Financial planning for young households*

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Abstract

We analyze the financial planning problems of young households whose main decisions are how to finance the purchase of a house (liabilities) and how to allocate investments in pension savings schemes (assets). The problems are solved using a multi–stage stochastic programming model where the uncertainty is described by a scenario tree generated from a vector auto-regressive process for equity returns and interest rate evolution. We find strong evidence of the importance of taking into account the multi–stage nature of the problem, as well as the need to consider the asset and liability sides jointly.

Keywords: Asset-liability management, mortgage, pension, multi-stage stochastic programming, scenario tree, vector auto-regression.

JEL: C61, G11

1 Introduction

Buying a house and saving for retirement are the two largest financial transactions for most households. Traditionally these two problems have been treated separately. In this paper we investigate the effects of considering mortgage and pension decisions in an integrated fashion. As an application we study the Danish mortgage and pension market.

A number of papers have taken a stochastic programming approach to the mortgage choice problem of the individual home owner, for instance Nielsen and Poulsen (2004), Rasmussen and Clausen (2007), and Rasmussen and Zenios (2007). Our reason for focusing on the Danish mortgage market is not solely home-bias on the part of the authors. This market has number of interesting features:

– It is transparent. The mortgage-backed bonds are publicly traded at the Copenhagen Stock Exchange.

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– It is large. The market value of mortgage bonds is about 30% larger than yearly Danish gross domestic product.

– Its individual borrowers face a non-trivial choice between competitively priced loan types, most noticeably between the traditional callable fixed rate mortgages (FRM) and adjustable rate mortgages (ARM).

For more on the Danish mortgage market we refer to Frankel et al. (2004) and Svenstrup and Willemann (2006).

Most research within pension investments has been concentrated on the corporate side, for instance Drijver (2005) and Klein Haneveld et al. (2010) investigate asset-liability management (ALM) models for pension funds. However, in the recent years the literature on asset and liability modeling for individual investors has been increasing. Consiglio et al. (2007, 2004), Geyer et al. (2009), Medova et al. (2008) have all developed ALM models for individual investors.

In this paper we propose a multi–stage extension of the single–stage stochastic model introduced by Pedersen et al. (2010). As driving stochastic model we use a use first-order vector auto-regression (VAR(1)) process. Similarly to Boender et al. (2005), we combine the evolution of interest rates — given by Nelson/Siegel parameters — with equity returns. We integrate the wealth accumulation model for the pension saving problem of a household, taking transaction costs and tax effects into consideration with the mortgage model of Rasmussen and Zenios (2007) and suggest a number of combined mortgage loan and pension investment strategies for the individual household. As a contribution of this paper we show how to generalize the model in Ferstl and Weissensteiner (2011) for cases where the frequency of the decision process differs from the frequency of the data generating process. We provide a detailed numerical example involving a typical young household. This shows that it is important to consider the multi–stage nature of the planning problem; using static portfolios costs 0.6-1.3% (more the less risk-aversion is) per year in expected wealth terms.

We should stress that our analysis is limited to the young household. It is, thus, not a full life-cycle analysis (as opposed to Cocco et al. (2005) and more recently Kraft and Munk (2011)), but rather an investigation of what we believe to be a highly relevant ALM sub-problem.

The rest of the paper is organized as follows: In Section 2 we formulate the multi-stage optimization model (with the mainly technical parts being relegated to the Appendix), in Section 3 we describe and estimate the empirical model for assets and liabilities, and in Section 4 we report the results (and robustness checks) when the model is applied to a typical household.

2 The model

The two most important decisions in the financial plan of a young household are how to finance the purchase of a house now and how to invest the pension savings. Another decision could be *how much* to invest in pension savings but in this paper, we consider young households with liquidity constraints that mean that they only want to invest what is mandatory in pension savings (about 17% of wages). By making decisions on what mortgage loan portfolio to have and which asset allocation of pension savings to choose at different time stages, the household seeks to maximize its expected wealth at the end of the planning horizon, while still considering risk.

The household can choose between callable fixed-rate mortgages (FRMs) and adjustable rate mortgages with annual refinancing (ARMs). Different mortgage loans are available at the different time steps depending on the evolution of interest rates. On the pension side, investments can be made in various asset classes, specifically stocks (we use a broad index) and bonds with different times to maturity. The stochastic development of interest rates and stock returns are represented by a scenario tree. Each node in the scenario tree consists of a realization of the stock returns and zero-coupon interest rates. Based on the interest rates, the prices of the callable mortgage bonds are found using the approximation from Nielsen and Poulsen (2004).

Figure 1 shows an example of a scenario tree with two stages, ten nodes and six scenarios. We define by $\mathcal{N}_t$ all the nodes at time stage t in the tree. Thus, $\mathcal{N}_1 = \{n2, n3, n4\}$. Every node $n \in \mathcal{N}$ has a unique parent denoted by $a(n)$. In our example, we have for instance that $a(n7) = n3$. Each path leading from the root node $n1$ to one of the leaf nodes $\{n5, n6, \ldots, n10\}$ is called a scenario. In the following $path(n, node)$ is used to check whether the nodes n and $node$ are on the same path.

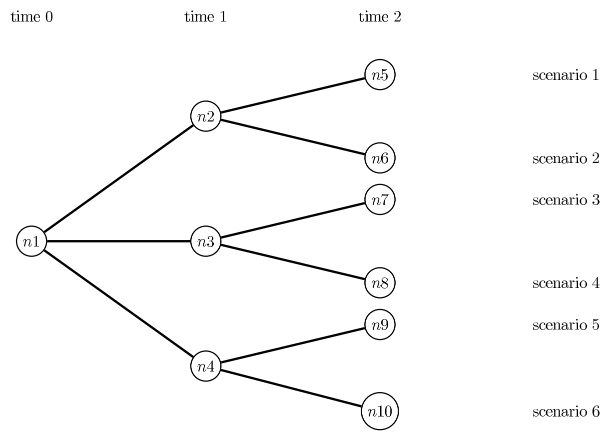

Figure 1: An example of a scenario tree

2.1 Model formulation

To ease the exposition we have moved all formal variable and parameter definitions as well as most budget/balance/wealth/transport equations ("a rose by any other name ...") to the Appendix.

Given a scenario tree, we formulate the household's over-all problem as:

$$\max \quad \sum_{n \in \mathcal{N}_T} p_n \cdot \left[d_T \cdot W_n + d_T \cdot CF_{Tn} + \sum_{t=1}^{T} d_t \cdot C_{tn} \right] \tag{1}$$

subject to

$$
\begin{aligned}
CVaR_\alpha &\geq minCVaR && (2)\\
VarDev_n &\geq VaR - \left[d_T \cdot W_n + d_T \cdot CF_{Tn} + \sum_{t=1}^{T} d_t \cdot C_{tn} \right] \quad \forall n \in \mathcal{N}_T && (3)\\
CVaR_\alpha &= VaR - \frac{\sum_{n\in\mathcal{N}_T} p_n \cdot VaRDev_n}{1-\alpha} && (4)\\
VaRDev_n &\geq 0 \quad \forall n \in \mathcal{N}_T && (5)\\
W_n &= VH_n + IA - PP_n \qquad \forall n \in \mathcal{N}_T && (6)\\
PP_n &= \sum_{i\in U} (OD_{iTn} \cdot K_{in}) \qquad \forall n \in \mathcal{N}_T && (7)
\end{aligned}
$$

The objective function (1) maximizes the expected present value of final wealth plus aggregated consumption subject to a constraint on the conditional-value-at-risk (CVaR; also known as tail-expected-value and several other names) of that same quantity. This is specified by equation (2), which then represents the household's aversion to risk. As we are working with net wealth, "small numbers are bad". In Ogryczak and Ruszczynski (1999) the relation between risk-measures (such as CVaR), stochastic dominance and the maximization of expected utility is investigated. CVaR$_\alpha$ is defined as the average over the α, say 5%, worst scenarios and it is a coherent risk measure, which basically means — to paraphrase one of our favorite descriptions of good interest rate models from Rogers (1995) — that it doesn't do silly things. As the definition of CVaR involves a quantile it is perhaps (or perhaps not) surprising that a constraint on CVaR can be formulated through linear constraints, see Rockafellar and Uryasev (2000), which is done with (3)-(5). The optimization is done by choosing a dynamic asset allocation and mortgage refinancing strategy — more on which later. The final wealth in (6) is given by the final value of the pension savings plus the equity in the house which is the initial value of the house IA minus the costs of prepaying the loan at horizon PP_n, i.e. we assume the price of the house does not change in the future. Our criterion function looks different from the one used in for instance Rasmussen and Zenios (2007) where a convex combination of expected payments and CVaR is minimized. However, it is simply a reparametrization of the problem; to solve the problem (conceptually, at least) the CVaR constraint (equation (2)) goes directly into the Lagrange function. By solving the problem for a range of CVaR constraint levels we get an efficient frontier in (expected payments, CVaR)-space (and associated optimal strategies). To find levels of $minCVaR$ that are of reasonable scale for a particular problem, we do the following: (1) Solve the problem without any constraint on CVaR and calculate the CVaR associated with this solution, $lowerCVaR$. (2) Find the maximal value of $minCVaR$ such that the problem has a feasible solution, $upperCVaR$; this can be done by maximizing CVaR subject to the remaining constraints. (3) Solve the problem for a range (say 10) of equidistant $minCVaR$-values between $lowerCVaR$ and $upperCVaR$. We follow this procedure for the numerical results in Section 4.

The rest of the model naturally splits into three parts; the liability side, the asset side, and the tax system that connects them.

Liability side. The basic risk/return trade-off here is whether to use variable rate loans or callable fixed-rate loans — and in case the latter are used, when to exercise the embedded American option (to refinance at lower rates). A particular feature of the Danish mortgage market is the so-called delivery option; the borrower can retire his debt by paying the (well-defined, observable) market price of the underlying bond. This is particularly relevant with short planning horizon, as captured by the PP_n-terms in equations (6)-(7). The purchase of the house is funded by issuing (or selling) mortgage loans (equation (18) in the Appendix). The outstanding debt at a given stage is equal to the outstanding debt at the previous stage minus principal payment plus/minus any extra outstanding debt originating from rebalancing the mortgage loan portfolio; equations (19)-(22) keep track of this. The cost of rebalancing includes both a fixed and a variable component, this is handle by equations (24)-(26). The total payment on the loan portfolio at a given time and node in the scenario tree is given in (27) and (28).

Asset side. Every year, a constant fraction β (in our example 0.17; effectively stipulated by Danish law) of the household's income (I_t) is contributed to the pension savings schemes, and it must chose how to allocate the investment between assets classes (captured by the κ-variables in (31)). Based on this allocation the final value of the pension savings before taxes is calculated by (32).

Taxes etc. In Denmark the returns on pension savings are only liable to a fixed tax rate tax_{pal} and are not connected to household income. However, when the pension savings are paid out, they are taxed as income. The rules for taxation of the payments from the pension savings schemes are rather complex and depend on among others the size of the payments and the pension scheme. In this paper, we assume that the payments at the end of horizon are of such a size that they are taxed with low bracket tax rate (33). The amount left after the contribution to pension savings is taxed and should cover the loan payments and consumption. It it is (through (37)-(39)) possible to deposit any remaining money on a bank account, or, if needed, withdraw accumulated cash, e.g. to pay for increased loan payments (34)-(35). Danish taxation is progressive, as captured by tax-bracket limits $ba.$'s and households get a tax deduction on the (net) interest rate payments (in the low bracket). Equation (44) registers whether the household has positive or negative capital income. If the capital income is positive it is taxed as income, again subject to some bracketing, with equations (41) and (42).

3 Modeling uncertainty

3.1 Danish Data Sources

For (monthly) stock returns we use the Danish OMX index[1] of blue-chips stocks. This is calculated by the Copenhagen Stock Exchange (or commercially correctly: NASDAQ OMX Group, Inc.) and it is the value-weighted average of the prices of the twenty largest Danish stocks, where size is measured by a combination of market capitalization and trading volume, and the weights are revised every six months.[2] Details aside, the calculation of the OMX index is similar to, for instance, that of the S&P500 index. So as with most indices, the OMX index calculation does not include or adjust for dividends. For the purposes of this paper it is, however, important to include dividends for potentially two reasons. First, dividend payments account for a significant proportion of the total return from stock investment; the yearly dividend yield is typically between two and four

[1] Formerly known as the C20 or the KFX index

[2] Further descriptions are given in Bechmann (2004) and at `https://indexes.nasdaqomx.com/docs/methodology_OMXC20.pdf`

percent. Not including dividends would distort the balance between stock and bond returns, in particular over long horizons. risk-aversion. Second, some studies find dividends to have a short-term predictive effect on stock returns, which will in turn affect optimal investment decisions (e.g., Campbell et al., 2003). Dividend adjustment is non-trivial and/or tedious, but fortunately the calculations (assuming stock reinvestment of dividends) have been carefully performed by Belter et al. (2005), and data for total return and standard indices covering the period 1986 to 2009 are conveniently accessible at `www.tanggaard.com`. We follow Goyal and Welch (2008) and adjust dividends for seasonality by using as price/dividend ratio the cumulative dividends over the last 12 months relative to the current index value.

For our term structure model we use Danish zero-coupon yield curves. All data are at a monthly frequency.

3.2 Term structure of interest rates

To model the evolution of the yield curve, we propose to use the Nelson/Siegel model for two reasons. On the one hand, this parsimonious parametric model can represent the entire yield curve by only a few parameters, restricting the size of the scenario tree and ensuring computational tractability. On the other hand, some advanced approaches may not be superior to model the dynamic evolution of interest rates due to their potential over-fitting of in-sample data (see e.g. Diebold and Li, 2006). Furthermore, we do not consider affine term structure models. The demanding parameter estimation of such models, combined with the long persistence of interest rates results in a high parameter uncertainty which neglects promising out-of-sample forecasts, see Ang et al. (2007).

The three-factor model for the spot rates can be written as:

$$y(\boldsymbol{\beta}_t, m) = \beta_{1,t} + \beta_{2,t}\left(\frac{1 - e^{-\lambda_t m}}{\lambda_t m}\right) + \beta_{3,t}\left(\frac{1 - e^{-\lambda_t m}}{\lambda_t m} - e^{-\lambda_t m}\right), \tag{8}$$

where $y(\boldsymbol{\beta}_t, m)$ indicates the (continuously compounded) spot rate for maturity m at stage t given the parameter vector $\boldsymbol{\beta}_t = [\beta_{1,t}, \beta_{2,t}, \beta_{3,t}]^\top$ for level, slope and curvature of the term structure of interest rates. Following Diebold and Li (2006), we fix λ_t at 0.6876, which minimizes the mean squared error in our data set. In this way, the estimation of the remaining parameters $\beta_{1,t}$, $\beta_{2,t}$ and $\beta_{3,t}$ simplifies to an ordinary least square (OLS) regression. The estimated Nelson/Siegel parameters are then included in the estimation of the VAR(1) process presented below.

3.3 Pricing Callable Mortgage Bonds

In the traditional “fixed” rate mortgage loans, the borrower has the right to repay her loan at remaining outstanding principal. In option pricing terms, she is equipped with American-type optionality. This makes the connection between zero-coupon yield curves (and the dynamics of these) and model — or even worse: market — prices of the mortgage bonds tricky. A sizeable literature exists on this, where either an option pricing approach is taken (based on partial differential equations or nowadays more often simulation; the ”bounded rationality” model from Stanton (1995) is an example) or a more empirical approach is taken to model prepayment, typically

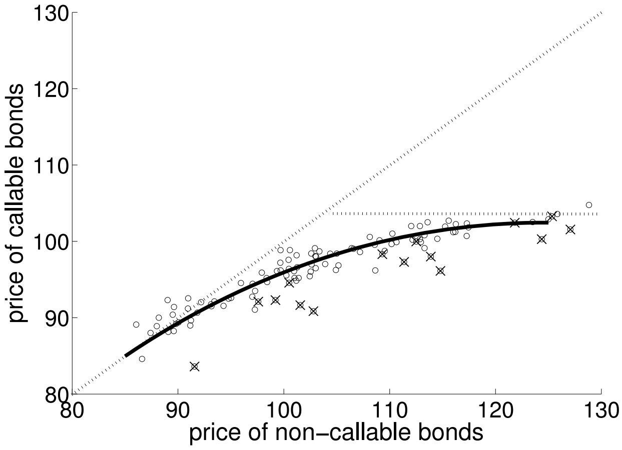

Figure 2: The callable bond price model. The circles are observed pairs of (non-callable, callable) bond prices between 2004 and 2009 (recorded every three months across available coupon rates). The fully drawn curve is the relationship postulated by the model from Nielsen and Poulsen (2004). The circles with X's through them are from after the Lehman default (mid-September 2008).

based on a required gains criterion, see e.g. Schwartz and Torous (1989). In this paper, we take a more simple approach by following Nielsen and Poulsen (2004, Section 3.2). The idea here is to estimate — based on genuine empirical data — a parameterized functional relationship between prices of non-callable bonds and prices of callable bonds.[3]

Figure 2 shows that even when adapting verbatim the parameters and functional form from Nielsen and Poulsen (2004),[4] who used 1999 to 2002 data, we get a quite satisfactory relation between callable and non-callable bond price.[5]

3.4 Time-varying investment opportunities

We model time-varying investment opportunities with an unrestricted, stationary[6] VAR(1)-process (for an application in asset allocation decisions see e.g. Barberis, 2000; Campbell et al., 2003; Brandt et al., 2005) and combine the evolution of interest rates (represented by the coefficient vector $\boldsymbol{\beta}_t$) with equity returns, see, e.g., Boender et al. (2005); Ferstl and Weissensteiner (2011). We use the following $(K \times 1)$ parameter vector $\boldsymbol{\xi}_t$ (with $K = 4$):

$$\boldsymbol{\xi}_t = \begin{bmatrix} r_t^1 \\ \boldsymbol{\beta}_t \end{bmatrix}, \tag{9}$$

where $r_t^1 \equiv \log(R_t^1)$ refers to the log equity return and $\boldsymbol{\beta}_t$ to the (3×1) to the vector of Nelson/Siegel parameters. Opposed to other countries (see e.g.Campbell et al. (2003) and the literature mentioned therein), the dividend-price ratio has no statistically significant predictive power for the Danish market, neither for equities nor for yield curve, and is not driven significantly by the other parameters.[7] Therefore we decided not to include this

[3]In a one-factor model, such a one-to-one correspondence is indeed theoretically correct.

[4]The lower right part of the forked expression on page 1276 defining f_{30} should read "$x \geq c + (ab)^{1/(1-b)}$".

[5]To be fair, the "post-Lehman" period does post problems. And one could contemplate adding an idiosyncratic noise factor. But we leave those issues for future work.

[6]Stationarity refers to time-invariant expected values, variances, and covariances.

[7]In Denmark, almost all stocks pay yearly dividends. And almost all pay in April or May. This may be the reason we are unable to detect any (short-term) predictive effects.

ratio explicitly in the VAR-process (but of course dividends were included when returns were calculated or simulated). The functional form of the VAR(1) process can be written as:

$$\boldsymbol{\xi}_t = \mathbf{c} + \mathbf{A}\boldsymbol{\xi}_{t-1} + \mathbf{u}_t, \tag{10}$$

where $\mathbf{c}$ is the $(K \times 1)$ vector of intercepts, $\mathbf{A}$ is the $(K \times K)$ matrix of slope coefficients and $\mathbf{u}_t$ the $(K \times 1)$ vector of i.i.d. innovations with $\mathbf{u} \sim N(0, \boldsymbol{\Sigma})$. The covariance of the innovations $\boldsymbol{\Sigma}$ is given by $\mathbb{E}(\mathbf{u}\,\mathbf{u}^\top)$. Thus, we allow the shocks to be cross-sectionally correlated, but assume that they are homoscedastic and independently distributed over time. If all eigenvalues of $\mathbf{A}$ have modulus less than one, as in our empirical example below, the stochastic process in equation (10) is stable with unconditional mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Gamma}$ for the steady state at $t = \infty$ (see e.g. Lütkepohl, 2005):

$$\boldsymbol{\mu} := (\mathbf{I} - \mathbf{A})^{-1}\mathbf{c} \tag{11}$$

$$\text{vec}(\boldsymbol{\Gamma}) := (\mathbf{I} - \mathbf{A} \otimes \mathbf{A})^{-1}\text{vec}(\boldsymbol{\Sigma}), \tag{12}$$

where $\mathbf{I}$ refers to the identity matrix, the symbol $\otimes$ is the Kronecker product and "vec" transforms a $(K \times K)$ matrix into a $(K^2 \times 1)$ vector by stacking the columns.

To estimate the intercepts and slope parameters of the VAR(1) process via OLS we use monthly data described in Section 3.1. Equity returns refer to the OMX performance index, while Nelson/Siegel parameters are estimated from the Danish zero-coupon yield curve. The corresponding parameters are reported in Table 1 (values for the t-statistics in parenthesis), where $|z|$ indicates the modulus of the eigenvalues of the characteristic polynomial. The monthly Nelson/Siegel coefficients $\beta_{i,t-1}$ are highly persistent as well as statistically significant.

Table 1: VAR(1) parameters and t-statistics () for monthly data 1986.M6–2009.M1

	c	r^1_{t-1}	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$	R^2
r^1_t	-0.0159	0.0455	0.1343	-0.1574	-0.4314	0.0481
	(-1.4510)	(0.7433)	(0.9527)	(-0.8383)	(-2.6415)	
$\beta_{1,t}$	0.0018	0.0081	0.9833	-0.0023	0.0382	0.9702
	(2.0228)	(1.7531)	(92.3128)	(-0.1603)	(3.0956)	
$\beta_{2,t}$	-0.0001	0.0040	0.0004	0.9426	0.0326	0.9088
	(-0.0571)	(0.6235)	(0.0246)	(47.2813)	(1.8796)	
$\beta_{3,t}$	-0.0049	0.0283	0.0142	0.0416	0.8108	0.6616
	(-1.7388)	(1.9496)	(0.4240)	(0.9332)	(20.8968)	
$\|z\|$		0.0612	0.7841	0.9880	0.9487	

By comparing the parameters $\beta_{1,t-1}$ with $\beta_{2,t-1}$ we see, in line with the general empirical evidence, that the yield level dynamic is more persistent than the spread dynamic. The in-sample R^2 for equities is 4.81%, below the 8.6% in Campbell et al. (2003) where the dividend-price ratio shows significant predictive power.

While Table 2 illustrates monthly standard deviations (multiplied by 100) on the main diagonal and cross correlations of residuals above it, in Table 3 we indicate the unconditional expected mean $\boldsymbol{\mu}$ of the VAR parameters in the steady state. The expected return per annum for equities equals 5.4%, while the term

Table 2: Cross correlations and standard deviations of residuals for monthly data 1986.M6–2009.M1

	r^1_{t-1}	$\beta_{1,t-1}$	$\beta_{2,t-1}$	$\beta_{3,t-1}$
r^1_{t-1}	5.6178	-0.2085	-0.0348	-0.0833
$\beta_{1,t-1}$	–	0.4243	-0.3879	-0.3029
$\beta_{2,t-1}$	–	–	0.5967	0.1510
$\beta_{3,t-1}$	–	–	–	1.3342

structure of interest rates (continuously compounded) given by the Nelson/Siegel parameters is (for longer maturities) increasing and concave as shown in Figure 3.

Table 3: Unconditional expected values $\boldsymbol{\mu}$ for the steady state

r^1	β_1	β_2	β_3
0.0045	0.0573	-0.0141	-0.0238

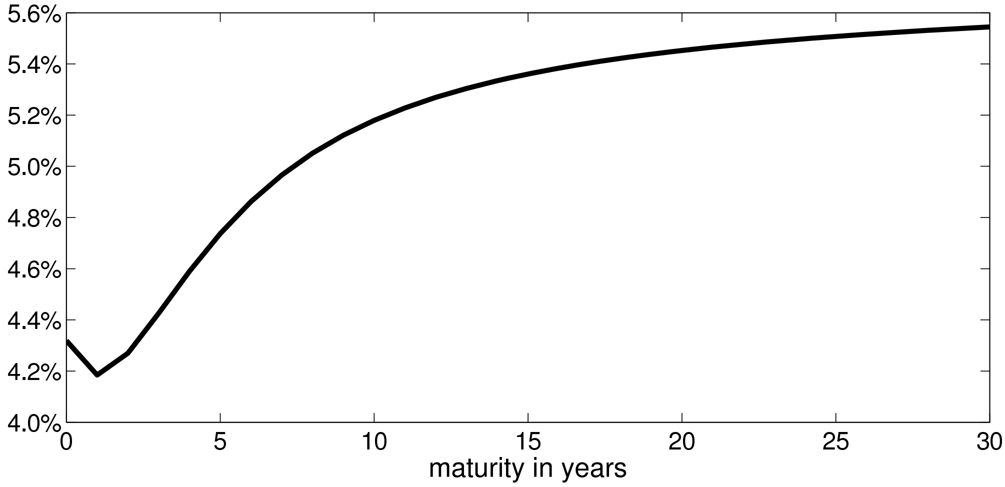

Figure 3: Term structure of interest rates for the steady state

3.5 Scenario generation

The multivariate process in equation (10) evolves in discrete time. In order to reduce the complexity of the optimization problem, a so-called scenario tree approximates the probability distribution of the process with a few mass points. Therefore, we use the moment matching technique proposed by Høyland and Wallace (2001) and Høyland et al. (2003).[8] To uncouple our results from a particular root note, here we decided to start our tree construction from the unconditional expected values as done, e.g., by Campbell et al. (2003) and Ferstl and Weissensteiner (2011). In addition to matching moments and co-moments of the conditional distributions of our stochastic process well, asset returns implied by our discrete scenario tree must satisfy the no-arbitrage principle. Without such a condition the results will be unbounded or biased (for a discussion see, e.g. Geyer et al., 2010). To ensure this requirement, we apply the arbitrage-check proposed by Klaassen (2002).

[8]For our numerical example in Section 4 we use a constant branching factor of ten

While the estimated stochastic process (10) in Section 3.4 represents the time-discrete *monthly* co-evolution of the equity returns and the Nelson/Siegel parameters, for the optimization task we choose *annual* decision stages to reallocate our portfolio. Given that the decision intervals differ from the frequency of our stochastic process, additional aspects must be considered for the scenario generation and the following arbitrage check. On the on hand, we have to account for the annual aggregated monthly asset returns between two decision stages. Barberis (2000) shows how to calculate expectations and covariances for cumulated asset returns under a VAR process. On the other hand, in addition to pure asset returns our stochastic process in equation (10) includes also the evolution of the Nelson/Siegel parameter vector, which cannot be cumulated in the sense of Barberis (2000). However, as changes in the yield curve drive the market price of the different bond holdings, the potential bond returns have to be considered for the arbitrage check. To sum up: The annual equity returns are given by the cumulated monthly returns, whereas the annual bond returns can be calculated by changes in the yield curve between the two decision stages.

For the generation of the scenario tree we proceed in two steps. First, we calculate the conditional distribution of *cumulated* monthly equity returns with (*non-cumulated*) Nelson/Siegel parameters at the future decision stages and approximate these multivariate distribution by a few mass points using Høyland et al. (2003). Second, having the discrete realizations of the yield curves at two sequent stages we calculate the annual returns of our tradeable bonds and include them, together with the annual equity returns, in the arbitrage check.

We begin to describe the *first* step, and generalize the model in Ferstl and Weissensteiner (2011) for cases where the frequency of the decisions differs from the frequency of the data generating process. For notation brevity, we define $\boldsymbol{\zeta}_\tau$ as the vector of *cumulated* equity returns and Nelson/Siegel parameters[9] and introduce a $(K \times K)$ indicator matrix $\mathbf{J} = \mathrm{diag}(1, 0, 0, 0)$. We show how to calculate the expectation and the covariance of $\boldsymbol{\zeta}_\tau$ for two time steps (i.e., months) of (10) ahead, and generalize afterwards. $\boldsymbol{\zeta}_2$ is given by:

$$\boldsymbol{\zeta}_2 = \underbrace{(\mathbf{I} + \mathbf{A})\,\mathbf{c} + \mathbf{A}^2\,\boldsymbol{\xi}_0 + \mathbf{A}\,\mathbf{u}_1 + \mathbf{u}_2}_{\boldsymbol{\xi}_2} + \mathbf{J}(\underbrace{\mathbf{c} + \mathbf{A}\,\boldsymbol{\xi}_0 + \mathbf{u}_1}_{\boldsymbol{\xi}_1}). \tag{13}$$

The expected value of $\boldsymbol{\zeta}_2$ results as:

$$\mathbb{E}(\boldsymbol{\zeta}_2) = (\mathbf{I} + \mathbf{A} + \mathbf{J})\,\mathbf{c} + (\mathbf{J}\,\mathbf{A} + \mathbf{A}^2)\,\boldsymbol{\xi}_0,$$

and the corresponding covariance as:

$$\mathbb{V}(\boldsymbol{\zeta}_2) = \boldsymbol{\Sigma} + (\mathbf{J} + \mathbf{A})\boldsymbol{\Sigma}(\mathbf{J} + \mathbf{A})^\top.$$

Expanding the approach in (13) for more (T) discrete steps, by collecting terms the following general result can be obtained:

$$\mathbb{E}(\boldsymbol{\zeta}_T) = \left(\left(\sum_{i=1}^{T-1} (\mathbf{I} + \mathbf{J}(T-i))\,\mathbf{A}^{i-1} \right) + \mathbf{A}^{T-1} \right) \mathbf{c} + \left(\mathbf{A}^T + \sum_{i=1}^{T-1} \mathbf{J}\,\mathbf{A}^i \right) \boldsymbol{\xi}_0, \tag{14}$$

[9] The difference to $\boldsymbol{\xi}$ is given by first row, see (9). While in $\boldsymbol{\xi}$ equity returns are on a monthly basis, $\boldsymbol{\zeta}_\tau$ cumulates τ monthly equity returns. The Nelson/Siegel parameter vector is the same in both cases.

and

$$
\begin{aligned}
\mathbb{V}(\boldsymbol{\zeta}_T) = &\ \boldsymbol{\Sigma} \\
&+ (\mathbf{J}+\mathbf{A})\,\boldsymbol{\Sigma}\,(\mathbf{J}+\mathbf{A})^\top \\
&+ \left(\mathbf{J}+\mathbf{J}\,\mathbf{A}+\mathbf{A}^2\right)\boldsymbol{\Sigma}\left(\mathbf{J}+\mathbf{J}\,\mathbf{A}+\mathbf{A}^2\right)^\top \\
&+ \dots \\
&+ \left(\mathbf{A}^{T-1}+\sum_{i=1}^{T-1}\mathbf{J}\,\mathbf{A}^{i-1}\right)\boldsymbol{\Sigma}\left(\mathbf{A}^{T-1}+\sum_{i=1}^{T-1}\mathbf{J}\,\mathbf{A}^{i-1}\right)^\top .
\end{aligned} \tag{15}
$$

For each predecessor node we approximate the conditional distribution at the next decision stage given by (14) and (15) with a finite number of successor nodes using the moment matching technique.

Now we present the *second* step of our scenario generation, i.e. the calculation of annual bond returns, which have to be included in the arbitrage check together with the cumulated equity returns. We define $P^s(t,m)$ as the market price of the m-year maturity zero bond at stage t and scenario s. The gross return after a holding period Δt is given by:

$$
R^s(t+\Delta t, m) = \frac{P^s(t+\Delta t, m-\Delta t)}{P^s(t,m)} = \frac{e^{my(\boldsymbol{\beta}_t^s, m)}}{e^{(m-\Delta t)y(\boldsymbol{\beta}_{t+\Delta t}^s, m-\Delta t)}}, \tag{16}
$$

where $y(\boldsymbol{\beta}_t^s, m)$ defines the term structure at stage t and scenario s, see (8). Moving to log holding-period returns results in:

$$
r^s(t+\Delta t, m) = my(\boldsymbol{\beta}_t^s, m) - (m-\Delta t)y(\boldsymbol{\beta}_{t+\Delta t}^s, m-\Delta t). \tag{17}
$$

For a more general result with coupon-paying bonds see Campbell et al. (1997).[10] We include these annual bond returns together with the cumulated equity returns from the first row of $\boldsymbol{\zeta}$ in the arbitrage-check proposed by Klaassen (2002) to ensure the no-arbitrage condition.

Finally, to investigate the plausibility of the yield curves in our scenario tree, in Table 4 we give percentiles of spot rates with different maturities for cumulative probabilities p_z, with $p_z \in \{0.025, 0.5, 0.975\}$ at the final decision state. In addition, to show that most yields lie within a meaningful range, Table 4 illustrates that the variability is higher for short-rate yields – a well-known empirical fact.

Table 4: Percentiles for the term structure of interest rates at $m_t = 3$

	1Y	5Y	10Y	15Y	20Y	30Y
$p_z = 0.025$	0.01164	0.02075	0.02668	0.02877	0.03013	0.03150
$p_z = 0.5$	0.04211	0.04743	0.05170	0.05323	0.05413	0.05523
$p_z = 0.975$	0.07221	0.07277	0.07614	0.07805	0.07928	0.08043

[10] Given the yearly decision stages in our numerical exhibition of Section 4, to calculate annual returns of the bonds in our asset menu we set Δt equal to one year.

4 Model validation and testing

We test our multi-period stochastic program formulation on a scenario tree with three one-year stages and a branching factor of ten at each stage. Hence, the tree structure is (10 × 10 × 10) which amounts in 1,000 scenarios. The model is implemented in the algebraic modeling language GAMS using CPLEX 12.1.0.

4.1 Problem description

As a sample case we consider a customer with an income of EUR 67,000. The income increases with 2% per year. She wants to buy a house worth EUR 270,000 but does not know which mortgage loan or loans to choose to finance the purchase; she can choose between fixed rate mortgages with coupons from 1% to 10 % (FRM1%-FRM10%) or an adjustable rate mortgage with annual refinancing (ARM1). The agent also wants to start saving for her retirement. She does not earn a lot so for now she is only interested in saving the mandatory 17% per year of the income. However, she wants to know which asset classes are optimal for her to invest in. The asset classes are: a broad equity index and bonds with 1, 5 and 10 years to maturity (hereafter denoted 1Y, 5Y, and 10Y bonds). In round, pre-tax numbers, her mortgage payments are 50% larger than her pension payments (18,000 vs. 12,000 per year).

4.2 Numerical results

Figure 4 shows the initial mortgage loan portfolio and the initial asset allocation for ten different levels of agent risk-aversion (in sense of the constraint (2)) ranging from risk-neutrality (on the left) to maximization of conditional value-at-risk. To understand the results let us start with the mortgage side. From size payment considerations, this should be the dominant one, and it means that we address "the elephant in the room" first.

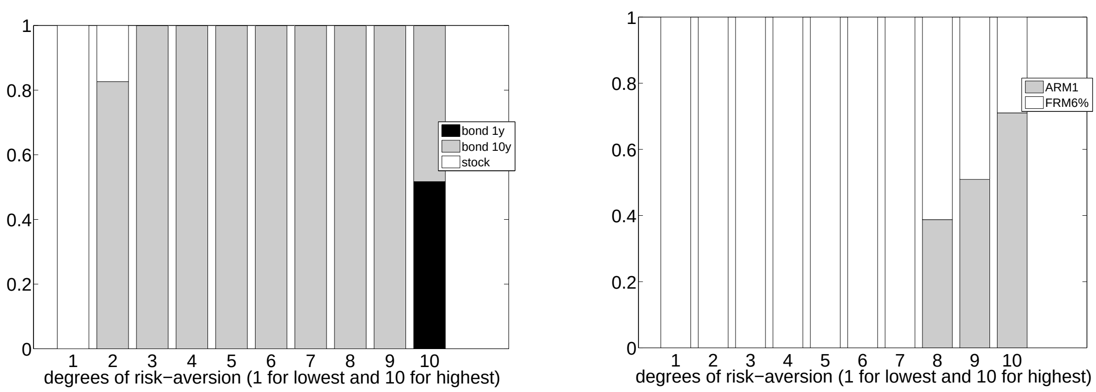

Figure 4: The initial optimal mortgage (right) and pension (left) portfolios. The optimal portfolios are shown for ten levels of agent risk-aversion; the left-most columns correspond to risk-neutrality, the right-most one to maximization of CVaR.

The more risk-neutral agents use callable fixed rate loans (white bars) to finance their mortgages, while the adjustable rate loans (gray bars) are used for the more risk-averse agents. This is counter-intuitive; one would think that the fixed rate should appeal to the risk-averse borrower, while the risk-neutral would use short-term financing because short-term interest rates are typically lower than long-term interest rates, as Figure 3 illustrates. This reasoning is incomplete, though. Firstly, conventional opinion is formed with short-term memory, and in recent years yield curves have been considerably steeper than the steady-state estimate we use. Secondly, when you have a three-year horizon (as do our borrowers), a 30Y fixed rate loan is not risk free. If you take a long-term fixed rate loan, interest rates fall, and you have to liquidate your position after a period that is significantly shorter than the maturity of the loan, then you will suffer a loss; you have to buy back the loan at a now higher price. This effect is alleviated by the loan's embedded call-feature (its price cannot go above par), but it is still in force. One might argue about our choice of a three year horizon; a longer one would make it easier to let the loan ride and diminish the liquidation price-risk. We may have picked the horizon on the short side, but there is little doubt that in the future there will be stricter running monitoring of borrower solvency, like the loan-to-value of the house. The last piece of the puzzle is the effect of the embedded call-option. Effectively, the borrower who uses callable fixed rate mortgages is (on top of a hypothetical non-callable fixed rate loan) investing in American call-options. And it is perfectly viable and in line with economic theory that these call-options have significantly positive expected excess returns, which makes them attractive to mortgagors with low risk-aversion.

Let us now turn to the pension/asset/investment side; the left panel in Figure 4. For this, it is important to look at Table 5 with descriptive statistics for the sample of total returns.

Table 5: Statistics on the total return (in%) along the scenarios

	Equity	Bond 1Y	Bond 5Y	Bond 10Y
min	-72.03	3.93	-2.03	-25.64
max	196.07	16.50	31.63	66.27
mean	14.97	11.07	13.98	14.65
std. dev.	38.58	3.00	6.33	14.91

Firstly, we see that the risk-neutral agent's asset portfolio is invested solely in stocks. This serves mostly as a sanity check on the implementation and optimization. The model is able to pick up the rather minuscule mean excess return of equity over the 10Y bond; 0.3% over three years. Secondly, we see that across a large range of risk-aversion it is optimal to invest the pension savings in the 10Y bond. This is surprising as the mean return on the 10Y bond is only marginally higher than that of the 5Y bond (0.7% over three years), while the 5Y bond return has a considerably lower return standard deviation (6.33% vs. 14.91%). The answer lies in remembering the joint nature of the problem, i.e. in considering the mortgage side. Here we found the optimal instrument to be the callable fixed rate mortgage over a large spectrum of risk-aversion levels. And while the 5Y bond is virtually uncorrelated with the callable fixed rate mortgage bond in-sample, the 10Y bond has correlation 0.36, and is thus the natural hedge instrument.[11]

To show that the optionality effect is really the driving force behind the seemingly strange liability side, and that it is *critical* that the borrower makes dynamic adjustments to her mortgage portfolio, i.e. that she

[11] The correlation is not perfect or constant across maturity for two reasons; firstly, we use a three-factor model, and secondly the callable bond prices are non-linearly related to prices of non-callable bonds, as Figure 2 demonstrates.

exercises her American options appropriately, let us look at an experiment similar to the one described above, except the agent's is restricted from dynamically adjusting her portfolio.[12] The results of that experiment are shown in Figure 5.

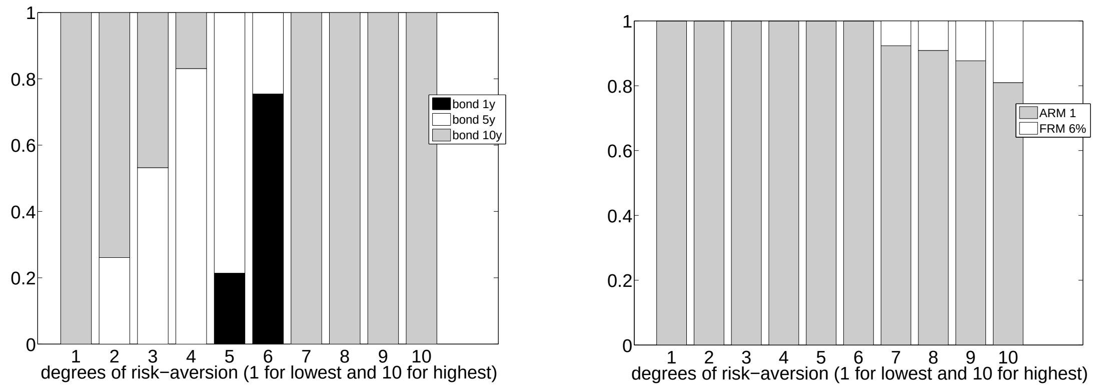

Figure 5: The mortgage and pension portfolios when readjusting portfolios is disallowed.

Here we see that the picture is completely reversed on the mortgage side; the risk-averse borrower uses fixed rate loans (to some extend), while the risk-neutral borrower use fully adjustable-rate financing; why pay for options that you aren't allowed to use? On the asset side the picture is "muddy"; at low risk-aversion levels 10Y bonds are attractive because of their high expected return, and at high risk-aversion levels they are attractive because of the role as natural hedge instruments against the liability side, while at intermediate levels 1Y and 5Y bonds are used.[13]

Table 6: Optimal asset portfolio compositions over time for a medium risk-averse agent

time, state	1Y bonds	5Y bonds	10Y bonds	stocks
0, steady state			1	
1, low long-rate		0.33	0.21	0.46
1, high long-rate	0	0	0.67	0.33

A different way to illustrate the multi–stage nature of the problem, i.e. that portfolios are indeed dynamically adjusted, is given in Table 6. Here we look at the optimal asset portfolios at times 0 and 1 for a medium risk-averse agent. The initial position is fully invested in the 10Y bond, as we know from Figure 4. We have stratified the time-1 portfolios (of which there are 10 — one for each time-1 node in the tree) according to the long-rate level and report average portfolio compositions over the three highest and the three lowest levels. We see a considerable reaction in the portfolio composition. For instance, if long rates are low (remember: we have steady-state from which we start, so saying "low" does make logical sense), investments shift away from the

[12]Because there are cashflows multiple dates, we take "fixed-mix strategy" to mean the following: All pension payments must be invested according to a fixed mix (e.g. 60% in stocks, 40% in 1Y bonds), debt can't be called, and switching between fixed and floating rate loans is not allowed.

[13]With Table 5 in mind it seems plain wrong that even the completely risk-neutral agent does not have stocks in her portfolio. But that is because Table 5 gives comparative statistics for returns of buy-and-hold strategies, while this agents' pension portfolio has (annuity-like) cash-inflows over time. With the equity premium as small as it is in the estimation period, this may have an effect in-sample. The reason the risk-neutral, dynamically adjusting agent does use stock initially is that there is an element of predictability in bonds returns, so she can and will only alter her portfolio later.

10Y bond towards stock and short-term bonds. This makes perfect sense: in this case long rates will (more likely than not) go up leading to negative returns on especially long-term bonds.

4.2.1 The Devil's advocate, I: Numerical reliability

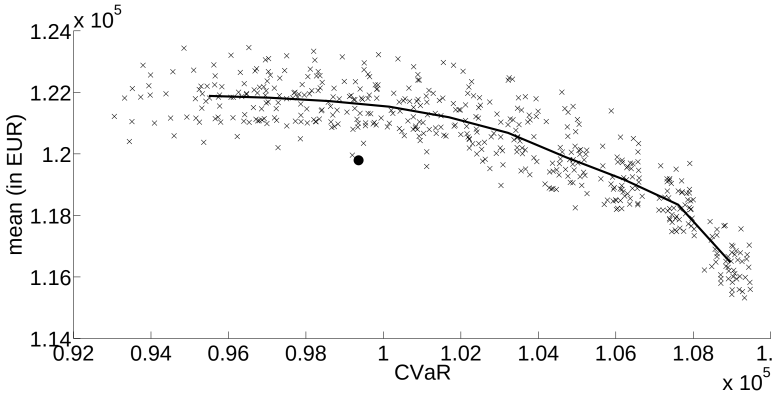

Figure 6: The x's show (CVaR,Mean)-optimal combinations for different simulated (10,10,10)-trees and different degrees of risk-aversion. The big dot is the optimal (CVaR,Mean)-point for a medium risk-averse agent in a (25,25,25)-tree.

Is the problem overly simplified? The results we report in the previous (and the following) sections are based on a simulated (10,10,10)-tree structure. Are we just seeing patterns where none exist or are results stable? As discussed in Consiglio et al. (2004), Consiglio et al. (2008) and Kaut et al. (2007) there is both an in-sample and an out-of-sample stability aspect of stability.

In-sample stability is easily investigated for our problem formulation. We simply simulate a number (50 in this case) of different (10,10,10)-trees, solve the optimization problem on each of these (for different risk-aversion levels) and compare the results. This is done in Figure 6, where the x's show optimal (CVaR,Mean)-combinations.[14] These fall neatly around the efficient frontier from the previous section (the fully drawn curve) with (estimated optimal) expected wealths not deviate more about 1%.

Out-of-sample stability means using the optimal strategy determined in one tree in another simulated (or otherwise constructed) tree. In the aforementioned references only one-period settings are considered, and it is not clear how to generalize the analysis to a multi-stage setting. However, what we have done is solve large instance, a (25,25,25)-tree, for a medium risk-averse agent. The estimated optimal (CVaR,Mean)-combination is shown (with the big dot) in Figure 6. As we would expect, optimization on the smaller trees lead to a positively biased estimate (the dot is under the curve) of what can be achieved in a more complicated model (represented by the bigger tree). But the bias is less than 2% in terms of expected wealth, an order of magnitude that we do not find alarming.

One might then swing the other way and ask: *Have you made the problem overly complicated?* In the next section we will show that both multi-period and joint (asset, liability) natures are important features to account for. But what about taxes and transaction costs? Are they what drive the results? Figure 7 shows that

[14]The x's do not fall on vertical lines because our risk-aversion is parametrized via in-sample CVaR.

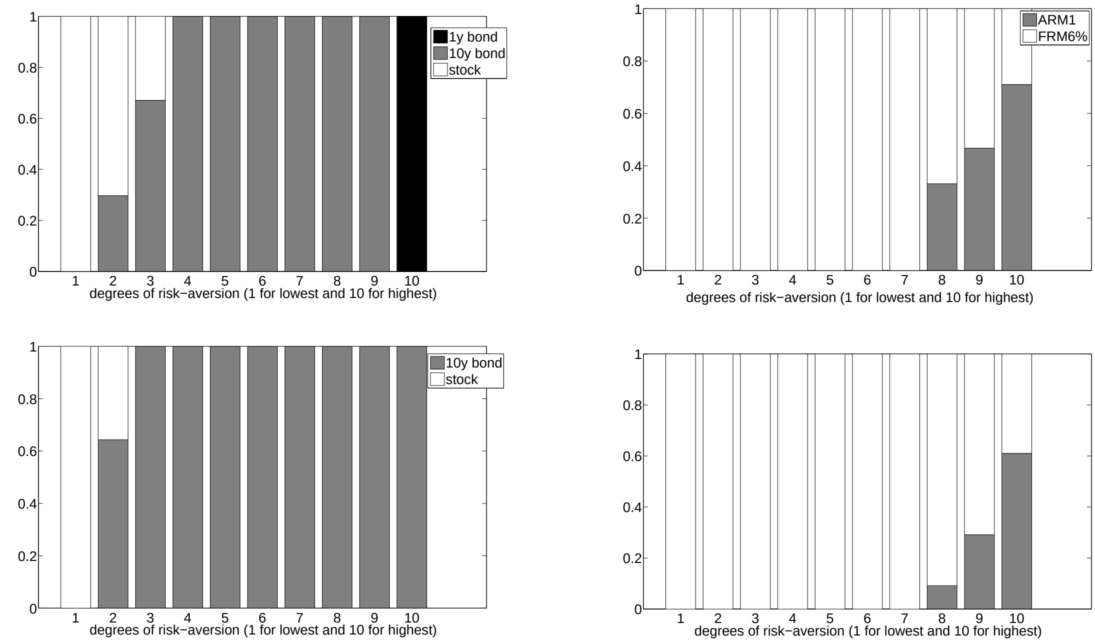

Figure 7: Optimal initial portfolios (assets on the left, liabilities/mortgage side on the right) without taxes (upper line) and without transaction costs (lower line).

they are not. The figure depicts optimal initial portfolio compositions in the case where taxes (upper graphs) or transaction costs (lower graphs) are set to 0. The differences in portfolio compositions are quite small — also when compared to the "full solution" in Figure 4. So why have taxes and transaction costs in the model formulation at all? Because we can. Or, bearing in mind the first author's affiliation: Because we have to.

4.3 The Devil's advocate, II: Economic significance and robustness

A comparison between the dynamic and the non-dynamic strategy in the previous section (see Figures 4 and 5 and Table 6) shows considerable variation in the composition of optimal portfolios. But are the differences actually economically significant? Figure 8 emphasizes that they are. Here we plot efficient frontiers: optimal combinations of expected net wealth and CVaR for varying degrees of risk-aversion. And we do that for both the dynamically adjusting agent and for the agent who is required to chose her investment and loan strategy at the initial date. The difference is clearly visible. And it is not just a graphical trick; at the same levels of CVaR, the benefit from the dynamic strategy in terms of expected wealth ranges from 0.6% to 1.3% per year (depending on risk-aversion).

In our historical sample the stocks performed badly as we included the financial crisis in the estimation of our stochastic process. While stock performed poorly during this period, the term structure of interest rates declined – with corresponding high returns for the long-maturity bonds. For other time intervals stocks may show a more attractive risk/return profile compared to bonds. Therefore, as sort of sensitivity analysis, here we increase the expected stock return by 2% per year. The initial mortgage loan and pension savings portfolios are shown in Figure 9. Unsurprisingly, stocks now comprise a bigger (though not completely dominant) part of

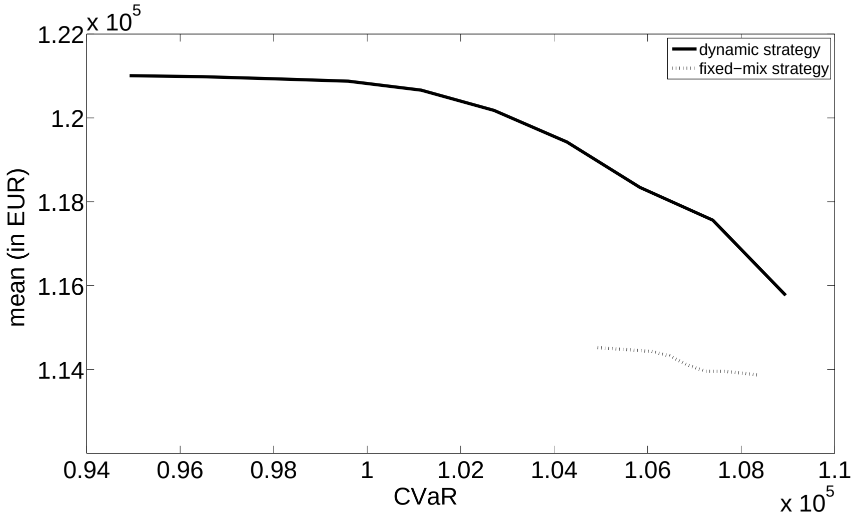

Figure 8: Efficient frontiers for a dynamic strategy and a fixed-mix strategy.

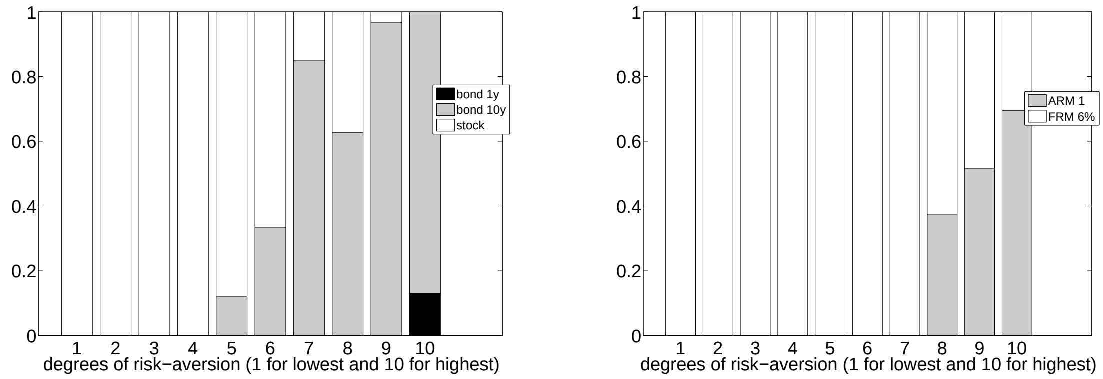

Figure 9: Optimal initial portfolios (across risk-aversion levels) when 2% is added to stock returns.

the pension savings portfolio, while the liability side is largely unchanged (compare the right panels of Figures 4 and 9).

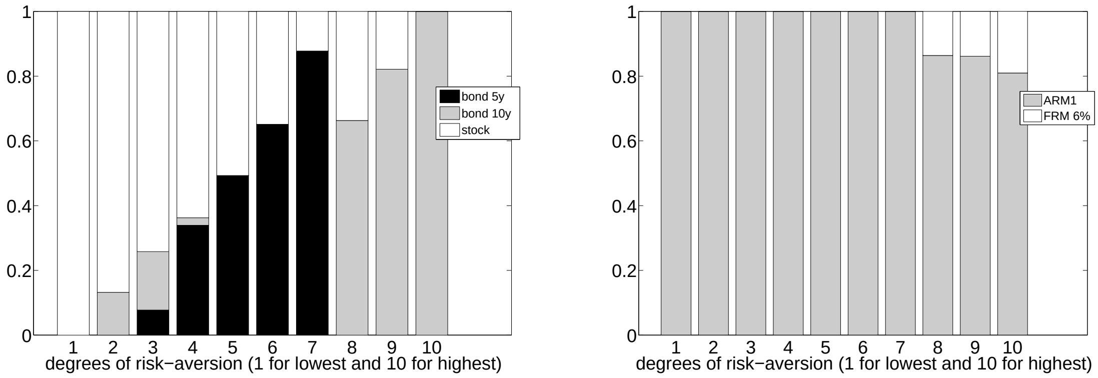

Figure 10: Optimal initial portfolios when 2% is added to stock returns but agents must fix their strategy at time 0.

Finally, we run the experiment with increased stock returns for the dynamically restricted agents. The liability side is again unchanged, but we now see stocks entering the asset portfolio for low degrees of risk-aversion. However, the optimal portfolio compositions appear a lot less robust across risk-aversion levels.

5 Conclusion

In this paper we formulated the mortgage choice and pension savings problem of a young household as the joint, multi-period asset liability management problem that it is. We focused on conditional value-at-risk as the measure of risk, and could then use stochastic linear programming to solve a number of instances of the problem. These instances were — under Danish conditions — realistic both empirically (stochastic models for the stock price and yield curve, mortgage bond pricing) and institutionally (cost structure, tax and pension laws). The numerical results were robust and clearly demonstrated the importance of considering both the joint and the dynamic nature of the problem. A typical difference in yearly expected wealth growth rates between households that, respectively, were and were not dynamically restricted in their portfolio choices was around 1%; a gain we definitely deem to be economically significant.

Finally, let us mention a few topics that we find to be highly relevant for future research. Firstly, introducing house prices a as new risk-factor. One could say “oh, but that’s just another thing to put on the asset side”. But the house would have to be treated quite differently than stock-investment and pensions; it is much less liquid (and quite indivisible) and one would conjecture its price to be quite interest rate sensitive (although one would not necessarily be right). A related extension would then be to make the size of the mortgage (i.e. how expensive a house you buy) endogenous; a choice variable. A second challenge lies in connecting this paper’s young household’s financial planning problem with that of the middle-age and old households, thus ending up with a full life-cycle analysis.

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A The full stochastic programming formulation

The following parameters and variables are used in the stochastic programming problem. (A "parameter" is something that can be specified/calculated/makes sense independently of any agent's optimization problem. A "variable" is something that is chosen by the agent — however trivially — as part of the optimization problem.)

<u>Parameters</u>

$minCVaR$: the minimal conditional value-at-risk the agent is willing to accept;

p_n: probability of being at node n;

d_t: discount factor for time t viewed from time 0;

r_{in}: coupon rate for loan i at node n,

PP_{in}: the net present value of prepayments from one unit of loan i at node n including any retirement of the debt at prevailing market prices;

r_{jtn}: annual return on asset instrument j at node n, time t;

K_{in}: price of prepaying loan i at node n. We have that $K_{in} = \min\{1, k_{in}\}$ for FRMs and $K_{in} = k_{in}$ for ARMs where k_{in} is the market price for the bond underlying loan i at node n;

I_t: the investor's gross income at time t;

IA: the cost of buying the house;

P_{in}: price of loan i at node n;

c: variable transaction cost for issuing a loan;

c_l: fixed transaction cost for issuing a loan;

c_f: fixed cost associated with the mortgage origination;

$admC$: administration fee rates given as a percentage of outstanding debt for bond i;

γ: tax reduction rate from interest payments;

tx_l: low-bracket tax rate;

tx_h: high-bracket tax rate;

ba_h: basic allowance in high-bracket tax;

α : confidence (quantile) level for VaR and CVaR ($\sim 5\%$ worst cases in our applications);

M: large constant (the big M);

$cmin$: minimum amount used for consumption;

tax_{pal}: tax rate on earnings from retirement investments;

β: fixed percentage of the household's income that is used for contribution to retirement savings;

rd_{tn}: interest rate on savings in the bank account at time t and node n.

Variables

y_{itn}: units sold of loan i at node n, time t;

x_{itn}: units bought of loan i at node n, time t;

$$Z_{itn}: \begin{cases} 1 & \text{if any amount of loan } i \text{ is issued/sold} \\ 0 & \text{otherwise.} \end{cases};$$

$cref_{tn}$: total costs of refinancing the loans at node n, time t;

$$LP_{tn}: \begin{cases} 1 & \text{if a loan is refinanced} \\ 0 & \text{otherwise.} \end{cases};$$

A_{itn}: principal payment of loan i at node n, time t;

BT_{tn}: total payments on all loans at node n, time t, including interest and principal payments and price of prepaying the loans at horizon;

κ_{jtn}: percentage of income used to invest in instrument j at node n, time t;

VP_n: value of retirement investments before tax;

VH_n: horizon value of retirement savings after tax;

W_n: total wealth at horizon at node n;

C_{tn}: amount needed for consumption at node n, time t;

$x1_{tn}$: part of gross income which is taxed with low-bracket tax rate;

$x2_{tn}$: part of gross income which is taxed with high-bracket tax rate;

OD_{itn}: outstanding mortgage debt of loan i at node n, time t;

IR^+_{tn}: interest payments at node n, time t;

IR^-_{tn}: interest earnings at node n, time t;

v^+_{tn}: part of interest earnings taxed as income;

v^-_{tn}: Part of interest payments deducted in tax;

CD_{tn}: deposit (or withdrawal) in the savings bank account at node n, time t;

CF_{tn}: balance of the savings bank account at node n, time t;

VaR: Value-at-risk;

$CVaR$: Conditional value-at-risk.

The liability (or mortgage) side

$$
\begin{aligned}
\sum_{i\in U} P_{i1}\cdot y_{i01} &\geq IA + c_l + \sum_{i\in U}(c\cdot y_{i01} + c_f\cdot Z_{i01}) && (18)\\
OD_{i01} - y_{i01} &= 0 \qquad \forall i\in U, \forall n\in\mathcal{N}_t && (19)\\
OD_{itn} &= OD_{i,t-1,a(n)} - A_{itn} + y_{itn} - x_{itn}\\
&\qquad \forall i\in U, \forall n\in\mathcal{N}_t, t=1,\ldots,T && (20)\\
A_{itn} &= OD_{i,t-1,a(n)}\cdot\left[\frac{r_{i,t-1,a(n)}}{1-(1+r_{i,t-1,a(n)})^{-T+t-1}} - r_{i,t-1,a(n)}\right]\\
&\qquad \forall i\in U, \forall n\in\mathcal{N}_t, t=1,\ldots,T && (21)\\
\sum_{i\in I} P_{in}\cdot y_{itn} &= cref_{tn} + \sum_{i\in I} K_{in}\cdot x_{itn} \quad \forall n\in\mathcal{N}_t, t=1,\ldots,T-1 && (22)\\
cref_{tn} &= c_l\cdot LP_{tn} + \sum_i (c_f\cdot Z_{itn} + c\cdot(y_{itn}+x_{itn}))\\
&\qquad \forall n\in\mathcal{N}_t, t=1,\ldots,T-1 && (23)\\
M\cdot Z_{itn} - y_{itn} &\geq 0 \qquad \forall i\in U, \forall n\in\mathcal{N}_t, t\in\mathcal{T} && (24)\\
LP_{tn} &\geq Z_{itn} \qquad \forall i\in U, n\in\mathcal{N}_t, t=1,\ldots,T && (25)\\
LP_{tn} &\leq \sum_i Z_{itn} \quad \forall n\in\mathcal{N}_t, t=1,\ldots,T && (26)\\
BT_{tn} &= IR^-_{tn} + \sum_{i\in U} A_{itn} \quad \forall n\in\mathcal{N}_t, t=1,\ldots,T && (27)\\
IR^-_{tn} &= \sum_{i\in U} OD_{i,t-1,a(n)}\cdot(r_{i,t-1,a(n)} + admC)\\
&\qquad \forall n\in\mathcal{N}_t, t=1,\ldots,T && (28)\\
x_{i01} &= 0 \qquad \forall i\in U && (29)\\
y_{iTn}, x_{iTn} &= 0 \quad \forall i\in U, n\in\mathcal{N}_T && (30)
\end{aligned}
$$

The asset (or pension) side

$$
\begin{aligned}
\sum_j \kappa_{jtn} &= 1 \qquad \forall n\in\mathcal{N}_t, t=1,\ldots,T && (31)\\
VP_{leaf} &= \sum_{j\in J}\ \sum_{\substack{n\in\mathcal{N}_t,\\ path(n,leaf)}}\ \sum_{t>1}^{T} \beta\cdot\kappa_{j,t-1,a(n)} I_{t-1}\\
&\qquad \prod_{\substack{\tau\geq t,\\ path(na,leaf)}} (1 + r_{j\tau na}\cdot(1-tax_{pal})) \quad \forall leaf\in\mathcal{N}_T && (32)\\
VH_n &= (1-tx_l)VP_n \qquad \forall n\in\mathcal{N}_t && (33)
\end{aligned}
$$

Taxes etc.

$$\begin{aligned}
(1-\beta)I_t &= x1_{tn} + x2_{tn} \qquad \forall n \in \mathcal{N}_t, \forall t = 1,\ldots,T && (34)\\
BT_{tn} + C_{tn} + CD_{tn} &= (1-tx_l)x1_{tn} + (1-tx_h)x2_{tn} + \gamma \cdot v^-_{tn} \\
&\quad -TaxvPlus_{tn} \quad \forall n \in \mathcal{N}_t, \forall t = 1,\ldots,T && (35)\\
C_{tn} &\geq cmin \qquad \forall n \in \mathcal{N}_t, t = 1,\ldots,T && (36)\\
CF_{tn} &= CF_{t-1,a(n)}(1+rd_{tn}) + CD_{tn} \\
&\quad \forall n \in \mathcal{N}_t, t = 1,\ldots,T && (37)\\
CF_{tn} &\geq 0 \qquad \forall n \in \mathcal{N}_t, \forall t = 1,\ldots,T && (38)\\
CF_{01} &= 0 && (39)\\
IR^+_{tn} &= CF_{t-1,a(n)} \cdot rd_{tn} \quad \forall n \in \mathcal{N}_t, t = 1,\ldots,T && (40)\\
v^+_{tn} &= xm1_{tn} + xm2_{tn} \quad \forall n \in \mathcal{N}_t, t = 1,\ldots,T && (41)\\
xm1_{tn} + x1_{tn} &\leq ba_h \quad \forall n \in \mathcal{N}_t, t = 1,\ldots,T && (42)\\
TaxvPlus_{tn} &= tx_l \cdot xm1_{tn} + tx_h \cdot xm2_{tn} \\
&\forall n \in \mathcal{N}_t, t = 1,\ldots,T && (43)\\
v^+_{tn} - v^-_{tn} &= IR^+_{tn} - IR^-_{tn} \quad \forall n \in \mathcal{N}_t, t = 1,\ldots,T && (44)
\end{aligned}$$