S_t$, we model the bid/ask-prices as

$$S_t(1 - f_n) \text{ and } S_t(1 + f_n),$$

where $f_n$ is the friction in the underlying. For major exchange rates (such as €/$ or $/¥$) the bid/ask-spreads are quite tight; reasonable values for $f_n$ are between 0.0001 and 0.0002, i.e. 1-2 basis points.

**Bid/Ask-spreads for vanilla options.** Wystup (2007, Section 3.2) describes how market makers typically determine spreads for vanilla options:

1. Let $C_{BS}(T, S_t, K, \sigma)$ denote the time-$\tau$ BlackScholes (or Garman-Kohlhagen) price of an expiry-$T$ strike-$K$ call-option, and let $\sigma^*$ denote the implied volatility for the at-the-money option. The vanilla spread is given by $C_{BS}(T, S_t, S_t, \sigma^*)(1 + f_n) - C_{BS}(T, S_t, S_t, \sigma^*)(1 - f_n)$, where $f_n$ is the vanilla friction.

2. This spread is used for all the vanilla options with the same expiry-date, i.e. an expiry-$T$ vanilla option with time-$\tau$ mid-price $C_t$ has bid and ask prices $C_t(1 - \text{vanilla spread}/2)$ and $C_t(1 + \text{vanilla spread}/2)$.

For vanilla options on major exchange rates, realistic values for the vanilla friction $f_n$ are between 0.01 and 0.02 (i.e. 1-2 percentage points) in the interbank market, twice that for institutional and corporate clients and even higher for retail clients.

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<th>Table 1: Default parameters for the simulation experiments.</th>
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4 Hedging of Barrier Options: Base Case

We now perform simulation experiments to compare static and dynamic hedging with bid/ask-spreads that are realistic for a major exchange rate market. As a target option we choose a daily monitored knock-out call option with 40 days to expiry, strike at-the-money and barrier 4% out-of-the-money. That is, at time $t$ it pays

$$P = \begin{cases} 0 & t < T, \\ (S_T - K)^+ 1_{S_T < B} & t = T, \end{cases}$$

where $T$ is either the time of the first barrier crossing or expiry, whichever comes first: $T = \min\{\min\{t \in \{0, dt, 2dt, \ldots, T\}; S_t \geq B\}, T\}$. We choose volatility $\sigma = 0.1$ and interest rate $r = 0$, and assume zero drift under both the real world and the risk neutral measure. These choices are not really important — the results from the simulations look very much the same for other values of $\sigma$ and $r$. With these parameters, the benchmark (fair) value of the barrier option is $P(0) = E[e^{-rT}P] = 0.35$.

**Risk-minimizing hedging with bid/ask-spreads.** We consider three risk measures: quadratic ($u(x) = x^2$), positive part ($u(x) = x^+$) and ES$_{5\%}$, the Expected 5% Shortfall.

Table 2 reports the risk-minimizing hedges corresponding to the different risk-measures and the budget restriction $H(0) \cdot c \leq P(0)$, for different values of the vanilla friction $f_v$; we clearly see how the hedge weights decrease with higher friction. This is not surprising: the time-0 transaction cost from a portfolio is $h \times (\text{vanilla spread})/2$, where $h$ is the sum of absolute hedge weights:

$$h = \sum_{k=1}^{M} |c_k| + |c_k'|.$$

A conceivable 20% mark-up on the barrier option corresponds to $h \approx 5$ for $f_v = 0.01$ and only $h \approx 2$ for $f_v = 0.02$. So static hedging may be very expensive in the presence of realistic transaction costs: Table 2

![Figure 1: Risk (not counting transaction costs) vs. expected total transaction cost for the dynamic hedges, for the values 0.01, 0.1, 1, 10, 100 of the risk-aversion parameter $\gamma$. The curves correspond to $f_v = 0.0001$ (solid) and 0.0002 (dashed), the stars correspond to $\gamma = 1$.](image-url)
Figure 2: Risk (including transaction costs incurred after time-0) vs. budget for the dynamic and risk-minimizing hedges, for different risk-measures. Dynamic hedges: $f_v = 0.0001$ (star) and $0.0002$ (circle). Static hedges: each curve corresponds to a particular level of vanilla friction, $f_u = 0, 0.005, \ldots, 0.04$.

Figure 3 shows that all the zero-friction risk-minimizing portfolios have $h \geq 15$.

To compare dynamic and static hedging in the presence of frictions, we evaluate the performance of the bandwidth hedging strategy from Whalley & Wilmott (1997). Varying the risk-aversion parameter $\gamma$ gives a curve in a cost/risk-space, see Figure 1. We fix $\gamma = 1$ in what follows.

Figure 2 shows the performance of the dynamic hedges for $f_u = 0.0001$ and $0.0002$ together with the performance of the risk-minimizing static hedges for different budget restrictions and vanilla friction $f_u = 0, 0.005, \ldots, 0.04$. Look at the performance of the hedges corresponding to the budget restriction (time-0 cost of hedge) $\leq P(0)$. Static hedging is superior to dynamic hedging as long as $f_u \leq 0.015$ for the quadratic risk-measure, but inferior for any $f_u \geq 0.005$ in the case of the positive part risk-measure. For the risk-measure $ES_{5\%}$ though, the static hedges are better than the dynamic hedges even for very large values of $f_u$. The reason is that $ES_{5\%}$ looks only at the tail: the risk-minimizing hedges manage to avoid very large losses, while the dynamic hedge does not. We conclude that whether or not it is better to hedge statically or dynamically depends both on the relative sizes of $f_v$ and $f_u$ and on the choice of risk-measure.

Figure 2 also illustrates that the performance of the risk-minimizing hedges improves when the budget restriction is relaxed, in all cases except one: when $f_u = 0$, the budget restriction is not binding for the hedge corresponding to the quadratic risk-measure. This is financially weird, and reflects the fact that the quadratic risk-measure punishes gains and losses symmetrically. Even worse, for high budgets, the quadratic risk-minimizing hedges perform better when $f_u = 0.005$ than in the no-frictions case. This is a malign consequence of the symmetry in this risk-measure: the hedger is reducing the risk by taking long and short positions in the same instruments, thus throwing away money by paying unnecessary transaction costs!

Figure 3 shows the points $(f_u, f_v)$ where static and dynamic hedging performs equally well. For $f_u = 0.02$, the static hedges are competitive only if $f_v \geq 0.0005$ for the quadratic and $f_v \geq 0.0008$ for the positive part risk-measure — in practice, the friction in the underlying is much smaller. The picture is different for the risk-measure $ES_{5\%}$: the dynamic hedge start to compete only for very high levels of vanilla friction.

5 Hedging of Barrier Options: Extensions

Finally, we run a couple of additional experiments.

**Static + dynamic hedging.** A natural idea is to do static hedging, and then hedge the residual dynamically. This is straightforward to implement: just view $P - H \cdot c$ as the time-$\tau$ payoff from a contingent claim.

However, dynamic hedging of the residual does not improve performance,
it may even make things worse. Figure 4 shows scatterplots of the time-$\tau$ value of the dynamic hedging portfolio against the residual $P - H \cdot c$. The explanation for the poor performance of the dynamic hedge is that this residual is quite nasty; even more so than much the original barrier option payoff. This is because $H \cdot c \approx P$ and we take the difference. Conceptually, the failure of the dynamic hedge tells us something about the optimality of the static hedge: if there was some structure left that was easy to delta hedge, it could be taken care of by improving the static portfolio.

**Beyond Black-Scholes.** As a simple empirical sanity check of the hedge results we exchange the Black-Scholes model’s normal return distribution. Specifically, we switch the daily returns to independent draws from the unconditional empirical distribution of daily $$/¥$ returns over the period 1991–2008. This distribution is skewed (skewness 0.5) and considerably more heavy-tailed (excess kurtosis 4.2) than the normal distribution.

We then look at the performance of the risk-minimizing hedges based on this empirical distribution (remember, as soon as we can simulate, we can compute optimal hedges) and compare it to that of a bandwidth hedger who uses $\sigma = 0.1$, which is extremely close to both the annualized standard deviation of the returns and the Black-Scholes implied at-the-money volatility. The results are shown in Figure 5. The risk is higher than in the Black-Scholes case for both the dynamic and the static hedges, but comparable: look at the left-most endpoints of the curves in Figure 2. As economists we are tempted say that there is a Pareto improvement of static hedges over dynamic hedges: never worse, sometimes better. For the quadratic risk-measure, the static hedges are competitive for larger values of $f_v$ (up to 0.03, compared to 0.015 with normal returns), while results are as before for the positive part risk-measure and $\text{ES}_{5\%}$.

**6 Conclusion**

We showed that with an optimization approach to static hedging it is easy to work with bid/ask-spreads. We then performed a simulation study to compare dynamic and static hedging of a reverse barrier option, assuming bid/ask-spreads similar to what is typically observed in major currency markets. The results showed that the playing field between dynamic and static hedging is now quite level: whether it is optimal to hedge statically or dynamically depends on the relative sizes of the spreads in the spot and for the vanilla options, and on the choice of

![Figure 4: Scatter plots of time-$\tau$ value of the dynamical hedge portfolio plotted against the residual $P - H \cdot c$, for the risk-measures $E[(\cdot)^2](\text{left}), E[|\cdot|^+]$ (middle) and $\text{ES}_{5\%}$ (right) — for a perfect hedge, all points would be on the solid line. The static hedges all have time-0 price $H(0) \cdot c = P(0)$ and the frictions are $f_v = 0.0005$ and $f_u = 0.0001$.](image)

![Figure 5: Vanilla friction vs. risk with an empirical $$/¥$ return distribution. The three panels correspond to different risk measures. The horizontal lines correspond to dynamic hedging with underlying frictions of 1 (dotted) and 2 (dashed) basis points, and the solid lines to the risk-minimizing static hedges, computed with the budget restriction $H(0) \cdot c \leq P(0)$.](image)
risk-measure. For the tail focusing risk-measure Expected Shortfall, static hedging with options is the way to go, but dynamical delta hedging is typically superior for the quadratic and the positive part risk-measures. The explanation is that while bid/ask-spreads are significantly larger (typically two orders of magnitude) for the vanilla options than for the spot, the static hedges are good at removing the very large hedge errors. The conclusion that hedging of large risk “goes well” with frictions is also reached in Selmi & Bouchaud (2003), although in a quite different setting. Of course, there are other problems with dynamic hedging of barrier options than the costs associated with the bid/ask-spreads: Taleb (1996) discusses in detail so-called liquidity holes and other problems that occur in practice.

9. The Expected a% Shortfall is the expected loss incurred in the a% worst cases, see Tasche (2002). This risk-measure is not immediately on the form $E[u(-\cdot)]$, but Rockafeller & Uryasev (2000 Theorem 1) show how minimization of $E_{a\%}$ can be formulated as a linear programming problem (at the cost of extra variables and constraints).

10. This has been suggested for instance in Avellaneda & Paras (1996).

11. We altered the mean return ever-so-slightly to make the underlying a martingale (so technically, we simulate under a U.S.-martingale measure with zero carry), and used simulations to consistently price the plain vanilla options.

FOOTNOTES & REFERENCES


2. The “worst case” approach in Wilmott (2004, Chapter 58) is a notable exception.

3. The idea of choosing a static hedge in an optimal way, however, goes back at least to Avellaneda & Paras (1996).

4. The hedge is not perfect, so the expectation should be taken w.r.t. the real-world probability measure.

5. One could imagine a benefit from letting options simply expire rather than liquidating; hedge accuracy would deteriorate, but costs would be saved. There is no such gain in the setting of Sections 4 and 5 according to our experiments (not reported).

6. This was the typical currency barrier option Danske Bank had on their books during 2004 and 2005.

7. Siven & Poulsen (2008) demonstrates that risk-minimizing static hedging works fine in the Bates model (stochastic volatility and jumps) as well as in an infinite intensity Levy model.

8. A detailed analysis of this distinction is beyond the scope of this paper — and the effects minor for the cases we consider — so our real world measure “just happens” to be the pricing martingale measure.


