Discrete Hedging

Throughout we consider the Black-Scholes model for a non-dividend-paying stock,

\[ dS = \mu Sdt + \sigma SdW. \]

We assume a deterministic (continuously compounded) interest rate of \( r \), and look at a strike-\( K \), expiry-\( T \) call-option on the stock. Default parameters are shown below.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Numerical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial stock price</td>
<td>( S(0) )</td>
<td>100</td>
</tr>
<tr>
<td>Stock price (proportional) drift</td>
<td>( \mu )</td>
<td>0.05</td>
</tr>
<tr>
<td>Volatility</td>
<td>( \sigma )</td>
<td>0.20</td>
</tr>
<tr>
<td>Interest rate</td>
<td>( r )</td>
<td>0.05</td>
</tr>
<tr>
<td>Strike/exercise price</td>
<td>( K )</td>
<td>105</td>
</tr>
<tr>
<td>Option maturity</td>
<td>( T )</td>
<td>1</td>
</tr>
</tbody>
</table>

Comment on “\( \mu = r \)”.

Recall that in order to replicate the call-option in the Black-Scholes model we have to hold

\[ \Delta(t) = \frac{\partial C}{\partial S} = \Phi \left( \frac{\ln(S(t)/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \right), \]

units of stock, and adjust our (stock,bank account)-portfolio (continuously) in a self-financing way. Of course, continuous adjustment is hard in practice, but how bad are we off if we do things discretely? To answer this, consider the following algorithm/procedure/plan:

- Chop the time interval between now (time 0) and expiry of the call-option (time \( T \)) into \( N \) pieces; denote the discretization points \( t_i \).
- Simulate stock-price paths, denote the \( j \)’th by \( (S^j(t_i))_0^N \) (i.e. values at the \( t_i \)’s).
Suppose that at time 0 somebody gives you an amount of money exactly equal to the Black-Scholes call-price. You use this to buy \( \Phi(d_1(S(0),0)) \) units of the stock (whatever extra money you need, you borrow in the bank).

- At time \( t_1 \) you adjust your portfolio such that you now hold \( \Phi(d_1(S^j(t_1),t_1)) \) units of the stock. You do this in such a way that extra funds needed (+/-) are borrowed at the bank. (Recall that money in the bank draws interest.)

- Do that all the way up to \( t_N = T \), where you liquidate the portfolio. Keep track of the value of the portfolio, say \( V^j(t_i) \), along each path. Compare \( V^j(t_i) \) to the true Black-Scholes call-price; call the difference \( \epsilon_{ij}^N(t_i) \); it may also be referred to as (running) hedge error, the profit/loss or simply “P/L”. Let \( \tilde{\epsilon}_{ij}^N(t_i) = e^{-r t i} \epsilon_{ij}^N(t_i) \) denote the discounted hedge error.

Run the simulation for \( M = 1000 \) paths and make a scatter-plot of the call-option payoff against the terminal portfolio value. Do this for (at least) hedge frequencies \( N = 12, 52, 250 \). What do you see?

What is \( E^Q(\tilde{\epsilon}_{ij}^N(t_N)) \)? What is \( E^Q_{t_l}(\tilde{\epsilon}_{ij}^N(t_m)) \) for any \( t_l < t_m \)? Hint: The \( Q \)-expected discounted value of a self-financing trading strategy (which?) is a ??? What can you say about \( \text{std.dev.}(\tilde{\epsilon}_{ij}^N(t)) \) as a function of \( t \)?

It is possible (but hard) to show that as a function of \( N \), \( \text{std.dev.}(\tilde{\epsilon}_{ij}^N(t_N)) \) behaves asymptotically as \( 1/\sqrt{N} \). Illustrate that numerically.

What happens if \( \mu \neq r \)? Illustrate and explain.

What happens if we hedge with a wrong volatility? Experiment and explain.

Do the discrete delta-hedging exercise again, but this time for an option whose pay-off is \( 1_{S(T) \geq K} \), i.e. a digital option. Now how does the standard deviation of the terminal hedge error behave as a function the hedging frequency?