Options on realized variance by transform methods: 
A non-affine stochastic volatility model

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Abstract

In this paper we study the pricing and hedging of options on realized variance in the 3/2 non-affine stochastic volatility model, by developing efficient transform based pricing methods. This non-affine model gives prices of options on realized variance which allow upward sloping implied volatility of variance smiles. Heston’s (1993) model, the benchmark affine stochastic volatility model, leads to downward sloping volatility of variance smiles — in disagreement with variance markets in practice. We show a robust method, using control variates, to express the Laplace transform of the variance call function in terms of the Laplace transform of realized variance. The proposed method works in any model where the Laplace transform of realized variance is available in closed form. Additionally, we apply a new numerical Laplace inversion algorithm which gives fast and accurate prices for options on realized variance, simultaneously at a sequence of variance strikes. The method is also used to derive hedge ratios for options on variance with respect to variance swaps.

keywords: options on realized variance, transform pricing, variance swaps, stochastic volatility, 3/2 model, Heston model

jel: C63, G12, G13.

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1 Introduction

The trading and risk management of variance and volatility derivatives requires models which both adequately describe the stochastic behavior of volatility as well as allow for fast and accurate numerical implementations. This is especially important for variance markets since their underlying asset, namely, variance, displays much more volatility than the corresponding stock or index, in the spot market. It is not uncommon for the volatility of variance to be several orders of magnitude higher than the volatility of the underlying stock or index. Many practical aspects relevant to variance and volatility markets are discussed in Bergomi (2005, 2008), Gatheral (2006) and Eberlein, Madan (2009).

Simple volatility derivatives, such as variance swaps, corridor variance swaps, gamma swaps and other similar variations, can be priced and hedged in a model free way and hence do not require the specification of a stochastic volatility model. Neuberger (1994) made a first contribution to this area by proposing the use of the log-contract as an instrument to hedge volatility risk. Due to their role in trading and hedging volatility, variance swaps have become liquidly traded instruments and have led to the development of other volatility derivatives. A comprehensive treatment of model free pricing and hedging of variance contracts can be found in Demeterfi, Derman, Kamal, Zou (1999) and Carr, Madan (2002).

More complicated volatility derivatives, particularly, options on realized variance and volatility, require explicit modeling of the dynamics of volatility. Important early stochastic volatility models studied in the literature include Scott (1987), Hull, White (1987) and Chesney, Scott (1989). Since no fast numerical methods are available to compute large sets of European option prices in these models, calibration procedures can become difficult. Heston (1993) proposed the use of an affine square root diffusion process to model the dynamics of instantaneous variance. The model has become widely popular due to its tractability and existence of a closed form expression for the characteristic function of log returns. The important result of Carr, Madan (1999) shows how to apply fast Fourier inversion techniques to price European options when the characteristic function is available in closed form.

In recent studies, Broadie, Jain (2008a) and Sepp (2008) have developed methods for pricing and hedging options on realized variance in the Heston model. Gatheral (2006) and Carr, Lee (2007) show how to use variance swap and volatility swap prices to fit a log-normal distribution to realized variance, thus arriving at Black-Scholes (1973) style formulas for prices and hedge ratios of options on variance. Several authors have considered the pricing of volatility derivatives in models with jumps; Carr, Geman, Madan, Yor (2005) price options on realized variance by assuming the underlying asset follows a pure jump Sato process, Broadie, Jain (2008b) determine the ef-
fect of jumps and discrete sampling on the prices of volatility and variance swaps and find that the well known convexity correction formula does not provide reliable results in models with jumps, Sepp (2008) augments the Heston model with simultaneous jumps in the underlying and in the volatility process and also provides prices for forward start options on realized variance.

In this paper we determine and compare the prices and hedge ratios of options on realized variance in the $3/2$ non-affine stochastic volatility model versus the Heston (1993) model. The $3/2$ model has been used previously by Ahn, Gao (1999) to model the evolution of short interest rates, by Andreasen (2003) as a default intensity model in pricing credit derivatives and by Lewis (2000) to price equity stock options. More recently, Carr, Sun (2007) discuss the $3/2$ model in the context of a new framework in which variance swap prices are modeled instead of the short variance process. Besides its analytical tractability, the $3/2$ diffusion specification also enjoys empirical support in the equity market. Using S&P100 implied volatilities, studies by Jones (2003) and Bakshi, Ju, Yang (2004) estimate that the variance exponent should be around 1.3 which favors the $3/2$ model over the $1/2$ exponent in the Heston (1993) model. Additionally, as we show in this paper, the $3/2$ and Heston models predict opposite dynamics for the short term equity skew as a function of the level of short variance and, more importantly, the Heston model wrongly generates downward sloping volatility of variance smiles, at odds with variance markets in practice.

We develop robust transform methods, based on control variates, to express the Laplace transform of the variance call function in terms of the Laplace transform of realized variance. Our approach works in any model where the Laplace transform of realized variance is available in closed form. We then apply a fast and accurate numerical Laplace inversion algorithm, recently proposed by Iseger (2006), which allows the use of the FFT technique of Cooley, Tukey (1965) to recover the variance call function at a sequence of strikes simultaneously. Finally, we show how the tools can be used to obtain hedge ratios for options on variance.

The paper is organized as follows. In section 2, we present general properties of the $3/2$ and Heston models and compare them from the standpoint of short variance dynamics, equity skew dynamics and fitting to vanilla options. Section 3 is the main section, where we develop our transform based tools and then apply them to pricing options on realized variance. In section 4, we discuss the derivation of hedge ratios with respect to variance swaps. Section 5 summarizes the main conclusions. All proofs not shown in the main text can be found in the appendix.
2 Model descriptions and properties

Two parametric stochastic volatility models are considered in this paper. The well-known Heston (1993) model assumes the following dynamics under the pricing measure $\mathbb{Q}$:

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sqrt{v_t}dB_t$$
$$dv_t = k(\theta - v_t)dt + \epsilon \sqrt{v_t}dW_t$$

where $r$ is the risk-free rate in the economy, $\delta$ is the dividend yield, and $B_t$, $W_t$ are one-dimensional standard Brownian Motions with correlation $\rho$. The parameters of the instantaneous variance diffusion have the usual meaning: $k$ is the speed of mean reversion, $\theta$ is the mean reversion level and $\epsilon$ is the volatility of volatility. The theoretical results in this paper allow the mean reversion level to be time dependent, but deterministic. This can be useful if the model user wants to interpolate the entire term structure of variance swaps. Therefore, we allow the short variance process to obey the following extended Heston dynamics:

$$dv_t = k(\theta(t) - v_t)dt + \epsilon \sqrt{v_t}dW_t$$ (1)

where $\theta(t)$ is a time-dependent and deterministic function of time. An alternative model, which forms the main focus of our study, is known in the literature as the 3/2-model. It prescribes the following dynamics under the pricing measure $\mathbb{Q}$:

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sqrt{v_t}dB_t$$
$$dv_t = kv_t(\theta(t) - v_t)dt + \epsilon v_t^{3/2}dW_t$$ (2)

where, as in the case of the extended Heston model, $\theta(t)$ is the time-dependent mean reversion level. However, it is important to note that the parameters $k$ and $\epsilon$ no longer have the same interpretation and scaling as in the Heston model. The speed of mean reversion is now given by the product $k \cdot v_t$, which is a stochastic quantity; in particular, we see that variance will mean revert more quickly when it is high. Also, we should expect the parameter $k$ in the 3/2-process to scale as $1/v_t$ relative to the parameter $k$ in the Heston model; the same scaling applies to the parameter $\epsilon$. These scaling considerations are useful when interpreting the parameter values obtained from model calibration.

We next address a couple of technical conditions needed to have a well defined 3/2-model. An application of Itô’s lemma to the process $1/v_t$ when $v_t$ follows dynamics (2) gives:

$$d\left(\frac{1}{v_t}\right) = k\theta(t)\left(\frac{k + \epsilon^2}{k\theta(t)} - \frac{1}{v_t}\right)dt - \frac{\epsilon}{\sqrt{v_t}}dW_t$$
which reveals that the reciprocal of the 3/2 short variance process is, in fact, a Heston process of parameters \( \left( k\theta(t), \frac{k+\epsilon^2}{k\theta(t)}, -\epsilon \right) \). Using Feller’s boundary conditions, it is known that a time-homogeneous Heston process of dynamics (1), with \( \theta(t) = \theta \), can reach the zero boundary with non-zero probability, unless:

\[
2k\theta \geq \epsilon^2.
\]

This result has been extended by Schlögl & Schlögl (2000) to the case of time-dependent piecewise-constant Heston parameters. We have seen that, if \( v_t \) is a 3/2-process, then \( 1/v_t \) is a Heston process. A non-zero probability of reaching zero for \( 1/v_t \) would imply a non-zero probability for the short variance process to reach infinity. For a piecewise constant \( \theta(t) \), applying the result of Schlögl & Schlögl (2000) to the dynamics of \( 1/v_t \), we obtain the non-explosion condition for the 3/2-process as:

\[
2k\theta(t) \cdot \frac{k+\epsilon^2}{k\theta(t)} \geq \epsilon^2
\]
or

\[
k \geq -\frac{\epsilon^2}{2}.
\]

(3)

In what follows we assume that \( k > 0 \) which will automatically ensure that the non-explosion condition is satisfied. Another technical condition necessary in the 3/2 model refers to the martingale property of the process \( S_t/\exp((r-\delta)t) \); Lewis (2000) shows that for this process to be a true martingale, and not just a local martingale, the non-explosion test for \( v_t \) must be satisfied also under the measure which takes the asset price as numeraire. Applying the results in Lewis (2000), leads to the additional condition on the 3/2 model parameters:

\[
k - \epsilon\rho \geq -\frac{\epsilon^2}{2}.
\]

(4)

If we require that the correlation parameter \( \rho \) be non-positive, this condition will be automatically satisfied. In practice, imposing the restriction \( \rho \leq 0 \) does not raise problems since market behavior of prices and volatility usually displays negative correlation. To summarize, conditions (3) and (4) together ensure that we have a well-behaved 3/2-model. They are both satisfied if we impose the sufficient conditions \( k > 0 \) and \( \rho \leq 0 \).

Of importance to our subsequent analysis will be the joint Fourier-Laplace transform of the log-price \( X_T = \log(S_T) \) and the annualized integrated variance \( V_T = \frac{1}{T} \int_0^T v_t dt \), where \( T \) denotes the maturity of interest. In both models, it is possible to derive a closed form solution for this joint transform. In particular, using the characteristic function of \( X_T = \log(S_T) \) it is possible to price European options by Fourier inversion using the method developed in Carr, Madan (1999). Also, in the next
section, we develop fast transform based methods to price options on realized variance using the Laplace transform of $V_T$. Propositions (2.1) and (2.2) below give the expression of the joint transforms in the two models. Below, we let $X_t = \log(S_t e^{(r-\delta)(T-t)})$ denote the log-forward price process.

**Proposition 2.1.** In the Heston model with time-dependent mean-reversion level, the joint conditional Fourier-Laplace transform of $X_T$ and the de-annualized realized variance $\int_t^T v_s ds$ is given by:

$$E \left( e^{iuX_T - \lambda \int_t^T v_s ds} \bigg| X_t, v_t \right) = \exp \left( iuX_t + a(t, T) - b(t, T) v_t \right)$$

where

$$a(t, T) = -\int_t^T k\theta(s) b(s, T) ds$$

$$b(t, T) = \frac{(iu + u^2 + 2\lambda) \left( e^{\gamma(T-t)} - 1 \right)}{(\gamma + k - i\epsilon\rho u) \left( e^{\gamma(T-t)} - 1 \right) + 2\gamma}$$

$$\gamma = \sqrt{(k - i\epsilon\rho u)^2 + \epsilon^2 (iu + u^2 + 2\lambda)}.$$

For the case when the mean reversion level $\theta(t), t \in [0, T]$, is a piecewise constant function it is possible to calculate explicitly the integral which defines $a(t, T)$ in Proposition (2.1). If we let $0 = t_0 < t_1 < \ldots < t_N = T$ be a partition of $[0, T]$ such that $\theta(t) = \theta_j$ on the interval $(t_j, t_{j+1})$, $j \in \{0, 1, 2, \ldots, N-1\}$, then the function $a(t, T)$ is given by:

$$a(t, T) = \sum_{j=0}^{N-1} \frac{k\theta}{\epsilon^2} \left[ (k - \gamma - i\epsilon\rho u) (t_{j+1} - t_j) - \right.$$ \[ -2 \log \left( \frac{\alpha e^{\gamma(t_{j+1})} + \beta e^{-\gamma(t_{j+1})}}{\alpha e^{\gamma(t_{j+1})} + \beta} \right) \left. \right]$$

where

$$\alpha = \gamma + k - i\epsilon\rho u$$

$$\beta = \gamma - k + i\epsilon\rho u$$

with $\gamma$ as defined in Proposition (2.1). A similar result which gives the closed form expression of the joint Fourier-Laplace transform can be obtained in the 3/2-model. The result is due to Carr, Sun (2007).

**Proposition 2.2** (Carr, Sun (2007)). In the 3/2-model with time-dependent mean-reversion level, the joint conditional Fourier-Laplace transform of $X_T$ and the de-annualized realized variance $\int_t^T v_s ds$ is given by:

$$E \left( e^{iuX_T - \lambda \int_t^T v_s ds} \bigg| X_t, v_t \right) = e^{iuX_t} \Gamma \left( \frac{\gamma - \alpha}{\Gamma(\gamma)} \right) \left( \frac{2}{\epsilon^2 y(t, v_t)} \right)^{\alpha} M \left( \alpha, \gamma, -\frac{2}{\epsilon^2 y(t, v_t)} \right)$$
Figure 1: Simultaneous fit of Heston model to 3-months (left) and 6-months (right) S&P500 implied volatilities on July 31st 2009. **Solid**: Heston implied volatilities, **Dashed**: Market implied volatilities. Heston parameters obtained: $v_0 = 25.56\%^2$, $k = 3.8$, $\theta = 30.95\%^2$, $\epsilon = 92.88\%$ and $\rho = -78.29\%$.

where

$$y(t, v_t) = v_t \int_t^T e^{\int_t^s k\theta(s)ds} du$$

$$\alpha = -\left( \frac{1}{2} - \frac{p}{\epsilon^2} \right) + \sqrt{\left( \frac{1}{2} - \frac{p}{\epsilon^2} \right)^2 + 2\frac{q}{\epsilon^2}}$$

$$\gamma = 2\left( \alpha + 1 - \frac{p}{\epsilon^2} \right)$$

$$p = -k + i\epsilon u$$

$$q = \lambda + \frac{iu}{2} + \frac{u^2}{2}$$

$M(\alpha, \gamma, z)$ is the confluent hypergeometric function defined as:

$$M(\alpha, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\gamma)_n} \frac{z^n}{n!}$$

and

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1).$$

**Proof** We refer the reader to Carr, Sun (2007).

Before looking at how the models differ in pricing exotic contracts, such as options on realized variance, we discuss, in the rest of this section, the pricing of European vanilla options. The calibration of the models to vanilla options is important because the prices of European options determine the values of variance swaps which, in turn, are the main hedging instruments for options on realized variance; for a broad introduction to variance swaps we refer to Carr, Madan (2002). Moreover, vanilla options are also often used to hedge the vega exposure of options on realized variance.
We begin by fitting both models to market prices of S&P500 European options; the fit is done simultaneously to two maturities: 3 months \((T = 0.25)\) and 6 months \((T = 0.5)\) on July 31 2009. The results are shown in figures (1) and (2). The parameters obtained are \((v_0, k, \theta, \epsilon, \rho) : (25.50\%, 22.84, 46.69\%, 8.56, -99.0\%)\) in the Heston model and \((24.50\%, 22.84, 46.69\%, 8.56, -99.0\%)\) in the 3/2 model.

We remark that, while both models are able to fit the two maturities simultaneously, the Heston model parameters violate the non-zero boundary condition. This usually happens when calibrating the Heston model in the equity markets; the empirical study of Bakshi, Chao, Chen (1997) also finds Heston parameters which violate the non-zero boundary condition. This occurs because the Heston model requires a high volatility-of-volatility parameter \(\epsilon\) to fit the steep skews in equity markets. On the other hand, the 3/2 parameters yield a variance process which does not reach either zero or infinity. We notice, however, that the 3/2 model requires a more negative correlation parameter \((-99.0\%)\) compared to the Heston model \((-78.29\%)\). We explain this by the much ‘wilder’ dynamics (also called ‘dynamite dynamics’ by Andreasen (2003)) of short variance in the 3/2 model – as illustrated in figure (3) – which can cause a decorrelation with the spot process.

Figure (3) illustrates two main differences between the evolution of instantaneous variance in the Heston model versus the 3/2 model: (a) the Heston variance paths spend much more time around the zero-boundary than the 3/2 paths and (b) the 3/2 model allows for the occurrence of extreme paths with periods of very high instantaneous volatility. From a

\footnote{The data were kindly provided to us by an international investment bank.}
trading and risk management perspective, both of these observations favor
the 3/2 model. It is hard to justify a vanishing variance process and the
nonexistence of high (or, extreme) volatility scenarios. In the next section
we see that these differences have a major effect on the prices of options on
realized variance.

Another important difference is in the behavior of implied volatility
smiles. Specifically, the steepness of the smile responds in opposite ways
to changes in the level of short variance in the two models. To show this, we
make use of the implied volatility expansion derived in Medvedev, Scaillet
(2007) for short times to expiration and near the money options. Letting
\( X = \log(K/S_0e^{(r-\delta)T}) \) denote the log-forward-moneyness corresponding to
a European option with strike \( K \) and maturity \( T \), we apply Proposition 1
in Medvedev, Scaillet (2007) to the case of the Heston and 3/2 models. One
obtains the following expansions for implied volatility \( I(X,T) \) in a neigh-
borhood of \( X = T = 0 \). For the Heston model:

\[
I(X, T) = \sqrt{v_0} + \frac{\rho \epsilon X}{4\sqrt{v_0}} + \left( 1 - \frac{5\rho^2}{2} \right) \frac{\epsilon^2 X^2}{24v_0^{3/2}} + \\
+ \left( \frac{k(\theta - v_0)}{4\sqrt{v_0}} + \frac{\rho \epsilon \sqrt{v_0}}{8} + \frac{\rho^2 \epsilon^2}{96\sqrt{v_0}} - \frac{\epsilon^2}{24\sqrt{v_0}} \right) T + \ldots
\]

which gives the at-the-money-forward skew

\[
\frac{\partial I}{\partial X}(X, T) \bigg|_{X=0} = \frac{\rho \epsilon}{4\sqrt{v_0}}, \quad T \to 0.
\]

And for the 3/2 model:

\[
I(X, T) = \sqrt{v_0} + \frac{\rho \epsilon \sqrt{v_0} X}{4} + \left( 1 - \frac{\rho^2}{2} \right) \frac{\epsilon^2 \sqrt{v_0} X^2}{24} + \\
+ \left( \frac{k(\theta - v_0)}{4\sqrt{v_0}} + \frac{\rho \epsilon \sqrt{v_0}}{8} + \frac{\rho^2 \epsilon^2}{96\sqrt{v_0}} - \frac{\epsilon^2}{24\sqrt{v_0}} \right) T + \ldots
\]

\[ + \left( \frac{k(\theta - \nu_0)}{4} + \frac{\rho \epsilon \nu_0}{8} - \frac{7\rho^2 \epsilon^2 \nu_0}{96} - \frac{\epsilon^2 \nu_0}{24} \right) \sqrt{\nu_0 T} + ... \]

which gives the at-the-money-forward skew

\[ \frac{\partial I}{\partial X}(X, T) \bigg|_{X=0} = \frac{\rho \epsilon \sqrt{\nu_0}}{4}, \quad T \to 0. \]

In figure (4) we see a very good agreement between the Medvedev, Scaillet (2007) expansion and the true implied volatilities calculated by Fourier inversion. Since the expansions are valid only for short expirations and close to the money, we chose a one month maturity and a [90%, 110%] relative strike range. Therefore, in the Heston model, the short term skew flattens when the instantaneous variance increases whereas, in the 3/2 model, the short term skew steepens when the instantaneous variance increases. It is important to realize that this implies very different dynamics for the evolution of the implied volatility surface. The Heston model predicts that, in periods of market stress, when the instantaneous volatility increases, the skew will flatten. Under the same scenario, the skew will steepen in the 3/2 model. From a trading and risk management perspective, since the magnitude of the skew is itself a measure of market stress, the behavior predicted by the 3/2 model appears more credible.
3 Transform pricing of options on realized variance

The main quantity of interest in pricing options on realized variance is the annualized integrated variance, given by:

\[ V_T = \frac{1}{T} \int_0^T v_t dt. \]

We study the prices of call options on realized variance; prices of put options follow by put-call parity. The payoff of a call option on realized variance with strike \( K \) and maturity \( T \) is defined as:

\[ \left( \frac{1}{T} \int_0^T v_t dt - K \right)_+ = (V_T - K)_+. \]

In a very important result Carr, Madan (1999) showed that, starting from the characteristic function of the log stock price \( \log(S_T) \), it is possible to derive a closed form expression for the Fourier transform of the (damped) call price viewed as a function of the log strike \( k = \log(K) \). Once the call price transform is known, fast inversion algorithms – such as the FFT method developed by Cooley, Tukey (1965) – can be applied to recover the call prices at a sequence of strikes simultaneously. This technique is now widely used in the literature to price stock options in non-Black-Scholes models which have closed form expressions for the characteristic function of \( \log(S_T) \). We next develop a similar idea for the problem of pricing options on realized variance. Specifically, we show that, starting from the Laplace transform of integrated variance, it is possible to derive in closed form the Laplace transform of the variance call price viewed as a function of the variance strike. This idea was first suggested by Carr, Geman, Madan, Yor (2005). We here provide a proof and also propose an important improvement of the result by the use of control variates. Additionally, we show the application of a new numerical Laplace inversion algorithm which gives prices of options on realized variance at a sequence of variance strikes simultaneously.

**Proposition 3.1.** Let \( L(\cdot) \) denote the Laplace transform of the annualized realized variance over \([0, T]\):

\[ L(\lambda) = E\left(e^{-\lambda \frac{1}{T} \int_0^T v_t dt}\right). \]

Then the undiscounted variance call function \( C : [0, \infty) \to \mathbb{R} \) defined by

\[ C(K) = E\left(\frac{1}{T} \int_0^T v_t dt - K\right)_+ \]

has Laplace transform given by

\[ L(\lambda) = \int_0^\infty e^{-\lambda K} C(K) dK = \frac{L(\lambda) - 1}{\lambda^2} + \frac{C(0)}{\lambda}. \]
Proof} Let \( \mu(dx) \) denote the probability law of the annualized realized variance \( \frac{1}{T} \int_0^T v_t \, dt \). We have to compute

\[
L(\lambda) = \int_0^\infty e^{-\lambda K} \int_K^\infty (x - K) \mu(dx) dK.
\]

Since the integrand in this double integral is non-negative, we can apply Fubini’s theorem to change the order of integration and we obtain

\[
\int_0^\infty \int_0^x e^{-\lambda K} (x - K) dK \mu(dx).
\]

The inner integral now follows easily by integration by parts

\[
\int_0^x e^{-\lambda K} (x - K) dK = e^{-\lambda x} - \frac{1}{\lambda^2} + \frac{x}{\lambda}.
\]

Finally, we can compute the Laplace transform as follows

\[
L(\lambda) = \int_0^\infty \left( \frac{e^{-\lambda x} - 1}{\lambda^2} + \frac{x}{\lambda} \right) \mu(dx) = \frac{\mathcal{L}(\lambda) - 1}{\lambda^2} + \frac{C(0)}{\lambda}
\]

where we have used that

\[
C(0) = E\left( \frac{1}{T} \int_0^T v_t dt \right).
\]

Relation (5) of Proposition (3.1) gives a closed form solution for the Laplace transform \( L(\lambda) \) of the variance call function \( C(K) \) in terms of the Laplace transform \( \mathcal{L}(\lambda) \) of the annualized realized variance \( V_T = \frac{1}{T} \int_0^T v_t dt \).

The closed form expression for \( \mathcal{L}(\lambda) \) is obtained from Proposition (2.1) and Proposition (2.2) by setting \( t = 0, u = 0 \) and \( \lambda = \frac{1}{T} \).

However, we notice that the following two, polynomially decaying terms, appear in expression (5):

\[
-\frac{1}{\lambda^2} + \frac{C(0)}{\lambda}.
\]

These vanish slowly as \( |\lambda| \to \infty \) affecting the accuracy of numerical inversion algorithms. The term \( \frac{C(0)}{\lambda} \) appears because the function has a discontinuity of size \( C(0) \) at 0, while the term \( \frac{1}{\lambda^2} \) appears because the first derivative has a discontinuity of size \(-1\) at 0. We propose to eliminate these slowly decaying terms by applying the idea of control variates. Specifically, we choose a proxy distribution for the realized variance which allows the calculation of the variance call function in closed form. Denote this control variate function by \( \tilde{C}(\cdot) \). If we choose the proxy distribution such that it has the same mean...
as the true distribution of realized variance, we have $C(0) = \tilde{C}(0)$. Then, by the linearity of the Laplace transform we obtain:

$$L_{C - \tilde{C}}(\lambda) = L_C(\lambda) - L_{\tilde{C}}(\lambda) = \frac{\mathcal{L}(\lambda) - \tilde{\mathcal{L}}(\lambda)}{\lambda^2}.$$ 

Both power terms have been eliminated since the difference $C(\cdot) - \tilde{C}(\cdot)$ is now a function which is both continuous and differentiable at zero. Differentiability comes from the fact that both functions have a left derivative at 0 equal to $-1$. This is seen in the following simple lemma.

**Lemma 3.1.** Let $V$ be a random variable such that $V > 0$ a.s. and $E(V) < \infty$. Then the function $C : [0, \infty) \rightarrow \mathbb{R}$ defined by:

$$C(K) = E(V - K)_+$$

satisfies

$$\lim_{K \downarrow 0} \frac{C(K) - C(0)}{K} = -1.$$ 

In summary, by making use of a control variate we can achieve smooth pasting at 0. In choosing the proxy distribution for realized variance, one appealing choice is the log-normal distribution. This would give Black-Scholes style formulas for the control variate function $\tilde{C}(\cdot)$. However, this choice does not work because the Laplace transform of the log-normal distribution is not available in closed form. Instead, we choose the Gamma distribution as our proxy distribution. The Laplace transform is known in closed form and the following lemma shows how to compute the control variate function $\tilde{C}(\cdot)$.

**Lemma 3.2.** Let the realized variance over $[0, T]$ follow a Gamma distribution of parameters $(\alpha, \beta)$. Specifically, assume the density of realized variance is:

$$\frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}, x > 0.$$ 

Then the control variate function $\tilde{C}(\cdot)$ is given by

$$\tilde{C}(K) = \alpha\beta (1 - F(K; \alpha + 1, \beta)) - K (1 - F(K; \alpha, \beta))$$

where $F(x; \alpha, \beta)$ is the Gamma cumulative distribution function of parameters $(\alpha, \beta)$.

To ensure that $\tilde{C}(0) = C(0)$, the only necessary condition on $\alpha$ and $\beta$ is that the mean of the Gamma distribution matches $C(0)$:

$$\alpha\beta = C(0).$$
Since we have two parameters, from a theoretical standpoint, we can choose one of them freely. Optionally, the extra parameter could be used to fix the second moment of the proxy distribution. For example, we can match the second moment of the model realized variance:

\[ \alpha \beta^2 + (\alpha \beta)^2 = E \left( \frac{1}{T} \int_0^T v_t \, dt \right)^2 = \left. \frac{\partial^2 \mathcal{L}(\lambda)}{\partial \lambda^2} \right|_{\lambda=0} \]

where \( \mathcal{L}(\cdot) \) is the Laplace transform of realized variance. In the Heston model, the second moment of realized variance is available in closed form; see Dufresne (2001). In the 3/2 model, however, we do not have a closed form formula for the second moment of realized variance; as shown later, even the calculation of the first moment requires the development of some additional results. In this case, the second moment could be approximated by using a finite difference to obtain the second derivative of \( \mathcal{L}(\cdot) \) at zero. Alternatively, an easier approach to choose a reasonable second moment for the control variate distribution is to match the second moment of a log-normal distribution of the form:

\[ C(0) e^{\sigma \sqrt{T}} N(0,1) - \frac{\sigma^2 T}{2} \]

where \( N(0,1) \) is a standard normal random variable and \( \sigma \) is a parameter of our choice – a sensible pick would have an order of magnitude that is representative for the implied volatility of variance. As the subsequent numerical results reveal, any choice for \( \sigma \) in the range, say, \([50\%, 150\%]\) would be reasonable. The second moment condition on \( \alpha \) and \( \beta \) reads:

\[ \alpha \beta^2 + (\alpha \beta)^2 = C(0)^2 e^{\sigma^2 T} \]

We obtain that a possible choice for the parameters of the proxy distribution is:

\[ \alpha = \frac{C(0)}{\beta} \quad \beta = C(0)(e^{\sigma^2 T} - 1). \]

To implement the above calculations one needs to be able to determine

\[ E \left( \int_0^T v_t \, dt \right) \]

in both models – Heston and 3/2. The computation is straightforward in the Heston model but is more complicated in the 3/2 model. We first show the calculation for the Heston model with a piecewise constant mean reversion level \( \theta(t) \), \( t \in [0,T] \). If we let \( \theta(t_i) = \theta_i \) on \( (t_i, t_{i+1}) \), \( i \in \{0, 1, 2, \ldots, N - 1\} \) we can write

\[ E \left( \int_0^T v_t \, dt \right) = \sum_{i=0}^{N-1} E \left( \int_{t_i}^{t_{i+1}} v_t \, dt \right) \]
where

\[ E \left( \int_{t_i}^{t_{i+1}} v_t \, dt \right) = \frac{e^{-kt_i} - e^{-kt_{i+1}}}{k} \cdot \left( v_0 + \sum_{j=0}^{i-1} \theta_j \left( e^{kt_{j+1}} - e^{kt_j} \right) \right) + \]

\[ + \frac{\theta_i}{k} \left( e^{-k(t_{i+1}-t_i)} - 1 + k \left( t_{i+1} - t_i \right) \right). \]

In the case of the 3/2 model, Carr, Sun (2007) (see Theorem 4 therein) show that

\[ E \left( \int_0^T v_t \, dt \right) = h \left( v_0 \int_0^T e^{k \int_0^t \theta(s) \, ds} \, dt \right) \]

where

\[ h(y) = \int_0^y e^{-\frac{z}{y}} \cdot z^{\frac{2k}{y^2}} \cdot \int_0^\infty \frac{2}{e^{t^2}} e^{\frac{2}{e^2} u \cdot \frac{2k}{y^2} - \frac{2k}{y^2}} \, dudz. \quad (6) \]

The integral appearing in the argument to the function \( h(\cdot) \) is straightforward to compute for a piecewise constant \( \theta(t) \):

\[ \int_0^T e^{k \int_0^t \theta(s) \, ds} \, dt = \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} e^{k \int_0^t \theta(s) \, ds} \, dt \]

where

\[ \int_{t_i}^{t_{i+1}} e^{k \int_0^t \theta(s) \, ds} \, dt = \frac{e^{k\theta_i(t_{i+1}-t_i)} - 1}{k\theta_i} \cdot \exp \left( k \sum_{j=0}^{i-1} \theta_j(t_{j+1} - t_j) \right). \]

However, the integral representation (6) of the function \( h(\cdot) \) is hard to use for fast and accurate numerical implementations. We prove an alternative representation, based on a uniformly convergent series whose terms are easy to calculate and the total error can be controlled a priori. The result is formulated in Proposition (3.2) and Lemma (3.3).

**Proposition 3.2.** The function \( h(\cdot) \) admits the following uniformly convergent series representation

\[ h(y) = \alpha \cdot \left( E \left( \frac{\gamma}{y} \right) + \sum_{n=1}^{\infty} \frac{F_{\frac{\gamma}{\beta}}(n)}{n(n-\beta+1)} \right) \]

where

\[ E(x) = \int_x^\infty e^{-t} \cdot t^{-1} \, dt, \quad x > 0 \]

\[ F_{\nu}(n) = P(Z \leq n), \quad Z \sim \text{Poisson}(\nu) \]

\[ \alpha = \frac{2}{e^2} \]

\[ \beta = \frac{-2k}{e^2}. \]
In Proposition (3.2) we recognize the special function $E(x)$ as the exponential integral which is readily accessible in any numerical package. The terms appearing in the infinite series are very fast and easy to compute. Moreover, as shown in Lemma (3.3) next, the total error arising from truncating the series can be controlled a priori.

**Lemma 3.3.** The infinite series of Proposition (3.2) has a remainder term

$$R_k = \sum_{n=k}^{\infty} \frac{F_{\alpha y}(n)}{n(n-\beta+1)}$$

which is positive and satisfies the following bounds

$$\frac{F_{\alpha y}(k)}{m+1} \left( \frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{k+m} \right) < R_k < \frac{1}{k}$$

where $m = \lceil -\beta \rceil$. If we let $\bar{R}$ the mid-point between the two bounds i.e.

$$\bar{R} = \frac{1}{2} \left( \frac{1}{k} + \frac{F_{\alpha y}(k)}{m+1} \left( \frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{k+m} \right) \right)$$

then we also have

$$|R_k - \bar{R}| < \frac{m + \alpha y}{4k^2} \quad (7)$$

The application of Lemma (3.3) proceeds as follows. To compute $h(y)$, for a given $y$, use bound (7) to determine the number of terms needed to achieve the desired precision and then set

$$h(y) \approx \alpha \left( \frac{E\left(\frac{\alpha y}{1-\beta}\right)}{1-\beta} + \sum_{n=1}^{k-1} \frac{F_{\alpha y}(n)}{n(n-\beta+1)} + \bar{R} \right).$$

Having completed our discussion about the determination of the Laplace transform of the variance call function, we now turn to the problem of choosing a fast and accurate Laplace inversion algorithm. Many numerical Laplace inversion algorithms have been proposed in the literature; some important early contributions in this area include Weeks (1966), Dubner, Abate (1968), Stehfest (1970), Talbot (1979) and Abate, Whitt (1992). In what follows, we apply the very efficient algorithm recently proposed by Iseger (2006). Extensive analysis and numerical tests indicate that this algorithm is faster and more accurate than the other methods available. For a detailed treatment of the numerical and mathematical properties of this new method we refer the reader to Iseger (2006). We next outline the main steps of the method.

Suppose we want to recover the difference between the variance call functions $C - \tilde{C}$ at a sequence of variance strikes $k\Delta$, $k = 0, 1, \ldots, M - 1$;
let \( g(k) = C(k\Delta) - \tilde{C}(k\Delta) \) and \( \hat{g} \) the Laplace transform of \( g \). The starting point of the method is the well-known Poisson summation formula which states that, for any \( v \in [0, 1) \), the following identity holds for the function \( g \):
\[
\sum_{k=-\infty}^{\infty} \hat{g}(a + 2\pi i (k + v)) = \sum_{k=0}^{\infty} e^{-ak} e^{-2\pi ikv} g(k)
\]
(8)

where \( a \) is a positive damping factor. The Poisson summation formula applies to functions of bounded variation and in \( L^1[0, \infty) \). To check these conditions for the function \( g \), we derive the simple Lemma (3.4).

**Lemma 3.4.** Let \( V \) be a random variable such that \( V > 0 \) a.s. and \( E(V^2) < \infty \). Then the function \( C : [0, \infty) \to \mathbb{R} \) defined by:
\[
C(K) = E(V - K)_+
\]

belongs to \( L^1[0, \infty) \) and is of bounded variation.

In both the Heston and the 3/2 model, the Laplace transform of integrated variance exists in a neighborhood of zero, which implies that all moments of integrated variance are finite. The same is true for our control variate distribution, Gamma. Applying Lemma (3.4), we conclude that functions \( C(\cdot) \) and \( \tilde{C}(\cdot) \) are in \( L^1[0, \infty) \) and of bounded variation. It follows that the function \( g \) satisfies the same conditions.

Equation (8) relates an infinite sum of Laplace transform values (the LHS) to a dampened series of function values (the RHS). This result also forms the basis of the method developed by Abate, Whitt (1992). The series of Laplace transform values usually converges slowly and Abate, Whitt (1992) proposed a technique, known as Euler summation, to increase the rate of convergence for this series. Iseger (2006) proposes a completely different idea for handling the infinite series of Laplace transform values. It constructs a Gaussian quadrature rule for the series on the LHS of (8). Specifically, the infinite sum is approximated with a finite sum of the form
\[
\sum_{k=1}^{n} \beta_k \cdot \hat{g}(a + i\lambda_k + 2\pi iv)
\]
where \( \beta_k \) are the quadrature weights and \( \lambda_k \) are the quadrature points. The exact numbers \( \beta_k \) and \( \lambda_k \) can be found in Iseger (2006) (see Appendix A therein) for various values of \( n \). It is found that a number of \( n = 16 \) quadrature points is sufficient for results attaining machine precision.

Having developed a fast and accurate approximation for the LHS of (8), we next turn to the dampened series of function values on the RHS. This series is much easier to handle. As shown in Iseger (2006) it is possible to choose the damping parameter \( a \) and a truncation rank \( M_2 \) to attain any desired level of truncation error. For double precision, the authors
recommend truncating the series after \( M_2 = 8M \) terms and using a dampening factor \( a = 44/M_2 \). Finally, applying the identity (8) repeatedly for all \( v \in \{0, \frac{1}{M_2}, \frac{2}{M_2}, \ldots, \frac{M-1}{M_2}\} \), we can recover each function value \( g(k) \) by inverting the discrete Fourier series on the RHS as follows:

\[
e^{\frac{a}{M}} \frac{e^{\frac{a}{M}}}{M} \sum_{j=0}^{M_2-1} \sum_{l=1}^{n} \beta_l \cdot \hat{g} \left( a + i\lambda_l + \frac{2\pi j}{M_2} \right).
\] (9)

A great advantage of this method is that these sums can all be calculated simultaneously for all \( k \in \{0, 1, 2, \ldots, M_2 - 1\} \) using the FFT algorithm of Cooley, Tukey (1965). In the end, we retain the first \( M \) values in which we are interested. For the FFT algorithm it is recommended that \( M \) be a power of 2. In the rest of the paper, we shall refer to the Iseger (2006) numerical inversion algorithm as the Gaussian-Quadrature-FFT algorithm or GQ-FFT, for short.

As an application of the tools developed so far, we now price options on realized variance in the Heston and 3/2 models. Similar to options on stocks, market practitioners express the prices of realized variance options in terms of Black-Scholes implied volatilities. Specifically, the undiscounted variance call price obtained from the model – Heston or 3/2, in our case – is matched to a Black-Scholes formula with zero rate and zero dividend yield:

\[
C(K) = C(0)N(d_1) - KN(d_2)
\]

where \( C(0) = E \left( \frac{1}{T} \int_0^T v_t \, dt \right) \) is the fair variance as seen at time 0, and

\[
d_1 = \log \left( \frac{C(0)}{K} \right) + \frac{\xi^2 T}{2}
\]
\[
d_2 = d_1 - \xi \sqrt{T}.
\]

are the usual Black-Scholes terms. The parameter \( \xi \), ensuring the equality between the model price and the Black-Scholes price, will be called the implied volatility of variance, corresponding to strike \( K \).

As a simple first numerical example, we take a standard choice for the Heston parameters from the existing literature: in an empirical investigation Bakshi, Cao, Chen (1997) estimated the following parameter set for the Heston model: \( v_0 = 0.0348, k = 1.15, \theta = 0.0348, \text{ and } \epsilon = 0.39 \). Gatheral (2006) also uses the parameter set of Bakshi et al. (1997) to analyze prices of options on realized variance. Here we apply our previous Laplace transform techniques to determine the prices of call options on 6-months realized variance. The left part of Figure (5) shows the (undiscounted) variance call function recovered over a wide strike range, from \( K = 0 \) to \( K = 0.48^2 \); the

\[\text{Note that Gatheral}(2006)\text{ use slightly modified values for } v_0 \text{ and } \theta.\]
call prices have been scaled by a notional of 10,000. A single run of the GQ-
FFT algorithm computes the variance call prices for the entire sequence of
strikes considered. As mentioned earlier, it is natural to convert these ab-
solute prices to implied volatilities of variance. The right part of Figure (5)
shows the implied volatilities of variance as a function of strike expressed as
a volatility.

Compared to the spot market, we see that volatilities of variance can be
several orders of magnitude higher than the volatility of the underlying stock
or index. Depending on the volatility strike and maturity, it is common to
see volatilities of variance in the range [50%, 150%]. For short maturities,
the implied volatilities of variance increase very quickly; this makes trading
sense, since, the shorter the period, the more uncertainty about the future
realized variance. We refer the reader to Bergomi (2005, 2008) where many
practical aspects of volatility markets are discussed. Figure (5) also reveals
the main drawback of pricing volatility derivatives in the Heston model. We
obtain a downward sloping smile for the volatility of variance whereas the
slope is strongly positive in practice; see, for example, Bergomi (2008). From
a trading and risk management perspective, it is clear that upside calls on
variance should be more expensive, since, during periods of market stress
when the volatility is high, the volatility of volatility is also very high. This
behavior cannot be captured by the Heston model.

In Figure (6), we see the implied volatilities of variance in the Heston
versus the 3/2 model, with the parameters calibrated in the previous section.
We price 3-months and 6-months options on realized variance. Notice that,
in both models, the volatilities are higher for 3-months variance than for
6-month variance; this is in line with our expectation. Most importantly,
Figure (6) shows that, unlike the Heston model, the 3/2 model generates
upward sloping volatility of variance smiles, thus capturing an important
Figure 6: Implied volatility of variance as a function of volatility strike. Left: Heston model. Right: 3/2 model. Maturities of 3 months (higher curve) and 6 months (lower curve).

feature of the volatility derivatives market.

To further investigate the differences between the Heston and 3/2 model, we compare the densities of realized variance. Since the GQ-FFT algorithm gives us the variance call function for a sequence of strikes simultaneously, we can apply the well-known Breeden, Litzenberger (1978) formula to obtain the density of realized variance as follows:

$$\phi_{RV}(K) = \frac{\partial^2}{\partial K^2} C(K).$$

Applying this formula, we obtain in the left part of figure (7) the densities of the 3-months realized variance in the Heston and 3/2 models. We see that the well behaved and accurate prices obtained by our Laplace methods, allow us to get a smooth and positive density for a wide range of variance values (in this case, from $K = 0$ to $K = 55\%^2$). We notice that the 3/2 density is much more peaked and puts less weight for variance near zero. In the equity markets, the Heston fits often violate the zero boundary test, thus assigning significant probabilities to low variance scenarios. To better explain the downward / upward sloping volatility of variance smiles, we plot, in the right panel of figure (7), the density of the log realized variance return:

$$\log \left( \frac{1}{T} \int_0^T v_t dt \right)$$

which we obtain from the density of the realized variance $\frac{1}{T} \int_0^T v_t dt$ as follows:

$$\phi_{LogRV}(k) = C(0) \cdot e^k \cdot \phi_{RV}(C(0)e^k).$$

We notice in the right panel of figure (7) that in the Heston model the variance log return is skewed to the left, while in the 3/2 model the variance
log return is skewed to the right. This explains why the variance smile is downward sloping in the Heston model and upward sloping in the 3/2 model.

4 Hedge ratios for options on realized variance

The Laplace transform method developed in the previous section can also be applied to compute hedge ratios for options on realized variance. The natural hedging instruments for options on realized variance are their underlying variance swaps. Similar to options on stocks, we need to determine a delta, here with respect to variance swaps. As shown in Broadie, Jain (2008), one can derive that the correct amount of delta, for a variance call of strike $K$, is given by:

$$\Delta_{V S} = \frac{\partial}{\partial v_0} \frac{C(K)}{\partial v_0} C(0)$$

where $v_0$ is the current value of the short variance process and $C(0) = E \left( \frac{1}{T} \int_0^T v_t dt \right)$ is the current fair variance swap rate for maturity $T$. The formula is intuitively clear: to hedge against the randomness in the short variance process, one needs to look at the ratio of the sensitivities of the two instruments — option on variance and variance swap — with respect to the value of short variance. From the inversion equation (9) of the previous section, it can be seen that in order to derive $\frac{\partial}{\partial v_0} C(K)$, we need to have the expressions for $\frac{\partial}{\partial v_0} \mathcal{L}(\lambda)$ in both models, where $\mathcal{L}(\lambda)$ is the Laplace transform of annualized realized variance. From section (2), $\mathcal{L}(\lambda)$ is obtained from Propositions (2.1) and (2.2) by setting $t = 0$, $u = 0$ and $\lambda = \frac{1}{T}$.

From Proposition (2.1), we obtain for the Heston model:

$$\frac{\partial}{\partial v_0} \mathcal{L}(\lambda) = -b(0, T) \cdot \exp \left( a(0, T) - b(0, T) v_0 \right)$$
Figure 8: Variance swap delta for call options on realized variance with six months (left) and one week (right) to expiry, as a function of option strike expressed as a volatility; Solid: 3/2 model, Dashed: Heston model.

with \( a(0,T) \) and \( b(0,T) \) as defined in Proposition (2.1). Similarly, using Proposition (2.2) and the property of the confluent hypergeometric function

\[
\frac{\partial}{\partial z} M(\alpha, \gamma, z) = \frac{\alpha}{\gamma} M(\alpha + 1, \gamma + 1, z)
\]

we obtain for the 3/2-model :

\[
\frac{\partial}{\partial v_0} L(\lambda) = -\frac{\alpha z^\alpha}{v_0} \frac{\Gamma(\gamma - \alpha)}{\Gamma(\gamma)} \left[ M(\alpha, \gamma, -z) - \frac{z}{\gamma} M(\alpha + 1, \gamma + 1, -z) \right]
\]

with \( \alpha, \gamma \) as defined in Proposition (2.2) and

\[
z = \frac{2}{\sigma^2 y(0,T)}.
\]

Using these results, we can apply the inversion tools developed previously to obtain the variance swap deltas for call options on realized variance at a sequence of strike simultaneously. In Figure (8) we show the results for the Heston and 3/2 models, calibrated in section (2). As expected, we notice a behavior of the variance swap delta similar to the delta of vanilla stock options. As we move from deep in the money calls to deep out of the money calls, the variance swap delta smoothly decreases from 1 to 0. Besides the six months maturity, we also show, in the right panel of Figure (8), a short maturity of one week and notice that delta approaches the expected digital behavior near expiry.

5 Conclusion

We have developed a fast and robust method for determining prices and hedge ratios for options on realized variance applicable in any model where
the Laplace transform of realized variance is available in closed form. The method was used to price options on realized variance in the 3/2 stochastic volatility model and in the Heston (1993) model. It has been shown that the 3/2 model offers several advantages for trading and risk managing volatility derivatives. Unlike the 3/2 model, the Heston model assigns significant weight to very low and vanishing volatility scenarios and is unable to produce extreme paths with very high volatility of volatility. Most importantly, the 3/2 model generates upward sloping implied volatility of variance smiles — in agreement with the way variance options are traded in practice. Finally, we have shown that the transform methods can be efficiently used to obtain hedge ratios for options on realized variance.

6 Appendix

Proof of Proposition (2.1) We are in a special case of the general multifactor affine framework introduced by Duffie, Pan, Singleton (2000). With $X_t = \log \left( S_t e^{(r-\delta)(T-t)} \right)$ denoting the log forward process, we have:

$$
\begin{align*}
\frac{dX_t}{v_t} &= -\frac{v_t}{2} dt + \sqrt{v_t} dB_t \\
\frac{dv_t}{v_t} &= k (\theta(t) - v_t) dt + \epsilon \sqrt{v_t} dW_t.
\end{align*}
$$

The state vector $(X_t, v_t)$ is a two-dimensional affine process as defined in Duffie et al (2000). Let

$$
\psi(X_t, v_t, t) = E \left( e^{iuX_T - \lambda \int_t^T v_s ds} \left| X_t, v_t \right. \right)
$$

denote joint Fourier-Laplace transform of $X_T$ and the de-annualized integrated variance $\int_t^T v_s ds$. Observing that the process

$$
e^{-\lambda \int_0^T v_s ds} \cdot \psi(X_t, v_t, t) = E \left( e^{iuX_T - \lambda \int_0^T v_s ds} \left| X_t, v_t \right. \right)
$$

is a martingale, an application of Ito’s Lemma gives the following partial differential equation for $\psi(x, v, t)$:

$$
\frac{1}{2} \epsilon^2 v \psi_{vv} + k (\theta(t) - v) \psi_v + \epsilon \rho v \psi_{vx} - \frac{1}{2} v \psi_{xx} - \frac{1}{2} v \psi_{xx} - \lambda \psi + \psi_t = 0
$$

with terminal condition

$$
\psi(x, v, T) = e^{iux}.
$$

Looking for a solution of the form

$$
\psi(x, v, t) = \exp (iu \alpha(T) - b(t) \psi)
$$
leads to the ODEs for \( a(\cdot, T) \) and \( b(\cdot, T) \):

\[
\begin{align*}
b' &= \frac{1}{2} e^2 b^2 + (k - i \epsilon \rho u) b - \left( \frac{1}{2} i u + \frac{1}{2} u^2 + \lambda \right), \\
a' &= k \theta(t) b
\end{align*}
\]

with terminal condition \( a(T, T) = b(T, T) = 0 \), and \( a' \), \( b' \) denoting derivatives with respect to \( t \). The complex valued ODE for \( b(\cdot, T) \) can be solved in closed form; see Cox et al. (1985) or Heston (1993). In our case, the solution for \( b(\cdot, T) \) is

\[
b(t, T) = \left( i u + u^2 + 2 \lambda \right) \left( e^{\gamma (T-t)} - 1 \right) \\
\end{align*}
\]

with

\[
\gamma = \sqrt{(k - i \epsilon \rho u)^2 + \epsilon^2 (i u + u^2 + 2 \lambda)}.
\]

Once the function \( b(\cdot, T) \) is known, the ODE for \( a(\cdot, T) \) gives:

\[
a(t, T) = - \int_t^T k \theta(s) b(s, T) ds.
\]

\[\square\]

**Proof of Lemma (3.1)\** Rewrite the limit as

\[
\lim_{K \downarrow 0} \frac{C(K) - C(0)}{K} = \lim_{K \downarrow 0} \frac{E(V - K)_+ - E(V)}{K} = \lim_{K \downarrow 0} E \left( \frac{(V - K)_+ - V}{K} \right).
\]

Since \( V > 0 \) a.s. we have

\[
\lim_{K \downarrow 0} \frac{(V - K)_+ - V}{K} = \lim_{K \downarrow 0} \frac{V - K - V}{K} = -1 \quad \text{a.s.}
\]

Using the obvious bound

\[
\left| \frac{(V - K)_+ - V}{K} \right| \leq 1
\]

we can apply the dominated convergence theorem to interchange the order of limit and expectation

\[
\lim_{K \downarrow 0} E \left( \frac{(V - K)_+ - V}{K} \right) = E \left( \lim_{K \downarrow 0} \frac{(V - K)_+ - V}{K} \right) = E(-1) = -1.
\]

\[\square\]

**Proof of Lemma (3.2)\** This result follows easily by direct integration. For \( V \sim \text{Gamma}(\alpha, \beta) \), we have

\[
\tilde{C}(K) = E(V - K)_+ = \int_{K}^{\infty} (x - K) \frac{1}{\Gamma(\alpha) \beta^\alpha} x^{\alpha - 1} e^{-\beta x} dx
\]

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Computing separately the integral corresponding to each term in the parentheses, we obtain

\[
\int_{K}^{\infty} \frac{1}{\Gamma(\alpha)\beta} x^{\alpha} e^{-\frac{x}{\beta}} dx = \int_{K}^{\infty} \frac{\alpha \beta}{\Gamma(\alpha+1)\beta^{\alpha+1}} x^{\alpha} e^{-\frac{x}{\beta}} dx = \alpha \beta (1 - F(K; \alpha + 1, \beta))
\]

and

\[
\int_{K}^{\infty} \frac{K}{\Gamma(\alpha)\beta} x^{\alpha-1} e^{-\frac{x}{\beta}} dx = K (1 - F(K; \alpha, \beta))
\]

where \(F(\cdot, \alpha, \beta)\) is the CDF of the \(\text{Gamma}(\alpha, \beta)\) distribution. \(\square\)

**Proof of Proposition (3.2)**

In

\[
h(y) = \int_{0}^{y} e^{-\frac{\alpha}{Z} z} \cdot \frac{2k}{\epsilon^2} \cdot \int_{z}^{\infty} \frac{2}{\epsilon^2} \cdot e^{\frac{2}{\epsilon^2}} \cdot u \cdot \frac{2k}{\epsilon^2} - 2 du dz.
\]

denote \(\alpha = \frac{2}{\epsilon^2} > 0\), \(\beta = \frac{2k}{\epsilon^2} < 0\) and rewrite the integral as

\[
\alpha \int_{0}^{y} e^{-\frac{\alpha}{Z} z} \int_{z}^{\infty} e^{\frac{2}{\epsilon^2}} u^{-2} du dz = \alpha \int_{0}^{y} e^{-\frac{\alpha}{Z} z} \int_{z}^{\infty} e^{\frac{2}{\epsilon^2}} \left(\frac{z}{u}\right)^{-\beta} u^{-2} du dz.
\]

In the inner integral, making the change of variable \(\frac{z}{u} = t\) we obtain

\[
\int_{z}^{\infty} e^{\frac{2}{\epsilon^2}} \left(\frac{z}{u}\right)^{-\beta} u^{-2} du = \frac{1}{z} \int_{0}^{1} e^{\frac{2}{\epsilon^2} t^{-\beta}} dt.
\]

Expanding \(e^{\frac{2}{\epsilon^2} t^{-\beta}}\) in a series gives

\[
\frac{1}{z} \int_{0}^{1} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\alpha}{z}\right)^n t^{-\beta} dt = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\alpha^n}{z^{n+1}} \frac{1}{n - \beta + 1}
\]

where we interchanged integration and summation as all terms are non-negative.

\[
h(y) = \alpha \int_{0}^{y} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\alpha^n}{z^{n+1}} \frac{1}{n - \beta + 1} dz = \alpha \sum_{n=0}^{\infty} \frac{1}{n!(n - \beta + 1)} \int_{0}^{y} \frac{\alpha^n e^{-\frac{\alpha}{Z}}}{z^{n+1}} dz
\]

Making the change of variable \(\frac{\alpha}{Z} = t\) we obtain

\[
\int_{0}^{y} \frac{\alpha^n e^{-\frac{\alpha}{Z}}}{z^{n+1}} dz = \int_{\frac{\alpha}{Z}}^{\infty} e^{-t} \cdot t^{-n-1} dt.
\]

For \(n = 0\) we recognize the exponential integral function

\[
E\left(\frac{\alpha}{y}\right) = \int_{\frac{\alpha}{y}}^{\infty} e^{-t} \cdot t^{-1} dt
\]

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and for \( n \geq 1 \) we obtain the upper incomplete Gamma function which satisfies

\[
\int_0^\infty e^{-t} \cdot t^{n-1} dt = (n-1)! \cdot e^{-\frac{\alpha}{y}} \cdot \sum_{k=0}^{n} \frac{\left(\frac{\alpha}{y}\right)^k}{k!} = (n-1)! \cdot F_{\frac{\alpha}{y}}(n)
\]

where \( F_{\nu}(\cdot) \) is the CDF of a Poisson random variable of parameter \( \nu \). Collecting the previous results we obtain

\[
h(y) = \alpha \cdot \left( E\left(\frac{\alpha}{y}\right) + \sum_{n=1}^{\infty} \frac{F_{\alpha y}(n)}{n(n-\beta+1)} \right)
\]

Uniform convergence follows from the bound

\[
\sum_{n=k}^{\infty} \frac{F_{\alpha y}(n)}{n(n-\beta+1)} < \sum_{n=k}^{\infty} \frac{1}{n(n-\beta+1)} < \sum_{n=k}^{\infty} \frac{1}{n(n+1)} = \frac{1}{k}.
\]

**Proof of Lemma (3.3)** The upper bound has been derived in the proof of Proposition (3.2). If we let \( m = \lceil -\beta \rceil \) (i.e. the smallest integer greater than or equal to \( -\beta \)), we can write

\[
R_k = \sum_{n=k}^{\infty} \frac{F_{\alpha y}(n)}{n(n-\beta+1)} > \sum_{n=k}^{\infty} \frac{1}{n(n-\beta+1)} \geq \frac{F_{\alpha y}(k)}{m+1} \sum_{n=k}^{\infty} \frac{1}{n(n+m+1)} = \frac{F_{\alpha y}(k)}{m+1} \left( \frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{k+m} \right).
\]

Let \( \bar{R} \) denote the mid-point between the two bounds i.e.

\[
\bar{R} = \frac{1}{2} \left( \frac{1}{k} + \frac{F_{\alpha y}(k)}{m+1} \left( \frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{k+m} \right) \right).
\]

and let \( p = F_{\alpha y}(k) \). We then have that \(|R_k - \bar{R}|\) must be less than or equal to half the difference between the two bounds i.e.

\[
|R_k - \bar{R}| \leq \frac{1}{2} \left( \frac{1}{k} - \frac{p}{m+1} \left( \frac{1}{k} + \frac{1}{k+1} + \cdots + \frac{1}{k+m} \right) \right).
\]

\[
= \frac{1}{2(m+1)} \sum_{j=0}^{m} \frac{j + (1-p)k}{k(k+j)}
\]

\[
< \frac{1}{2k^2(m+1)} \left( \frac{m(m+1)}{2} + (1-p)k(m+1) \right)
\]

\[
= \frac{m}{4k^2} + \frac{1-p}{2k}.
\]
Observing that

\[ 1 - p = e^{-\frac{\alpha}{\bar{y}}} \cdot \sum_{n=k+1}^{\infty} \frac{\left(\frac{\alpha}{\bar{y}}\right)^n}{n!} = e^{-\frac{\alpha}{\bar{y}}} \cdot \sum_{n=k+1}^{\infty} \frac{\left(\frac{\alpha}{\bar{y}}\right)^{n-1}}{n \cdot (n-1)!} < \frac{\alpha}{\bar{y}} e^{-\frac{\alpha}{\bar{y}}} \cdot \sum_{n=k}^{\infty} \frac{\left(\frac{\alpha}{\bar{y}}\right)^n}{n!} < \frac{\alpha}{\bar{y}}K \]

we finally obtain

\[ |R_k - \bar{R}| < \frac{m + 2\frac{\alpha}{\bar{y}}}{4k^2}. \]

\[ \square \]

**Proof of Lemma (3.4)** Bounded variation follows immediately by observing that \( C(K) = E(V - K)_+ \) is a monotone decreasing function of \( K \). Next we check that \( C(K) \in L^1[0, \infty) \) i.e.

\[ \int_0^{\infty} C(K) dK = \int_0^{\infty} E(V - K)_+ dK < \infty. \]

Since the integrand is positive, we can apply Fubini to change the order of integration and expectation:

\[ \int_0^{\infty} E(V - K)_+ dK = E \left( \int_0^{\infty} V - K dK \right) = E \left( \int_0^{V} V - K dK \right) = E \left( \frac{V^2}{2} \right) < \infty. \]

\[ \square \]

**References**


[23] Dufresne, D (2001), The integrated square-root process, research paper no. 90, University of Melbourne.


