A look inside the mysterious world of stock options.

Wall Street Profits, Arbitrage, and the Pricing of Stock Options

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Finance is sometimes called the study of arbitrage. A finance person defines arbitrage as the presence of a riskless profit. Suppose we can buy IBM stock today from Peter for $100 and immediately sell it to Paul for $105. This would represent a $5 profit with no risk! It is like finding $5 on the sidewalk. Of course, our capital markets are fully aware that equivalent assets should sell for the same price, that riskless assets should earn the riskless interest rate, and that five-dollar bills do not remain on the sidewalk for very long. Investors establish prices such that these conditions prevail and any arbitrage opportunities quickly disappear. Stock option pricing theory provides a fascinating illustration of how arbitrage arguments lead to a remarkable, counterintuitive conclusion, one that led to the 1997 Nobel Prize in Economics: The value of a stock option is independent of the expected rate of return on the stock. Although a derivation of the Black–Scholes option-pricing model is beyond the scope of this article, the intuition behind the model is interesting, especially given the counterintuitive result regarding the importance of expectations about the stock's future.

How Stock Options Work

There are two types of stock options. A call option gives its owner the right to buy shares of stock at a specified price called the exercise price. A put option gives its owner the right to sell at the exercise price. The person granting the option is the option writer. The option has an expiration date, or set lifetime, after which it ceases to exist. The buyer of an option pays a premium to the option writer, and the writer keeps this premium regardless of the future state of the world. Note that the premium is not a down payment on a delayed purchase; it is the cost of the option.

Anyone can buy or write puts or calls. Buying a call is like buying a ticket to a university football game. The ticket carries the right, but not the obligation, to attend the game. The ticket holder can use the option, sell it to someone else, or abandon it. The university wrote (created) the option and gets to keep the ticket price regardless of what the ticket holder ultimately does with the ticket.

A customer who buys something from an L. L. Bean catalog acquires a put option along with the purchase. Customers have the right to “put the merchandise back” to the company for a full refund at any time. L. L. Bean goods may seem expensive, but the buyer should recognize that there is a value to the money-back guarantee. L. L. Bean packages a put option along with every product the firm sells.

An individual might buy a $75 call option on 100 shares of stock XYZ for a
premium of $3 per share, or a total of $300. This option gives its owner the right to buy 100 shares at a price of $75 at any time prior to the expiration of the option. As with football tickets, before expiration the owner of the call has three alternatives — sell the option, exercise it and buy the shares, or abandon it and allow it to expire. The call holder will only exercise the option if the stock price rises above $75. If the call holder chooses to exercise the option, he would pay $7,500 to the option writer in exchange for the shares.

How Call Options Should Be Priced

We will create an imaginary capital market to illustrate rational option pricing. An investor can invest in U.S. government securities and earn 10% over the next year. (Financial theory refers to this benchmark as the riskless rate of interest. The 10% value is merely an assumption; the results hold regardless of the selected interest rate.) Stock XYZ currently sells for $75 per share. There are no transaction costs or taxes.

In our hypothetical market there are only two possible states of the world at year-end: either the stock rose to $100 or it fell to $50. There are call options for sale that give the owner the right to buy the stock a year from today at its current price of $75. If the stock rises to $100, the call would be worth $100 – $75 = $25, because it gives its owner the right to buy stock for $25 less than the market price. If the stock falls to $50, the option is unattractive and would expire worthless; no one would choose to pay $75 for stock worth $50. Figure 1 illustrates the scenario. For what price should such an option sell?

One seemingly logical approach to the problem is to determine the expected value of the stock in one year, from that determine the expected call value, and discount this amount back to a present value. Perhaps an optimistic investor believes there is a 90% chance the stock will rise and a 10% chance it will fall. With branch probabilities .9 and .1, the expected stock price is $95, the expected call value is $95 – $75 = $20, with a present value of $20/1.10 = $18.18.

This price, however, presents an arbitrage opportunity, and it could not prevail for long in a well-functioning marketplace. To see why, consider the steps an arbitrageur would quickly implement. First, buy a share of the stock, spending $75. Second, write and sell two of these calls at $18.18 each, receiving $36.36. The net investment, then, is $75 – $36.36 or $38.64.

If the stock falls, the options will expire worthless and the portfolio will contain one share of stock worth $50. If the stock rises, the share will be worth $100. The options would be valuable to their owner because each of them permits the purchase of shares worth $100 for only $75. The option holder would exercise this option; the option writer would have to sell two shares for $25 less than their current market value, thereby losing $50. Therefore, the ending portfolio value would be $100 (the value of the stock) minus the $50 loss on the options, or $50. This means that, regardless of whether the stock follows the upper or the lower branch in Fig. 1, the portfolio will be worth $50 in one year. Because the initial cash outlay was $38.64, the $50 terminal value translates into a certain one-year gain of 29.4% with no risk. This is inconsistent with a U.S. government risk-free rate of 10%, so $18.18 cannot be the correct value for the call option.

Suppose a different investor views XYZ's prospects differently and reverses the probabilities for the two branches: She/he feels that the stock has a 10% chance of appreciating and a 90% chance of falling. Using the prior logic, the stock's expected value is $55, and she/he therefore concludes that the option has no value because the expected future stock price is less than the price the option entitles you to pay. No one would choose to pay $75 for shares worth $55. The arbitrageur, however, offers to pay $1 a piece for these calls if the investor will write them. Thinking the arbitrageur has erred in his/her calculations, the investor agrees to write two such contracts for $1 each.

Having acquired these two calls, the arbitrageur will then sell one share of the stock short. Selling short involves borrowing a share, selling it, buying a replacement share from someone else at a later time, and replacing the borrowed share. Short sellers are, in essence, selling first and buying second. (They make money when prices decline.) Buying the two calls for $1 each and selling one share short at $75 results in a net cash inflow of $73. As in the prior example, at expiration the portfolio will be worth $50 regardless of the branch traveled. If the stock goes up, the arbitrageur loses $25 on the short position when he purchases shares at $100 to replace those borrowed. He/she profits on each call, however. The right to buy at $75 is worth $25 when the stock price is $100. Having paid $1, the gain is $24 on each call, or $48 on two of them. The net portfolio gain is $2 x ($25 – $1) – $25 = $23.

If the stock goes down, the calls expire worthless, but the arbitrageur makes $25 on the short sale for a net $23 gain, as before. In this case, the return is infinite, because there was no initial cost to this investment; it began with a net inflow. Clearly $1 is not an equilibrium value for the call either.

To find out what the call price must be, we can generalize the example and create an arbitrage portfolio as follows. If the stock rises, the call will be worth $25; it will be worth $0 if the stock falls. We can construct a portfolio of stock and options such that the portfolio has the same value regardless of the stock price after one year. One way to do this is to
$100-$25N

$75 - (N)(c)

$50

Today

One Year Later

Figure 2. Portfolio values.

buy one share of stock today and write a quantity of calls we will call N; this makes our portfolio worth 75 – N(c), where C is the cost of the option.

If the stock falls, the value of this portfolio will be $50. The share is worth $50 and the calls expire worthless. If the stock rises, the portfolio is worth $100 – $25N. The stock is worth $100 and the options are worth $25 apiece to their owner (or minus $25 to the person who wrote them). When they are exercised, the option writer will have to sell $100 stock for $75 per share, losing $25 per share. Figure 2 shows the possibilities.

We can solve for N such that the portfolio value in one year must be $50. Setting the two possible values equal, $100 - $25N = $50 and N = 2. This means that if we buy one share of stock today and write two calls we know the portfolio will be worth $50 in one year. In other words, the future value is known and riskless. (Note that this is not a prescribed investment strategy. No one would choose to engage in a strategy that is guaranteed to lose money. We are merely solving for a value that must exist in the absence of arbitrage.) Economic theory requires that an investment with a known future value must earn the riskless rate of interest, which is 10% in this example. Therefore, assuming no arbitrage opportunity exists we must have

\[(75 - 2C)(1.10) = 50\]

Solving for C, we find the option must sell for $14.77. This value is independent of the probability associated with the two branches! It makes no difference what the probabilities are that the investor assigns to the two branches. At any price above this value, a person who buys one share of stock and sells two call options will earn risk-free more than they would by putting money in the bank. Conceptually, this means people will be lining up to sell options and their price would be driven down to $14.77. Similarly if the price were lower there would be risk-free opportunities to option buyers. This discovery is an epiphany for students of derivatives: The price of an option is independent of the expected return on the stock.

Put Pricing in the Presence of Call Options

Knowing that this call option must sell for $14.77 we can now turn to its counterpart, the put option, and learn something about its equilibrium price. Suppose the put also sells for $14.77. The arbitrageur, observing that this value is incorrect given the call premium, engages in the transactions shown in Table 1.

We see that this series of transactions results in an initial cost of $0.00 and a future portfolio value of $7.50. In other words, we invest no money now but will receive $7.50 in one year. This is the proverbial "free lunch."

In fact, the arbitrageur need not even wait a year for this windfall. The arbitrageur could invest the discounted value of the striking price, or $75/1.10 = $68.18. Table 2 shows this would result in an initial cash flow of $6.82 and a portfolio value at option expiration of $0.00.

Table 2 shows that with the put and call both selling for $14.77 the arbitrageur could, with no investment and no risk, capture $6.82 today or, as Table 1 shows, capture the future value of this, $6.82(1.10) = $7.50 in one year. Given a call premium of $14.77, the put cannot also sell for this amount in an arbitrage-free market.

In fact, the put must sell for $14.77 – $6.82 = $7.95. This put price would result in an initial cash inflow of $0.00 in Table 2. We would expect something that we know to be worthless at all points in the future to also be worthless today. This relationship leads to what options users know as the put-call parity model:

\[C - P + K(1 + R)^T = 0\]

Where C = call premium, P = put premium, S = current stock price, K = exercise price, R = interest rate per period, and T = periods until option expiration. The call premium, put premium, stock price, and striking price form an interrelated securities complex. If you know all variables but one, you can solve for its arbitrage-free value.

The Effect of Volatility

Perhaps in our hypothetical market there is another firm whose stock also sells for $75 per share, but this stock is more volatile: In the next year it will either rise to $110 or fall to $40 as in Fig. 3.

Using the same pricing procedure as in the earlier example, a one-year, $75

<table>
<thead>
<tr>
<th>Activity</th>
<th>Cash Flow</th>
<th>Stock price = $100</th>
<th>Stock price = $50</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy call</td>
<td>-$14.77 (payment)</td>
<td>$25</td>
<td>0 (expires worthless)</td>
</tr>
<tr>
<td>Write put</td>
<td>+14.77 (receipt)</td>
<td>0 (expires worthless)</td>
<td>-$25</td>
</tr>
<tr>
<td>Sell short</td>
<td>+75.00 (receipt)</td>
<td>-100 (cost to close out)</td>
<td>-50 (cost to close out)</td>
</tr>
<tr>
<td>Invest $75 in T bills</td>
<td>-$75.00 (payment)</td>
<td>82.50 (receipt from T Bill investment)</td>
<td>82.50 (receipt from T-bill investment)</td>
</tr>
<tr>
<td>Totals</td>
<td>$0.00 (net receipt)</td>
<td>$7.50</td>
<td>$7.50</td>
</tr>
</tbody>
</table>
The accompanying article describes a simple setting with only the risk-free investment, one stock, and its options in which to invest, and only two future states of the world. In this sidebar we extend the arbitrage illustration given there to a more realistic setting.

Suppose we have \( n \) possible stocks (bets) in which to invest. Let \( x_i \) represent the amount we invest (bet) on the \( i \)th stock. Here each "stock" could be a stock, an option on a stock or any other investment. Let the set of future states of the world (the set of possible outcomes in the experimental sample space of investment outcomes) be represented by \( S = \{1, 2, \ldots, k\} \) and let \( r_i(j) \) represent the rate of return for a one-dollar investment (bet) on the \( i \)th stock if \( j \) is the future state of the world. With an amount \( x_j \) invested in stock \( j \) and a future state of nature \( j \), the total value of this investment would then be \( x_j r_i(j) \). Let \( x = (x_1, \ldots, x_n) \) represent the portfolio or vector of stock investments (positive, negative, or zero). If the experimental outcome is \( j \) the total return from the stock investment portfolio (betting scheme), \( x \) would be

\[
R(x) = \sum_{j=1}^{n} x_j r_i(j)
\]

The Arbitrage Theorem states that either there exists a probability vector \( p = (p_1, \ldots, p_k) \) on the set of possible future states of the world (outcomes) such that each stock’s expected return equals 0 or else one can always determine an investment portfolio that guarantees a positive return on investment (with probability 1). We state this formally as follows:

**The Arbitrage Theorem:** Exactly one of the following holds:

1. There exists a probability vector \( p = (p_1, \ldots, p_k) \) for which

\[
\sum_{j=1}^{k} p_j r_i(j) = 0 \quad \text{for} \quad i = 1, \ldots, n,
\]

or

2. There exists an investment (betting) portfolio \( x = (x_1, \ldots, x_n) \) for which

\[
\sum_{j=1}^{n} x_j r_i(j) > 0, \quad \text{for all} \quad j = 1, \ldots, k.
\]

The first condition (a) states that there exists a probability distribution on the set of future states of the world, \( S = \{1, 2, \ldots, k\} \), that makes the expected return equal to 0 for every stock. This condition must prevail if we want to avoid arbitrage. But notice that although the sum in (a) runs over the possible future states of the world or sample space of outcomes for every stock investment option, \( x_i \), we sum in (b) over the portfolio or vector of stock investment amounts for every possible future state or the world. Condition (b) represents the presence of arbitrage and assures us that, irrespective of the future state of the nature (sample space outcome), there exists a stock investment portfolio that guarantees a positive return on investment independent of the probability distribution on the set of future states of nature. Examples of arbitrage were exhibited in the body of the article with the call and put pricing options. Only with an appropriate price for a call and a put did we avoid arbitrage.

The Arbitrage Theorem’s proof follows from a theorem in linear algebra on the separating hyperplane and relates to the duality theorem of linear programming. In fact, linear programming may be employed to find an investment strategy that assures the greatest return on investment, too.

An interesting application that may be more familiar to readers concerns betting with posted odds as at a race track. Let us suppose there exists \( k \) possible bets on \( k \) possible outcomes. We may choose only one of the outcomes \( i \), \( i = 1, \ldots, k \), on any bet. Frequently the return from such a bet will be posted in terms of odds with the odds for outcome \( i \) quoted as equal to \( o_i \) (read this as "\( o_i \) to 1.") This means a unit bet will return \( o_i \) if the outcome \( i \) occurs. Write this as \( r_i(j) = o_i \) if \( j = i \) and \(-1 \) otherwise. The Arbitrage Theorem says we do not have a condition of arbitrage if the probability vector \( p = (p_1, \ldots, p_k) \) on the set of outcomes, \( i = 1, 2, \ldots, k \), yields an expected value of 0 for all the possible \( k \) bets:

\[
o_i p_i - 1(1 - p_i) = 0
\]

Thus, the \( p_i \) probability values must satisfy

\[
p_i = \frac{1}{1 + o_i}, \quad i = 1, 2, \ldots, k
\]

Because the total probability must sum to 1, the following condition must hold to avoid arbitrage:

\[
\sum_{i=1}^{k} \frac{1}{1 + o_i} = 1
\]

But if

\[
\sum_{i=1}^{k} \frac{1}{1 + o_i} \neq 1
\]

we can be assured of a winning betting strategy as the following example demonstrates.

Suppose we have the following posted outcomes and odds:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Odds</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
</tr>
<tr>
<td>D</td>
<td>5</td>
</tr>
</tbody>
</table>

We have

\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} > 1
\]

Thus, we can be sure of a winning betting scheme. Let us bet \(-5/2\) on A (we win 5/2 if any outcome other than A occurs and lose 5/2 if A occurs); bet \(-5/3\) on B (win 5/3 if B does not occur and lose 10/3 if it does); \(-1\) on C and finally \(-5/6\) on D. For outcome A we win \(-5/2 + 5/3 + 1 + 5/6 = 1\); for outcome B we win \(5/2 - 10/3 + 1 + 5/6\) = 1; for C we win \(5/2 + 5/3 - 4 + 5/6\) = 1; and likewise for D we win \(5/2 + 5/3 + 1 - 25/6\) = 1. Of course, at the racetrack this strategy would require us betting a negative amount of money, which is not allowed. It is the person setting the odds (the racetrack) that is earning the risk-free payoff!
striking price call option on this stock should sell for $19.32. This is the solution to the equation $(75 - 2C)/(1.10) = 40$. Using the put-call parity model, the equilibrium value of the corresponding put is $C - S + K/(1+r)$, or $19.32 - 75 + 68.18 = $12.50. The higher premium associated with the greater volatility shows why options on Internet stocks sell for a "higher price" than options on similarly priced retail food stores or electric utilities.

In both this example and in the Fig. 1 example, the no-arbitrage value of the call exceeds the no-arbitrage value of the put. This is always the case when the stock price is exactly equal to the option striking price and the stock pays no dividends over the life of the option.

From Binomial Pricing to Black–Scholes

The analysis described previously is what financial theorists refer to as binomial pricing. The fact that future security prices are not limited to only two values in no way attenuates the usefulness of the model. We can simultaneously make the size of the jumps very small and the time interval very short. There are theoretically an infinite number of future states of the world. By making the jumps infinitely small and the time span infinitely short, we move into the world of continuous time calculus and the Black–Scholes model for which its inventors received the Nobel Prize. Figure 4 shows how the one-period tree diagrams can be extended to multiple periods and how you might visualize a normal distribution about the range of future states of the world.

The arguments in the Black–Scholes model are the current stock price, the option striking price, the remaining time until option expiration, the interest rate, and the anticipated volatility of the underlying asset. The formula also assumes that the underlying stock return distribution is lognormal. This is a common assumption in financial research. Although we know that security returns are slightly skewed to the right and that they are a bit heavy-tailed, the lognormal assumption is quite accurate and constitutes only a modest departure from reality.

The pricing logic remains: A riskless investment should earn the riskless rate of interest. If this is not the case, arbitrageurs will quickly transact so as to move prices to their equilibrium relationship. As the earlier example showed, it is not necessary to know the expected rate of return on the stock to find the equilibrium value of the right to buy the stock. This is just as true in continuous as in discrete time.

Summary

Option pricing is one of the most important developments in the field of finance in the past 30 years. The discovery that there is an unambiguous way to value the right to buy or sell something substantially enriched the fields of corporate finance and investments. The valuation of contingent claims (like options) is largely responsible for the emigration of mathematicians to Wall Street. In finance, arbitrage pricing normally involves the relationship between two or more assets. We may not know the proper value for either security A or security B, but given the value of A, we can sometimes determine what the value of B should be. For example, we now know that put prices, call prices, stock prices, and interest rates form an interrelated securities complex and that one asset does not change price without affecting the others.

Reference and Further Reading