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Option Replication in Discrete Time with Transaction Costs

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ABSTRACT

Option replication is discussed in a discrete-time framework with transaction costs. The model represents an extension of the Cox-Ross-Rubinstein binomial option pricing model to cover the case of proportional transaction costs. The method proceeds by constructing the appropriate replicating portfolio at each trading interval. Numerical values of these prices are presented for a range of parameter values. The paper derives a simple Black-Scholes type approximation for the option prices with transaction costs and demonstrates numerically that it is quite accurate for plausible parameter values.

THE BLACK-SCHOLES MODEL ASSUMES perfect frictionless markets. A replicating portfolio can be constructed consisting of a long position in the risky asset and a short position in bonds which is equal in value to the price of a call option. As time passes, the weights of this portfolio are rebalanced so that it replicates the payoff of the option contract at maturity. Under perfect market assumptions this rebalancing is costless, but if we introduce transaction costs this is no longer the case. This paper examines the impact of transaction costs on option prices and option replication.

Several papers have discussed this issue in recent years.[1] Leland (1985) used a continuous-time framework and derived a Black-Scholes type approximation for the option price in the presence of proportional transaction costs. He constructs a replicating stock-bond portfolio which (almost) replicates the value of the option at maturity.

Merton (1990) sets up the problem in a true discrete-time framework and derives the current option value when there are proportional transaction costs on the underlying asset. He constructs a portfolio of the risky asset and

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[1] Gilster and Lee (1984), Leland (1985), Merton (1990), Shen (1990), and Hodges and Nueberger (1989) have examined this topic.

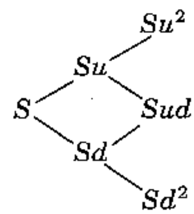
riskless bonds that precisely replicates the option value at expiration. The approach incorporates an allowance for the transaction costs arising from portfolio rebalancing. Our approach[2] is similar to Merton's, but we extend the analysis to several periods. We also employ a discrete-time framework and construct the portfolio to replicate a long and short European call.

We start by obtaining the long call price in a one-period model. This model is then extended to several periods and we develop a recursive procedure to obtain the replicating portfolio for the long call price. The procedure for obtaining the short call price is similar but not identical. The zero-transaction costs option values lie between the short call price and the long call price, as we would expect. Furthermore, as the transaction costs tend to zero, these prices converge to the Cox-Ross-Rubinstein option prices. We derive a closed-form expression for the long call price when there is a large number of revision times. In this case, the long call price can be approximated by the ordinary Black-Scholes formula with an adjusted variance. Our variance adjustment is similar to, but larger than, that derived by Leland (1985). We derive an analogous approximation for the short call price.

The layout of the present paper is as follows. In Section I we construct the replicating portfolio for a long call position when there are several periods. Section II derives an analytical expression for the long call price in the presence of transaction costs. Section III develops some convenient approximations and derives a Black-Scholes type expression for the long call price. In Section IV we construct the replicating portfolio for a short call position. We explore the numerical properties of these prices in Section V. Section VI concludes the paper.

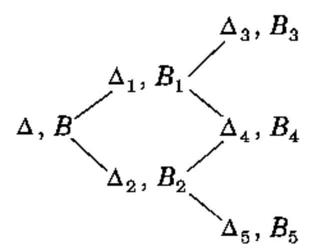
I. Option Replication in Discrete Time with Transaction Costs

We use no-arbitrage arguments to establish the cost of creating a long European call option by dynamic hedging when there are transaction costs. In the two-period case our model reduces to that obtained by Merton (1990) when allowance is made for differences in notation and convention. We use the multiplicative binomial lattice employed by Cox, Ross, and Rubinstein (1979) for the asset price



[2] Shen (1990) also employs a discrete-time framework to study the impact of transaction costs on option prices. His approach is similar to ours.

where we assume that $u > R > d$, with R equal to one plus the one-period riskless rate. A dynamic hedging strategy is employed to replicate the payoff to a European call option. The replicating portfolio will be constructed backward from the maturity date, i.e., if we know the portfolio at the points Su and Sd in the above diagram we will construct the portfolio at the point S. In order to take the transaction costs into account it is not enough to know the value of the replicating portfolio at each node; we also have to know how much is invested in the risky asset and how much is borrowed. The symbol Δ denotes the number of shares and B denotes the number of bonds. The following diagram gives the Δ's and B's at each node:



To introduce transaction costs, assume that proportional transaction costs are incurred when shares of the risky asset are traded.[3] Let k be the transaction costs rate measured as a fraction of the amount traded. We must select Δ and B so that the portfolio (Δ_1, B_1) can be bought if the up-state Su occurs and (Δ_2, B_2) can be bought if the down-state Sd occurs. This leads to the following two equations:

$$\Delta Su + BR = \Delta_1 Su + B_1 + k\,|\,\Delta - \Delta_1\,|\,Su \tag{1}$$

$$\Delta Sd + BR = \Delta_2 Sd + B_2 + k\,|\,\Delta - \Delta_2\,|\,Sd \tag{2}$$

Equation (1) indicates that the value of the portfolio in the up-state is exactly enough to buy the appropriate replicating portfolio corresponding to this state and to cover the transaction costs incurred in the rebalancing. Equation (2) has a similar interpretation for the down-state. Since we don't know whether a sale or purchase of the risky asset will be involved, we use the absolute value of $\Delta - \Delta_1$ and $\Delta - \Delta_2$. Equations (1) and (2) are two nonlinear equations in Δ and B, and it is not obvious whether a unique solution exists.

Theorem 1: *In the construction of a long European call option by dynamic hedging, equations* (1) *and* (2) *have a unique solution* (Δ, B). *Furthermore for this solution the following inequality holds*:

$$\Delta_2 \leq \Delta \leq \Delta_1. \tag{3}$$

[3] Proportional transactions costs on bonds can also be incorporated. However, the model becomes much more complicated without providing new insights.

This theorem enables us to rewrite equations (1) and (2) in the following form

$$\Delta S\bar{u} + BR = \Delta_1 S\bar{u} + B_1 \tag{4}$$

$$\Delta S\bar{d} + BR = \Delta_2 S\bar{d} + B_2 \tag{5}$$

where

$$\bar{u} = u(1 + k) \quad \text{and} \quad \bar{d} = d(1 - k). \tag{6}$$

Theorem 1 reduces the nonlinear equations to linear ones, which can be readily solved. These equations form the basis of an iterative procedure which can be used to obtain the composition of the replicating portfolio at inception. By working backward from the boundary we can compute the explicit portfolio weights at each node of the lattice. This procedure makes appropriate adjustment for transaction costs. We use this procedure in Section V to compute numerical values of the long call prices.

If we replace $\bar{u}$ by u and $\bar{d}$ by d in equations (4) and (5) we have the familiar equations for discrete-time option pricing without transaction costs. Hence, one might be tempted to calculate the current portfolio value with transaction costs by replacing u by $\bar{u}$ and d by $\bar{d}$ in the standard formula for the option price C (see Cox, Ross, and Rubinstein (1979) formula (6)). This is incorrect since the right-hand sides of equations (4) and (5) no longer represent the values of the call in the up-state and the down-state, as in the no-transaction cost case. The actual value of the call in the up-state is $\Delta_1 Su + B_1$ instead of $\Delta_1 S\bar{u} + B_1$ and similarly for the down-state.

We assume that the institution (or intermediary) creating the replicating portfolio does not have to buy the initial amount of the risky asset (Δ). Hence, we just take account of the additional transaction costs necessary to maintain the replicating portfolio. We assume that the replicating portfolio at option expiration for an in-the-money call option consists of one unit of the risky asset and a short position in riskless bonds equal to the exercise price. Our conventions correspond to those employed by Leland (1985) rather than Merton (1989).

II. The Replicating Portfolio as a Discounted Expectation

The value of a European call option without transactions costs can be expressed as a discounted expectation of the maturity value of the option, assuming that the risky asset price follows a certain risk-neutral binomial process. We will derive an analogous expected value formulation for the value of the replicating portfolio with transaction costs. From (4), (5), and (6) it follows for a two-period model

$$C = \Delta S + B = \frac{\bar{p}\left[(1 + k)\Delta_1 Su + B_1\right] + (1 - \bar{p})\left[(1 - k)\Delta_2 Sd + B_2\right]}{R}, \tag{7}$$

where C is the current value of the portfolio that exactly replicates the payoff to a long European call position (with transaction costs). This can be further simplified to

$$C = \Delta S + B = \big[\, \bar{p}\bar{p}_u\{(1+k)\Delta_3 Su^2 + B_3\} \\ + \bar{p}(1-\bar{p}_u)\{(1-k)\Delta_4 Sud + B_4\} \\ + (1-\bar{p})\bar{p}_d\{(1+k)\Delta_4 Sud + B_4\} \\ + (1-\bar{p})(1-\bar{p}_d)\{(1-k)\Delta_5 Sd^2 + B_5\}\big]/R^2, \quad (8)$$

where

$$\bar{p}_u = \frac{R(1+k) - \bar{d}}{(\bar{u} - \bar{d})} \quad \text{and} \quad \bar{p}_d = \frac{R(1-k) - \bar{d}}{(\bar{u} - \bar{d})}.$$

It immediately follows that $0 < \bar{p}_d < \bar{p}_u < 1$.

From (8) we see that the right-hand side can be interpreted as a discounted expectation. This gives rise to a new process, which we call the adjusted process and which differs from both the original asset price process and the risk-neutral price process. Under the adjusted process the probability of a particular state depends on whether the previous jump was upward or downward. After an up-jump the probability of another up-jump is $\bar{p}_u$ while just after a down-jump the probability of another up-jump is $\bar{p}_d$.

After a down-jump the probability of another down-jump is larger than in case of a preceding up-jump. This process can be formalized as follows: Let $X_1, X_2, X_3, \ldots, X_n$ be a Markov process with two states and values $\log_e u$ and $\log_e d$. The transition matrix is given by:

$$\bar{P} = \begin{pmatrix} \bar{p}_u & \bar{p}_d \\ 1 - \bar{p}_u & 1 - \bar{p}_d \end{pmatrix} \quad (9)$$

The first column of $\bar{P}$ represents the probability distribution of X_{j+1} if $X_j = \log_e u$ and the second column represents the probability distribution if $X_j = \log_e d$. The initial distribution for X_1 is given by $\bar{p} = (\bar{p}, (1-\bar{p}))^T$ (^{T} means the transposed vector). The following theorem can be proved by an induction argument.

Theorem 2: *The current cost of creating a long European call option in the presence of proportional transaction costs can be expressed as follows*:

$$C = \frac{E\big[\big((1+\bar{X}_n k)Se^Y - K\big) I_{Se^Y \geq K}\big]}{R^n}, \quad (10)$$

where n is the number of periods to option expiration,

$$Y = \sum_{i=1}^{n} X_i \quad \text{and} \quad \bar{X}_n = 1 \quad \text{if } X_n = \log_e u \quad \text{and} \quad \bar{X}_n = -1 \quad \text{if } X_n = \log_e d$$

and the expectation is with respect to the adjusted process.

Apart from the $\bar{X}_n k$ factor, the portfolio value is the discounted expectation of the value at maturity. However, in this case the expectation is based on a different stochastic process from that used in the no-transaction cost case. To obtain the corresponding formula for the standard no-transaction costs case we put $k = 0$. The transition matrix, P now will have identical columns reflecting the fact that the distribution of X_{j+1} no longer depends on X_j. In this case we are back to the familiar binomial process. Expression (10) also shows us that the cost of replicating a long call position with transaction costs is greater than the cost of replicating a call without transaction costs. Since after an up-jump the possibility of another up-jump is larger, there is a higher probability of a whole sequence of up-jumps leading to a higher probability for the high value of Y. The same holds for downward moves leading to a higher probability for low values of Y. Hence, the variance of Y is larger than the corresponding variance in the case of a call without transaction costs. The higher variance leads, in turn, to a higher price.

It is convenient to define the constants and matrices:

$$\theta = \frac{Rk}{\bar{u} - \bar{d}}, \quad A = \begin{pmatrix} 1 \\ -1 \end{pmatrix}(1, -1) \quad \text{and} \quad P = \begin{pmatrix} \bar{p} \\ 1 - \bar{p} \end{pmatrix}(1, 1) \tag{11}$$

we have the following matrix identity:

$$\bar{P} = \begin{pmatrix} \bar{p} + \theta & \bar{p} - \theta \\ 1 - \bar{p} - \theta & 1 - \bar{p} + \theta \end{pmatrix} = P + \theta A \tag{12}$$

The difference between our stochastic Markov process and a process with independent increments is given by the matrix θA. With $k = 0$, θ equals zero and we are back to the no transaction costs case.

III. An Approximation for the Long Call Option Price with Transaction Costs when the Number of Periods is Large

For numerical computations of the long call portfolio it is convenient to use the recursive formulae: Equations (4) and (5). Equation (10) is less useful for practical computations. However, it can be used to develop a Black-Scholes type approximation to the long call price when there are transaction costs. We now sketch the derivation of this approximation.

We will use the standard binomial tree as in Cox and Rubinstein (1985) with parameters u, d, and R[4] given by

$$u = e^{\sigma\sqrt{h}}, \qquad d = e^{-\sigma\sqrt{h}}, \qquad R = e^{rh} \tag{13}$$

where $h = T/n$, σ is the volatility of the risky asset, and r is the riskless continuous interest rate. We assume that the time to maturity of the option, T, is one year.

[4] Clearly the parameters u, d, and R depend on n. We suppress this dependence for convenience.

If we consider the Markov process described in the previous section, we have the following result.

Lemma 1: *The variance of the random variable Y of Theorem 2 has the following behavior for large n and small k*

$$\mathrm{Var}(Y) = \sigma^2\left(1 + O(k^2) + \left\{\frac{2k}{\sigma} + O(k^3)\right\}\sqrt{n}\right) + O\left(\frac{1}{\sqrt{n}}\right). \tag{14}$$

This result is proved in the Appendix.

The next step is to establish that the asset price process is risk-neutral under the new Markov process. To do this we compute an expression for the expected value of Y in Lemma 2.

Lemma 2: *The expected value of the random variable Y of Theorem 2 has the following behavior for large n and small k*:

$$E(Y) = r - \frac{1}{2}[\mathrm{Var}(Y)] + O\left(\frac{1}{\sqrt{n}}\right) + O(k^2) \tag{15}$$

where $\mathrm{Var}(Y)$ *is given by equation* (14). *This lemma is proved in the Appendix.*

Lemma 3: *For large n and small k we have the following result for the random variable* $\overline{X}_n k$:

$$E(\overline{X}_n k) = -k\{k + O(1/\sqrt{n})\} \tag{16}$$

$$\mathrm{Cov}(\overline{X}_n k, Y) = 4k^2 + O(1/\sqrt{n}) \tag{17}$$

The proof of this lemma is similar to the proofs of the other two lemmas and is omitted.

It follows from Lemma 3 that the factor $\overline{X}_n k$ in (10) can be neglected for large n and small k. Hence, (10) can be approximated by

$$C = \frac{E[(Se^Y - K)I_{Se^y \geq K}]}{R^n}. \tag{18}$$

For large n the distribution of the random variable Y tends[5] to the normal distribution with mean and variance given by Lemmas 1 and 2. Hence, the distribution of the stock price tends to the corresponding lognormal distribution. We can use the standard Black-Scholes methodology to compute expression (18). This leads to the following theorem.

Theorem 3: *For large n and small k the initial value of the hedge portfolio under a dynamic strategy that replicates a call option at the maturity date and is self-financing inclusive of transaction costs, is approximately equal to the*

[5] See Billingsley (1979), Example 25.5 and Theorem 27.5, for the justification of using this limit.

Black-Scholes value but with modified variance given by

$$\sigma^2\left(1 + \frac{2k\sqrt{n}}{\sigma\sqrt{T}}\right)$$

where T is the time to option maturity.

Theorem 3 provides a very convenient method to compute the long call price. As noted above we will illustrate the accuracy of the approximation formula in Section V.

IV. The Short Call Option Price

We now investigate[6] the replication of a short call in the presence of proportional transaction costs. To obtain the short call value we compute the cost of creating a self-financing replicating portfolio which has exactly the same value at expiration as a short position in a European call. The dynamic hedging strategy takes account of the transaction costs incurred at each trading date. The method is similar to that used to derive the long call value but there are some technical differences.

First consider a one-period model. Since we are replicating a short position in a call option the replicating portfolio at expiration will consist of a short position in the risky asset plus a long position in the riskless asset (or else zero shares of each security). If the call is in the money at expiration the value of the replicating portfolio at expiration will be negative. The replicating portfolio at the start of the period also involves a short position in the risky security. The recursive equations for the replicating portfolio which has a payoff equal to the short position in the call option are exactly the same as (1) and (2). One difference is that the sign of the holdings in the risky asset is now negative (or zero) on the boundary. Corresponding to Theorem 1 we have:

Theorem 4: *In the construction of a short call position by dynamic hedging there is a unique solution* (Δ, B) *to equations* (1) *and* (2) *if*

$$u(1-k) \geq R(1+k) \tag{20}$$

and

$$d(1+k) \leq R(1-k) \tag{21}$$

Furthermore, the Δ *corresponding to the unique solution always lies in the interval* (Δ_1, Δ_2) *if none of the terminal stock prices in the tree lies in the interval* $[K/(1+k), K/(1-k)]$.

[6] We are grateful to Fischer Black for suggesting we examine this issue.

The proof is given in the Appendix. Conditions (20) and (21) imply

$$u(1 - k)^2 > d(1 + k)^2 \tag{22}$$

from which it follows that $u(1 - k) \geq d(1 + k)$. In fact we will show that there exists a unique solution if this last condition is satisfied. However, we will still impose (20) and (21) since, as will be shown in the Appendix, they are necessary for proving that $\Delta_1 \leq \Delta \leq \Delta_2$. Furthermore, if (20) and (21) are not satisfied there will be a strategy that dominates the replicating strategy.

The final condition in Theorem 4 is necessary to rule out certain pathological cases, as will be explained in the proof, but it is not of great consequence. As long as (20) and (21) are satisfied it follows from (22) that the distance on a logarithmic scale of two adjacent final stock prices is at least $2[\ln(1 + k) - \ln(1 - k)]$ while the interval has only a logarithmic length of $[\ln(1 + k) - \ln(1 - k)]$. Hence in at least 50% of the cases the condition will be satisfied and if we fix S and K and vary n the condition will be satisfied for at least every second n as long as (20) and (21) hold. If this condition is not satisfied the value of a replicating short position always can be calculated directly using equations (1) and (2).

If the conditions are met and n is large we can use the method of the previous section to show that the value of replicating a short call can be approximated by a Black-Scholes formula with a modified variance given by

$$\sigma^2\left(1 - 2k\sqrt{n}/\sigma\sqrt{T}\right) \tag{23}$$

i.e., we replace k by $-k$. Since the option price is continuous in K, this is also a good approximation if the final stock price condition is not met. However, conditions (20) and (21) are essential for the approximation to be valid as we will now explain. Equation (22) states that $u(1 - k)^2 > d(1 + k)^2$. Hence by taking logarithmics and substituting the expressions for u and d we find that this is equivalent to

$$\sigma\sqrt{h} + 2\ln(1 - k) > -\sigma\sqrt{h} + 2\ln(1 + k) \tag{24}$$

which, for small k, is approximately equivalent to $2\sigma\sqrt{h} > 4k$, i.e.,

$$\sigma^2\left(1 - 2k\sqrt{n}/\sigma\sqrt{T}\right) > 0$$

Thus, if (20) and (21) are not satisfied the approximation might well lead to a negative modified variance.

V. Numerical Calculations

In this section we compute short and long call option prices for a range of parameter values.[7] In addition to illustrating the comparative statistics, these computations form a basis for comparison with the approximation we introduced earlier. For all our simulations, we take the current price of the

[7] Our parameter values correspond to those assumed by Leland.

risky asset to be 100, the time to option expiry one year, and the (effective) interest rate to be 10% p.a. For our base case assumptions the standard deviation of the return on the risky asset is 20% p.a. We examine the sensitivity of the option prices to the strike price and the transaction cost rate. The zero transaction cost case corresponds to the Cox-Ross-Rubinstein case and is used as a benchmark.

Our approach is to present the results for the long call case first. Table I provides the long call values for a range of transaction cost and strike price assumptions. The first panel, corresponding to $k = 0$, contains the zero-transaction costs benchmark prices. As we would expect the influence of the transaction costs increases with the frequency of trading and also with the magnitude of the costs. As the strike price increases, the size of the spread[8] caused by transaction costs increases, reaches a maximum, and then decreases. The long call spread is highest when the current asset price is equal to the discounted strike price. This corresponds to the case when the option's time value is highest. In a discrete-time model we can see the intuition behind this result. Consider a call which is very deep in the money so that there is no chance it will mature out of the money. The dynamic replicating portfolio in this case is certain to consist of a long position in the underlying asset and a short position in the discounted strike price. If this portfolio is maintained throughout the lattice there will be no transactions required. At the other extreme, consider an option which is so far out of the money that there is zero chance that it will end up in the money. In this case the option value at expiration will be zero so that the hedge portfolio is degenerate consisting of no risky asset and no bonds. To maintain such a portfolio throughout the lattice costs nothing and so transaction costs have no impact on the call's price (of zero). As the strike price moves away from either of these extremes the importance of transaction costs increases, reaching a maximum when the option's time value attains its maximum.

In Table II we compare the long call prices generated by our exact discrete model to the continuous-time approximation given in Theorem 3. We see that the approximation formula is very accurate and that the accuracy increases as the number of trading intervals increases. Recall that our maintained assumption is that the true asset return process follows a multiplicative binomial process as in Cox-Ross-Rubinstein.

In Table III we compare the long call prices produced by our exact discrete-time model with those of Leland for corresponding parameter values. In Leland's approach the dynamic portfolio strategy is not self-financing, since he uses a continuous model with discrete revision times. In our discrete approach the replicating portfolio is self-financing and therefore more costly. (It produces higher option values.) Since both Leland's formula and our approximation are of the Black-Scholes type with an adjusted variance and since our approximation is very accurate (according to the previous table),

[8] The spread is the difference between the option price with transaction costs and the price with no transaction costs.

Table I

European Long Call Prices

Long call prices are computed in a discrete-time setting using the recursive procedure based on equations

$$\Delta S\bar{u} + BR = \Delta_1 S\bar{u} + B_1 \tag{4}$$

$$\Delta S\bar{d} + BR = \Delta_2 S\bar{d} + B_2 \tag{5}$$

$$\bar{u} = u(1+k) \quad \text{and} \quad \bar{d} = d(1-k) \tag{6}$$

where Δ is the number of shares of stock in the initial period and Δ_1 represents the number of shares next period in the up-state and Δ_2 corresponds to the down-state. The corresponding holdings of riskless bonds are B, B_1, and B_2. One plus the one-period riskless interest rate is denoted by R; u is the one-period stock return in the up-state; d is the one-period stock return in the down-state; and k is the transaction cost rate on stocks.

Parameters: asset price = 100, standard deviation = 20% p.a., time to expiry = 1 year, interest rate = 10% p.a. effective.

	Number of Revision Times (n)			
Strike Price	6	13	52	250
		$k = 0\%$		
80	27.703	27.701	27.665	27.675
90	19.821	19.740	19.667	19.674
100	12.655	13.093	12.953	12.984
110	8.129	8.026	7.972	7.965
120	4.216	4.427	4.548	4.551
		$k = 0.125\%$		
80	27.735	27.747	27.753	27.876
90	19.894	19.842	19.865	20.103
100	12.770	13.248	13.256	13.630
110	8.254	8.205	8.324	8.715
120	4.329	4.595	4.882	5.269
		$k = 0.5\%$		
80	27.837	27.894	28.047	28.574
90	20.113	20.149	20.453	21.346
100	13.106	13.699	14.111	15.339
110	8.618	8.721	9.300	10.649
120	4.663	5.084	5.820	7.161
		$k = 2\%$		
80	28.297	28.563	29.409	31.568
90	20.983	21.346	22.643	25.524
100	14.358	15.333	16.966	20.413
110	9.965	10.555	12.469	16.192
120	5.926	6.859	8.950	12.750

our higher option values should be confirmed by the adjusted variance. Indeed, the two expressions for the variance are very similar, but where Leland has a factor of $\sqrt{(2/\pi)}$ we have unity. Since $\sqrt{(2/\pi)} \approx 0.8$ our model leads to higher option values than Leland's.

The comparative statistics for the short call case are very similar to those for the long call case. Table IV provides short call values for the same set of

Table II

Comparison of Accurate Long Call Prices Based on Our Discrete-Time Model with the Black-Scholes Type Approximation

Accurate long call prices (BV) with transaction costs are computed using the recursive procedure based on equations

$$\Delta S\bar{u} + BR = \Delta_1 S\bar{u} + B_1 \tag{4}$$

$$\Delta S\bar{d} + BR = \Delta_2 S\bar{d} + B_2 \tag{5}$$

$$\bar{u} = u(1+k) \quad \text{and} \quad \bar{d} = d(1-k) \tag{6}$$

where Δ is the number of shares of stock in the initial period, Δ_1 represents the number of shares next period in the up-state, and Δ_2 corresponds to the down-state. The corresponding holdings of riskless bonds are B, B_1, and B_2. One plus the one-period riskless interest rate is denoted by R; u is the one-period stock return in the up-state; d is the one-period stock return in the down-state; and k is the transaction cost rate on stocks.

The Black-Scholes approximation (A) uses the modified variance given by

$$\sigma^2\left(1 + \frac{2k\sqrt{n}}{\sigma\sqrt{T}}\right) \tag{19}$$

Parameters: asset price = 100, standard deviation = 20% p.a., time to expiry = 1 year, interest rate = 10% p.a. effective, BV = our discrete model, A = approximation value, AD = absolute difference.

	Number of Revision Times (n)								
Strike Price	6 BV	6 A	6 AD	52 BV	52 A	52 AD	250 BV	250 A	250 AD
				k = 0%					
80	27.703	27.675	0.028	27.665	27.675	0.010	27.675	26.675	0.000
90	19.821	19.675	0.147	19.667	19.675	0.008	19.674	19.675	0.000
100	12.655	12.993	0.338	12.953	12.993	0.040	12.984	12.993	0.008
110	8.129	7.966	0.164	7.972	7.966	0.006	7.965	7.966	0.000
120	4.216	4.555	0.339	4.548	4.555	0.007	4.551	4.555	0.004
				k = 0.125%					
80	27.735	27.705	0.031	27.753	27.764	0.010	27.876	27.876	0.000
90	19.894	19.741	0.153	19.865	19.870	0.006	20.103	20.102	0.001
100	12.770	13.096	0.326	13.256	13.292	0.036	13.630	13.636	0.006
110	8.254	8.086	0.167	8.324	8.316	0.008	8.715	8.714	0.001
120	4.329	4.670	0.341	4.882	4.889	0.007	5.269	5.272	0.004
				k = 0.5%					
80	27.837	27.797	0.040	28.047	28.056	0.009	28.574	28.572	0.002
90	20.113	19.940	0.173	20.453	20.451	0.002	21.346	21.342	0.004
100	13.106	13.397	0.291	14.111	14.135	0.023	15.339	15.340	0.001
110	8.618	8.438	0.180	9.300	9.286	0.014	10.649	10.645	0.004
120	4.663	5.006	0.343	5.820	5.826	0.006	7.161	7.162	0.002
				k = 2%					
80	28.297	28.207	0.090	29.409	29.398	0.011	31.568	31.549	0.019
90	20.983	20.724	0.260	22.643	22.603	0.040	25.524	25.498	0.025
100	14.358	14.513	0.155	16.966	16.941	0.025	20.413	20.389	0.024
110	9.965	9.715	0.250	12.469	12.418	0.050	16.192	16.166	0.026
120	5.926	6.246	0.320	8.950	8.933	0.017	12.750	12.733	0.017

Table III

Comparison of Long Call Prices Generated by Our Discrete-Time Model and Those Obtained from Leland's Model

Long call prices (BV) are computed using the recursive procedure based on equations

$$\Delta S\bar{u} + BR = \Delta_1 S\bar{u} + B_1 \tag{4}$$

$$\Delta S\bar{d} + BR = \Delta_2 S\bar{d} + B_2 \tag{5}$$

$$\bar{u} = u(1+k) \quad \text{and} \quad \bar{d} = d(1-k) \tag{6}$$

where Δ is the number of shares of stock in the initial period, Δ_1 represents the number of shares next period in the up-state, and Δ_2 corresponds to the down-state. The corresponding holdings of riskless bonds are B, B_1, and B_2. One plus the one-period riskless interest rate is denoted by R; u is the one-period stock return in the up-state; d is the one-period stock return in the down-state; and k is the transaction cost rate on stocks.

Leland's modified variance is:

$$\sigma^2\left(1 + \frac{2k\sqrt{2n}}{\sigma\sqrt{\pi T}}\right)$$

Parameters: asset price = 100, standard deviation = 20% p.a., time to expiry = 1 year, interest rate = 10% p.a. effective, L = via Leland's modification, BV = our discrete model, AD = absolute difference.

	Number of Revision Times (n)					
	6	6	6	52	52	52
Strike Price	L	BV	AD	L	BV	AD
			k = 0%			
80	27.675	27.703	0.028	27.675	27.665	0.010
90	19.675	19.821	0.147	19.675	19.667	0.008
100	12.993	12.655	0.338	12.993	12.953	0.040
110	7.966	8.129	0.164	7.966	7.972	0.006
120	4.555	4.216	0.339	4.555	4.548	0.007
			k = 0.125%			
80	27.698	27.735	0.037	27.745	27.753	0.008
90	19.728	19.894	0.167	19.831	19.865	0.034
100	13.075	12.770	0.305	13.232	13.256	0.024
110	8.062	8.254	0.192	8.246	8.324	0.077
120	4.647	4.329	0.318	4.822	4.882	0.060
			k = 0.5%			
80	27.771	27.837	0.066	27.974	28.047	0.073
90	19.887	20.113	0.227	20.296	20.453	0.157
100	13.317	13.106	0.210	13.915	14.111	0.196
110	8.344	8.618	0.274	9.035	9.300	0.265
120	4.916	4.663	0.254	5.582	5.820	0.238
			k = 2%			
80	28.091	28.297	0.206	29.026	29.409	0.383
90	20.515	20.983	0.468	22.052	22.643	0.592
100	14.225	14.358	0.133	16.253	16.966	0.714
110	9.388	9.965	0.577	11.659	12.469	0.809
120	5.926	5.926	0.000	8.172	8.950	0.778

Table IV

European Short Call Option Prices

Short call prices are computed in discrete time setting using the recursive procedure based on equations

$$\Delta Su + BR = \Delta_1 Su + B_1 + k\,|\,\Delta - \Delta_1\,|\,Su \quad (1)$$

$$\Delta Sd + BR = \Delta_2 Sd + B_2 + k\,|\,\Delta - \Delta_2\,|\,Sd \quad (2)$$

where Δ is the number of shares of stock in the initial period, Δ_1 represents the number of shares next period in the up-state, and Δ_2 corresponds to the down-state. The corresponding holdings of riskless bonds are B, B_1, and B_2. One plus the one-period riskless interest rate is denoted by R; u is the one-period stock return in the up-state; d is the one-period stock return in the down-state; and k is the transaction cost rate on stocks.

Parameters: asset price = 100, standard deviation = 20% p.a., time to expiry = 1 year, interest rate = 10% p.a. effective.

Strike Price	Number of Revision Times (n)			
	6	13	52	250
		k = 0%		
80	27.703	27.701	27.665	27.675
90	19.821	19.740	19.667	19.674
100	12.655	13.093	12.953	12.984
110	8.129	8.026	7.972	7.965
120	4.216	4.427	4.548	4.551
		k = 0.125%		
80	27.671	27.656	27.582	27.502
90	19.749	19.638	19.469	19.246
100	12.538	12.935	12.637	12.286
110	8.003	7.843	7.604	7.136
120	4.102	4.256	4.202	3.773
		k = 0.5%		
80	27.582	27.534	27.383	27.273
90	20.531	19.333	18.889	18.221
100	13.168	12.445	11.597	9.684
110	7.614	7.269	6.374	3.647
120	3.754	3.726	3.077	0.879
		k = 2%		
80	27.327	27.276	*	*
90	18.697	18.281	*	*
100	10.323	10.115	*	*
110	5.845	4.311	*	*
120	2.266	1.266	*	*

* Signifies that inequality (25) or (26) is violated for these parameter values.

parameters as Table I. The short call spread increases with the frequency of trading and also with the size of the transaction costs. However, for certain parameter combinations the necessary conditions for the validity of the first part of Theorem 4 are violated and we cannot use our recursive procedure to compute the short call prices. These combinations are denoted with an asterisk in Table IV.

VI. Conclusions

This paper derived a procedure for computing option prices in a discrete-time model when there are proportional transaction costs. The long call price corresponds to the current value of a portfolio which exactly replicates the payoff to a long call. The corresponding short call price can be established by finding the cost of replicating a short call position. We explored the numerical values of these prices and concluded that the impact of transaction costs can be substantial expecially if the number of revision times is large. While our analysis just dealt with European call options it can be extended to cover European put options. We could derive the corresponding recursive equations for the put case and develop an algorithm for the multi-period case. It is more convenient to derive the put values from put-call parity.

We also demonstrated that the long call price can be expressed as a discounted expectation under a new Markov process. This leads to an approximation for the long call price in terms of a modified Black-Scholes formula. This modification involves increasing the variance as described in Theorem 3. The accuracy of the approximation increases with the number of trading intervals. We noted some interesting asymmetries between the properties of the long call price and the short call price.

Our approach assumes that the frequency of transactions is specified exogenously. To derive the prices we have assumed that the replication is exact and that there will be no hedging errors at maturity. To ensure this, trading occurs at each trading date. Risk-averse economic agents would be willing to tolerate less than perfect hedging for a reduction in transaction costs. This leads to the possibility of determining the transaction frequency endogenously and some progress in this direction has been made in recent papers.

It is clear from our results that market-makers should both buy and sell calls at the same time. If they just sell call options they should charge higher prices to their clients in order to be able to hedge properly. However, this paper suggests why options with the same maturity but different exercise prices have different implied volatilities. For example, out of the money puts often have a higher implied volatility than at the money puts with the same maturity date. This can be due to the fact that there is a strong demand from customers for the out of the money puts. Hence market-makers have a net-selling position and should hedge.

Appendix

Proof, Theorem 1[9]: We prove Theorem 1 by backward induction. By induction we may assume that $\Delta_4 \leq \Delta_1 \leq \Delta_3$ and $\Delta_5 \leq \Delta_2 \leq \Delta_4$. Thus $\Delta_2 \leq \Delta_1$. Subtracting (2) from (1), transferring everything to the right-hand side, and

[9] We would like to thank Stuart Turnbull for shortening the original proof of Theorem 1.

introducing the function $f(\Delta)$ we get

$$f(\Delta) := \Delta S(u-d) - \Delta_1 Su + \Delta_2 Sd - B_1 + B_2 \\ - k\,|\Delta - \Delta_1|\,Su + k\,|\Delta - \Delta_2|\,Sd = 0 \quad \text{(A1)}$$

The function $f(\Delta)$ is continuous and piecewise linear; i.e., it is a linear function on $(-\infty, \Delta_2)$, (Δ_2, Δ_1) and (Δ_1, ∞) with constant derivatives on each interval with values $[(1+k)u - (1+k)d]S$, $[(1+k)u - (1-k)d]S$ and $[(1-k)u - (1-k)d]S$, respectively. Since all of these numbers are strictly positive, $f(\Delta)$ is a monotonically increasing piecewise linear function. Hence, it has a unique zero. This proves the first claim of Theorem 1. For the second part it is enough to show that

$$f(\Delta_2) \le 0 \quad \text{and} \quad f(\Delta_1) \ge 0 \quad \text{(A2)}$$

since this implies that $\Delta \in [\Delta_2, \Delta_1]$. Now

$$f(\Delta_2) = (\Delta_2 - \Delta_1)Su(1+k) - B_1 + B_2 \quad \text{(A3)}$$

$$f(\Delta_1) = (\Delta_2 - \Delta_1)Sd(1-k) - B_1 + B_2 \quad \text{(A4)}$$

Since by induction $\Delta_4 \le \Delta_1 \le \Delta_3$, we know that one of the equations from which Δ_1 has been deduced reads as follows:

$$\Delta_1 Sud + B_1 R = \Delta_4 Sud + B_4 + k(\Delta_1 - \Delta_4)Sud. \quad \text{(A5)}$$

Similarly, since $\Delta_5 \le \Delta_2 \le \Delta_4$ we have:

$$\Delta_2 Sud + B_2 R = \Delta_4 Sdu + B_4 + k(\Delta_4 - \Delta_2)Sdu. \quad \text{(A6)}$$

Subtracting (A5) from (A6) and dividing by R gives

$$(\Delta_2 - \Delta_1)Sdu/R + B_2 - B_1 = k\big[(\Delta_4 - \Delta_2) - (\Delta_1 - \Delta_4)\big]Sdu/R. \quad \text{(A7)}$$

Using this we can derive

$$\begin{aligned} f(\Delta_2) &= (\Delta_2 - \Delta_1)Su(1+k) - B_1 + B_2 \\ &\le (\Delta_2 - \Delta_1)Sdu(1+k)/R - B_1 + B_2 \\ &= k\big[(\Delta_4 - \Delta_2) - (\Delta_1 - \Delta_4)\big]Sdu/R + k(\Delta_2 - \Delta_1)Sdu/R \\ &= 2k(\Delta_4 - \Delta_1)Sdu/R \le 0. \end{aligned} \quad \text{(A8)}$$

Similarly $f(\Delta_1) \ge 0$ and thus we have proved (A2).

To start the induction we consider the option at maturity. At maturity there are two possible portfolios: $\Delta = 1$ and $B = -K$, if the asset price is above the exercise price and $\Delta = 0$ and $B = 0$, otherwise. Hence, at maturity we always have $\Delta_1 \ge \Delta_2$ in the notation of this appendix. One period before maturity there are three different cases. First, $\Delta_1 = \Delta_2 = 1$ in which case $\Delta = \Delta_1$ and $B = -K/R$ is the unique solution, which indeed has $\Delta_2 \le \Delta \le \Delta_1$. Second, $\Delta_1 = \Delta_2 = 0$, in which case $\Delta = 0$ and $B = 0$ is the unique solution which indeed has $\Delta_2 \le \Delta \le \Delta_1$. Finally, $\Delta_1 = 1$, $\Delta_2 = 0$. In this case the unique solution is

$$\Delta = \frac{(S\bar{u} - K)}{(S\bar{u} - S\bar{d})}.$$

Hence, $\Delta_2 = 0 < \Delta < 1 = \Delta_1$. This completes the first step of the induction proof.

Proof, Lemma 1: To calculate the variance of Y we first introduce the vector $\nu^T = (\log_e u \log_e d)$. It follows from properties of transition matrices like $\bar{P}$ that:

$$EX_i = \nu^T \bar{P}^{i-1} \hat{p} \tag{A9}$$

$$EX_i X_{i+j} = \nu^T \bar{P}^j \begin{pmatrix} \log_e u & 0 \\ 0 & \log_e d \end{pmatrix} \bar{P}^{i-1} \hat{p} \tag{A10}$$

Hence,

$$\begin{aligned} \operatorname{Cov}(X_i, X_{i+j}) &= EX_i X_{i+j} - EX_i EX_{i+j} \\ &= \nu^T \bar{P}^j \left\{ \begin{pmatrix} \log_e u & 0 \\ 0 & \log_e d \end{pmatrix} - \bar{P}^{i-1} \hat{p} \nu^T \right\} \bar{P}^{i-1} \hat{p} \end{aligned} \tag{A11}$$

Since

$$(1, 1) \left\{ \begin{pmatrix} \log_e u & 0 \\ 0 & \log_e d \end{pmatrix} - \bar{P}^{i-1} \hat{p} \nu^T \right\} = 0 \tag{A12}$$

and $PA = 0$ we can reduce (A11) to

$$\begin{aligned} \nu^T (\theta A)^j \left\{ \begin{pmatrix} \log_e u & 0 \\ 0 & \log_e d \end{pmatrix} - \bar{P}^{i-1} \hat{p} \nu^T \right\} \bar{P}^{i-1} \hat{p} &= \theta^j 2^{j-1} (\log_e u - \log_e d) \\ &\cdot (1, -1) \left\{ \begin{pmatrix} \log_e u & 0 \\ 0 & \log_e d \end{pmatrix} - \bar{P}^{i-1} \hat{p} \nu^T \right\} \bar{P}^{i-1} \hat{p}. \end{aligned} \tag{A13}$$

If we denote $\bar{P}^{i-1} \hat{p} = (p_i \;\, 1 - p_i)^T$, we find that

$$\operatorname{Cov}(X_i, X_{i+j}) = p_i (1 - p_i) \theta^j 2^j (\log_e u - \log_e d)^2 \tag{A14}$$

To calculate $\mathrm{Var}(Y) = \mathrm{Var}(\sum X_i)$ we simply have to add all the covariances, i.e., $\sum \mathrm{Var}(X_i) + 2\sum\sum \mathrm{Cov}(X_i, X_{i+j})$. We thus find as the total variance

$$(\log_e u - \log_e d)^2\left[\sum_{i=1}^{n} p_i(1-p_i)\left[2\sum_{j=0}^{n-i}(2\theta)^j - 1\right]\right]$$

$$= (\log_e u - \log_e d)^2\left[\sum_{i=1}^{n} p_i(1-p_i)\left(2\left(\frac{1-(2\theta)^{n-i+1}}{1-2\theta}\right) - 1\right)\right] \quad \text{(A15)}$$

Furthermore,

$$\bar{P}^{i-1}\hat{p} = (P + A\theta)^{i-1}\hat{p} = \sum_{j=0}^{i-1}(\theta A)^j P^{i-1-j}\hat{p} = \sum_{j=1}^{i-1}(\theta A)^j\hat{p} + \hat{p}$$

$$= \sum_{j=1}^{i-1}\theta^j 2^{j-1}(2\bar{p} - 1)\begin{pmatrix}1\\-1\end{pmatrix} + \hat{p}$$

$$= \theta\left(\frac{1-(2\theta)^{i-1}}{1-2\theta}\right)(2\bar{p} - 1)\begin{pmatrix}1\\-1\end{pmatrix} + \hat{p} \quad \text{(A16)}$$

where we have used that $PA = 0$ and $P\hat{p} = \hat{p}$. Hence,

$$p_i(1-p_i) = \left(\bar{p} + \theta\left(\frac{1-(2\theta)^{i-1}}{1-2\theta}\right)(2\bar{p}-1)\right)$$

$$\cdot\left(1 - \bar{p} - \theta\left(\frac{1-(2\theta)^{i-1}}{1-2\theta}\right)(2\bar{p}-1)\right) \quad \text{(A17)}$$

Substituting (A17) in (A15) and simplifying the resulting expression leads to[10]

$$(\log_e u - \log_e d)^2\left\{n\left[\left(\frac{\bar{p}-\theta}{1-2\theta}\right)\left(\frac{1-\bar{p}-\theta}{1-2\theta}\right)\left(\frac{1+2\theta}{1-2\theta}\right) - \frac{2(2\bar{p}-1)^2\theta(2\theta)^n}{(1-2\theta)^3}\right]\right.$$

$$+\left(\frac{\bar{p}-\theta}{1-2\theta}\right)\left(\frac{1-\bar{p}-\theta}{1-2\theta}\right)\left(\frac{-4\theta}{1-2\theta}\right)\left(\frac{1-(2\theta)^n}{1-2\theta}\right)$$

$$\left.+\frac{(2\bar{p}-1)^2}{(1-2\theta)^4}\left\{\theta + \theta^2 - \theta(2\theta)^n - \theta^2(2\theta)^{2n}\right)\right\}. \quad \text{(A18)}$$

[10] A detailed derivation is available from the authors.

To prove Lemma 1 we remark that

$$(\log_e u - \log_e d)^2 = \frac{4\sigma^2}{n} \tag{A19}$$

The next step is to examine each of the terms in expression (A18) to determine their dependence on n. In expression (A18) the major contribution will arise from the first term in square brackets:

$$\frac{\bar{p} - \theta}{1 - 2\theta} = \frac{e^{r/n} - e^{-\sigma/\sqrt{n}}(1 - k) - e^{r/n}k}{e^{\sigma/\sqrt{n}}(1 + k) - e^{-\sigma/\sqrt{n}}(1 - k) - 2e^{r/n}k} =$$

$$\frac{(1 - k)[1 + r/n + O(1/n^2)] - (1 - k)[1 - \sigma/\sqrt{n} + \sigma^2/2n + O(1/n\sqrt{n})]}{(1 + k)[1 + \sigma/\sqrt{n} + \sigma^2/2n + O(1/n\sqrt{n})] - (1 - k) \cdot [1 - \sigma/\sqrt{n} + \sigma^2/2n + O(1/n\sqrt{n})] - 2k[1 + r/n + O(1/n^2)]}$$

$$= \frac{(1 - k)\sigma/\sqrt{n} + (1 - k)(r - \sigma^2/2)/n + O(1/n\sqrt{n})}{2\sigma/\sqrt{n} - 2k(r - \sigma^2/2)/n + O(1/n\sqrt{n})}$$

$$= \left(\frac{(1 - k)}{2}\right)\left\{\frac{1 + (r - \sigma^2/2)/\sigma\sqrt{n} + O(1/n)}{1 - k(r - \sigma^2/2)/\sigma\sqrt{n} + O(1/n)}\right\}$$

$$= \left(\frac{1 - k}{2}\right)\{1 + (1 + k)(r - \sigma^2/2)/\sigma\sqrt{n} + O(1/n)\} \tag{A20}$$

which, if less precision is required, can be written as:

$$\left(\frac{1 - k}{2}\right)\{1 + O(1/\sqrt{n})\}$$

In a similar way one can derive:

$$\frac{1 - \bar{p} - \theta}{1 - 2\theta} = \left(\frac{1 + k}{2}\right)\{1 - (1 - k)(r - \sigma^2/2)/\sigma\sqrt{n} + O(1/n)\}$$

$$\text{or, } \left(\frac{1 + k}{2}\right)\{1 + O(1/\sqrt{n})\} \tag{A21}$$

$$\frac{1 + 2\theta}{1 - 2\theta} = \sqrt{n}\,\{2k/\sigma + (1 + 2k^2(r - \sigma^2/2)/\sigma^2)/\sqrt{n} + O(1/n)\},$$

$$\text{or } \sqrt{n}\,\{2k/\sigma + O(1/\sqrt{n})\} \tag{A22}$$

$$\frac{\theta}{1-2\theta} = \sqrt{n}\left[k/2\sigma + O(1/\sqrt{n})\right] \tag{A23}$$

$$\frac{1}{1-2\theta} = \sqrt{n}\left[k/\sigma + O(1/\sqrt{n})\right], \tag{A24}$$

$$\frac{2\bar{p}-1}{1-2\theta} = -k + \frac{(1+k)(1-k)(r-\sigma^2/2)}{\sigma\sqrt{n}} + O(1/n) \quad \text{or} \quad -k + O(1/\sqrt{n}). \tag{A25}$$

Also,

$$\left|(2\theta)^n\right| = \left|\frac{2^n k^n e^r}{e^{\sigma\sqrt{n}}\{(1+k)-(1-k)e^{-2\sigma/\sqrt{n}}\}^n}\right| \tag{A26}$$

The expression on the right-hand side of (A26) can be shown to be less than

$$e^r e^{-[\sigma+\log_e(1+\alpha)]\sqrt{n}},$$

where

$$\alpha = (1-k)\sigma(1-\sigma)/k.$$

From this last inequality we see that we can skip all terms with a factor $(2\theta)^n$ in expression (A18) if we want to calculate the limit behavior of (A18) for $n \to \infty$. Substituting (A20)–(A25) in what remains of (A18) and using alternatives in (A20)–(A22) depending on the required precision we derive

$$\begin{aligned}
&\frac{4\sigma^2}{n}\left\{n\left(\frac{1-k^2}{4}\right)\sqrt{n}\left(2k/\sigma + 1/\sqrt{n} + 6k^2(r-\sigma^2/2)/\sigma^2\sqrt{n} + O(1/n)\right)\right. \\
&\quad + (\sqrt{n})^2\left\{-\left(\frac{1-k^2}{4}\right)\frac{2k^2}{\sigma^2} + O(1/\sqrt{n})\right\} \\
&\quad \left. + (\sqrt{n})^2\left\{\frac{3k^4}{4\sigma^2} + O(1/\sqrt{n})\right\}\right\} \\
&= \sigma^2\sqrt{n}(1-k^2)\left(2k/\sigma + \left[1 + 6k^2(r-\sigma^2/2)/\sigma^2 - 2k^2/\sigma^2\right.\right. \\
&\quad \left.\left. + 3k^4/(1-k^2)\sigma^2\right]/\sqrt{n} + O(1/n)\right) \\
&= \sigma^2\left(1 + O(k^2) + \left\{\frac{2k}{\sigma} + O(k^3)\right\}\sqrt{n}\right) + O\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

This completes the proof of the Lemma 1.

Proof, Lemma 2: We use the same type of approach here as was used in establishing Lemma 1. From equation (A9) and the definition of $E(Y)$ we have

$$E(Y) = \sum_{i=1}^{n} \nu^T \bar{P}^{i-1} \hat{p} = \sum_{i=1}^{n} \nu^T \left[\theta\left(\frac{1-(2\theta)^{i-1}}{1-2\theta}\right)(2\bar{p}-1)\begin{pmatrix}1\\-1\end{pmatrix} + \hat{p}\right]$$

$$= (\log_e u - \log_e d)\sum_{i=1}^{n}\theta\left(\frac{1-(2\theta)^{i-1}}{1-2\theta}\right)(2\bar{p}-1)$$

$$+ \sum_{i=1}^{n}\left[\bar{p}\log_e u + (1-\bar{p})\log_e d\right]$$

$$= (\log_e u - \log_e d)\frac{n\theta(2\bar{p}-1)}{1-2\theta} - (\log_e u - \log_e d)$$

$$\cdot\left(\frac{\theta(2\bar{p}-1)}{1-2\theta}\right)\left(\frac{1-(2\theta)^n}{1-2\theta}\right) + n\bar{p}\log_e u + n(1-\bar{p})\log_e d$$

$$= 2\sigma\sqrt{n}\,\theta\left(\frac{2\bar{p}-1}{1-2\theta}\right) - \frac{2\sigma}{\sqrt{n}}\left(\frac{\theta}{1-2\theta}\right)\left(\frac{2\bar{p}-1}{1-2\theta}\right) + 2\sigma\sqrt{n}\,\bar{p} - \sigma\sqrt{n}$$

$$= 2\sigma\sqrt{n}\,\theta\left(\frac{2\bar{p}-1}{1-2\theta}\right) - \frac{2\sigma}{\sqrt{n}}\left(\frac{\theta}{1-2\theta}\right)\left(\frac{2\bar{p}-1}{1-2\theta}\right)$$

$$+\sigma\sqrt{n}\left(\frac{2\bar{p}-1}{1-2\theta}\right)(1-2\theta)$$

$$= \left(\frac{2\bar{p}-1)}{1-2\theta}\right)\sigma\sqrt{n} - \frac{2\sigma}{\sqrt{n}}\left(\frac{\theta}{1-2\theta}\right)\left(\frac{2\bar{p}-1}{1-2\theta}\right)$$

$$= \left(\frac{2\bar{p}-1}{1-2\theta}\right)\left\{\sigma\sqrt{n} - \frac{2\sigma}{\sqrt{n}}\left(\frac{\theta}{1-2\theta}\right)\right\}$$

$$= \left(-k + \frac{(1-k^2)(r-\sigma^2/2)}{\sigma\sqrt{n}} + O\left(\frac{1}{n}\right)\right)\left[\sigma\sqrt{n} - k + O\left(\frac{1}{\sqrt{n}}\right)\right]$$

$$= -k\sigma\sqrt{n} + k^2 + (1-k^2)\left(r - \frac{\sigma^2}{2}\right) + O\left(\frac{1}{\sqrt{n}}\right)$$

$$= r - \frac{\sigma^2}{2}\left[1 + \frac{2k\sqrt{n}}{\sigma}\right] + k^2\left(1 - r + \frac{\sigma^2}{2}\right) + O\left(\frac{1}{\sqrt{n}}\right)$$

$$= r - \frac{1}{2}\mathrm{Var}(Y) + k^2\left(1 - r + \frac{\sigma^2}{2}\right) + O\left(\frac{1}{\sqrt{n}}\right) - O(k^2),$$

where we have used the results of Lemma 1. Hence

$$E(Y) = r - \frac{1}{2}\mathrm{Var}(Y) + O(k^2) + O\left(\frac{1}{\sqrt{n}}\right).$$

This completes the proof of Lemma 2.

Proof, Theorem 4: The proof of Theorem 4 is similar to that of Theorem 1. We still have the function:

$$f(\Delta) = \Delta S(u-d) - \Delta_1 Su + \Delta_2 Sd - B_1 + B_2 - k\,|\,\Delta - \Delta_1\,|\,Su + k\,|\,\Delta - \Delta_2\,|\,Sd = 0$$

There are two possible cases: $\Delta_1 \leq \Delta_2$ and $\Delta_1 > \Delta_2$.

When $\Delta_1 \leq \Delta_2$, $f(\Delta)$ is linear function on $(-\infty, \Delta_1)$, (Δ_1, Δ_2), and (Δ_2, ∞), with constant derivatives on each interval with values

$$(1+k)(u-d)S, \quad [(1-k)u - (1+k)d]S, \quad \text{and} \quad (1-k)(u-d)S,$$

respectively. If $(1-k)u \geq (1+k)d$, all of these derivatives are positive, and $f(\Delta)$ is an increasing function.

When $\Delta_1 > \Delta_2$, $f(\Delta)$ is linear function on $(-\infty, \Delta_2)$, (Δ_2, Δ_1), and (Δ_1, ∞), with constant derivatives on each interval with values

$$(1+k)(u-d)S, \quad [(1+k)u - (1-k)d]S, \quad \text{and} \quad (1-k)(u-d)S,$$

respectively. All of these derivatives are strictly positive, and therefore, $f(\Delta)$ is an increasing function. Since the function is piecewise linear, it must have a zero.

To prove that $\Delta_1 \leq \Delta \leq \Delta_2$ we may assume by induction that $\Delta_3 \leq \Delta_1 \leq \Delta_4$ and $\Delta_4 \leq \Delta_2 \leq \Delta_5$. Thus $\Delta_1 \leq \Delta_2$. It only remains to show that $f(\Delta_1) < 0$ and $f(\Delta_2) \geq 0$. Now by condition (20).

$$f(\Delta_2) = (\Delta_2 - \Delta_1)Sd(1+k) - B_1 + B_2 \leq (\Delta_2 - \Delta_1)Sdu(1-k)/R - B_1 + B_2 \quad \text{(A27)}$$

As in the proof of Theorem 1 we can use the induction assumption to derive an equation equivalent to (A7):

$$(\Delta_2 - \Delta_1)Sdu/R - B_1 + B_2 = -k[(\Delta_4 - \Delta_2) - (\Delta_1 - \Delta_4)]Sdu/R \quad \text{(A28)}$$

substituting this in (A27) we derive

$$f(\Delta_1) \leq -k[(\Delta_4 - \Delta_2) - (\Delta_1 - \Delta_4)]Sdu/R - k(\Delta_2 - \Delta_1)]Sdu/R = 2k(\Delta_1 - \Delta_4)Sdu/R \leq 0 \quad \text{(A29)}$$

Similarly $f(\Delta_2) \geq 0$.

To start the induction we only have to consider the case where the stock price is such that the option ends in the money if it goes up and out of money if it goes down. In that case $\Delta_1 = -1$ and $\Delta_2 = 0$. Similar to the proof of Theorem 1

$$\Delta = -\left(\frac{Su(1-k) - K}{Su(1-k) - Sd(1+k)}\right)$$

Now by our conditions (20) and (21) the numerator is always positive and by the final condition of the theorem that both Su and Sd are not within the interval $[K/(1+k), K/(1-k)]$, it follows that $Su(1-k) \geq K \geq Sd(1+k)$, i.e., $-1 \leq \Delta \leq 0$.

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