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SESSION TOPIC: OPTIONS

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THE VALUATION OF AMERICAN PUT OPTIONS

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I

WHILE THE PROBLEM OF PRICING European put options on non-dividend paying stocks was solved under certain conditions by Black-Scholes [2] in their seminal article on option pricing, no closed form solution exists for the valuation of American put options which permit exercise prior to maturity, except for the case of a perpetual put option on a non-dividend paying stock: this was treated by Merton [6]. In this paper we present an algorithm for the put pricing problem when the put has a finite life and may or may not be protected against dividend payments on the underlying stock. The algorithm is then used to evaluate the pricing of put contracts traded in the New York dealer market. Black and Scholes [1] have previously examined the pricing of calls in this market, while Gould and Galai [3] have documented violations of put call parity. A recent paper by Parkinson [7] applies numerical integration to the pricing of puts.

II

Define:

S—the market price of one share of stock on which the put is written
E_t—the exercise price of the put at time t.
$P(S,t)$—the value at time t of a put to sell one share of stock at the exercise price E_τ, $(\tau = t,\ldots,T)$ until expiration, T.
r—the continuously compounded risk free rate of interest.
D_t—the amount of the discrete dividend payment on the underlying stock at time t.

In between dates of dividend payments the stock price is assumed to follow the stochastic process

$$\frac{dS}{S} = \mu\, dt + \sigma\, dz \tag{1}$$

where dz is a Gauss-Wiener process. Then, as shown by Black-Scholes [2] and Merton [6], arbitrage considerations dictate that the value of the put must obey the partial differential equation

$$\tfrac{1}{2}\sigma^2 S^2 P_{SS} + rSP_S - rP + P_t = 0 \tag{2}$$

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where the subscripts denote partial differentiation.

In addition, $P(S,t)$ must satisfy certain further conditions:

$$P(S,T)=\max[E_T-S,0] \tag{3}$$

The value of the put at expiration is equal to the greater of its exercise value and zero.

$$P(S,t)\geqslant \max[E_t-S,0] \tag{4}$$

The possibility of early exercise prevents the value of an American put falling below the exercise value.

$$P(S,t)\leqslant E_t \tag{5}$$

(5) holds if the exercise price is a non-decreasing function of time to maturity. Then the maximum value the put can attain is the current exercise price, if the stock price falls to zero. Simple considerations of stochastic dominance indicate that the value of the put can never exceed this maximum value.

$$P(S,t)\geqslant 0 \tag{6}$$

Since the put contract is an option its value can never fall below zero, as shown by (4).

Further, like a warrant, the value of a put is a convex function of the stock price. Writing the put value explicitly as a function of the stock price and the exercise price, the put price must satisfy.

$$\lambda P(S_1,E)+(1-\lambda)P(S_2,E)\geqslant P(\lambda S_1+(1-\lambda)S_2,E), \qquad 0\leqslant\lambda\leqslant 1 \tag{7}$$

To see this, let $S_1=h_1E, S_2=h_2E$. Then we wish to establish that

$$\lambda P(h_1E,E)+(1-\lambda)P(h_2E,E)\geqslant P(\lambda h_1E+(1-\lambda)h_2E,E) \tag{8}$$

Assuming that the option price is homogeneous of degree one in the stock price and the exercise price (Cf. Merton [6, p. 146]), then (8) is equivalent to

$$\lambda h_1P(E,E/h_1)+(1-\lambda)h_2P(E,E/h_2)\geqslant(\lambda h_1+(1-\lambda)h_2)P(E,E/(\lambda h_1+(1-\lambda)h_2)) \tag{9}$$

Then consider forming a portfolio consisting of λh_1 puts with an exercise price of E/h_1, and $(1-\lambda)h_2$ puts with an exercise price of E/h_2. The value of this portfolio when the stock price is equal to E is given by the left hand side of (9). Correspondingly, the right hand side of (9) is the value of a portfolio of $(\lambda h_1+(1-\lambda)h_2)$ puts with an exercise price of $E/(\lambda h_1+(1-\lambda)h_2)$, when the stock price is E. Then, following Smith [8] it is readily shown that the returns on the first portfolio exhibit first degree stochastic dominance over the returns on the second portfolio. Hence the value of the first portfolio must exceed that of the second which implies (9) and (7).

The convexity of the put price together with the upper and lower bounds on the value of the put given by (4) and (5) imply:

$$\lim_{S\to\infty} P_S(S,t)=0 \tag{10}$$

Since the put is a convex function of the stock price, it may be shown that the equilibrium put price is an increasing function of the riskiness of the stock. The proof of this proposition is not offered here since it parallels Merton's [6] proof that the value of a warrant is an increasing function of the riskiness of the stock.

Figure 1 illustrates the boundary conditions which must be satisfied by the put price prior to maturity and shows the relationship between the put price and the stock price for two stocks with different variance rates. S_c is the critical stock price, to be discussed below.

Finally, the possibility of discrete dividends on the underlying stock and of changes in the exercise price introduce a further boundary condition. Thus, suppose that at time t there is change in the exercise price, and a discrete dividend D_t is paid. Let t^- denote the instant before the dividend/exercise price change, and t^+ the instant after. Then the put value must satisfy

$$P(S,t^-)=\max[E_{t^-}-S,P(S-D_t,t^+)] \tag{11}$$

where $P(S,t^+)$ is the put value when the exercise price changes to E_{t^+}. Equation (11) reflects the fact that the put value before the dividend/exercise price change is equal to the greater of its immediate exercise value $(E_{t^-}-S)$ and the value after the dividend/exercise price change, $P(S-D_t,t^+)$. If a dividend is paid with no offsetting change in the exercise price,[1] then it will never pay the put holder to exercise immediately before the dividend for then

$$P(S-D_t,t^+)\geqslant E-(S-D_t)>E-S$$

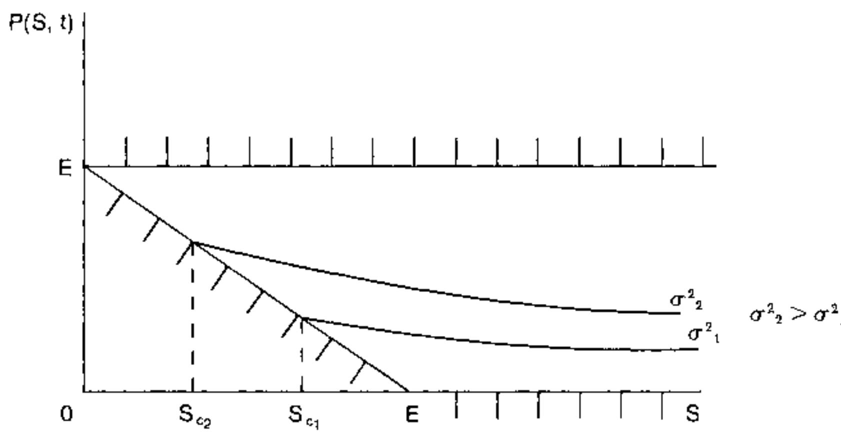

FIGURE 1. Relationship Between Put Price and Stock Price

1. Cf. Merton [6] for a discussion of the exercise price adjustment which will leave the put-holder indifferent to the amount of the dividend.

and the ex-dividend value of the put exceeds its pre-dividend exercise value.

The problem of valuing the put is then that of solving the differential equation (2) subject to the boundary conditions (3), (4), (5), (6), (10) and (11). A numerical procedure is employed to obtain an approximate solution to the equation. First re-write (2) using the variable τ, time to maturity, instead of t, calendar time

$$\tfrac{1}{2}\sigma^2 S^2 P_{SS} + rSP_S - rP - P_\tau = 0 \qquad (2')$$

Then approximating the partial derivatives in (2′) by finite differences, (2′) may be written as[2]

$$a_i P_{i-1,j} + b_i P_{i,j} + c_i P_{i+1,j} = P_{i,j-1}; \qquad i = 1, \ldots, (n-1), \quad j = 1, \ldots, m \qquad (12)$$

where

$$a_i = \tfrac{1}{2} r\, ki - \tfrac{1}{2}\sigma^2 ki^2$$

$$b_i = 1 + rk + \sigma^2 ki^2$$

$$c_i = -\tfrac{1}{2} r\, ki - \tfrac{1}{2}\sigma^2 ki^2$$

$$P(S,\tau) = P(S_i, \tau_j) = P(ih, jk) = P_{i,j}$$

h and k are the discrete increments in the stock price and time to maturity respectively. m and n are the number of discrete increments in the time to maturity and stock price respectively; the former corresponds to the time to expiration of the put, while the latter is chosen so that the boundary condition (10) is well approximated at the highest stock price value considered.

The boundary condition (10) which holds for all values of j is approximated by

$$P_{n-1,j} - P_{n,j} = 0 \qquad (j = 1, \ldots, m) \qquad (13)$$

(12) and (13) constitute a set of n linear equations in the $(n+1)$ unknowns $P_{i,j}$ $(i = 0, 1, \ldots, n)$ and with the addition of one further equation enable us to solve for $P_{i,j}$ in terms of $P_{i,j-1}$. Since $P_{i,0}$ is given by the condition governing the value of the put at expiration (3), the whole set of $P_{i,j}$ may be solved for by repeated solution of the set of equations.

Note that the solutions to the differential equation must satisfy the boundary condition (4) which may be written as

$$P_{i,j} \geqslant E_j - ih \qquad (i = 0, 1, \ldots, n) \qquad (14)$$

where E_j is the exercise price ruling at time increment j $(\tau = jk)$; the differential equation holds only for those values of $P_{i,j}$ for which (14) holds as a strict inequality. The maximum stock price for which (14) holds as an equality, the "critical stock price" is the price at which the put should rationally be exercised. The problem then is to determine the value of i, i_c, corresponding to the critical stock price $S_c (i_c h = S_c)$. The system of equations (12–13) then holds for values of $i \geqslant i_c$.

First, note that by successive subtraction of each equation of (12–13) from a

2. See McCracken and Dorn [5] for further discussion.

suitable multiple of its predecessor (i.e. subtract the last equation from a multiple of equation n; subtract the transformed n^{th} equation from equation $(n-1)$ etc.), the system (12–13) may be transformed into:

$$e_i P_{i-1,j} + f_i P_{i,j} = g_i \qquad (i = 1, \ldots, n) \tag{15}$$

where e_i, f_i, g_i are coefficients of the transformed system.

Let $P_{0,j} = E_j$ from (4) and (5), and use the first equation of (15) to solve for $P_{1,j}$. If $P_{1,j} < E_j - h$, let $P_{1,j} = E_j - h$ and solve for $P_{2,j}$; if $P_{1,j} \geqslant E_j - h$ solve the remaining equations for $P_{i,j}$ $(i = 2, \ldots, n)$. Continue in this manner until a complete set of $P_{i,j}$ are obtained which satisfy the differential equation and the boundary condition. The maximum value of i for which $P_{i,j} = E_j - ih$ is the critical value of i, i_c and the critical stock price is $S_c = i_c h$.

Finally, on the date of a dividend, D, or change in conversion terms, condition (11) is written

$$\begin{aligned} P_{i,j^-} &= P_{i-D/h,j^+}, \qquad \text{for} \quad P_{i-D/h,j^+} > E_{j^-} - ih \\ &= E_{j^-} - ih, \qquad \text{for} \quad P_{i-D/h,j^+} < E_{j^-} - ih \end{aligned} \tag{16}$$

In the application of the model to empirical data on 6 month puts the time increment k was set equal to one day, and the stock price increment, k, was set equal to 1% of the striking price. As a crude test of the accuracy of the above procedure several puts were valued assuming a zero rate of interest, and the result compared to the corresponding Black-Scholes solution.[3] The numerical method was accurate to within about 0.1%, at option prices corresponding to the exercise price.

III

The model was used to evaluate 55 put options traded in the New York dealer market between May 1966 and May 1969.[4] These were all the put contracts for which a call contract of the same maturity was traded on the same day, and for which the rates of return on the underlying security were available on the Wells Fargo Daily Return file. The risk free rate of interest for each contract was constructed by taking the price of the Treasury Bill maturing closest to the expiration date of the option and computing the continuously compounded rate of return per trading day. A synthetic stock price series was constructed by taking the closing stock price on the day the option was written, and treating the return relatives on the Wells Fargo file as price relatives to generate a stock price series throughout the life of the contract. This procedure is equivalent to assuming that no dividends were paid and is appropriate if the options are perfectly protected against dividends by adjustment of the strike price. In fact, as Merton [6] has pointed out, the actual adjustment of the strike price is not quite the correct one.

3. If the interest rate is non-positive the value of a European put always exceeds its exercise value prior to maturity, so that it is equivalent to that of an American put. To see this note that the put-call parity theorem holds for European options [2], so that $p = c - S + Ee^{-rt} \geqslant E - S$ if $r \leqslant 0$ where p and c are the values of a European put and call.

4. The authors are most grateful to Myron Scholes for making this data available to them.

1. Model and Market Put Prices

The only parameter of the valuation model which is not directly observable is the variance rate on the stock. While this may be estimated from historical data, such estimates are necessarily subject to error. Therefore in this paper we restricted our sample to put contracts for which an equivalent call contract was written on the same day. The put pricing model employed here is based on exactly the same set of assumptions as the Black-Scholes option pricing model which may be used to price call contracts. Therefore, if these assumptions correctly describe the option pricing process, the put and the call should be priced on the basis of the same variance estimate.[5] Therefore for each put contract we used the associated call contract to estimate the implied variance rate, which is defined as that variance rate for which the Black-Scholes valuation of the call is equal to the market price.[6] We also obtained an historical variance estimate using the prior 250 trading day returns and an estimated variance rate over the life of each contract (the "actual variance rate").

Then, for each of the 55 put options an estimate of the equilibrium price at the time of issue was computed using the numerical method described earlier and the three different estimates of the variance. Additionally, the options were valued as if they were European options by using the Black-Scholes [2] model for pricing European put options. Table I contains the frequency distributions of the ratios of the model prices to the market prices at time of issue, where the model price is derived using both the numerical solution technique (NS) and the Black-Scholes model (BS) for all three variance estimates.

TABLE I

FREQUENCY DISTRIBUTION OF MODEL TO MARKET PUT PRICE RATIOS

Ratio of Model Price to Market Price	Historical Variance BS 1	Historical Variance NS 2	Implied Variance BS 3	Implied Variance NS 4	Actual Variance BS 5	Actual Variance NS 6
>1.80	10.91	12.73	7.27	10.91	7.27	7.27
>1.70	5.45	5.45	5.45	1.82	0.0	0.0
>1.60	3.64	3.64	9.09	10.91	3.64	5.45
>1.50	3.64	3.64	9.09	7.27	7.27	9.09
>1.40	7.27	5.45	1.82	3.64	9.09	7.27
>1.30	7.27	12.73	27.27	30.91	5.45	5.45
>1.20	14.55	12.73	18.18	21.82	10.91	16.36
>1.10	7.27	9.09	18.18	9.09	23.64	21.82
>1.00	9.09	9.09	0.0	0.0	12.73	9.09
>0.90	18.18	12.73	1.82	1.82	10.91	12.73
>0.80	7.27	7.27	0.0	0.0	3.64	1.82
<0.80	5.45	5.45	1.82	1.82	5.45	3.64
Std. Err.	0.44	0.46	0.47	0.50	0.41	0.43
Mean ratio	1.257	1.287	1.390	1.420	1.249	1.278
t-statistic	4.33	4.63	6.16	6.23	4.51	4.80

5. Even if the Black-Scholes model is incorrect, both options will appear to be priced on the basis of the same variance if the options are European and the put-call parity theorem holds.

6. Latane and Rendleman [4] also derived implied variances from the Black-Scholes model.

Using the implied variance rate, the numerical solution model value exceeds the market price by 42% on average, and the t-statistic for the null hypothesis that the mean model to market put price ratio is unity is 6.23; the model put value is less than the actual market value in less than 4% of the cases. Based on this evidence it is easy to reject the hypothesis that puts and calls are priced consistently and in accordance with this model, and our results confirm the findings of Gould and Galai [3] of frequent violations of put-call parity.[7] The puts appear to be significantly underpriced relative to the calls.

Even when the historical or actual variance estimate is used in the numerical solution, the model prices the puts significantly higher than the market price, the mean excess of the model price over the market price being 28.7% and 27.8% respectively. The t-ratios for the null hypothesis that the mean ratio is unity are 4.63 and 4.80, and the model overprices 75% and 82% of the contracts depending on whether the historical or actual variance rate is used.

Much the same distribution of price ratios is obtained when the Black-Scholes model for European options is used to price the contracts. Indeed the magnitude of the discrepancies between the Black-Scholes (European) option values and the numerical solution (American) option values is surprisingly small as shown in Table 2. As is to be expected, the European option value is always less than the corresponding American option value, but the small differences in the computed values suggest that the right to early exercise contained in the American option is not of great economic value, so that the Black-Scholes model may reasonably be used to value 6-month dividend protected American puts. The discrepancies will of course be greater for unprotected puts, for puts of longer maturity or when the stock is selling for below the striking price.

The model prices were further compared with the market prices by cross-section regression of the market price of the put on the model prices, both prices being

7. To see the relationship between these results and those of Gould and Galai consider the second column of their Table 5 (p. 118) which gives values of ϵ, the violation of put-call parity, defined by:

$$\epsilon = \frac{C-P}{V} - \frac{i}{(1+i)}$$

Taking as the model price, the put-call parity price, $\hat{P}$, defined by

$$\frac{C-\hat{P}}{V} - \frac{i}{(1+i)} = 0$$

and combining these two expressions, the ratio of model price to market price is

$$\frac{\hat{P}}{P} = \epsilon\frac{V}{P} + 1$$

Using the average ϵ from Table 5 (.0349) and the average P/V (.085) we obtain

$$\frac{\hat{P}}{P} = 1.41$$

Now the model price derived from put call parity should be identical to that derived using BS model with the implied variance. Hence the mean ratio of model to market price for BS (implied variance) should be about 1.41. In fact from our Table 1 the mean ratio is 1.39. The differences may be accounted for by sample correlation between ϵ and V/P and by the fact that our sample contains 6 more observations than Gould and Galai.

TABLE 2

DISTRIBUTION OF DIFFERENCES (D) BETWEEN EUROPEAN (E) AND AMERICAN (A) PUT VALUES (HISTORICAL VARIANCE ESTIMATE)

$D \equiv (E-A)/A$

$D \geqslant -.01$	$D \geqslant -.02$ $D < -.01$	$D \geqslant -.03$ $D < -.02$	$D \geqslant -.04$ $D < -.03$	$D < -.04$
0	17	27	5	6
Mean Difference	$-.025$			

TABLE 3

REGRESSION OF NORMALIZED MARKET PRICE, P/S, ON NORMALIZED MODEL PRICE USING HISTORICAL VARIANCE $NS(H)$, IMPLIED VARIANCE $NS(I)$, AND ACTUAL VARIANCE $NS(A)$

$$P/S = \underset{(6.80)}{.059} + \underset{(3.15)}{.245}\, NS(H)/S \qquad R^2 = .16$$

$$P/S = \underset{(3.56)}{.031} + \underset{(6.35)}{.454}\, NS(I)/S \qquad R^2 = .43$$

$$P/S = \underset{(5.85)}{.050} + \underset{(4.30)}{.331}\, NS(A)/S \qquad R^2 = .26$$

11 groups on $NS(I)/S$

$$P/S = \underset{(2.51)}{.039} + \underset{(3.02)}{.431}\, NS(H)/S \qquad R^2 = .50$$

11 groups on $NS(H)/S$

$$P/S = \underset{(3.78)}{.036} + \underset{(5.16)}{.411}\, NS(I)/S \qquad R^2 = .78$$

11 groups on $NS(H)/S$

$$P/S = \underset{(6.60)}{.043} + \underset{(6.43)}{.390}\, NS(A)/S \qquad R^2 = .82$$

t-ratios in parenthesis.

measured as a fraction of the striking price. The results are given in Table 3. While these regressions should be evaluated with care, since we have no assurance that the relationship is linear or that the errors are normally distributed, it is clear that the slope coefficient is far from its theoretical value of unity. One reason for this is that the independent variable, the numerical solution value, impounds errors in the variance estimate. To mitigate this problem, the observations were grouped on an instrumental variable, the numerical solution based on one of the other variance estimates: even with the grouping procedure the slope coefficient is still less than 0.5. We have argued that both puts and calls should be priced on the basis of the same variance, so that the problem of errors-in-variables bias does not arise with the implied variance, and indeed we find that the slope coefficient does not improve with grouping when the implied variance is used to derive the model value. These results indicate that not only are market put prices in general lower than the model prices but that, accepting the model prices as equilibrium prices, put prices on high variance stocks (those with a high ratio of put price to striking price) are systematically underpriced relative to put prices on low variance stocks. This latter result is in accord with the Black-Scholes [1] finding that call prices on high variance stocks are underpriced relative to those on low variance stocks. However, while Black-Scholes found some slight evidence that call contracts in general tended to be overpriced, our evidence thus far is that puts tend to be underpriced, at least relative to model prices. However the real test of the model is its ability, not to predict market prices, but to yield prices which are equilibrium in the sense that they yield no arbitrage profits: such tests are discussed below. First however we discuss the optimal exercise strategy.

2. *The Exercise Strategy*

An integral part of the numerical algorithm for valuing American put options is the determination of the optimal exercise strategy to be followed. This is described by a time series of critical stock prices, such that the put is exercised optimally as soon as the stock price drops below the critical stock price. Figure 2 shows the time series of critical stock prices expressed as a fraction of the striking price for a 131 day put on Atlas Chemical Industries, using the three different variance estimates. Since the equilibrium put price is an increasing function of the variance rate, the critical stock price is a decreasing function of the variance rate as shown in Figure 1; consequently, the historical variance rate, which is the lowest of the three variance estimates for this company, yields the uniformly highest critical stock price series. The general pattern of critical stock prices indicates that the optimal strategy is to pursue an aggressive policy at first by requiring a high exercise value (low stock price) before exercising, but that as time runs out on the contract to reduce rapidly the minimum required exercise value so that immediately before expiration it becomes optimal to exercise so long as the exercise value is positive.

Table 4 shows the distributions of exercise dates relative to maturity for the sample of 55 put options. The first column shows the timing of exercise by the actual purchasers of the options as recorded in the option dealer's diary. Only 15 of the 55 contracts were actually exercised and of these, 14 were exercised at maturity. We then calculated when the options would have been exercised had the optimal policy been followed, by comparing the time series of critical stock prices with the time series of actual stock prices adjusted for dividends, and assuming that the option was exercised the first time the stock price fell below the critical stock price. The second and third columns of Table 4 show the distribution of optimal exercise times assuming that the optimal exercise policy is determined using the historical and implied variance estimates respectively. Under either estimate of the optimal policy 20 contracts would have been exercised, of which only seven would have been exercised at maturity. It is of interest to note that the optimal policy using the historical variance resulted in a higher exercise value than was actually achieved by the purchasers of the puts for 16 contracts, while the reverse was true for only 3 contracts; when the optimal policy was derived from the implied variance, the corresponding figures were 13 contracts and 3 contracts.

As a further measure of the departure of the actual exercise strategies followed from the optimal strategies of the model, we calculated for each contract a "maximum return differential." The return differential for a contract on a particular day is defined as the difference between the return the option purchaser would have earned if the contract were exercised at the prevailing stock price and the return the purchaser would have earned if the contract were exercised at the critical

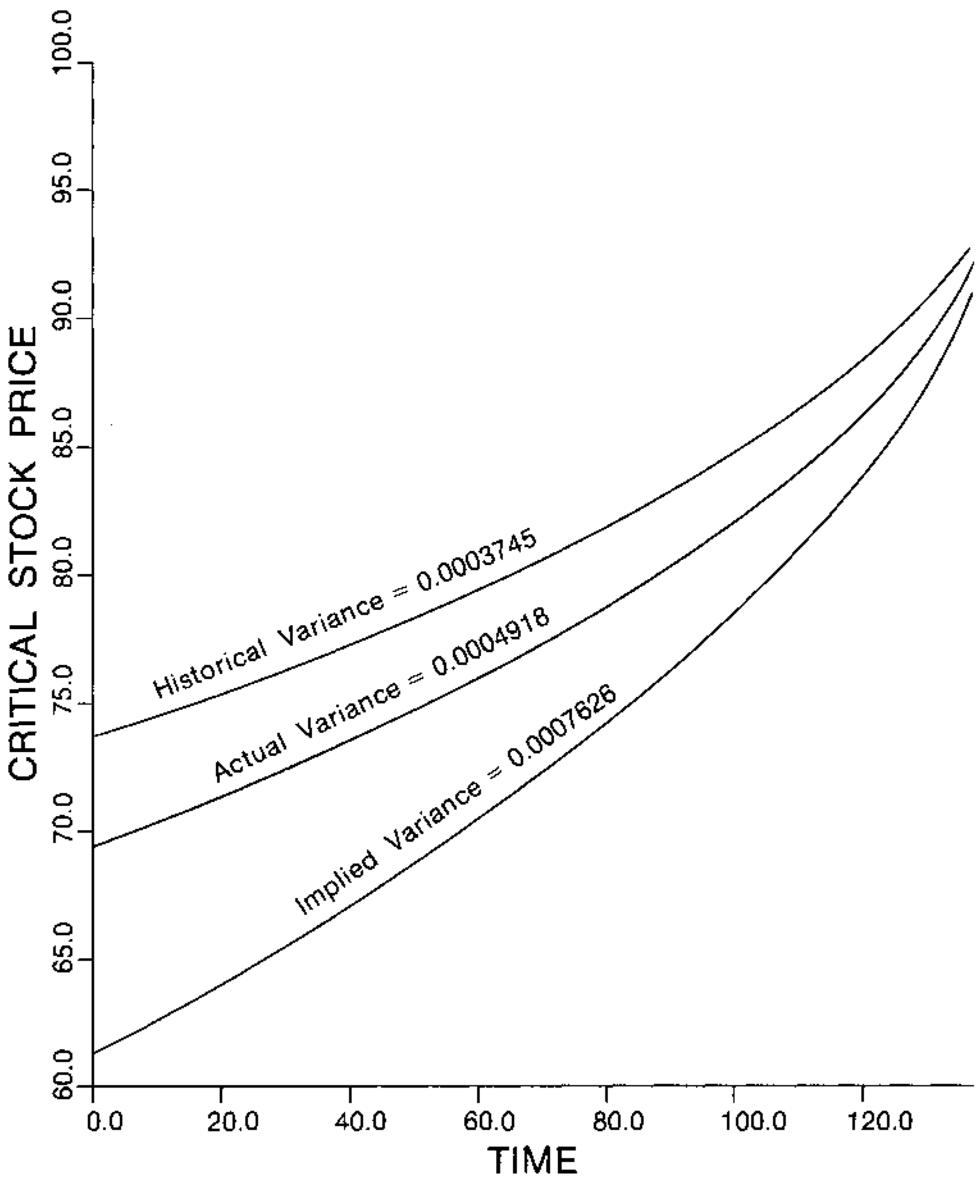

FIGURE 2. Critical Stock Prices for Atlas Chemical Industries

TABLE 4

DISTRIBUTION OF EXERCISE DATES RELATIVE TO MATURITY

	Actual Policy	Policy Historical Variance	Optimal Implied Variance
At expiration	14	7	7
1–5 days before expiration		0	2
6–10 days before expiration	1	1	2
11–20 days before expiration		6	2
21–40 days before expiration		3	5
>40 days before expiration		3	2
Number of contracts exercised	15	20	20

stock price for that day. For example, if the put were originally sold for \$5, the striking price were \$50, the critical stock price were \$40 and the stock price were only \$35, the return if the put were exercised at the prevailing stock price would be $(50-35)/5=300\%$, and the return if the put were exercised at the critical stock price would be $(50-40)/5=200\%$, so that the return differential would be 100%. The maximum return differential is the maximum of these daily return differentials over the time the contract was unexercised. The maximum return differential will be zero if the contract is exercised optimally, and positive if it is exercised too late. The magnitude of the positive differential is an indication of the holder's reluctance to exercise relative to the optimal policy. The distribution of the maximum return differentials for the 55 contracts is shown in Table 5 where the optimal policy is derived using both the historical variance estimate and the implied variance.

TABLE 5

DISTRIBUTION OF "MAXIMUM RETURN DIFFERENTIAL"

	Historical Variance	Implied Variance
No discrepancy	39	39
<10%	1	3
11–30%	1	1
31–50%	5	3
51–100%	3	4
101–150%	4	3
>150%	2	2

The evidence from the distributions of both the exercise dates and the maximum return differentials indicates that the option purchasers were markedly reluctant to exercise their options prior to maturity even though the model indicated that it was optimal to do so. This may be attributable, first, to the failure of the model to value the puts correctly: this seems unlikely since, as we have seen, the model values the puts more highly than does the market and would therefore tend to set the critical stock price too high, delaying exercise. Secondly, it may be attributable to gamblers' greed which causes put purchasers to hold out against exercise in the hope of higher profits in the future. However, a more probable explanation is the tax system with its preferential treatment for long term capital gains.

3. *The Model Prices as Equilibrium Prices*

We have found that the model tends to systematically over-value the put contracts relative to the observed market prices. The question then arises as to whether the model prices are equilibrium prices in the sense that they yield no arbitrage profits, for if they are, then the market prices are not equilibrium and present the opportunity for arbitrage profits, at least in a world without transaction costs. To determine whether or not the model prices are equilibrium prices we employ a test procedure based on that used by Black and Scholes [1]. That is, we simulated a strategy of purchasing all the put contracts at the model prices and selling short each day $P_S(S,t)$ shares of the underlying stock to obtain a hedged position. Values of $P_S(S,t)$ were obtained from the matrix of $P(S,\tau)$ values generated by the valuation model. Excess dollar returns were computed daily for each contract from the date it was written to the date it was exercised according to the optimal exercise policy, in a manner similar to that described by Black-Scholes. The excess dollar returns were aggregated each day over all contracts outstanding that day to yield a series of portfolio returns for 710 trading days from June 10, 1966 to April 16, 1969.[8] To avoid giving excessive weight to contracts on high priced stocks we assumed that instead of purchasing one put contract for 100 shares, we purchased a put contract with a striking price of \$1,000; so if the striking price was \$50 we assumed that we bought 1/5 of a contract (\$5,000/\$1, 000).

In addition to computing the total dollar returns per day from this strategy we computed the dollar returns per day per contract by dividing the total dollar returns for each day by the number of unexercised contracts outstanding that day. The total dollar returns per day and the dollar returns per day per contract were then regressed on the rate of return on the market portfolio to allow for systematic risk in the returns due to imperfect hedging. The systematic risk estimates were all insignificantly different from zero and in Table 6 we report only the constant terms from these regressions as estimates of the excess returns. To overcome the problem of heteroscedasticity caused by the different numbers of contracts outstanding on each day, we also employed a weighted regression procedure, which weighted the total dollar returns by $\sqrt{1/N_t}$ and the dollar returns per contract by $\sqrt{N_t}$, where N_t was the number of contracts outstanding on day t. The weighted regression results are reported in the lower half of Table 6.

From the weighted regression results we see that the excess total dollar return per day using the implied variance was $-$\$0.59 while the excess dollar return per contract was $-$\$0.086 per day. The associated t-statistics indicate that these are both significantly different from zero and that therefore the model values obtained using the implied variance are in excess of the equilibrium values. Assuming that the average contract was outstanding for about 125 days the daily loss per contract

8. The sample period was actually two days shorter when the implied variance was used, since the optimal strategy derived using the implied variance caused the final put contract to be exercised two days earlier.

TABLE 6

EXCESS RETURNS FROM PURCHASING CONTRACTS AT MODEL PRICES AND FOLLOWING OPTIMAL EXERCISE STRATEGY

	Total Dollar Returns per day						Dollar Returns per day per contract					
	Implied Variance			Historical Variance			Implied Variance			Historical Variance		
	α	$t-\alpha$	ρ	α	$t-\alpha$	ρ	α	$t-\alpha$	ρ	α	$t-\alpha$	ρ
Unweighted				$x10^{-1}$			$x10^{-1}$			$x10^{-2}$		
Total Period	−1.02	−4.76	.11	−.37	−.17	.11	−.53	−2.05	.15	.68	.27	.12
First Half	0.04	0.25	.17	.97	.64	.08	.82	1.89	.14	6.90	1.67	.12
Second Half	−2.08	−5.31	.07	−1.73	−.42	.11	−1.88	−7.10	.09	−5.57	−1.99	.11
First Quarter	.66	2.96	.10	4.40	2.22	.06	2.45	3.03	.09	16.64	2.17	.10
Second Quarter	−.58	−2.76	.16	−2.33	−1.01	.08	−.79	−2.81	.18	−2.68	−.89	.16
Third Quarter	−2.30	−4.61	−.02	−2.30	−.44	−.02	−1.30	−4.98	−.00	−1.64	−.58	−.01
Fourth Quarter	−1.88	−3.11	.12	−1.44	−.23	.20	−2.48	−5.40	.11	−9.61	−1.98	.15
Weighted												
Total Period	−.59	−5.18	.12	−.10	−.09	.11	−.86	−2.27	.16	−.12	−.03	.12
First Half	.18	1.67	.17	1.78	1.75	.08	.42	.74	.14	4.19	.78	.12
Second Half	−1.75	−9.01	.07	−2.93	−1.45	.11	−1.66	−3.37	.10	−2.84	−.55	.11
First Quarter	.63	3.69	.10	4.19	2.76	.06	2.40	2.36	.09	15.97	1.69	.10
Second Quarter	−.56	−4.40	.16	−2.07	−1.50	.08	−.78	−1.74	.18	−2.93	−.60	.16
Third Quarter	−2.30	−9.66	−.02	−2.64	−1.04	−.02	−1.28	−2.35	−.00	−1.42	−.24	−.01
Fourth Quarter	−1.46	−4.77	.12	−3.31	−1.05	.20	−2.23	−2.72	.11	−5.20	−.60	.15

ρ—Serial Correlation of Residuals.

per day corresponds to a total loss per contract of \$10.57. This measure of the extent to which the implied variance model overprices the contract may be compared with a mean model price of \$118.77 which is therefore about 9% above the equilibrium price.

However, when we simulate a policy of buying at the model prices derived from the historical variance and hedging on the same basis, the excess total dollar returns and dollar returns per contract rise to −\$0.01 and −\$0.0012 per day respectively, and the associated t-values become very small, so that we cannot reject the null hypothesis that the model values are equilibrium values. Note that the excess dollar returns per contract per day are equivalent to only about −\$0.15 over a 125 day life contract; this may be compared with an average model put premium of \$106.68 on a contract with a striking price of \$1,000. For individual subperiods the magnitude of the absolute excess dollar returns per contract per day are somewhat larger, but in no case are they statistically significant. The largest excess dollar return per contract per day is \$0.016 which is equivalent to \$2.00 over the life of the average contract. Hence the average model price obtained using the historical variance estimate appears to be very close to the equilibrium price.

Since the market prices tend to be below the model prices we may infer that the market prices are below the equilibrium prices, and therefore give rise to profit opportunities. To evaluate these profit opportunities further, we calculated the incremental profit to be derived by purchasing the puts at the market prices rather than at the model prices derived from the historical variance estimate. This was done by converting the dollar profits (losses) for all the contracts due to the difference between market and model prices into an equivalent daily annuity over the whole sample period. The incremental daily profit was $1.68 per day or $0.19 per day per contract.[9] These incremental profits are respectively 170 and 160 times the average daily losses on the hedge portfolio constructed using the model prices. Combining the annuitized profits from purchasing at market rather than model prices with the daily returns on the hedge portfolio, the resulting returns yield t-ratios of 14.5 on a total dollar basis and 4.6 on a dollar per day per contract basis.

While further tests could be employed to evaluate the ability of the model to discriminate between under-priced and over-priced contracts, the above results leave little doubt that this sample of put contracts was on average under-priced and offered the opportunity for profit by following the naive strategy of purchasing all the put contracts. However, as Black and Scholes [1] have pointed out the very high level of transactions costs in this market would almost certainly render it impossible to profit by this evidence of market inefficiency.

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9. $0.19 per day per contract is equivalent to $23.80 over the whole life of an average contract, compared with an average market price per contract of $84.97.