NATURVIDENSKABELIG KANDIDATEKSAMEN VED KØBENHAVNS UNIVERSITET

MATHEMATICAL FINANCE THEORY

TAKE-HOME EXAM

Handed out 9 am (9:00) Tuesday January 11, 2005.
Answers must be handed in no later than noon (12:00) Wednesday January 12, 2005. You can hand in answers to me (room E-411 or electronically) or to the secretary in room E-301. Please hand in answers in duplicate (ie. two copies; photo-copies are fine) and remember to put your name and/or exam number on your answers.

This is an individual exam; you are not allowed to work in groups. So if you have questions concerning the exam, ask me (`rolf@math.ku.dk`). But please look at the course homepage before you ask.

Remark: There are several remarks in this exam set. They contain no questions and no direct hints, but provide supplementary information. So you might learn something from reading them. Whether or not you think that is a wise endeavor to pursue during an exam is up to you.

EXERCISE 1; STOCHASTIC CALCULUS

In this exercise W denotes a 1-dimensional (standard) Brownian motion.

1a [10%]
Show that the process defined by

$$X(t) = \left(1 + \frac{1}{3}W(t)\right)^3$$

solves the stochastic differential equation (SDE)

$$dX(t) = \frac{1}{3}X^{1/3}(t)dt + X^{2/3}(t)dW(t), \quad X(0) = 1.$$

Show that the process defined by

$$Y(t) = \cos(W(t))$$

solves SDE

$$dY(t) = -\frac{1}{2}Y(t)dt - \sqrt{1 - Y^2(t)}dW(t), \;\; Y(0) = 1.$$

1b[10%]
Consider the Cox/Ingersoll/Ross SDE

$$dX(t) = \kappa(\theta - X(t))dt + \sigma\sqrt{X(t)}dW(t), \quad X(0) = x.$$

Show that

$$m(t) := \mathbf{E}(X(t)) = X(0)e^{-\kappa t} + \theta(1 - e^{-\kappa t}).$$

Hint: Write the SDE on integral form, take mean, interchange (don't worry about technicalities; it is allowed), and observe that the m-function solves an ordinary differential equation (ODE).

Derive an ODE for the second moment, $h(t) := \mathbf{E}(X^2(t))$.
Hint: Use Ito on X^2 and do as in the first part of the question.

Remark: You are *not* asked to solve the second moment ODE. If you (or MAPLE) do, and manipulate the results a little, you'll find that

$$\text{var}(X(t)) = X(0)\frac{\sigma^2}{\kappa}(e^{-\kappa t} - e^{-2\kappa t}) + \theta\frac{\sigma^2}{2\kappa}(1 - e^{-\kappa t})^2.$$

Because X is time-homogeneous Markov process conditional moments are found by shifting the formulas/equations.

EXERCISE 2: BLACK/SCHOLES OPTION PRICING (20%)

In this exercise we consider the base-case Black/Scholes model.

Let (the power) $p \in \mathbb{R}$, (the strike) $K \in \mathbb{R}_+$, and (the expiry date) $T \in \mathbb{R}_+$ be given and consider the *powered call*, which is a simple contingent claim that that pays

$$(S^p(T) - K)^+ \text{at time } T.$$

Derive a closed-form expression for the arbitrage-free price of the powered call.
Hint: "Brute force" works, but using Ito on S^p and remembering how the Black/Scholes formula looks for dividend-paying stocks can save you calculations.

Assume $T = 1$, $K = 10$, $p = 1/2$, $S(0) = 100$, $r = 0.02$ and $\sigma = 0.2$. **Calculate** (numbers, please) the time-0 price of the powered call.

Explain how the powered call can be dynamically hedged using the stock and the bank-account. **How many** units of stock must you hold?

EXERCISE 3: OPTION PRICING WITH A TWIST (20%)

In this exercise we consider a general arbitrage-free stock price model in a world where there exists a bank-account on which the interest rate is 0, i.e.

$$dS(t) = \sigma(t)S(t)dW^Q(t),$$

where σ is an arbitrary (up to regularity conditions, that you needn't worry about) stochastic process.
Consider a simple contingent claim on S with pay-off function g, ie. it pays $g(S(T))$ to the holder at time T.
Let $\widetilde{\sigma} \in \mathbb{R}$ be given and define the function $F : [0;T] \times \mathbb{R} \mapsto \mathbb{R}$ as the solution to the (familiar) partial differential equation (subscripts denote diffentiation)

$$F_t(t,x) + \frac{1}{2}\widetilde{\sigma}^2 x^2 F_{xx}(t,x) = 0 \text{ for } t < T, \quad F(T,x) = g(x).$$

Consider, finally, a trading strategy h that holds $h_1(t) = F_x(t, S(t))$ units of the stock and $h_2(t) = F(t, S(t)) - S(t)F_x(t, S(t))$ units of the bank account at time t.

Show that this trading strategy replicates the pay-off of the g-claim, ie. that its value process, say V^h, satisfies $V^h(T) = g(S(T))$.

Show that the self-financing condition for this trading strategy boils down to the equation

$$\frac{1}{2}(\sigma^2(t) - \widetilde{\sigma}^2)S^2(t)F_{xx}(t, S(t)) = 0 \tag{1}$$

holding (almost everywhere, in an appropriate sense, that you needn't worry about).

Argue that "usual results" are obtained when $\sigma(t) = \widetilde{\sigma}$, and that in general the h-strategy can be interpreted as "trying to replicate as if it were the Black/Scholes model".

Remark: The result in (1) may alternatively be formulated by saying that h has an extra financing need of

$$\frac{1}{2}\int_0^T (\sigma^2(t) - \widetilde{\sigma}^2)S^2(t)F_{xx}(t, S(t))dt,$$

and is sometimes called the "1st fundamental theorem of derivative trading". It has consequences for hedging in misspecified models: *If* we consider a convex claim (in the sense that $F_{xx} > 0$) and *if* there is an upper bound on the σ-process, *then* a super-replicating strategy is achieved by "Black/Scholes Δ-hedging with the upper bound".

<u>EXERCISE 4: LONG RATES</u> (20%)

This exercise is very much inspired by Björk's exercise 22.7.

In term structure modelling it is often considered a reasonable request that the forward rate curve has a horizontal asymptote, ie. $\lim_{T\to\infty} f(t,T)$ exists for all t.

Remark: Björk says that "obviously the limit will depend on $r(t)$ and t". To me seems more obvious that the limit – if it exists – does *not* depend on $r(t)$ and t. Neither is true, but it can be shown that if the process $f^{\infty}(t) := \lim_{T\to\infty} f(t,T)$ is well-defined, then it is increasing. And that there are models – fairly strange ones, though – where the limit depends non-trivially on $r(t)$.

<u>4a</u> [10%]
Show that in the Vasicek model, the request is fulfilled, ie. there is indeed a horizontal asymptote. **Find it.**

<u>4b</u> [10%]
Consider a Ho/Lee-model calibrated (via θ as usual) to an initially observed forward rate curve, say $f^*(0,T)$.
Show that

$$f(t,T) = f^*(0,T) + \sigma^2 t\left(T - \frac{1}{2}t\right) + \sigma W^Q(t) \quad \text{for all } T \geq t \geq 0.$$

Use this to **show that** even if the initial forward rate curve has a horizontal asymptote, none of the forward rate curves that the model subsequently generates have this property.

Remark: The result in question 4b is an example of a modelling inconsistency across the calender time and maturity time domains. It is a subject that Björk (with co-authors) has studied a great deal. A strange result (obtained with doses of stochastic geometry and Lie-brackets) is that the Nelson/Siegel-curve (which you may or may not have heard of) is inconsistent with any finite dimensional (which you haven't heard of anything that isn't) arbitrage-free term structure model.

EXERCISE 5: OPTIONS ON COUPON BONDS (20%)

In this exercise we consider the Vasicek model,

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW^Q(t),$$

assume (as always) that $\kappa > 0$, and let A and B denote the functions such that $P(t,T) = \exp(A(t,T) - B(t,T)r(t))$.
We now consider a *coupon bond* that makes deterministic positive payments $\alpha_1, \ldots, \alpha_N$ at dates $T_1, \ldots, T_N$. Clearly the price of this coupon bond is

$$\pi^C(t) = \sum_{i|T_i>t} \alpha_i P(t, T_i).$$

(It is strict inequality, ">", to keep in line with prices always being ex-dividend.)
The last ingredient we need is a (positive) strike-K, expiry-T European call-option on the coupon bond.

Show that there exists a unique $r^* \in \mathbb{R}$ such that $\pi^C(T) \geq K$ if and only if $r(T) \leq r^*$.
Hint: Use the Vasicek assumption and ask yourself "when is $\pi^C(T) = K$?"

Define the *adjusted strikes* via

$$K_i = \exp(A(T, T_i) - B(T, T_i)r^*).$$

Show that the pay-off of the call can be written as

$$(\pi^C(T) - K)^+ = \sum_{i|T_i>T} \alpha_i (P(T, T_i) - K_i)^+.$$

Hint: Two things can happen to the right hand side. Investigate these separately.

Explain how (given results known from for instance Björk) this leads to a closed-form (up to knowledge of r^*) expression for the price of the call on the coupon bond.

Assume

- $\theta = 0.05$, $\kappa = 0.2$, $\sigma = 0.01$, $r_0 = 0.05$
- $T = 1$, $N = 2$, $T_i = i + 1$ and $\alpha_i = 1$ for $i = 1, 2$, $K = 1.856$.

Calculate the time-0 price of the call. (Numbers, please; this will involve solving an equation numerically.)

Remark: The exact same trick (due to Farshid Jamshidian) reduces a call on a portfolio in the Cox/Ingersoll/Ross to a portfolio of ZCB calls.
Unfortunately, the trick does not work in multidimensional models.