Unspanned stochastic volatility and fixed income derivatives pricing

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Abstract

We propose a parsimonious ‘unspanned stochastic volatility’ model of the term structure and study its implications for fixed-income option prices. The drift and quadratic variation of the short rate are affine in three state variables (the short rate, its long-term mean and variance) which follow a joint Markov (vector) process. Yet, bond prices are exponential affine functions of only two state variables, independent of the current interest rate volatility level. Because this result holds for an arbitrary volatility process, such a process can be calibrated to match fixed income derivative prices. Furthermore, this model can be ‘extended’ (by relaxing the time-homogeneity) to fit any arbitrary term structure. In its ‘HJM’ form, this model nests the analogous stochastic equity volatility model of Heston (1993) [Heston, S.L., 1993. A closed form solution for options with stochastic volatility. Review of Financial Studies 6, 327–343]. In particular, if the volatility process is specified to be affine, closed-form solutions for interest rate options obtain. We propose an efficient algorithm to compute these prices. An application using data on caps and floors shows that the model can capture very well the implied Black spot volatility surface, while simultaneously fitting the observed term structure.

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1. Introduction

As with many topics in modern finance, the origin of term structure modeling can be traced back to a footnote in Merton’s work. There Merton develops the first so-called ‘short-rate’ model of the term structure of interest rates. Many refinements followed, most notably Vasicek (1977), Richard (1978), Cox et al. (1981, 1985b), Langetieg (1980), Brennan and Schwartz (1979), Longstaff and Schwartz (1992), and Duffie and Kan (1996). These papers propose increasingly more sophisticated dynamics for the instantaneous risk-free rate process and, imposing absence of arbitrage, derive cross-sectional restrictions for bond prices across maturities.

Subsequently, starting with the work of Ho and Lee (1986) and Heath, Jarrow and Morton (HJM, 1992), one strand of the literature shifted from explaining the cross-section of bond prices to the pricing of fixed-income derivatives. These so-called HJM-models relax the assumption of time-homogeneity (by effectively expanding the number of parameters) in order to fit the initial term structure perfectly. The insight of HJM is that given an initial term structure and some specification for the volatility structure (of forward rates or bond prices), the risk-neutral dynamics of the short-rate process are completely determined. One difficulty that arises, however, is that for most volatility structure specifications the resulting short-rate process is not Markov. Often, this renders the pricing of derivatives (especially for those securities with early exercise features) computationally challenging. As a result, a large body of literature has been devoted to restricting volatility structures to functional forms which generate a simple Markov structure for the short rate (Carverhill, 1994; Jeffrey, 1995; Cheyette, 1995; Ritchken and Sankarasubramaniam, 1995; de Jong and Santa-Clara, 1999). One particular HJM specification that is often convenient is the so-called ‘extended’ form of standard short-rate models. This consists of using a short-rate model where one or a few parameters are specified to be time-dependent and are chosen so that bond prices (which possess closed-form solutions) fit the initial term structure. This approach makes clear the one-to-one mapping between the short-rate models and the HJM models: any HJM model gives rise to a (possibly non-Markov and time-inhomogeneous) short-rate model. Conversely, any (finite-factor) short-rate model gives rise to an HJM model with a specific forward rate volatility structure. As an example, Hull and White (1990) derive the extended Vasicek model. The equivalence to the corresponding HJM model is shown in Rogers (1995).

Recently, a number of papers have investigated whether the existing models can capture the joint dynamics of term structure and plain-vanilla derivatives such as caps and swaptions. Simple, model-independent evidence based on principal component analysis suggests that there are factors driving Cap and Swaption implied volatilities that do not drive the term structure (Collin-Dufresne and Goldstein (CDG),

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1 See footnote 42 in Merton’s (1973) seminal paper on option pricing.
2 For a recent survey of the developments of term structure literature, see, e.g., Dai and Singleton (2003).
CDG call this feature ‘unspanned stochastic volatility’. Simply put, it appears that fixed-income derivatives such as Caps and Swaptions cannot be perfectly replicated by trading (even in a very large number of) the underlying bonds. That is, in contrast to the predictions of standard short-rate models, bonds do not span the fixed income market. The evidence in Longstaff et al. (2001), Jagannathan et al. (2003) suggests that it is indeed difficult to capture the joint dynamics of fixed income derivatives and forward rates within models where all factors affect the term structure. Subsequent empirical studies find evidence in favor of USV (Collin-Dufresne et al., 2002; Han, 2004; Li and Zhao, forthcoming; Jarrow et al., 2004). Practitioners also seem to find that USV is useful to capture joint dynamics of derivatives and term structure. (Andersen and Brotherton-Ratcliffe (2001), for example, propose a stochastic volatility LIBOR market model where the changes in the forward LIBOR rate are independent of changes in its volatility.)

In this paper we propose an extended USV model of the term structure of interest rates. This model by construction fits the observed yield curve. Furthermore, it admits a low dimensional three-factor Markov representation for forward rates (and hence for the yield curve) in terms of the short rate, its volatility and its long-term mean. We obtain closed-form solutions for standard European style fixed income securities, such as Caps and Floors. Further, because the model is Markov, American-style claims can be priced using standard PDE techniques. We also present the HJM-version of the model. Interestingly, in its HJM form, this model is remarkably similar to the Heston (1993) equity option model with stochastic volatility. The model possesses the interesting property that it can be calibrated separately to the term structure of interest rates as well as to the term structure of cap/floor volatilities. Indeed, since the volatility is unspanned, its parameters do not affect the bond market. In this sense, the model is effectively ‘doubly-extended’. That is, as in Hull and White’s extended Vasicek model, we let the long-term mean of the short rate be time-dependent and we use it to fit the term structure of interest rates. Then, we also let the long-term mean of the volatility process be time dependent and use this to fit the term structure of ATM cap volatilities. The remarkable USV feature allows these two calibration procedures to be performed independently, which is computationally very advantageous, and contrasts with other stochastic volatility models such as Longstaff and Schwartz (1992), for example. Indeed, any change in a parameter driving the volatility in a standard model would also affect the term structure. We further show that in a USV model the correlation coefficient between the short rate and its volatility governs the skew of the implied volatility smile – a result analogous to that found by Heston (1993) for stock options. In particular, using data from Swaps, Caps/Floors we show that the model can capture very well the observed shape of the smile across maturity and strikes (by construction in its ‘doubly extended form’ it can fit perfectly the term structure of swap rates and the ATM implied Cap volatilities).

Closely related is the paper by Andreasen et al. (1997) who proposed an HJM framework with USV for derivative pricing. Unfortunately, their model does not

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4 This is not unanimous, however, Fan et al. (2001) argue against USV.
admit simple explicit solutions for derivatives or simple Markov representation for
the term structure. Thus they have to resort to numerical techniques (non-recombin-
ing trees) to obtain derivative prices.\footnote{de Jong and Santa-Clara (1999) and Collin-Dufresne and Goldstein (2002a) also investigate related frameworks.}

The rest of the paper is as follows: Section 1 presents the short-rate USV model. Section 2 presents the HJM version of the model and identifies the similarity to Heston’s equity model. Section 3 gives the closed-form option prices. Section 4 presents calibration results using panel data on cap/floors and swap rates. Section 5 concludes.

2. An example of USV term structure model

In this section we present the simplest extension to the one-factor short-rate
model of Vasicek (1977) that displays USV. First we derive an explicit solution
for bond prices in that framework. Then we show that this model can be extended
in the sense of Hull and White (1990) to fit any observed term structure of interest
rates. Finally, we identify the equivalent HJM form of this model and demonstrate
the similarity to Heston’s (1993) equity stochastic volatility model.

2.1. A short-rate USV model

Probability the most widely used one-factor short-rate model is that of Vasicek (1977). However, it is well-known that multiple factors are needed to capture the
dynamics of the term structure (Litterman and Scheinkman, 1991). Further, there
is substantial evidence that short-rate volatility is stochastic (Andersen and Lund,
1997; Brenner et al., 1996). Thus we consider the following three-variable, two-factor
extension of the one-factor mean-reverting short-rate model. We assume there exists
a standard filtered probability space \((\Omega, F, \{F_t\}, P)\) where the filtration is the natural
filtration generated by a two-dimensional vector of independent Brownian motions
\((z_r, z_v)\). We further assume that there exists a measure \(Q\) equivalent to \(P\) under which
discounted security prices are martingales. It is well-known that this is sufficient to
rule out arbitrage opportunities (e.g., Duffie, 2001). We assume the short rate has
the following dynamics under the risk-neutral measure:

\[
dr_t = \kappa_r(\theta_t - r_t)dt + \sqrt{\nu_t}dz^Q_r(t),
\]

\[
d\theta_t = \left(\gamma_\theta(t) - 2\kappa_r\theta_t + \frac{\nu_t}{\kappa_r}\right)dt,
\]

\[
d\nu_t = \mu_v(v_t, t)dt + \sigma_v(v_t, t)dz^Q_v(t),
\]

where \(\gamma_\theta(\cdot)\) is a deterministic function of time, and \(dz^Q_r(t)dz^Q_v(t) = 0\). The drift and
volatility components of the \(v\)-dynamics \(\{\mu_v(\cdot, \cdot), \sigma_v(\cdot, \cdot)\}\) are completely arbitrary
so long as they satisfy sufficient regularity conditions for the system of stochastic differential equations to admit a unique strong solution (see Duffie, 2001, Appendix D). We note that this model nests the simple one-factor model of Vasicek (1977) (which corresponds to constant $\theta, v$), as well as the Fong and Vasicek (1991) stochastic volatility model (square-root volatility process, constant $\theta$). Note also that even though this is a two-factor model (i.e., there are two Brownian motions) the short rate is Markov in three state variables ($r, \theta, v$). Collin-Dufresne and Goldstein (2002a) show that, in a time-homogeneous framework, at least three state variables are needed to obtain USV. Here we demonstrate that this model exhibits USV.

**Proposition 1.** Suppose the risk-neutral dynamics of the short rate are given by Eqs. (1)–(3) above. Then bond prices take the form 

$$
P_T(t, r_t, \theta_t) = \exp \left\{ A(t, T) - B_r(t)r_t - B_\theta(t)\theta_t \right\},
$$

(4)

where

$$
B_r(t) = \frac{1}{\kappa_r} (1 - e^{-\kappa_r t}),
$$

(5)

$$
B_\theta(t) = \frac{1}{2\kappa_r} (1 - e^{-\kappa_r t})^2,
$$

(6)

$$
A(t, T) = -\int_t^T ds \gamma_\theta(s) B_\theta(T - s).
$$

(7)

**Proof.** Before providing a formal proof, note that if we posit that the solution takes the form as in Eq. (4), and then use

$$
r = \frac{1}{dt} E^Q \left[ \frac{dP}{P} \right] = P_t + P_t \kappa_r(\theta_t - r_t) + \frac{1}{2} P_t \kappa_r v_t + P_t \left( \gamma_\theta(t) - 2\kappa_r \theta_t + \frac{v_t}{\kappa_r} \right),
$$

(8)

and then collect terms that are linear in $\{r, \theta, v\}$ and terms independent of all three state variables, we obtain the four equations:

$$
0 = -1 + B_r' + B_r \kappa_r,
$$

(9)

$$
0 = B_\theta' - B_r \kappa_r + 2\kappa_r B_\theta,
$$

(10)

$$
0 = -\frac{1}{\kappa_r} B_\theta + \frac{1}{2} B_r^2,
$$

(11)

$$
0 = \gamma_\theta(t) B_\theta.
$$

(12)

Note that our proposed solution in Eqs. (5)–(7) satisfy these four equations.

More formally, we claim that in order to prove the proposition it is sufficient to show that $\exp(\int_0^T r_s ds)P_T(t)$ is a $Q$-martingale for $P_T(t)$ as defined in Eq. (4).
Indeed, in that case it follows that \( \exp(-\int_0^t r_s \, ds)P^T(t) = E^Q_t[\exp(-\int_0^T r_s \, ds)P^T(T)] \), which implies
\[
P^T(t) = E^Q_t\left[ \exp\left(-\int_0^T r_s \, ds\right) \right],
\]
(13)
since Eqs. (5)–(7) imply \( P^T(T) = 1 \). To show that \( \exp(-\int_0^t r_s \, ds)P^T(t) \) is a \( Q \)-martingale we apply Itô’s lemma to Eq. (4). Using the system of ODE satisfied by \( A(\cdot), B_\alpha(\cdot) \) and \( B_\beta(\cdot) \), we find that
\[
E^Q_t[\partial P^T(t) - r_s P^T(t) \, dt] = 0.
\]
(14)
A standard argument then shows that \( \exp(-\int_0^t r_s \, ds)P^T(t) \) is a \( Q \)-martingale.

We make a few remarks about this model. First, it provides a simple counterexample to the conjecture that there is a one-to-one relationship between affine short-rate dynamics and exponential affine bond prices. This result holds, as proved by Duffie and Kan (1996), for the one-factor case. However, the example above shows that one can have non-affine state vector and yet obtain an exponential affine model for bond prices. In other words, we have shown that

\[ \text{[Affine short-rate model]} \Leftrightarrow \text{[Exponential affine bond prices].} \]

Indeed, the dynamics of the volatility of the short rate can be arbitrarily chosen in the model we proposed. (In particular, it could be chosen so that the characteristic function of the short rate is not exponentially affine in the state vector – the defining characteristic of affine models – Duffie et al. (2001)).

Second, one cannot ‘rotate’ the state vector from \( \{r, \theta, v\} \) to a collection of three yields (i.e., this is not a ‘yield factor model’ in the sense of Duffie and Kan (1996)), since all yields depend only upon \( \{r, \theta\} \), and hence are independent of \( v \). That is, the model exhibits USV in the sense that bond prices are insensitive to volatility-risk, and hence cannot be used to hedge volatility-risk. This is most easily seen by deriving the risk-neutral bond price dynamics:
\[
\frac{dP^T(t)}{P^T(t)} = r(t)dt - B_r(T-t)\sqrt{v_t}d\phi^Q(t),
\]
(15)
\[
dv(t) = \mu_v(v_t,t)dt + \sigma_v(v_t,t)d\phi^Q_v(t).
\]
(16)
We see that changes in bond prices, which are driven solely by \( d\phi^Q(t) \), an uncorrelated with changes in volatility, which are driven by \( d\phi^Q_v(t) \). In fact, neither the volatility drift or its diffusion \( (\mu_v, \sigma_v) \) affect the shape of the term structure. In that sense, the volatility process is derivative-specific. That is, they cannot be pinned down by using information on the term structure alone. In a sense, USV models also offers a possible economic rational for the existence of fixed income derivatives: even in

\[ \text{Merton (1992, Chap. 14) discusses the well-known Hakansson (1979) paradox of derivative securities in the context of contingent claim analysis.} \]
the presence of continuous trading they are necessary to complete fixed income mar-
kets. In this model, dynamic trading strategies in a continuum of bonds are not suf-
ficient to span all sources of risk.

2.2. Extended USV model: Calibration of the term structure

Following the insights of Hull and White (1990), we demonstrate in this section
that the short-rate model introduced in the previous section can be calibrated to
fit the initial term structure by an appropriate choice of the deterministic function
\( \gamma_0(\cdot) \). Suppose that we observe the instantaneous term structure of forward rates
\( f(0, t) \forall t \). By definition, forward rates and zero-coupon bond prices are related by

\[
P^T(t) = \exp \left( - \int_t^T f(t, s) \, ds \right).
\]

We seek a function \( \gamma_0(\cdot) \) so that model-implied forward rates fit the initially observed
term structure of interest rates. We obtain the following result.

**Proposition 2.** Suppose the function \( \gamma_0(\cdot) \) is given by

\[
\gamma_0(T) = \frac{1}{\kappa_r} \hat{f}_{TT}(0, T) + 3\hat{f}_T(0, T) + 2\kappa_r \hat{f}(0, T).
\]

Then the forward rates generated by the short-rate model (1)–(3) equal the observed
forward rates, i.e.,

\[
f(0, T) = -\partial_T (\log P^T(0)) = \hat{f}(0, T) \quad \forall T.
\]

**Proof.** Using Eq. (4) we find that model implied forward rates are given by

\[
\hat{f}(0, T) = -A_T(0, T) + (r_0 + \theta_0)e^{-\kappa_r T} - \theta_0 e^{-2\kappa_r T}.
\]

Differentiating this equation two times (assuming that the observed forward curve is
sufficiently smooth) we find

\[
\hat{f}_T(0, T) = -A_T(0, T) - \kappa_r (r_0 + \theta_0)e^{-\kappa_r T} + 2\kappa_r \theta_0 e^{-2\kappa_r T},
\]

\[
\hat{f}_{TT}(0, T) = -A_{TT}(0, T) + \kappa_r^2 (r_0 + \theta_0)e^{-\kappa_r T} - 4\kappa_r \theta_0 e^{-2\kappa_r T}.
\]

It is convenient to define the integrals

\[
I_1 = \int_0^T dt \gamma_0(t)e^{-\kappa_r (T-t)},
\]
\[ I_2 = \int_0^T \gamma_0(t)e^{-2\kappa_r(T-t)}dt. \] (24)

Note that \( A(0, T) \) has been determined in Eq. (7). Furthermore, one can time-differentiate Eq. (7) to obtain

\[
A_T(0, T) = -I_1 + I_2, \tag{25}
\]

\[
A_{TT}(0, T) = +\kappa_r I_1 - 2\kappa_r I_2, \tag{26}
\]

\[
A_{TTT}(0, T) = -\gamma_0(T)\kappa_r - \kappa_r^2 I_1 + 4\kappa_r^2 I_2. \tag{27}
\]

Eqs. (25)–(27) can be combined to eliminate \( I_1 \) and \( I_2 \) producing

\[
\gamma_0(T) = -\frac{1}{\kappa_r}A_{TTT} - 2\kappa_r A_T - 3A_{TT}. \tag{28}
\]

Then using Eqs. (20)–(22) to eliminate \( A_T, A_{TT}, \) and \( A_{TTT} \), we obtain Eq. (18). \( \square \)

We note that the short-rate model is now time-inhomogeneous in that the drift depends explicitly on time. Indeed, the dynamics of the short rate can now be specified as

\[
dr_t = \kappa_r(\theta_t - r_t)dt + \sqrt{\nu_r}d\zeta^r(t), \tag{29}
\]

\[
d\theta_t = \left( \frac{1}{\kappa_r}f_{TT}(0, t) + 3f_T(0, t) + 2\kappa_r f(0, t) - 2\kappa_r \theta_t + \frac{1}{\kappa_r} \nu_t \right)dt, \tag{30}
\]

\[
d\nu_t = \mu_v(\nu_t, t)dt + \sigma_v(\nu_t, t)d\zeta^\nu(t). \tag{31}
\]

In this derivation of the model, the dynamics of this short-rate model may appear a bit contrived.\footnote{This is necessary so that the four equations (9)–(12) can be satisfied even though there are only three ‘unknowns’ \( \{A, B_r, B_0\} \).} However, in the next section we show that such dynamics arise naturally in a HJM framework once the absence of arbitrage restriction derived by HJM (1992) is imposed.

### 2.3. Equivalence to HJM model

Suppose that instead of starting within the short-rate model, our starting point is some arbitrage free dynamics for the futures curve, \( \{f(T, t)\}_{0 \leq t \leq T} \) which satisfies the following:

1. An initial condition \( f(0, T) = \hat{f}(0, T) \\forall T \) (i.e., the model fits the initial term structure by construction), and
2. some dynamics:
Absence of arbitrage requires that the drift be equal to the risk-free rate which implies
\[ \mu_f(t,T) = \sigma_f(t,T) \int_t^T \sigma_f(t,s) ds \]
(32)
which as shown by HJM (1992) must satisfy \( \mu_f(t,T) = \sigma_f(t,T) \int_t^T \sigma_f(t,s) ds \) to be arbitrage-free.8

Furthermore, consider an HJM model where the forward rate volatility structure is a product of a state variable \( v \) driving stochastic volatility, and an exponential time decay structure:
\[ df(t,T) = v_t \left( \frac{1}{K_r} \right) \left( e^{-\kappa_r(T-t)} - e^{-2\kappa_r(T-t)} \right) dt + \sqrt{v_t} e^{-\kappa_r(T-t)} dz^O(t), \]
(33)
\[ dv_t = \mu_v(v_t,t) dt + \sigma_v(v_t,t) dz^O(t). \]
(34)
It is easy to verify that this model satisfies the HJM (1992) no-arbitrage restriction. Indeed, applying Itô’s lemma to zero-coupon bond prices given by \( P^T(t) = \exp(-\int_t^T f(t,s) ds) \) we obtain their risk-neutral dynamics:
\[ \frac{dP^T(t)}{P^T(t)} = r(t) dt - B_r(T-t) \sqrt{v_t} dz^O(t), \]
(35)
\[ dv(t) = \mu_v(v_t,t) dt + \sigma_v(v_t,t) dz^O(t). \]
(36)
Since \( E^Q\left[ \frac{dP^T(t)}{P^T(t)} \right] = rd_t \), it follows that discounted bond prices are martingales under the risk-neutral measure, which is sufficient to rule our arbitrage (Harrison and Kreps, 1979). Further, we see that by construction this is a USV model in that shocks to volatility are uncorrelated with those driving bond prices. Indeed, comparing the dynamics with those derived in the previous model (Eq. 15), we see that they are identical, which suggests the equivalence between this HJM model and the extended short-rate model of the previous section. We prove this equivalence by showing that this HJM framework generates the same process for the risk-free rate (and thus bond prices and all other fixed income derivatives). We have the following result:

**Proposition 3.** Consider the HJM model given by Eqs. (33) and (34) above (and some appropriate initial condition \( f(0,T) = \hat{f}(0,T) \forall T \)). Then, the short rate generated in the model is identical to that given in Eqs. (29)–(31).

---

8 This is easily seen by applying Itô to the bond price \( P^T(t) = \exp(-\int_t^T f(t,s) ds) \) Then
\[ \frac{dP^T(t)}{P^T(t)} = \left( r_t - \int_t^T \mu_f(t,s) ds + \frac{1}{2} \left( \int_t^T \sigma_f(t,s) ds \right)^2 \right) dt + \left( \int_t^T \sigma_f(t,s) ds \right)^2 dz^O(t). \]

Absence of arbitrage requires that the drift be equal to the risk-free rate which implies
\[ \int_t^T \mu_f(t,s) ds = \frac{1}{2} \left( \int_t^T \sigma_f(t,s) ds \right)^2. \]
Differentiating the latter with respect to \( T \) gives the result.
Proof. By definition we have
\[ r_t = f(t, t) = f(0, t) + \int_0^t df(s, t) = f(0, t) + H(t) - I(t), \] (37)
where we have defined
\[ H(t) = \int_0^t \frac{v_s}{\kappa_r} ds + \int_0^t \frac{v_s}{\kappa_r} \sqrt{v_r} dz_r(t), \] (38)
\[ I(t) = \int_0^t \frac{v_s}{\kappa_r} e^{-2\kappa_r (t-s)} ds. \] (39)

Applying Itô’s lemma we obtain the following dynamics:
\[ dH(t) = \left( \frac{v_t}{\kappa_r} - \kappa_r H(t) \right) dt + \sqrt{v_r} dz_r(t), \] (40)
\[ dI(t) = \left( \frac{v_t}{\kappa_r} - 2\kappa_r I(t) \right) dt. \] (41)

Hence, from Eq. (37) we have
\[ dr(t) = f(0, t) dt + dH(t) - dI(t) = \kappa_r (\theta_t - r_t) dt + \sqrt{v_r} dz_r(t), \] (42)
where
\[ \theta_t = \frac{1}{\kappa_r} f(0, t) + I(t) + f(0, t). \] (43)

By applying Itô’s lemma and using Eq. (41), it follows that the dynamics of \( \theta_t \) are identical to those given in Eq. (30). The result thus follows. \( \square \)

We have thus provided a simple two-factor HJM model which displays unspanned stochastic volatility, whose equivalent short-rate model is Markov in three state variables. Note that remarkably, all of this is performed for arbitrary volatility dynamics \( \mu_v, \sigma_v \)! This result emphasizes that in this model volatility is derivative-specific, i.e., the current value of the volatility state variable, along with parameters controlling its drift and its volatility, all need to be backed-out from derivative prices. The corresponding bond price dynamics exhibited in Eqs. (35) and (36) show that the model developed here is the fixed income derivative counterpart to the Hull and White (1987) stochastic volatility equity model. Indeed, shocks to volatility of bonds are independent from shocks to bond prices themselves. This has two immediate implications: First, for pricing derivatives similar techniques can be used. In particular, we note that given the Markov representation, it is possible to price derivatives using standard PDE techniques. Second, for hedging derivatives at least one additional derivative is necessary, since bond prices do not span volatility-risk.
3. USV term structure modeling: Extensions

As is well-known, one parameter that is crucial for explaining observed skews in the implied volatility surface of option prices is the correlation between equity prices and volatility (Heston, 1993). The simple model presented in the previous section is restrictive in that it assumes no correlation between innovations in volatility and bond prices. We chose to present that model since it is the simplest extension to the Vasicek (1977) model that also displays USV. In this section we present a more general three-state variable model that displays USV and can capture correlation between bond price dynamics and its volatility. In particular, we consider the following short-rate model:

\[
\begin{align*}
\text{d}r_t &= \kappa_r (\theta_t - r_t) \text{d}t + \sqrt{\vartheta_r + \psi_t} \text{d}z_r^O(t) + \sigma_{rv} \sqrt{\vartheta_t} \text{d}z_v^O(t) + \sigma_{r\theta} \text{d}z_{\theta}^O(t), \\
\text{d}\theta_t &= \left( \gamma_{\theta}(t) - 2 \kappa_r \theta_t + \left( \frac{1 + \sigma_{rv}^2}{\kappa_r} \right) \psi_t \right) \text{d}t + \sigma_{\theta} \text{d}z_{\theta}^O(t), \\
\text{d}v_t &= \gamma_v(r_t, \theta_t, v_t, x_t, t) \text{d}t + \sigma_v(r_t, \theta_t, v_t, x_t, t) \text{d}z_v^O(t), \\
\text{d}x_t &= \gamma_x(r_t, \theta_t, v_t, x_t, t) \text{d}t + \sigma_x(r_t, \theta_t, v_t, x_t, t) \text{d}z_x^O(t).
\end{align*}
\]

Here, the state variable \( v \) and the vector \( x \) follow an arbitrary Markov process with coefficients \( \gamma_v(r_t, \theta_t, v, x, t) \), \( \sigma_v(r_t, \theta_t, v, x, t) \), \( \gamma_x(r_t, \theta_t, v, x, t) \), \( \sigma_x(r_t, \theta_t, v, x, t) \) which satisfy standard Lipschitz and growth conditions.\(^9\) Clearly this model nests the example analyzed in the previous section. In particular, it can accommodate correlation between short rate and volatility through the parameter \( \sigma_{rv} \), allows for a stochastic long-term mean, which translates in stochastic slope factor (Collin-Dufresne et al., 2002) and allows for multi-factor volatility process. We can prove the analogous to Proposition 1.\(^10\)

**Proposition 4 (Short-rate model).** Suppose the short-rate model is given by Eqs. (44)–(47), then bond prices take the form \((T - t) \equiv \tau)\):

\[
P^T(t, r_t, \theta_t) = \exp \{ A(t, T) - B_r(T - t) r_t - B_{\theta}(\tau) \theta_t \}, \tag{48}
\]

independent of \( v \) and \( x \), where

\[
\begin{align*}
B_r(\tau) &= \frac{1}{\kappa_r} \left( 1 - e^{-\kappa_r \tau} \right), \tag{49} \\
B_{\theta}(\tau) &= \frac{1}{2\kappa_r} \left( 1 - e^{-\kappa_r \tau} \right)^2, \tag{50}
\end{align*}
\]

\(^9\) These conditions are necessary for the SDE to admit a unique strong solution. See Duffie (2001, Appendix D).

\(^10\) Since the proofs in this section are analogous to those in the previous section we state the proposition without proof.
The model thus displays unspanned stochastic volatility. As before, we can select the function $\gamma_0$ to fit the observed term structure of yields. Indeed we can prove the following:

**Proposition 5** (Extended short-rate model). *Suppose that in the short-rate model of Eqs. (44)–(47) the function $\gamma_0(\cdot)$ is chosen so as to satisfy the following:

$$
\gamma_0(T) = \frac{1}{k_r} \hat{f}_{TT}(0, T) + 3 \hat{f}_T(0, T) + 2k_r \hat{f}(0, T) + 2k_r \beta_0 + 6\beta_3 k_r^2 e^{-3k_r T}
+ 24 \beta_4 k_r^2 e^{-4k_r T},
$$

(52)

where $\beta_0 - \beta_4$ are coefficients given in Appendix A. Then the model implied term structure of forward rates $f(0, T) = -\partial_T \log P^T(t)$ equals the forward rates $f(0, T)$ observed at date 0 for all maturities $T$.

Finally, we obtain the following equivalent HJM model:

**Proposition 6** (Equivalent HJM model). *The extended short-rate model of Proposition 4 is equivalent to the HJM model defined by (i) initial conditions $f(0, T) = \hat{f}(0, T)$ $\forall T$ (equivalently $P^T(0) = \tilde{P}^T(0) \forall T$), and (ii) bond price and forward rate dynamics given by

$$
\frac{dP(t, T)}{P(t, T)} = r(t, T) dt - B_r(T - t) \left( \sqrt{x_r + v_r \sigma_0^2(t)} + \sigma_r \sqrt{v_r} \sigma_0^2(t) + \sigma_{\sigma_0} \sigma_0^2(t) \right)
- \sigma_{\sigma_0} \sigma_0^2(t) \right),
$$

(53)

$$
\frac{df(t, T)}{f(t, T)} = \mu_f(t, T) dt - e^{-\kappa_r(T-t)} \left( \sqrt{x_r + v_r \sigma_0^2(t)} + \sigma_r \sqrt{v_r} \sigma_0^2(t) + \sigma_{\sigma_0} \sigma_0^2(t) \right)
- e^{-2\kappa_r(T-t)} \sigma_{\sigma_0} \sigma_0^2(t),
$$

(54)

$$
dv_i = \gamma_e(v_i, x_i, t) dt + \sigma_e(v_i, x_i, t) ds_i(t),
$$

(55)

$$
dx_i = \gamma_x(v_i, x_i, t) dt + \sigma_x(v_i, x_i, t) ds_i(t),
$$

(56)

where absence of arbitrage imposes that

$$
\mu_f(t, T) = (\kappa_r + (1 + \sigma_r^2) v_r) (e^{-\kappa_r(T-t)} - e^{-2\kappa_r(T-t)})
+ (\sigma_{\sigma_0} e^{-\kappa_r(T-t)} - \sigma_0 e^{-2\kappa_r(T-t)}) \int_t^T (\sigma_{\sigma_0} e^{-\kappa_r(s-t)} - \sigma_0 e^{-2\kappa_r(s-t)}) ds.
$$

(57)
Analogous to the previous section, we can demonstrate the equivalence between the short-rate model and the HJM model where one takes as a starting point the forward or bond price dynamics given in (53) and (54). The remarkable feature of this model is that the cross-section of the term structure is basically pinned down independently of the volatility dynamics which can be freely calibrated to derivative prices. In particular, since the model also allows for correlation between volatility and interest rates, this model offers a framework that can possibly cope with the implied interest rate volatility skews, analogously to Heston’s (1993) model.

4. Option pricing and implied volatility smile

In this section, we specialize the USV model’s volatility dynamics to be affine and show that in this framework the model admits closed-form solutions for European zero-coupon options (and hence Cap and Floors). We then apply the model to cap/floor implied volatilities and show that the model can capture reasonably well the surface of implied at-the-money cap volatilities.

4.1. Pricing zero-coupon bond options (i.e., caps)

Specifically, we consider the two-factor USV model (special case of Proposition 6) with volatility dynamics that follow a square root process:

\[
\frac{dP^T(t)}{P^T(t)} = r_t dt - B_r(T - t)\left(\sqrt{\gamma_r(t)} + v_t dz^O_r(t) + \sigma_{tr}\sqrt{v_t}dz^O_v(t)\right),
\]

(58)

\[
dv_t = (\gamma_v(t) - \kappa_v v_t) dt + \sigma_v\sqrt{v_t}dz^O_v(t).
\]

(59)

We allow \(\gamma_r(\cdot)\) to be a deterministic function of time. Since volatility parameters are derivative-specific in the USV framework, \(\gamma_r(\cdot)\) will be determined from option prices. Specifically, as we demonstrate below, it is helpful for capturing the term structure of ATM cap volatilities. Second, the parameter \(\sigma_{rv}\) governs the correlation between bond prices and volatility which in turn governs the skew of implied volatility. We note that this stochastic volatility term structure model is rather analogous to Heston’s (1993) stochastic volatility equity model. One main difference is that in the bond case, the stochastic differential equation is specified for a continuum of bonds. Further, as our previous results show, this model is a two-factor, three-variable Markov process of the short rate.

First, we build on Heston (1993), Duffie et al. (2001), Chacko and Das (2002), and Collin-Dufresne and Goldstein (2002a,b,c) to show that explicit solutions for pricing of call and put options on zero-coupon bond prices can be obtained, based on Fourier transform techniques.

The payoff of a European bond-option (or caplet) with exercise date \(T_0\) is

\[
C_p(t = T_0) = (P^T_0(T_0) - K)\mathbf{1}_{\{P^T_0(T_0) > K\}}.
\]

(60)
The price of the bond-option at date-\(t\) before expiration can be expressed as
\[
C_P(t) = E^Q_t \left[ \exp \left( - \int_t^{T_0} ds r_s \right) (P^{T_1}(T_0)) - K \right] \mathbf{1}_{\{P^{T_1}(T_0) > K\}}
\]
\[
= E^Q_t \left[ \exp \left( - \int_t^{T_0} ds r_s \right) e^{\log P^{T_1}(T_0)} \mathbf{1}_{\{\log P^{T_1}(T_0) > \log K\}} - KE^Q_t \left[ \exp \left( - \int_t^{T_0} ds r_s \right) \mathbf{1}_{\{\log P^{T_1}(T_0) > \log K\}} \right] \right]
\]
\[
= \Psi_{t,1}(\log K) - K \Psi_{t,0}(\log K),
\]
where we have defined
\[
\Psi_{t,a}(k) \equiv E^Q_t \left[ \exp \left( - \int_t^{T_0} r_s ds \right) \exp(a \log P^{T_1}(T_0)) \mathbf{1}_{\{\log P^{T_1}(T_0) > k\}} \right].
\]
To evaluate this expression, we introduce the following transform:
\[
\psi_t(\alpha) \equiv E^Q_t \left[ \exp \left( - \int_t^{T_0} r_s ds \right) e^{a \log P^{T_1}(T_0)} \right]
\]
for some (complex valued) \(\alpha\).

Now, note that
\[
\frac{\Psi_{t,a}(k)}{\psi_t(\alpha)} = E^Q_t \left[ \exp \left( - \int_t^{T_0} r_s ds \right) \exp(a \log P^{T_1}(T_0)) \right] \mathbf{1}_{\{\log P^{T_1}(T_0) > k\}} \psi_t(\alpha)
\]
\[
\equiv E^p_t \mathbf{1}_{\{\log P^{T_1}(T_0) > k\}} = \tilde{P}_t(\log P^{T_1}(T_0) > k),
\]
where the probability measure \(\tilde{P}\) equivalent to \(Q\) is defined by the following Radon–Nykodim derivative:
\[
\frac{d\tilde{P}}{dQ} = \frac{\exp \left( - \int_t^{T_0} r_s ds \right) \exp(a \log P^{T_1}(T_0))}{\psi_t(\alpha)}. \quad (65)
\]
Following Heston (1993), Duffie et al. (2000), and others, we can therefore use the Fourier inversion theorem for the random variable \((\log P^{T_1}(T_0))\) to obtain
\[
\Psi_{t,a}(k) = \frac{\psi_t(\alpha)}{2} + \pi \int_0^\infty \frac{\text{Im}[\psi_t(\alpha + iv)e^{-ik}]}{v} \, dv. \quad (66)
\]

Note that evaluating \(\Psi_{t,a}(k)\) requires only a single numerical integral to be performed, even though the derivative may be written on an arbitrary number of bond-yields, and even though each bond-yield is driven by a risk-factor that cannot be hedged by positions in other bonds. In fact, following Carr and Madan (1999) we show below how to use a Fast Fourier Transform algorithm to efficiently perform the inversion for option prices of multiple strikes at the same time.

To evaluate option prices the only requirement is thus to obtain an explicit expression for the transform given in Eq. (63). We prove the following:
Proposition 7. Let \( \psi_t(a) \) be the transform defined in Eq. (63) above:

\[
\psi_t(a) = E_t^Q \left[ \exp \left( - \int_t^{T_0} r_s \, ds \right) \exp(a \log \frac{P_{T_1}(t)}{P_{T_0}(t)}) \right]
\]

for some (complex valued) parameter \( a \). The transform admits the following simple solution:

\[
\psi_t(a) = P_{T_0}(t) \exp \left( M(t) + N(t)v(t) + a \log \frac{P_{T_1}(t)}{P_{T_0}(t)} \right),
\]

where the deterministic functions \( \{M(\cdot), N(\cdot)\} \) in Eq. (67) satisfy the system of ODEs given by

\[
0 = N'(t) + \frac{\sigma_r^2}{2} N(t)^2 - N(t)(\kappa_v + \sigma_v \sigma_{rv}(aB_r(T_0 - t) + (1 - a)B_r(T_1 - t))) + \frac{1}{2} \sigma_v^2 V(t),
\]

(68)

\[
0 = M'(t) + N(t)\gamma_v(t) + \frac{\sigma_v}{2} V(t),
\]

(69)

subject to the boundary conditions \( M(T_0) = N(T_0) = 0 \). Here, \( V(t) \) is defined as

\[
V(t) = (aB_r(T_1 - t) + (1 - a)B_r(T_0 - t))^2 - aB_r(T_1 - t)^2 - (1 - a)B_r(T_0 - t)^2.
\]

(70)

Proof. The proof consists in showing that the process

\[
X(t) = \exp \left( - \int_t^T r_s \, ds \right) P_{T_0}(t) \exp \left( M(t) + N(t)v(t) + a \log \frac{P_{T_1}(t)}{P_{T_0}(t)} \right)
\]

is a \( Q \)-martingale. Indeed, if this is the case then we have

\[
X(t) = E_t^Q[X(T_0)]
\]

\[
= E_t^Q \left[ \exp \left( - \int_0^{T_0} r_s \, ds \right) P_{T_0}(T_0) \exp \left( M(T_0) + N(T_0)v(T_0) + a \log \frac{P_{T_1}(T_0)}{P_{T_0}(T_0)} \right) \right]
\]

\[
= \exp \left( - \int_0^{T_0} r_s \, ds \right) e^{a \log P_{T_0}(T_0)},
\]

(71)

where we have used the boundary conditions \( M(T_0) = N(T_0) = 0 \). Dividing by \( \exp(- \int_0^{T_0} r_s \, ds) \) we obtain the desired result

\[
P_{T_0}(t) \exp \left( M(t) + N(t)v(t) + a \log \frac{P_{T_1}(t)}{P_{T_0}(t)} \right)
\]

\[
= E_t^Q \left[ \exp \left( - \int_t^{T_0} r_s \, ds \right) e^{a \log P_{T_0}(T_0)} \right].
\]
Hence, all that is left to show is that \( X(t) \) is a \( Q \)-martingale. To that effect, note that
\[
X(t) = \exp(M(t) + N(t)v(t) + (1 - a)y(t)) + ay(t),
\]
where \( y_i(t) = \log(\exp(-\int_0^t r_s ds)P_i(t)) \) satisfies the following SDE:
\[
dy_i(t) = -\frac{1}{2}B_i(T_i - t)(z_t + (1 + \sigma^2_v)\nu(t))dt - B_i(T_i - t)\left(\sqrt{z_t + \nu(t)}dZ_i(t) + \sigma_v\sqrt{\nu(t)}dZ_v(t)\right).
\]

Applying Itô’s lemma and using the dynamics of \( y_i \), \( i = 0, 1 \) above we find that \( E_Q^t[dX(t)] = 0 \) if \( M, N \) satisfy the ODE given in Eqs. (69) and (68) above. A standard argument allows us then to conclude that \( X(t) \) is a \( Q \)-martingale. \( \square \)

We note that in our framework log bond prices are affine in the state variables \( r, \theta \) and thus the transform would be exponentially affine in the three state variables \( r, \theta, \nu \). However, since in the HJM framework the initial term structure is an input, there is a tremendous advantage to working with bond prices as state variables. As shown in Collin-Dufresne and Goldstein (2002b), this insight can be used in a much more general setting to extend the Markov time-homogeneous framework we use here to an infinite dimensional string model.

4.2. The fractional Fourier transform algorithm

The integrand of Eq. (66) has a singularity at \( \nu = 0 \), which prohibits the direct usage of standard numerical integration techniques. To avoid this singularity, Carr and Madan (1999) (CM) propose multiplying the transform by a factor \( e^{\omega \nu} \) where \( \omega \) is a positive damping constant. The Fourier Inversion Theorem applied to the damped transform yields
\[
\Psi_{t,a}(k) = \frac{e^{-\omega k}}{\pi} \int_0^\infty \text{Im} \left[ \frac{\psi_t(a + \omega + i\nu)e^{-ik\nu}}{\nu - i\omega} \right] d\nu.
\]
Eq. (72) is equivalent to Eq. (66) but is well defined at \( \nu = 0 \) when \( \omega > 0 \). CM show how to discretize the right-hand side integral of the above equation so as to use a Fast Fourier Transform (FFT) technique to evaluate it. The forward FFT is an efficient implementation of the following discrete Fourier transform:
\[
X_m = \sum_{n=1}^N e^{-2\pi i/n(n-1)(m-1)}x_n \quad \text{for} \quad m = 1, \ldots, N.
\]
The FFT algorithm is fast, because reduces the complexity from an order of \( N^2 \) to that of \( N \log\{N\} \). Let \( k_m = -k_1 + \lambda(m - 1) \) be the discretization of the (log) strike for \( m = 1, \ldots, N \). The FFT algorithm returns a vector \( X_m \) of size \( N \). Each of the elements of \( X_m \) contains the summation in Eq. (73) for different (log) strikes \( k_m \). CM price equity options and determine \( \lambda \) and \( k_1 \) such that a conventional FFT scheme can be implemented. They find that \( \lambda \) has to be \( \frac{2N}{\Delta} \), where \( \Delta \) is the step of the discretized integral of Eq. (72), and set \( k_1 \) such that the range of \( k_m \) is symmetric around 0.
Unfortunately, fixing $\lambda$ has dreadful implications for the true efficiency of the FFT implementation because it is the step between any two consecutive (log) strikes. It is impossible to reconcile a precise approximation of the integral in Eq. (72) (i.e., a small $\Delta$) with a fine grid for the strikes $k_m$ (i.e., a large $\Delta$). For pricing interest rate options the selection of a small $\lambda$ (or a large $\Delta$) is even more critical than for equity options, which makes the FFT approach very inefficient. For example, for reasonable values of the numerical parameters $N$ and $\Delta$, say, $N = 2^{13}$ and $\Delta = 0.2$, only 6 strikes out of the $2^{13}$ returned by the FFT scheme correspond to interest rate strikes between 4% and 9%. This means that more than 99.9% of the elements of $X_m$ are useless.

A very appealing way of breaking the relation between $\lambda$ and $\Delta$ is by using the Fractional Fourier Transform (FRFT) scheme of Bailey and Swarztrauber (BS, 1991). This highly efficient algorithm has been implemented in many scientific areas where Fourier transforms are used. The FRFT scheme builds upon the conventional FFT algorithm to solve the following transform in an efficient way:

$$\hat{X}_m = \sum_{n=1}^{N} e^{-2i\pi \alpha (n-1)(m-1)} \hat{x}_n \quad \text{for } m = 1, \ldots, N. \quad (74)$$

In the FRFT setting the spacing between any two consecutive $k_m$ is $\hat{\lambda} = \frac{2\pi}{\Delta} \alpha$, where the new parameter $\alpha$ can be any complex number. In our example above, a correct selection of $\alpha$ can yield the whole range of the $2^{13}$ strikes between 4% and 9%. The FRFT algorithm involves the solution of three conventional FFT (two forward and one inverse FFT of size $2N$), but increases considerably the precision of the solution, or alternatively, decreases significantly the time of achieving high accuracy. 12

Let $v_n = (n - 1)\Delta$ for $n = 1, \ldots, N$. Eq. (72) can be discretized as follows:

$$\Psi_{t,\alpha}(k_m) \approx \frac{e^{-\omega k_m}}{\pi} \text{Im} \left[ \sum_{n=1}^{N} \frac{\psi_i(a + \omega + i v_n)e^{-iv_n k_m}}{v_n - i\omega} \right] \quad (75)$$

$$\approx \frac{e^{-\omega k_m}}{\pi} \text{Im} \left[ \sum_{n=1}^{N} e^{-i\hat{\lambda}(n-1)(m-1)} \frac{\psi_i(a + \omega + i v_n)e^{i(n-1)\hat{k}_1}}{v_n - i\omega} \right] \quad (76)$$

$$\approx \frac{e^{-\omega k_m}}{\pi} \text{Im} \left[ \sum_{n=1}^{N} e^{-2i\pi \alpha (n-1)(m-1)} \hat{x}_n \right], \quad (77)$$

where $\hat{\lambda}$ and $\hat{k}_1$ are chosen such that the (log) strikes are inside the desired range. Finally, we consider Simpson’s rule weightings as in Carr and Madan (1999) and define the proper input vector as

11 Upon completion of this paper we became aware of Chourdakis (2004) who applies the BS (1991) algorithm to equity options.

12 Bailey and Swarztrauber (1991) show with a simple example that the FRFT scheme can achieve similar accuracy compared to the FFT scheme in less than 10% of the time.
\[ \hat{x}_n = \psi_j(a + \omega + i v_n) e^{i(n-1)i \lambda_k} \frac{A}{3} (3 + (-1)^n - \delta_{n-1}) \quad \text{for } n = 1, \ldots, N, \quad (78) \]

where \( \delta \) is the Kronecker delta function. Application of the BS (2001) algorithm then delivers an efficient estimate of the discretized integral on the right-hand side of Eq. (77).

4.3. Empirical results

In this section we compare the ability of various models nested in the general framework presented above at capturing the implied cap/floor volatility surfaces. In particular, we are interested in the impact of USV on the predicted shape of the smile. By analogy with the equity option literature it is natural to expect that stochastic volatility and, in particular, the correlation between volatility and bond prices can help fit the skew in the observed implied volatility surface (Rubinstein, 1994; Heston, 1993). We first describe the various models used, then we describe the data and methodology. Finally, we discuss the numerical results.

4.3.1. Calibration

In the following we calibrate and compare different specifications of the general HJM model in Proposition 6.

**Model 1:** A one-factor Gaussian model. This specification corresponds to a one-factor Vasicek model that is extended to fit the observed term structure of interest rates. The bond price dynamics in this model is

\[ \frac{dP(t, T)}{P(t, T)} = r_t dt - B_r(T - t) \sqrt{\sigma_r} d\omega_r^Q(t). \quad (79) \]

It is well known from Jamshidian (1989) that this model poses analytical solutions for zero-coupon bond options.

**Model 2:** A two-factor Gaussian model. This model adds an extra factor to the above specification. The long-term interest rate \( \theta_l \) follows a mean-reverting Gaussian dynamic that can be correlated with the short rate. The bond price dynamics in this model is

\[ \frac{dP(t, T)}{P(t, T)} = r_t dt - B_r(T - t) \left( \sqrt{\sigma_r} d\omega_r^Q(t) + \sigma_{r\theta} d\omega_{\theta}^Q(t) \right) - B_\theta(T - t) \sigma_{\theta\theta} d\omega_{\theta}^Q(t). \quad (80) \]

Under this Gaussian specification, it is well-known that zero-coupon bond options also admit a closed-form solution (Jamshidian, 1989).

**Model 3:** A two-factor USV model. This model augments the Vasicek model to include stochastic volatility in a USV framework. The volatility factor follows a time-homogeneous square root process that can be correlated with interest rates. Bond-price and volatility dynamics are given by
\[
\frac{dP(t,T)}{P(t,T)} = r_t dt - B_t(T-t)\left(\sqrt{z_t + \nu_t dz^O_t(t) + \sigma_{\nu_t}\nu_t dz^O_v(t)}\right),
\]
(81)

\[
d\nu_t = (\gamma + \kappa\nu_t)dt + \sigma\nu_t dz^O(t).
\]
(82)

This model is a special case of the one presented in Section 4.1. To price zero-coupon bond options, we use Proposition 7 and apply a FFT scheme.

**Model 4**: A two-factor USV model with \(\sigma_{\nu} = 0\). This is model 3, where we restrict correlation between volatility and interest rates to be zero. This model is used as a benchmark to analyze the effect of this correlation.

**Model 5**: An extended two-factor USV model. This model extends the volatility dynamics of the two-factor USV model to provide a perfect fit for the observed ATM volatility term structure (i.e., we allow \(c_v(t)\) to be a deterministic function of time).\(^{13}\) Bond-price and volatility dynamics are shown in Eqs. (58) and (59). The solution of zero-coupon bond option prices are presented in Section 4.1.

For each model we find the parameters that fit best (in a least square sense) the observed implied volatility surface for a specific day. By construction, all of these models fit perfectly the term structure of yields.

### 4.3.2. Data

Since we are only interested in illustrating the models predictions for the shape of the smile, we use data on caplet implied volatilities and Libor rates for one particular day. We obtain the implied volatility surface on July 18, 2000, from Peterson et al. (2002). These caplets are assumed to be written on 6-month forward Libor rates.

**Table 1** presents the caplet Black volatility surface for different strikes and maturities. This surface is also shown in the upper-left plot of **Fig. 1**. Across strikes, the curves have typically a U-shape or smile that flattens with maturity. The volatility is higher for deep in-the-money caplets and most of the times it is lower for ATM options (strike \(\approx 7\%\)). The term structure of volatilities is always hump-shaped. For example, for ATM options, the implied volatility is increasing with maturity in the first three years and decreasing for longer maturities.

The corresponding forward Libor rates are backed out from quoted swap rates obtained from Datastream (they are shown in **Table 2**).

### 4.3.3. Numerical results

**Table 3** shows the parameters that give the best mean square fit of the Black-caplet implied volatility surface for the five models presented above. **Fig. 1** contrasts observed vs. model implied volatilities. **Fig. 2** compares the observed and model

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\(^{13}\) In the analysis we refer to this model as the ‘extended’ USV model, though all models in Proposition 6 are extended to fit the short-rate term structure perfectly. To be more precise we could have named this model ‘doubly extended’ since \(c_v(t)\) is a deterministic function of time.
volatility term structure for ATM caplets. To quantify the fit of each model we present the mean square errors in Tables 4 and 5.

4.3.3.1. Fitting the implied volatility surface. It is well known that Gaussian models are unable to reproduce the observed implied volatility surface. Our results confirm this. The upper-right plot in Fig. 1 shows almost linear and monotonically decreasing volatility curves for the one-factor model. Thus, this model is incapable of capturing the observed skew in the data (upper-left plot of Fig. 1). Moreover, this model is unable to reproduce a higher curvature for shorter maturity curves. The errors in Table 4 confirms that the one-factor model has more problems fitting the implied volatilities of the first three years. Fig. 2 shows that the volatility term structure in this model is also monotonic and decreasing across maturities, thus it is unable to generate the humped shape observed in the data. The two-factor Gaussian model provides a slightly better fit than the one-factor model. Table 4 shows a MSE of 9.6 in the two-factor model compared to 10.6 in the one-factor model. This improvement is mainly due to the fact that this model can generate humped curves across maturity albeit these are flatter than observed in the data (see Fig. 2). Finally, the curves across strikes have no curvature, implying that the volatility hump does not vary for different strikes.

The two-factor USV model with time-homogeneous volatility process does a much better job at fitting the volatility surface (see middle-right plot in Fig. 1). The USV model can replicate most of the stylized facts in the data. In particular, it can disentangle the varying curvature of the volatility curves across maturities. Table 4 shows that the USV model has a good fit of shorter maturity curves (i.e., from 1- to 3-year maturity) and at the same time improves the fit of longer maturity curves. Also, it can produce U-shaped curves or smiles across strikes. Finally, this model generates more pronounced humps in the ATM implied volatility curve than the two-factor Gaussian model.

4.3.3.2. Correlation between volatility and interest rates. The correlation coefficient ($\sigma_{ri}$) drives the skewness of implied volatility smiles. For the USV model the correlation is positive and $\sigma_{ri}$ is 0.4290 (see Table 3). To understand the importance of this coefficient for the smile predictions, we restrict the correlation to 0 and compare the fit of the implied volatility surface.
The lower-left plot of Fig. 1 shows the volatility curves for the restricted case. Table 5 shows that the volatility of out-of-the-money caplets (strike $C_25\%$) worsen more than for any other option (i.e., the MSE increases from 8.4 in the unconstrained case to 15.5). Clearly, the constrained model restricts the skewness of the distribution of interest rates and is thus unable to fit shape of the implied volatility.
across strikes. However, ATM volatility term structure in the restricted model are not majorly affected when compared to the unrestricted case (see Fig. 2).

4.3.3.3. Fitting the ATM implied volatility curve. Finally, we consider an extended version of the two-factor USV model. We show that by allowing \( \gamma_v(t) \) to be a deterministic function of time that fits the ATM volatility term structure, the fit of the surface is almost perfect. Table 4 shows that the total mean square error decreases from 3.5 in the two-factor USV case to 1.3 in the extended case. As usual, the model fits better longer than shorter maturity volatilities. Nevertheless, the highest MSE in Table 4 for the extended USV model is 2.6 for 1-year maturity caplets (volatilities), which is comparable to the lowest MSE for the two-factor USV model (2.5 for the 7-year maturity curve). The extended version of the USV model also fits considerably better the volatility of out-of-the-money options (see Table 5), which was one of the main shortcomings of the other models.

### Table 2
**Swap rates**

<table>
<thead>
<tr>
<th>Maturity (yrs)</th>
<th>Swap rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.08</td>
</tr>
<tr>
<td>2</td>
<td>7.14</td>
</tr>
<tr>
<td>3</td>
<td>7.16</td>
</tr>
<tr>
<td>4</td>
<td>7.17</td>
</tr>
<tr>
<td>5</td>
<td>7.19</td>
</tr>
<tr>
<td>6</td>
<td>7.21</td>
</tr>
<tr>
<td>7</td>
<td>7.23</td>
</tr>
<tr>
<td>8</td>
<td>7.25</td>
</tr>
<tr>
<td>9</td>
<td>7.27</td>
</tr>
<tr>
<td>10</td>
<td>7.29</td>
</tr>
<tr>
<td>12</td>
<td>7.31</td>
</tr>
<tr>
<td>15</td>
<td>7.31</td>
</tr>
</tbody>
</table>

Term structure of US swap rates.

### Table 3
**Parameter estimates**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>One-factor Gaussian</th>
<th>Two-factor Gaussian</th>
<th>Two-factor USV</th>
<th>Two-factor USV (( \sigma_{rv} = 0 ))</th>
<th>Two-factor USV (extended)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa_r )</td>
<td>0.0223</td>
<td>0.1278</td>
<td>0.0825</td>
<td>0.0563</td>
<td>0.7469</td>
</tr>
<tr>
<td>( \alpha_r )</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0001</td>
</tr>
<tr>
<td>( \sigma_{\theta r} )</td>
<td>0.0096</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_{rv} )</td>
<td></td>
<td>0.4290</td>
<td></td>
<td>0.5123</td>
<td></td>
</tr>
<tr>
<td>( \sigma_{\theta} )</td>
<td>0.0169</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \gamma_v )</td>
<td>0.0002</td>
<td>0.0007</td>
<td></td>
<td>(extended)</td>
<td></td>
</tr>
<tr>
<td>( \kappa_v )</td>
<td>1.4799</td>
<td>4.8397</td>
<td>4.0648</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma_v )</td>
<td>0.0646</td>
<td>0.0323</td>
<td>0.2005</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Parameter estimates for five models that fit best the caplet Black implied volatilities in Table 1. All models fit the term structure of interest rates perfectly.
Fig. 2. At-the-money caplet volatility term structure for July 18, 2000 is shown. The volatilities are calculated using parameters in Table 3. All models fit the term structure of interest rates perfectly.

Table 4
Model comparison across maturities

<table>
<thead>
<tr>
<th>Maturity (yrs)</th>
<th>One-factor Gaussian</th>
<th>Two-factor Gaussian</th>
<th>Two-factor USV</th>
<th>Two-factor USV ($\sigma_{rv} = 0$)</th>
<th>Two-factor USV (extended)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>26.5</td>
<td>20.1</td>
<td>5.4</td>
<td>0.6</td>
<td>2.6</td>
</tr>
<tr>
<td>1.5</td>
<td>9.2</td>
<td>10.4</td>
<td>2.8</td>
<td>2.7</td>
<td>1.4</td>
</tr>
<tr>
<td>3</td>
<td>14.8</td>
<td>10.0</td>
<td>3.8</td>
<td>9.5</td>
<td>1.3</td>
</tr>
<tr>
<td>5</td>
<td>5.1</td>
<td>6.8</td>
<td>3.4</td>
<td>5.2</td>
<td>0.7</td>
</tr>
<tr>
<td>7</td>
<td>4.7</td>
<td>5.8</td>
<td>2.5</td>
<td>4.3</td>
<td>1.0</td>
</tr>
<tr>
<td>10</td>
<td>3.3</td>
<td>4.6</td>
<td>2.9</td>
<td>4.3</td>
<td>1.0</td>
</tr>
<tr>
<td>Total MSE</td>
<td>10.6</td>
<td>9.6</td>
<td>3.5</td>
<td>4.8</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Mean square errors ($\times 10^5$) of model vs. quoted implied volatilities for five different models using parameters in Table 3. The table shows the MSE for all caplet maturities in the dataset. All models fit the term structure of interest rates perfectly.

Table 5
Model comparison across strikes

<table>
<thead>
<tr>
<th>Strike rate (%)</th>
<th>One-factor Gaussian</th>
<th>Two-factor Gaussian</th>
<th>Two-factor USV</th>
<th>Two-factor USV ($\sigma_{rv} = 0$)</th>
<th>Two-factor USV (extended)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.0</td>
<td>12.2</td>
<td>15.9</td>
<td>2.4</td>
<td>2.8</td>
<td>1.6</td>
</tr>
<tr>
<td>5.5</td>
<td>2.0</td>
<td>3.6</td>
<td>2.0</td>
<td>3.3</td>
<td>1.4</td>
</tr>
<tr>
<td>6.0</td>
<td>6.7</td>
<td>6.2</td>
<td>3.2</td>
<td>4.5</td>
<td>2.6</td>
</tr>
<tr>
<td>6.5</td>
<td>7.5</td>
<td>5.2</td>
<td>2.1</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>7.0</td>
<td>13.5</td>
<td>9.5</td>
<td>2.7</td>
<td>2.0</td>
<td>0.0</td>
</tr>
<tr>
<td>8.0</td>
<td>21.6</td>
<td>17.3</td>
<td>8.4</td>
<td>15.5</td>
<td>1.3</td>
</tr>
<tr>
<td>Total MSE</td>
<td>10.6</td>
<td>9.6</td>
<td>3.5</td>
<td>4.8</td>
<td>1.3</td>
</tr>
</tbody>
</table>

Mean square errors ($\times 10^5$) of model vs. quoted implied volatilities for five different models using parameters in Table 3. The table shows the MSE for all caplet strike rates in the dataset. All models fit the term structure of interest rates perfectly.
5. Conclusions

We have proposed a parsimonious model of the term structure which displays unspanned stochastic volatility. In its short-rate form, the model is Markov in three state variables: the short rate, its long-term mean and an its volatility (even though it may be driven by only two Brownian motions). Interestingly bond prices are obtained in closed-form as exponential affine function of only two of the state variables: they do not depend on the interest rate volatility, which can follow an arbitrary process. The volatility process is thus ‘derivative specific’. The model can be ‘extended’ (by relaxing the time-homogeneity) to fit any arbitrary initial term structure. In its ‘HJM’ form the model nests the fixed-income analogous of the Heston (1993) stochastic volatility model. In particular, if the volatility process is specified to be affine, we obtain closed-form solutions for interest rate options. These rely on Fourier transform inversion as in Heston (1993) and Duffie et al. (2001). We apply the algorithm of Bailey and Swarztrauber (1991) which significantly improves the efficiency of the FFT algorithm proposed in Carr and Madan (1999) to perform the inversions. This proves especially useful in the fixed-income case where the range of strike prices typically used is relatively narrow. An application using data on caps and floors, shows that the model can capture very well the implied Black spot volatility surface, while by construction fitting the observed term structure. In its ‘doubly-extended’ form the model also fits the ATM cap volatility structure perfectly while the correlation coefficient between interest rate and volatility determines the skew across strikes.

Recent evidence in Collin-Dufresne et al. (2002) suggests that USV models provide a good description of the swap term structure dynamics. An open question is how well USV models (and, in particular, those developed here) of the term structure account for the joint dynamics of bond and fixed income derivative prices. We leave this for future research.

Appendix A. Fitting the term structure

\[
\hat{f}(0, T) = -A_T(0, T) + (r_0 + \theta_0)e^{-\kappa_r T} - \theta_0 e^{-2\kappa_r T}, \quad (A.1)
\]

\[
\hat{f}_T(0, T) = -A_T(0, T) - \kappa_r (r_0 + \theta_0)e^{-\kappa_r T} + 2\kappa_r \theta_0 e^{-2\kappa_r T}, \quad (A.2)
\]

\[
\hat{f}_{TT}(0, T) = -A_{TT}(0, T) + \kappa_r^2 (r_0 + \theta_0)e^{-\kappa_r T} - 4\kappa_r^2 \theta_0 e^{-2\kappa_r T}, \quad (A.3)
\]

where

\[
A(0, T) = \beta_0 T + \beta_1 (1 - e^{-\kappa_r T}) + \beta_2 (1 - e^{-2\kappa_r T}) + \beta_3 (1 - e^{-3\kappa_r T})
+ \beta_4 (1 - e^{-4\kappa_r T}) - \frac{1}{2\kappa_r} \int_0^T dt \gamma_0(t) (1 - 2e^{-\kappa_r (T-t)} + e^{-2\kappa_r (T-t)}), \quad (A.4)
\]
\[ A_T(0, T) = \beta_0 + \beta_1 \kappa e^{-\kappa_T} + 2\beta_2 \kappa e^{-2\kappa_T} + 3\beta_3 \kappa e^{-3\kappa_T} + 4\beta_4 \kappa e^{-4\kappa_T} \]
\[ - \int_0^T dt \gamma_\theta(t)(e^{-\kappa(T-t)} - e^{-2\kappa(T-t)}), \] (A.5)

\[ A_{TT}(0, T) = -\kappa_r^2(\beta_1 e^{-\kappa_T} + 4\beta_2 e^{-2\kappa_T} + 9\beta_3 e^{-3\kappa_T} + 16\beta_4 e^{-4\kappa_T}) \]
\[ + \kappa_r \int_0^T dt \gamma_\theta(t)(e^{-\kappa_r(T-t)} - 2e^{-2\kappa_r(T-t)}), \] (A.6)

\[ A_{TTT}(0, T) = \kappa_r^2(\beta_1 e^{-\kappa_T} + 8\beta_2 e^{-2\kappa_T} + 27\beta_3 e^{-3\kappa_T} + 64\beta_4 e^{-4\kappa_T}) - \gamma_\theta(t)\kappa_r \]
\[ - \kappa_r^2 \int_0^T dt \gamma_\theta(t)(e^{-\kappa_r(T-t)} - 4e^{-2\kappa_r(T-t)}). \] (A.7)

Plugging in for \( A_T(0, T), A_{TT}(0, T), A_{TTT}(0, T) \) into Eqs. (11)–(13), we find

\[ \gamma_\theta(t) = \frac{1}{\kappa_r} f_{TT} + 3f_T + 2\kappa_r f + 2\kappa_r \beta_0 + 6\beta_2 \kappa_r e^{-3\kappa_T} + 24\beta_4 \kappa_r^2 e^{-4\kappa_T}, \] (A.8)

where

\[ \beta_0 = \left(\frac{\sigma_v}{2\kappa_r^2} + \frac{\sigma_\theta^2}{8\kappa_r^2} + \frac{\sigma_\theta \sigma_{\theta \theta}}{2\kappa_r^2}\right), \] (A.9)

\[ \beta_1 = \left(\frac{1}{\kappa_r} \right) \left( -2 \left( \frac{\sigma_v}{2\kappa_r^2} \right) - 4 \frac{\sigma_\theta^2}{8\kappa_r^2} - 3 \frac{\sigma_\theta \sigma_{\theta \theta}}{2\kappa_r^2} \right), \] (A.10)

\[ \beta_2 = \left(\frac{1}{2\kappa_r} \right) \left( \frac{\sigma_v}{2\kappa_r^2} \right) + 6 \frac{\sigma_\theta^2}{8\kappa_r^2} + 3 \frac{\sigma_\theta \sigma_{\theta \theta}}{2\kappa_r^2}, \] (A.11)

\[ \beta_3 = \left(\frac{1}{3\kappa_r} \right) \left( -3 \frac{\sigma_\theta^2}{8\kappa_r^2} - \frac{\sigma_\theta \sigma_{\theta \theta}}{2\kappa_r^2} \right), \] (A.12)

\[ \beta_4 = \left(\frac{1}{4\kappa_r} \right) \left( \frac{\sigma_\theta^2}{8\kappa_r^2} \right). \] (A.13)

References


