FinKont2: Exercises for Thursday March 14

Exercise 6.1: Skews in interest rate options
Assume that the Vasicek-model’s arbitrage-free prices describe caps in some market. Choose reasonable parameters and plot curves (across strikes for different contract lengths) of flat (Black) volatilities in the sense of Björk’s Definition 27.3.

Exercise 6.2: Markovian representation of HJM-models
Consider a Heath-Jarrow-Morton-setup as in Björk’s Chapter 22 and 25 with the notational clarification that we write $\sigma_f(t,T)$ for the forward rate volatilities and $\sigma_P(t,T)$ for the zero-coupon bond volatilities. For simplicity, we assume that the driving Brownian motion is one-dimensional and that all stated dynamics are under $Q$.

From which result in Björk do we conclude that
$$\sigma_P(t,T) = -\int_t^T \sigma_f(t,u)du \quad ? \quad (1)$$

From which result in Björk do we conclude that
$$f(t,T) - f(0,T) = -\int_0^t \sigma_f(s,T)\sigma_P(s,T)ds + \int_0^t \sigma_f(s,T)dW(s) \quad ? \quad (2)$$

Assume now that
$$\sigma_f(t,T) = \sigma e^{-\kappa(T-t)},$$
where $\sigma$ and $\kappa$ are positive constants. Differentiate (2) wrt. $T$ (indicated by “subscript $T$”) and use Leibniz’ rule, (1), and (2) (from right to left) to conclude that
$$f_T(t,T) - f_T(0,T) = \int_0^t \sigma_f^2(s,T)ds + \kappa(f(0,T) - f(t,T)) \quad \text{for all } T. \quad (3)$$

Define
$$\phi(t) = \int_0^t \sigma_f^2(s,t)ds,$$
which there is no particular reason to calculate more explicitly.
Evaluate (3) at “$T = t$” and use Björk’s Prop. 22.5 to show that the dynamics of the short rate can be written as

$$dr(t) = \left( \kappa(f(0, t) - r(t)) + \phi(t) + f_T(0, t) \right) dt + \sigma dW(t).$$

Suppose the forward rate volatilities are generalised to be of the form

$$\sigma_f(t, T) = \sigma(r(t))e^{-\int_t^T \kappa(u) du},$$

where $\sigma$ and $\kappa$ are now functions, not constants, while $\phi$ is defined in the same way as before.

Recycle previous arguments to show that

$$dr(t) = \left( \kappa(t)(f(0, t) - r(t)) + \phi(t) + f_T(0, t) \right) dt + \sigma(r(t))dW(t),$$

and use Leibniz’ rule to show that

$$d\phi(t) = (\sigma^2(r(t)) - 2\kappa(t)\phi(t))dt.$$ 

This shows that the 2-dimensional process $(r, \phi)$ is Markovian (wrt. its own filtration), but that the 1-dimensional process $r$ isn’t (necessarily). HJM-models with this Markov representable structure often have the name “Cheyttte” attached.

**Exercise 6.3: Bermudan swaptions**

On the course homepage I have put the paper Andersen (2000). Questions:

- What is a Bermudan swaption?
- Why is the Bermudan swaption difficult to value?
- What does Leif suggest? How would things work in the (simpler) case of a put-option in the Black-Scholes-model