Can Home-Owners Benefit from Stochastic Programming Models? — A Study of Mortgage Choice in Denmark*

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Abstract

The Danish mortgage market is large and sophisticated. However, most Danish mortgage banks advise private home-owners based on simple, if sensible, rules of thumb. In recent years a number of papers (from Nielsen & Poulsen (2004) over Rasmussen & Zenios (2007) to Pedersen, Weissensteiner & Poulsen (2013)) have suggested a model-based, stochastic programming approach to mortgage choice. This paper gives an empirical comparison of performance over the period 2000-2010 of the rules of thumb to the model-based strategies. While the rules of thumb slightly outperform a passive benchmark on average and are less risky than pure adjustable rate loans, we find considerable gains from using the model-based strategies. Using a strategy that minimizes conditional-value-at-risk lowers average effective yearly interest rate over a 10-year horizon by 0.3-0.9%-points (depending on the borrower's level of conservatism) compared to the rules of thumb without increasing the risk. The answer to the question in the title is thus affirmative.

1 Introduction

One of the factors that triggered the global financial crisis that broke out in 2008 and still reverberates to this day, was the burst of the house price bubble in the US in 2007. Irrespective of what you think caused the bubble — predatory lending or predatory borrowing; whether it was a "bubble" at all, or a rational phenomenon; Shiller (2009) lists some unlearned lessons, primarily "land isn't scarce" — the crisis does show the need for better risk-management in the mortgage market. And where better to start than with the advice that banks give to individual borrowers regarding their choice of mortgage loan and refinancing?

For that, the Danish mortgage market makes an ideal testing ground. The market has a large participation rate (about 60% home-ownership, with many households actively monitoring and managing their debt), it is transparent (the bonds underlying the loans are traded at the Copenhagen Stock Exchange; the outstanding volume of mortgage bonds is about 1.3 times the Danish GDP), it is efficient (individual home-owners borrow at rates that are only about 1% above rates on government bonds), and it is sophisticated (a variety of products to choose from).

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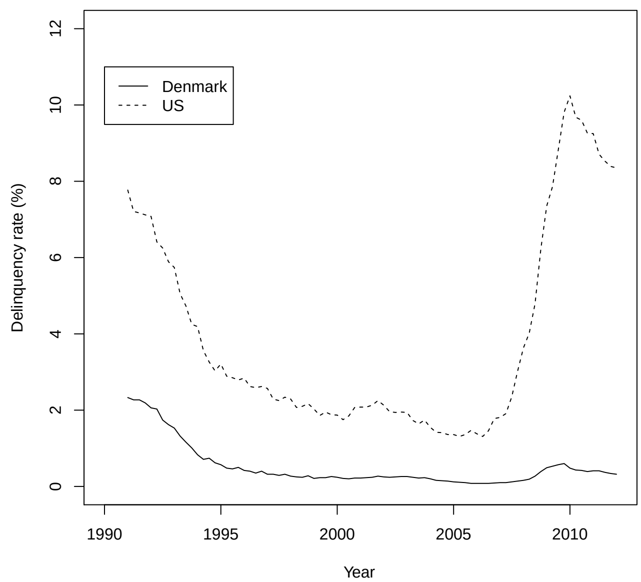

Figure 1: Delinquency rates for the period 1991 to 2012 in the USA and Denmark. The graphs are based on delinquency data from the homepages of Realkreditrådet for Denmark, and the Federal Reserve Board for the USA.

The Danish housing market experienced a similar bubble and burst to the US. However, all Danish mortgage banks survived the crisis, some with help from the government. A fundamental economic reason for this is that the extent of foreclosures in Denmark was not nearly as high as in the US, as shown in Figure 1. There are several institutional and regulatory differences between the Danish and the US mortgage systems that explain the difference. First, Danish borrowers are personally liable to repay their debt, regardless of the value of the collateral, so Danish banks need not worry about the so-called ”jingle mail”, where the home-owners mail there keys to the bank and walk away, as tends to happen in the US with so-called non-recourse loans. Second, despite introduction of several new mortgage products in Denmark in the period 1997 to 2004, some of the most risky loan products, such as balloon loans (with no initial payments for the first few years) were not introduced. (Interest-only loans were introduced; when interest rates are low, these are arguably ”same ballpark, different seats” in comparison to balloon loans. These loans may have exacerbated the rise in the house prices, but that discussion is somewhat beyond the scope of this paper.) Third, the Danish system offers complete transparency due to the practice of the principle of balance, which is a cornerstone of the Danish mortgage system, as practiced and enforced by the Mortgage Loan Act since the introduction of the Danish mortgage system in 1797. The principle of balance is no longer enforced by law due to the covered bond legislation of 2004 imposed by the EU. However, also after the 2004 changes, Danish mortgage banks use the principle of balance for practically all mortgage loans.

The principle of balance says that there has to be an exact match between the funding side and the loan side at any given time. This is why the Danish mortgage system is also characterized as a match funding or a pass-through system. The few specialized mortgage banks in Denmark are intermediaries in between the investors in the one side and the borrowers in

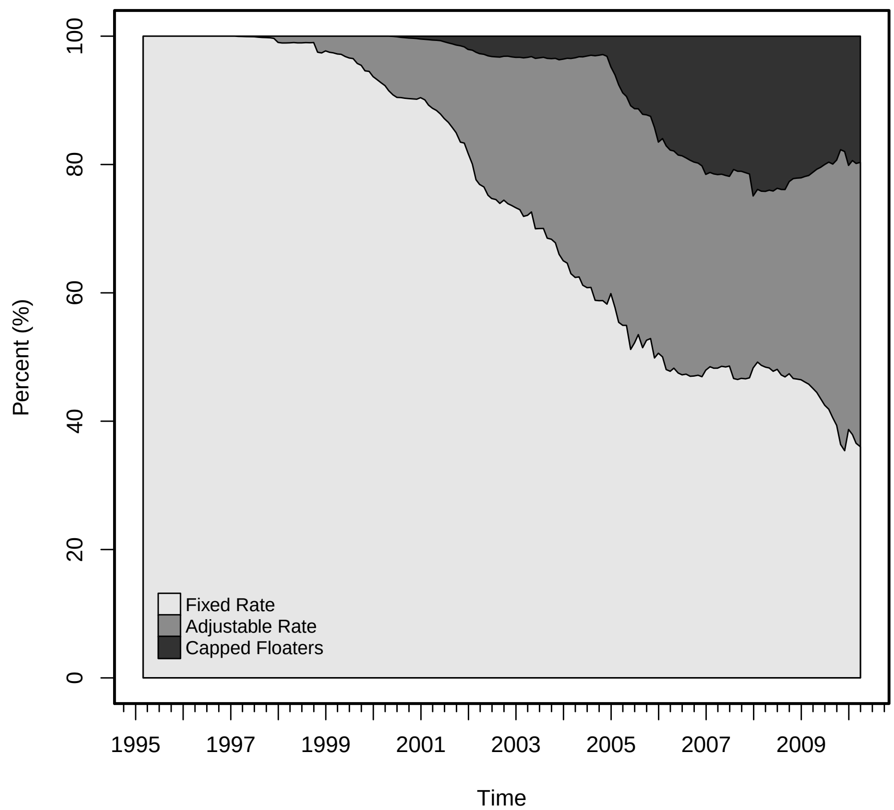

Figure 2: Development of the distribution of the underlying bonds used as funding instruments behind the Danish mortgage loans in the period 1995 to 2010.

the other side. Danish borrowers essentially issue mortgage bonds via the mortgage banks, and the bonds are sold to investors. The borrowers pay interest and principle payments on the issued bond plus a markup of around 0.5-0.7% as a risk premium and administration fee to the mortgage bank. The mortgage bank assumes only the default risk on the side of the borrowers, i.e. there is no asset/liability mismatch caused by short term funding of long term borrowing. With a maximum loan-to-value level of 80% for private houses, the value of the collateral should fall by more than 20%, before the mortgage banks encounter any losses in cases of foreclosures.

The Danish mortgage market offered borrowers only (callable) fixed rate mortgages during the first 200 years of its history. Several new mortgage loan products have been introduced since 1997, Figure 2 shows the distribution of mortgage loans in Denmark since 1995. The market is dominated by the adjustable rate mortgages (mostly with annual adjustments) and the traditional long term fixed rate callable mortgages.

In short, Denmark has a robust and well-functioning mortgage market, see Frankel, Gyntelberg, Kjeldsen & Persson (2004), Svenstrup & Willemann (2006), and — for a non-Danish angle — Campbell (2012)) for further description and discussion. But there is room for improvement, in particular when it comes to advising home-owners on their choice of mortgage and the following refinancing opportunities. The advisory practice has not followed the product development since the mid-90'ies. There is no consistent practice when it comes to risk management (on the borrower side) of the very popular short term adjustable rate mortgages. The post-crisis low interest rate regime makes development of such methods an urgent matter, since the new loans issued in Denmark dominated adjustable rates with annual adjustments. If interest rates increase sharply, or even moderately, within the next few years, there is a serious risk that the foreclosure rates in Denmark increase sharply as well, contributing to another wave of house price declines and other adverse effects on the whole economy.

This paper focusses on improving advisory standards for mortgage choice and refinancing for the Danish market. In particular the focus is on improving and extending the rules of thumbs used for refinancing. We do that by introducing and testing an optimization- and model-based framework for mortgage choice. Applying that framework we document that the model-based strategies offer a clear improvement over the rules of thumb. The rest of this paper is organized in the following way. Section 2 explains the existing rules of thumb for refinancing in Denmark. Section 3 introduces the stochastic modelling framework for individual mortgage choice and refinancing. Section 4 compares the performance of the model-based mortgage choice to the rules of thumb. Finally, section 5 concludes.

2 Refinancing rules of thumb

To formulate a quantitative benchmark, we interviewed a number of highly placed analysts from Danish mortgage banks; analysts that advise home owners on their choice of mortgage loan and refinancing. This is summarized in the following rules of thumb for refinancing of mortgage loans.

Refinancing from a fixed rate loan to another fixed rate loan with lower coupon-rate:

- The coupon-rate of the new loan should be 2% lower than that of the existing loan.
- The quarterly payments of the new loan should be at least 5% lower than those from the existing loan.
- The outstanding debt of the existing loan should be more than DKK 500,000 ($\sim$ EUR 67,000).
- The maturity of the existing loan should be greater than 10 years.
- The market price of bond underlying the new loan should be above 95.

This is the "classical" type of refinancing. Interest rates have fallen and the borrower uses the embedded Bermudan-type call feature to refinance to a lower rate. The first two bullet points aim to ensure that the option being exercised is sufficiently in-the-money. The 3rd and 4th bullet points reflect that there must be "sufficient mass" to warrant a refinancing. The final bullet point means that the call-feature of the new loan also has significant value; that the loan is not effectively non-callable, fixed rate. (Bullet points 3-5 remedy some dubious practices during the first waves of refinancing in the early 90'ies.)

Refinancing from a fixed rate loan to another fixed rate loan with higher coupon-rate:

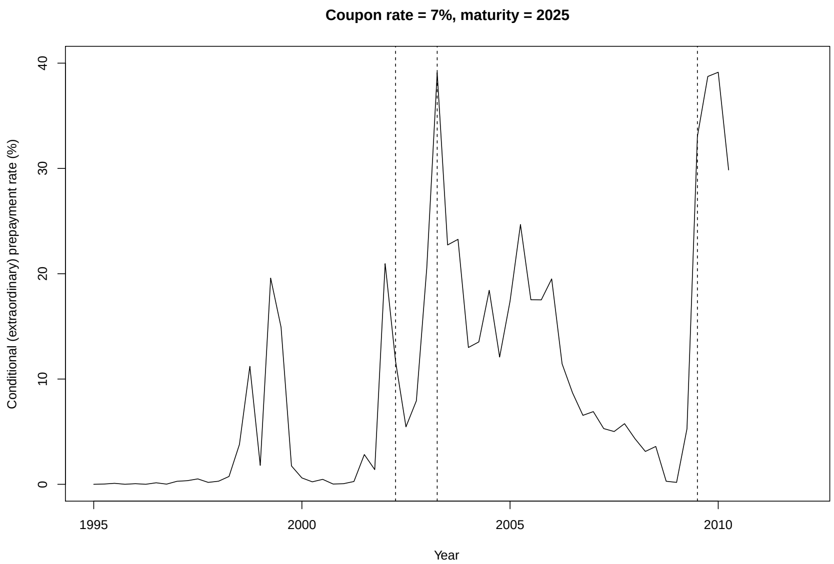

Figure 3: The graph shows the percentage of remaining outstanding debt for the coupon-7%, maturity-2025 bonds issue that was extraordinarily repaid (i.e. called) at different dates 1995-2012. The vertical lines indicate dates where the rules of thumb dictated the debt to be called.

- At least 10% reduction in outstanding nominal debt.
- The outstanding nominal debt of the existing loan should be more than DKK 500,000.
- The maturity of the existing loan should be greater than 10 years.
- The market price of bond underlying the new loan should be above 98.

This kind of refinancing (to a higher coupon-rate) is possible Denmark, since the borrower has the right to redeem the mortgage loan by buying back the underlying bonds at the prevailing market price. This is referred to as the "buy back" or "delivery option", even though this is not really an option in a financial sense, but rather the access to the features of the underlying bond. This type of debt restructuring can have a speculative purpose (the borrower hopes gain from future declines in interest rates) or a risk-management purpose (he limits future increases in the market value of his debt; he avoids lock-in effects), and is partially tax-aided (a larger percentage of his quarterly payments are labeled as "interest payment" and thus tax-deductable).

Figure 3 shows that historically observed refinancing activity have been high at around the dates where the rules of thumb suggest that borrowers should refinance. The graph indicates that the refinancing rules of thumb (at least the one for refinancing from to a lower coupon) are indeed known to borrowers and that they act upon them — albeit with some noise.

Refinancing from fixed rated to adjustable rate:
This is normally done as a means of reducing the payments or as an alternative to refinancing to higher coupon-rates. There are, however, no rules of thumb governing this type of refinancing. The advisors are reluctant to advise borrowers to refinance to adjustable rate mortgages. This despite the "fixed or floating?"-question entailing the fundamental risk/reward trade-off in interest rate modelling: short-term interest rates are typically lower than long-term ones,

but with short term-financing, the home-owner doesn't know his future payments. Regardless of the lack of advice, more than half of all mortgage loans in Denmark are adjustable rate mortgages with annual adjustments — commonly (and subsequently in this paper) called F1 loans.

3 Optimization Model

The main purpose of this paper is to test the performance of the refinancing rules of thumb described in the last section. In order to do that a few alternatives to the rules of thumbs are considered. One obvious alternative is to do nothing at all, a so called issue-and-hold strategy, where the borrower just chooses a loan to begin with and stays with the loan for the whole period until the loan is redeemed due to other reasons than refinancing, for example selling the house.

Another, very unrealistic, alternative is to refinance at all the right times, so that the maximum refinancing potential is gained. Obviously this alternative is very unlikely to be realized in real life, since it requires full foresight about future interest rate movements. Nevertheless this is a useful benchmark which provides an upper bound for the refinancing potential. If this potential is not much more than what is gained by following the rules of thumb or by not refinancing at all, there will be then no reason to try to develop more efficient methods for refinancing.

In this section therefore, a deterministic optimization model with full foresight is developed first. This model is referred to as the crystal ball benchmark. After that, the full foresight assumption is relaxed and a stochastic model is developed to generate realistic ex-ante refinancing strategies.

3.1 The crystal ball benchmark model

The crystal ball benchmark model described in this section finds the optimal refinancing strategy in the presence of full information about the future. Historical data on bonds are used to find the strategy that minimizes the borrower's total costs over a given period. The reason why this is not a trivial problem is the costs associated with refinancing. In particular, the fixed transaction costs (over EUR 1,000 per refinancing) makes this into a combinatorial optimization problem. (Another, "low practical", reason for looking at the crystal ball model is that it is helpful in detecting errors in program code.) The model is inspired by the formulations given in Nielsen & Poulsen (2004), Rasmussen & Clausen (2004) and Rasmussen & Zenios (2007).

The model is formulated on a time line with points t, where refinancing is allowed (in practice every quarter). Another fundamental quantity is the borrower's time horizon, the last point on the time line, τ. The horizon shouldn't be too short (people don't change their housing market participation in a major way every year), but planning with a 30 year horizon may leave you very vulnerable to risks outside the model (such as changes in job or family status). As a pragmatic compromise, we use τ =10 years.

Input data:
At any given time point t, the following information on any loan i is given:

r_{ti}, coupon rate for loan i at time t,

P_{ti}, origination price for loan i at time t,

K_{ti}, redemption price for loan i at time t. The distinction between the origination price P_{ti} and the redemption price K_{ti} is due to the call option embedded in the Danish mortgage bonds. The following relation holds between the two prices: $K_{ti} = \min\{1, P_{ti}\}$ for the fixed rate callable bonds and $K_{ti} = P_{ti}$ for the adjustable non-callable bonds,

γ_t, tax reduction rate, given as a percentage of negative capital income,

c_a, administration costs, given as a percentage of the loans outstanding debt,

c, variable transaction cost, given as a percentage of the loans outstanding debt,

c_f, fixed transaction costs associated with a refinancing,

M, the big M constant, used when a binary variable is set to 1 in the case of refinancing, ensuring that the fixed costs are incurred.

Variables:

z_{ti}, outstanding debt for loan i at time t,

y_{ti}, amount of originated bonds for funding loan i at time t (required non-negative),

x_{ti}, amount of redeemed bonds for repaying loan i at time t (required non-negative),

$Z_{ti} = \begin{cases} 1, & \text{if loan } i \text{ should be originated or redeemed at time } t, \\ 0 & \text{otherwise,} \end{cases}$

A_{ti}, principal payments for loan i at time t,

F_t, total loan payment at time t,

R_τ, the market value of outstanding debt at horizon τ (i.e. the sum of outstanding debts times bond prices).

Objective function:

The objective is to minimize the present value of the total loan costs in the given analysis period:

$$\min \quad \Big[\sum_{t=1}^{\tau} d_t F_t\Big] + d_\tau R_\tau, \tag{1}$$

where d_t is a discount factor. This could stem from the initial yield curve, it could be based on an agent-specific discount rate, or it could — for legal reasons — be identically 1. In practice we use $d(t) = \exp(-0.02t)$, but results are insensitive to this. From a formal programming point of view, the minimization is performed over all Variables. However, as we will see in the following some of these are very strongly linked, so essentially the decision variables are the x and y, the amounts of debt originated and redeemed. (The transaction costs means that we must keep buying and selling separate.) Notice the liquidation cost term in the objective function, $d_\tau R_\tau$; this appears because the bonds used have longer maturities than the planning horizon. This reflects that if the borrower issues long term, low coupon (i.e. almost non-callable) debt, he runs the risk of having to realize a capital loss.

Initialization:

To begin with enough bonds must be originated to raise the cash, V_0, needed by the borrower to purchase his house (this amount is exogenous to the model) as well as pay the costs associated with the origination:

$$\sum_{i \in U} P_{0i} y_{0i} \geq V_0 + \sum_{i \in U} \Big(c y_{0i} + c_f Z_{0i}\Big), \tag{2}$$

We then set the outstanding debt equal to originated amount of bonds at the beginning, $z_{0i} = y_{0i}$ for all $i \in U$.

Balance equations:
Equation (3) is a balance equation, that sets the outstanding debt for loan i at time t equal to the outstanding debt from time $t-1$ less the principal payment in case of no refinancing. If a refinancing takes place the equation initializes the outstanding debt in the new loan, and deducts the redeemed amount from the old loan:

$$z_{ti} = z_{t-1,i} - A_{ti} - x_{ti} + y_{ti}, \qquad \text{for all } i \in U, t = 1, \ldots, \tau. \tag{3}$$

Equation (4) ensures that the redeemed amount and the costs of refinancing come from originating new loan:

$$\sum_{i \in U} (P_{ti} y_{ti}) = \sum_{i \in U} \big(K_{ti} x_{ti} + c(y_{ti} + x_{ti}) + c_f Z_{ti} \big), \qquad t = 1, \ldots, \tau - 1. \tag{4}$$

Annuity loan payments:
Repayments of principal are defined in equation (5):

$$A_{ti} = z_{t-1,i} \Big[\frac{r_{t-1,i}(1 + r_{t-1,i})^{-T+t-1}}{1 - (1 + r_{t-1,i})^{-T+t-1}} \Big], \quad \text{for all } i \in U, t = 1, \ldots, \tau. \tag{5}$$

The total loan payment at time t is then defined in equation (6) as the sum of principal payments as well as interest payments and administration cost. The last two payments deducted by the amount which is given as a tax rebate. The total repayment cost at the end of the analysis period is given in equation (7).

$$F_t = \sum_{i \in U} \big(A_{ti} + (1 - \gamma)(r_{t-1,i} + c_a) z_{t-1,i} \big), \qquad t = 1, \ldots, \tau. \tag{6}$$

$$R_\tau = \sum_{i \in U} (z_{\tau i} K_{\tau i}) \tag{7}$$

Fixed costs:
The following equation ensures that the binary variable Z_{ti} is set to 1 when there is refinancing. If $Z_{ti} = 1$ then the fixed costs are counted as part of the total cost of refinancing.

$$M Z_{ti} - y_{ti} \geq 0, \qquad \text{for all } i \in U, t = 0, \ldots, \tau. \tag{8}$$

The optimization problem is solved using CPLEX and GAMS (version 23.7, specifically) on a standard pc. But before we do numerical comparisons (in section 4), we look at a more realistic model for personal mortgage choice.

3.2 Model-based refinancing strategies

The optimization model which is used in this section is a variation of the model introduced in Rasmussen & Zenios (2007). If the results of any model, in this case refinancing strategies, are to be used in real world, its creators need to substantiate

Factor	$\overline{\beta}$	$K_{i,i}$	standard deviation/correlation		
			Level	Slope	Curvature
Level	0.064 [0.012]	0.995 [$< 10^{-4}$]	0.002	-0.07	-0.08
Slope	0.027 [0.008]	0.987 [$< 10^{-4}$]		0.0028	0.30
Curvature	0.004 [0.004]	0.954 [$\sim 10^{-4}$]			0.0042

Table 1: Estimates for the auto-regressive model (9) used on weekly Danish data 1995-2010. The first numeric column gives the average factor-values over the sample period. The numbers in square brackets are the standard errors of the estimators.

that the strategies will indeed work better than the existing practice, in this case the refinancing rules of thumb. It is not justifiable to advise borrowers to follow a new set of refinancing strategies, before these strategies have been tested in a simulated framework, and before enough insight is gained to justify trying the strategies in the real world. This paper describes the simulation framework for testing the model-based refinancing strategies and discusses the insights gained.

3.2.1 The stochastic interest rate model

As driving stochastic model for interest rates, a Gaussian three-factor model is parameterized in terms of the three factors level, slope and curvature of the zero-coupon yield curve — collected in the vector $\beta = (\beta_1, \beta_2, \beta_3)$. The dynamic model for the factors is a first-order vector auto-regression, i.e.

$$\beta(t) = c + K\beta(t-1) + \epsilon(t), \tag{9}$$

where K is a 3×3 diagonal matrix and the $\epsilon(t)$'s are independent and $N_3(0, \Sigma)$-distributed. The fat curves in Figure 4 show the behavior of four different points on the zero-coupon yield curve 1995-2010, and Table 1 gives estimates of the model parameters.

The model then links the factor-values to the full yield curve via a Nelson-Siegel-type equation. Specifically with $y(t, \tau)$ denoting the time-t yield of a zero-coupon bond with time-to-maturity τ, we assume

$$y(t, \tau) = \beta_1(t) - \beta_2(t) \left(\frac{1 - e^{-\lambda\tau}}{\tau\lambda} \right) + \beta_3(t) \left(\frac{1 - e^{-\lambda\tau}}{\tau\lambda} - e^{-\lambda\tau} \right). \tag{10}$$

Models of this type fit the data quite well, are simple to work with, and have recently had a revival in the literature, e.g. Ferstl & Weissensteiner (2011) and Christensen, Diebold & Rudebusch (2011).

Simulating scenarios from such a model is straightforward; the clouds of grey curves in Figure 4 show examples. In our experimental set-up (which we will explain in detail later; for simulations we use the parameters in Table 1), there aren't any arbitrage-problems in the simulated model. There are 200 scenarios and the number of assets is less than a tenth of that. We test for non-existence of arbitrage opportunities after the scenarios are generated.

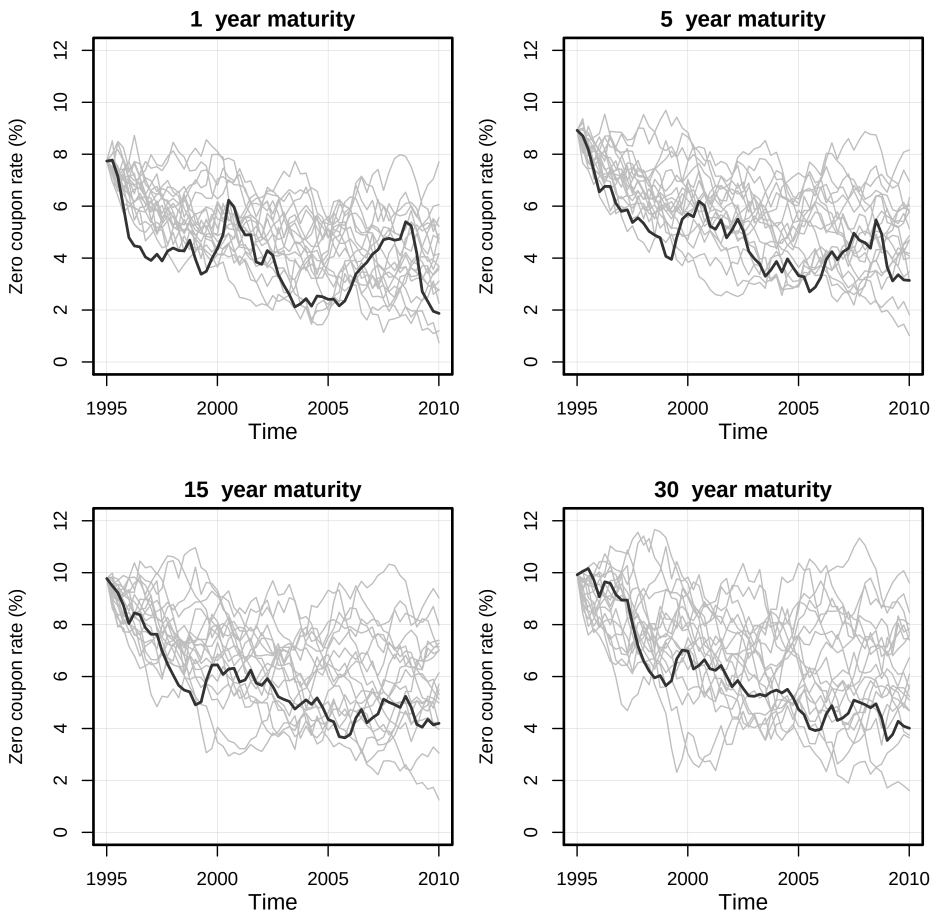

Figure 4: The fat curves show the behavior of Danish zero-coupon yields for four different maturities 1995-2010 (one in each panel). The grey clouds consist of simulated paths (started at the 1995 observed data) from the model auto-regressive model in equation (9) with the parameter-values given in Table 1

For each simulated zero-coupon yield curve, we need prices of callable mortgage bonds. We calculate these with a required gains model, *gevinstkravsmodel* in Danish, as suggested in e.g. Schwartz & Torous (1989) and the standard Danish reference Jakobsen (1992); our implementation is as Madsen (2000).

We do not include hybrid products (such as capped loans), as the markets for these is quite illiquid and prices rarely competitive.

3.2.2 The idea behind the optimization model

The optimal refinancing model resembles, in its overall structure, the crystal ball model. There is one main difference though: No information about future is given, instead a number of scenarios are generated.

The optimization model is equipped with the following information:

1. One loan or a portfolio of loans at the beginning of the period.

2. A number of alternative loans (and the corresponding bond prices) that the borrower may refinance the existing loan with.
3. Information about costs related to refinancing.
4. Information about bond prices in the 200 simulated future scenarios.

The model tries to find the optimal (loan) portfolio composition with respect to some criterion function that measures the trade-off between risk (scenarios with high payments) and reward (low average payments). Conceptual differences to the rule of thumb are:

- Time horizon is explicitly taken into account.
- Risk (as well as reward) is explicitly measurable and managed.
- Mixing of loans is allowed.
- The strategies are personal - borrower specific-input, e.g. a risk-aversion level, is taken into account.

The simulation framework is made of two sets of scenarios. The first set of scenarios are to be considered as alternative historical data. These are coherent alterations to historical data for a given period — 2000 to 2010 for this paper; the paths in Figure 4 are such scenarios. We analyze 250 paths each with a length of 10 years. The reason for simulating historical data is to avoid only performing back tests on one single period. So alternative likely "historical" data are generated for interest rates and bond prices.

The second set of scenarios are those used for the optimization model. They are generated along each of the 250 "historical" scenarios in steps of a quarter of a year, allowing the optimization model to come up with refinancing suggestions along the way. There are 200 of these scenarios at each step and they all have a horizon at 2010. Figure 5 demonstrates the analysis setup by depicting only one of the "historical" scenarios and three of the optimization scenario sets for three randomly picked quarters.

3.2.3 Measuring risk

The risk of refinancing is modeled here as the average of the 5% highest total period costs for any given optimization scenario set. This risk measure is known as conditional-value-at-risk (CVaR) (or as expected shortfall, which we actually find more descriptive), and is illustrated in Figure 6. CVaR is a coherent risk measure (see Artzner et al. (1999)) and in an optimization setting can be formulated linearly (see Rockafellar & Uryasev (2000)). Both properties are violated by other well-known risk measures such as value-at-risk and variance. Besides, CVaR measures the tail risk - the idea is that if total costs after a given refinancing are lower than total costs of staying in the existing loan for the tail of the scenario set, the refinancing may be considered as robust - beneficial almost no matter what happens. That is obviously a function of how reasonable the scenarios are built — a topic that will be studied in section 4.

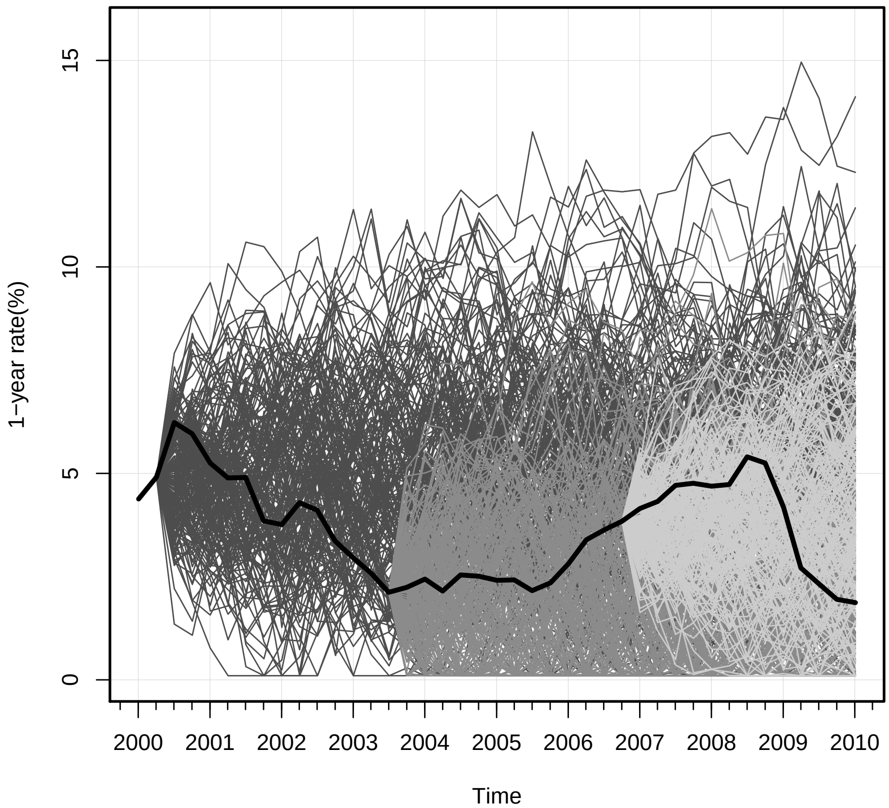

Figure 5: In each quarter there is a scenario tree of interest rates and bond prices. Three examples (out of 40) of such scenario trees are shown here for a simulation over the 10 year rate for the period 1995- 2005. The graph is meant to illustrate the point with the simulation optimization framework only.

3.3 The mathematical formulation of the optimization model

The model for generating model-based refinancing strategies is described in the following. It is a single period (two-stage) stochastic program where the scenario structure is defined on a starting point t_0, and a set of scenarios s. At any given starting point t_0 (start of a new quarter) there are a set of loans i. The model is given an existing loan that it might either maintain or refinance to another loan.

Scenario data:

λ, risk weight, has values between 0 and 1. For $\lambda = 0$, the risk neutral model, the average of total costs is minimized. For $\lambda = 1$ the CVaR of period costs is minimized.

α, is the confidence level for CVaR calculations - a 95% confidence level is used in this paper,

$P_{t_0,i}$, origination price for loan i at time t_0,

$K_{t_0,i}$, redemption price for loan i at time t_0,

$I_{t_0,i}$, outstanding debt for the existing loan i at time t_0.

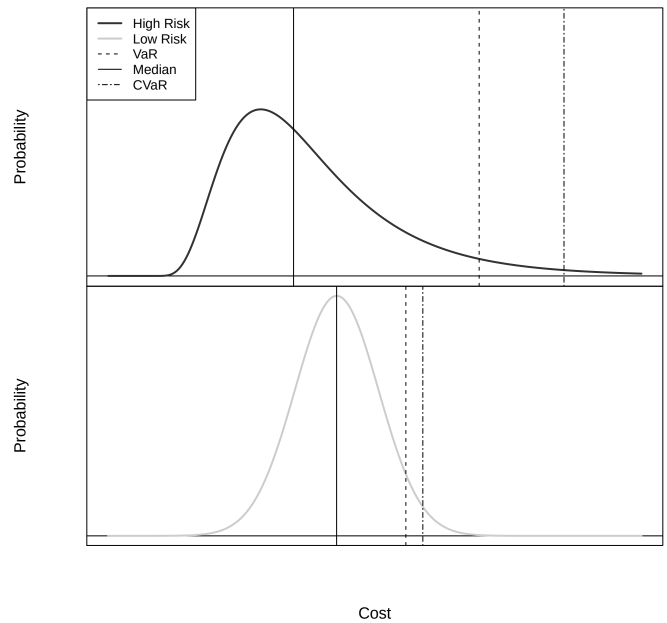

Figure 6: An illustration of the two risk measures CVaR and VaR. VaR is the biggest loss that can happen with a given probability. CVaR expresses the average loss for the scenarios where a loss higher than VaR is materialized.

p^s, probability of scenario s,
O_i^s, total period costs for loan i at scenario s, for an initial outstanding debt of 1,
c, variable transaction costs as a percentage of the loans outstanding debt,
c_f, fixed transaction cost in connection with a refinancing,
M, The Big M constant.

Variables:
$z_{t_0,i}$, outstanding debt for loan i at the starting point t_0,
$y_{t_0,i}$, amount of originated bonds for funding loan i at the starting point t_0 (required non-negative),
$x_{t_0,i}$, amount of redeemed bonds for repaying loan i at the starting point t_0 (required non-negative),

$$Z_{t_0,i} = \begin{cases} 1, & \text{if loan } i \text{ should be issued at the starting point } t_0, \\ 0 & \text{otherwise,} \end{cases}$$

ξ, Value-at-Risk (VaR) for a $100\alpha\%$ confidence level,
CVaR$(y;\alpha)$, Conditional Value-at-Risk for a portfolio of loans $y = (y_{t_0,i})$ for a $100\alpha\%$ confidence level,
ξ_+^s, the amount of the total costs on top of ξ for a scenario s. If the total costs do not exceed ξ, the value of ξ_+^s will be zero.

Objective function:
The objective of the optimization model is to minimize the average period cost of the

loan for a predetermined risk level, defined as a weight on CVaR:

$$\min \quad (1-\lambda)\Big[\sum_i \sum_s p^s y_{t_0,i} O_i^s\Big] + \lambda \text{CVaR}(y;\alpha) \tag{11}$$

Refinancing equation:
Equation (12) and (13) together ensure that refinancing can take place at time t_0:

$$z_{t_0,i} = I_{t_0,i} - x_{t_0,i} + y_{t_0,i}, \qquad \text{for all } i \in U. \tag{12}$$

$$\sum_{i\in U}(P_{t_0,i} y_{t_0,i}) = \sum_{i\in U}\big(K_{t_0,i} x_{t_0,i} + c(y_{t_0,i} + x_{t_0,i}) + c_f Z_{t_0,i}\big). \tag{13}$$

Equation (12) sets the outstanding debt for loan i equal to the amount of existing debt in case of no refinancing. If there is a refinancing the existing loan must be repaid. In this case equation (13) ensures that the repayment is funded by originating a new loan. The outstanding debt for the new loan is registered then in equation (12).

Fixed costs:
Requiring $MZ_{t_0,i} - y_{t_0,i} \geq 0$ for all $i \in U$ ensures that the binary variable $Z_{t_0,i}$ is set to 1, if there is a refinancing. If $Z_{t_0,i} = 1$ then the fixed costs are entered as part of the total costs at the refinancing.

CVaR constraints:
The two constraints 14 and 15 combined define CVaR for period costs. The linear formulation of CVaR in a CVaR minimizing context is due to Rockafellar & Uryasev (2000).

$$\xi_+^s \geq \big[\sum_i y_{t_0,i} O_i^s\big] - \xi, \qquad \text{for all } s \in S. \tag{14}$$

$$\text{CVaR}(y;\alpha) = \xi + \frac{\sum_s p^s \xi_+^s}{1-\alpha} \tag{15}$$

We solve the stochastic programming problem above using, again, CPLEX and GAMS. (Technically, GAMS keeps track of the algebraic structure of the model and then it calls the state-of-the-art linear optimizer CPLEX to solve the optimization problem.)

With the modelling framework now in place, the next section delves into numerical results and comparisons.

4 Results

In this section, we analyze the results from applying the rules of thumb, the crystal ball benchmark, and various model-based strategies on 250 simulated scenarios (including the actual historical data as one of the scenarios) for the period 2000-2010. We consider borrowers with a loan with an initial value of DKK 2,000,000. (approximately EUR

270,000).

First, strategies are compared for the actual historical data. Figure 7 shows (in its three top graphs) the development in outstanding debt, loan payments and the period's effective loan costs in percent (PEL) for, respectively, the CVaR-minimizing borrower, the rules of thumb-borrower, and borrower with a crystal ball (but without access to adjustable rate F1 loan — in order not to mess up the scaling on the x-axis.).

We first see that with effective loan costs (PEL) of 4.8% per year, the CVaR-minimizing strategy outperforms the rules of thumb, that has a PEL of 5.4%. We also see that there is — unsurprisingly, but nice as a "sanity check" — a considerable advantage to having perfect foresight; the PEL for the crystal ball strategy is 3.9%. The bottom graph in Figure 7 shows the optimal loan portfolio composition for the CVaR-minimizing borrower, who refinances much more often (6 times in 10 years) than the rules of thumb (one time in 10 years). We also see that the CVaR-minimizing borrower makes frequent use of adjustable rate loans (his catchy formulation could be "I'm risk-averse, I'm not stupid"), mixes loan types, and refinances when the interest rates increase in order to reduce outstanding debt. So, qualitatively, the rules of thumb are too passive time-wise, but too aggressive product-wise. Results (not shown, but can be found in Rasmussen, Madsen & Poulsen (2011)) are similar when the period 1995-2005 is analyzed (PEL's of 5.8%, 4.8%, and 4.4% for the three strategies/borrower types).

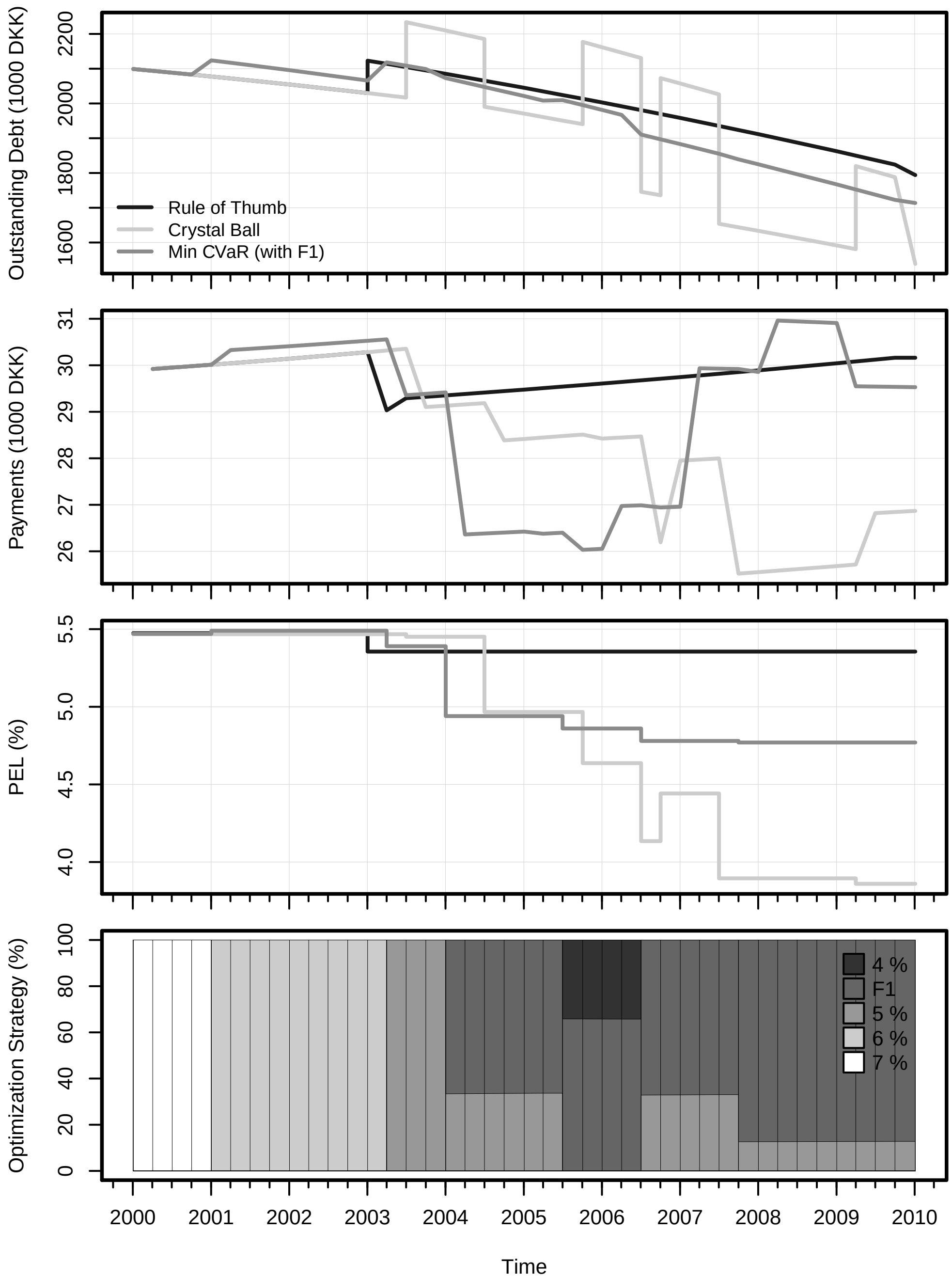

Figure 7: The development in outstanding debt, payments, and the period costs of the loan in percent are given here for the historical data from the period 2000-2010. The performance of the rules of thumb, the model-based strategy for the CVaR-minimizing borrower, and the crystal ball benchmark are compared.

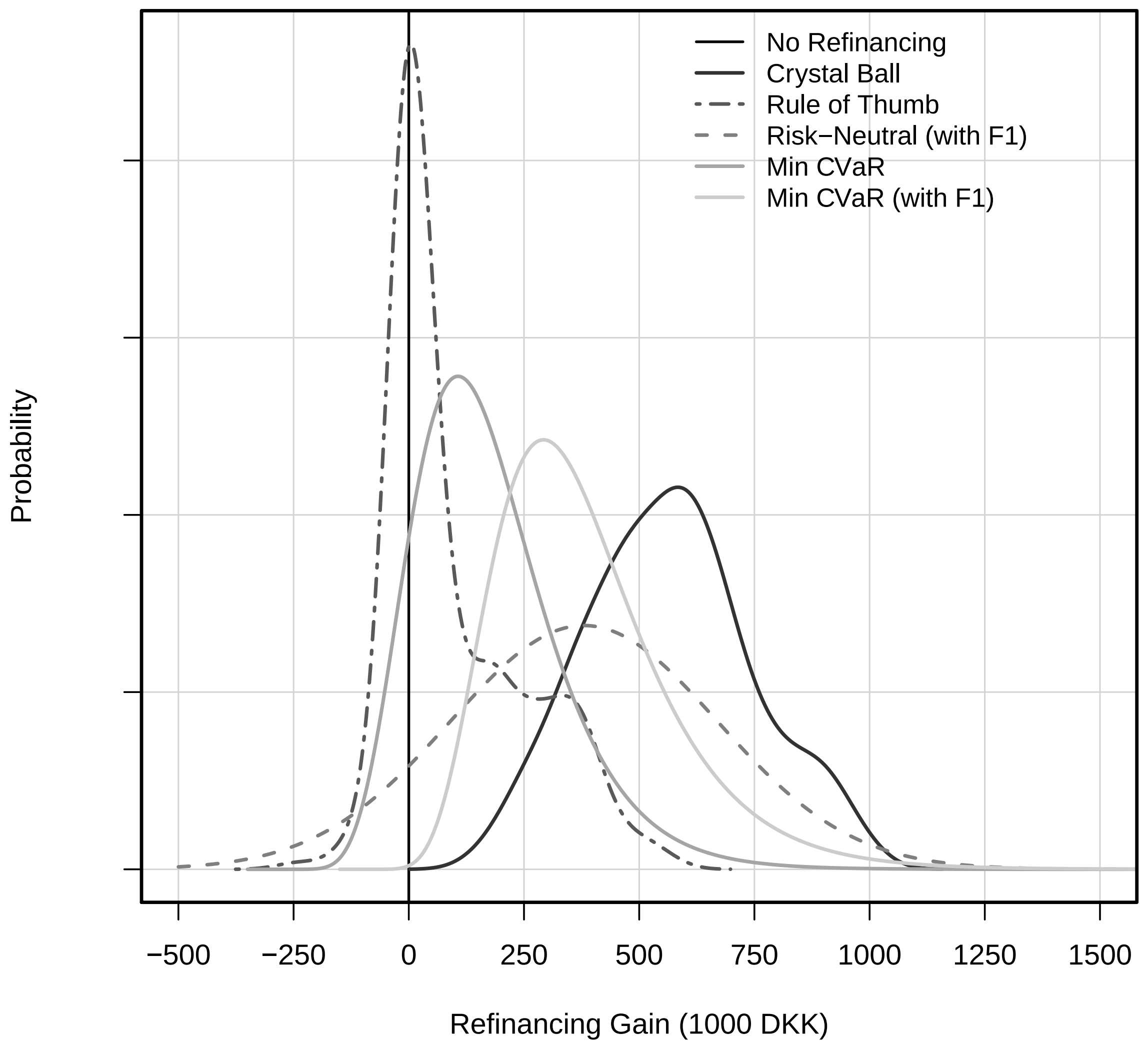

Figure 8: Distribution of refinancing gains (over fixed rate issue-and-hold) for different borrower types across the simulated scenarios for the period 2000-2010 (i.e. with simulations started at market data from year 2000).

One or two scenarios — even if they are as real as can be — do not tell us a lot; maybe the CVaR-minimizers just got lucky over the rules of thumb. (The CVaR-strategy is ex-ante optimal, not ex-post.) So next, strategies are investigated along 250 simulated scenarios to see if the results and insights from Figure 7 are robust. Figure 8 shows distributions of refinancing gains (the total cost of an issue-and-hold strategy in a fixed-rate bond minus the total cost of a particular strategy) for six different strategies:

1. The issue-and-hold strategy with a fixed rate bond (a vertical sight-line at 0, of course).
2. The crystal ball.
3. The rules of thumb.
4. The model based optimal strategy for a risk-neutral borrower.
5. The model based optimal strategy for CVaR-minimizing borrower who does not have access to adjustable rate (F1) loans.
6. The model based optimal strategy for CVaR-minimizing borrower who has access to adjustable rate (F1) loans.

The realized gains corresponding to the strategies in Figure 8 are (in DKK 1000's) 39 for the rules of thumb, 219 for the CVaR-minimizing strategy and 486 for the crystal ball. The figure and some calculations on the numbers used to produce it tell us several things:

- Benchmarking against the issue-and-hold strategy (liquidated at market value after 10 years) ensures that strategies are not being rewarded for operating along a path of generally declining interest rates, as was the case actually observed rates 2000-2010 (see Figure 4). Note that the realized gains are in the left tail of the gains distribution.
- Unless we have perfect foresight, there is always a risk (albeit possibly a small one) that the strategy ends up being outperformed by a simple issue-and-hold strategy.
- The rules of thumb outperform the issue-and-hold strategy on average (difference in average PEL's 0.3%), but with an increased risk (in 18% of cases issue-and-hold is better). So the rules of thumb are only slightly better than a passive benchmark. (Which isn't actually as bad as it sounds — many investment strategies do not pass this seemingly simple test.)
- The risk-neutral borrower's strategy (which mostly, but not exclusively, uses adjustable rate loans) has the highest expected gains over the issue-and-hold strategy (with the obvious exception of the crystal ball), but is also markedly riskier than all other strategies — including the rules of thumb.
- Even without adjustable rate loans, the CVaR-minimizing borrower's strategy dominates the rules of thumb. The left tail of the distributions (the losses) is below that from the rules of thumb, and the mean and median when CVaR-minimizing are visibly higher; the average PEL is about 0.3%-points lower. Reverse engineering and choosing a risk/reward weighting, λ, in equation (11) that

leads to the optimal strategy having a CVaR equal to that of the rules of thumb gives an improvement (over the rules of thumb) in average PEL of slightly above 0.5%-points.

- The performance of the CVaR-minimizing borrower's strategy improves significantly upon the inclusion of adjustable rate loans; the average PEL is 0.9% lower than for the rules of thumb, and the risk in terms of CVaR is lower (in terms of variance, the risk is about the same). So even if borrowers are risk-averse, they should not shy away from adjustable rate loans — it depends on the market conditions.

5 Conclusion

In this paper we have demonstrated that while the rules of thumb for mortgage choice used by Danish mortgage banks are not catastrophic, they are dominated by what can be achieved with a model-based approach that optimizes a risk/reward criterion. And this gain was achieved even in a single-stage stochastic programming model; Pedersen et al. (2013) indicates that there are further gains (but also considerable added computational complexity) in using multi-stage formulations.

The existing mortgage products offer considerable potential for diversification and individualization in terms of risk and cost characteristics of the loans. This potential is however not taken advantage of due to lack of effective and personal consultancy tools and services. The framework suggested in this paper suggests a foundation for such tools and services.

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