Static Hedging of Timing Risk

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Many exotic options involve a payoff that occurs at the first time the stock price crosses a constant barrier. Although the amount to be paid is known, the time at which it is paid is not.

This article shows how a static position in European options can be used to hedge against this timing risk. The simulation results show that this approach outperforms dynamic hedging with the underlying.

The authors show how these results can be used to price any barrier option.

Barrier options are the most popular form of the second generation exotic options. To soften the impact of hitting a barrier, many knock-out options include rebates that partially compensate the holder for the loss of the option at the first passage time. These rebates differ from European option payoffs in that the underlying's price at the payoff time is known, assuming that the price process is continuous. The time of the payoff, however, is unknown.

In the Black-Scholes model, this timing risk can be hedged by dynamically trading in the underlying assets. The objective of this article is to show that this timing risk can also be hedged by employing static positions in plain vanilla options.

While both hedges work perfectly under the model's assumptions, static hedges may offer some benefits over dynamic hedging in practice. First, static hedging is likely to be less sensitive to the assumption of zero transaction costs. These transaction costs include not only commissions and spreads, but also the cost of paying individuals to monitor a position continuously.

Second, static hedges enjoy vegas and gammas close to those of the barrier option, while dynamic positions in the underlying have zero vega and gamma. Hence, changes in volatility or price that arise in a fast moving market or over daily market closings have a greater impact on the replication error of a dynamic stock strategy than of a static options position.

Finally, when negative exposure to the stock is required, it is generally easier to buy puts or write calls than to short stock. Similarly, when negative exposure to the riskless asset is required, it is generally easier to buy calls or sell puts than to borrow using the stock as collateral.

There is a growing literature on the static hedging of barrier options. Using the assumptions of the Black [1976] model, Bowie and Carr [1994] first showed how to construct static hedges for the sale of single barrier options and look-back options. Subsequent researchers extended these results to a non-zero underlying carry rate (Carr and Chou [1996] and El Karoui and JeanBlanc-Piqué [1997]), nonlognormal distributions (Carr, Ellis, and Gupta [1998] and Aparicio and Clewlow [1997]), and to multiple barrier options (Carr and Chou [1997]).

These writers all use diffusion models for the evolution of the underlying asset, and generate analytic solutions for the values and hedges of the barrier option. In general, the hedging strategy involves static positions in a continuum of options of all strikes, with a common maturity matching that of the exotic option. Derman, Ergener, and Kani [1994, 1995] pioneered a second approach that uses a discrete-time model to generate numerical solutions for the values and hedges of barrier options. The hedging strategy uses a set of options maturing with and before the barrier option. The options maturing strictly before the barrier option are all struck at the barrier.

A recent article by Chou and Georgiev [1998] links these two approaches, and shows how a static position involving a single maturity and multiple strikes can be converted into a static position involving a single strike and multiple maturities.

Several analyses use Monte Carlo simulation to compare the effectiveness of static hedging with dynamic hedging in realistic environments. Tompkins [1997] simulates dynamic hedging and the Bowie and Carr [1994] static hedge, which assumes zero carrying cost for the underlying. Assuming zero interest rates and dividends, Tompkins finds that imposing discrete trading opportunities, positive transaction costs, and stochastic volatility raises the standard deviation of the profit and loss (P&L) when delta-hedging, with more pronounced effects holding for a down-and-in call than a down-and-out call.

The standard deviation of the PL in the static hedge is by contrast virtually immune to the introduction of these features, leading Tompkins to conclude, perhaps prematurely, that:

> These results not only confirm the effectiveness of the put-call symmetry principle for these barriers, but also imply that this approach is a vastly superior approach to dynamically covering these products [1997, p. 27].

Thomsen [1998] extends Tompkins' simulations to underlyings with non-zero (but constant) carrying rates, but restricts attention to a comparison of dynamic hedging with the Carr and Chou [1996] static hedge of down-and-in calls. Defining a perfect market setting as the Black-Scholes model with discrete trading opportunities and strikes, he concludes:

> The static hedge did best — not only under the perfect market setting —but also under transactions costs and mis-specification did it show better — although not perfect — hedging results than the dynamic hedge did [1998, p. 92].

As in Tompkins, the misspecification involves introducing stochastic volatility, which has a greater impact on the replication error for dynamic hedging than static hedging.

Finally, Toft and Xuan [1998] test the Derman et al. [1994, 1995] static hedge on up-and-out calls with rebates by assuming that Heston's stochastic volatility model describes market option prices. Although they do not compare dynamic hedging, they conclude that the static hedge is effective if volatility is moderate or if the rebate is set to equal the intrinsic value of the corresponding standard call. If the payoff at the barrier differs from this value, however, the quality of this static hedge deteriorates rapidly when volatility is high.

We develop a new static hedging methodology that combines elements of previous approaches to static replication. Following the Carr and Chou approach, the Black-Scholes model is used, allowing explicit analytic solutions for the values and hedges. We also use options maturing at the same date as the barrier option as part of the hedge. Like Derman et al., we use options struck at the barrier and maturing before the barrier option as part of the hedging strategy.

Unlike previous researchers, however, our hedge generally involves options maturing before the barrier option and struck at all levels beyond the barrier. Given that a continuum of strikes and maturities is not available in practice, the fact that our approach uses a larger set of strikes and maturities suggests the procedure might work better in practice.

The contribution of our work is threefold. First, we emphasize the role of American binary options, which pay one dollar at the first passage time. To value these exotic options, we highlight the role of *stationary securities*, which have the unusual property that their values are the same function of the stock price as their payoff. It turns out that an American binary call can be created out of static positions in digital calls on these stationary securities, which can in turn be spanned by plain vanilla options.

Second, we develop a state pricing density that indexes states by hitting times rather than by terminal price levels. In principle, this state pricing density can be observed from vanilla option prices and can be used to price and hedge any single barrier option. It can also be used to obtain the market's "risk-neutral" forecast of the hitting time of a barrier.

Third, we conduct the first simulation tests of the efficacy of our static hedging approach for American binary calls. We find that our approach significantly outperforms dynamic hedging for these relatively common exotic instruments.

I. STATIONARY SECURITIES

We assume the Black-Scholes model holds with a constant interest rate $r \geq 0$, a constant dividend yield $\delta \geq 0$, and a constant volatility rate $\sigma > 0$. We determine a set of payoffs occurring at a fixed time T, which will be called *stationary*. By definition, stationary payoffs have the property that they engender arbitrage-free values that do not vary with time $t \in [0, T]$, although they still vary with the stock price. In other words, these stationary securities have no time decay. As a consequence, the values of stationary payoffs are given by the same function of the stock price as the payoff.

To determine whether such payoffs exist, we set the time derivative in the Black-Scholes PDE to zero:

$$\frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2}V^s(S,t) +$$

$$(r-\delta)S\frac{\partial}{\partial S}V^s(S,t) - rV^s(S,t) = 0 \tag{1}$$

Substituting a guess that $V^s(S, t) = S^p$ yields a quadratic equation for the power p:

$$\frac{\sigma^2}{2}p(p-1) + (r-\delta)p - r = 0 \tag{2}$$

Using the quadratic root formula, the two solutions are $p = \gamma + \varepsilon$, and $p = \gamma - \varepsilon$ where:

$$\gamma \equiv \frac{1}{2} - \frac{r-\delta}{\sigma^2} \quad \text{and} \quad \varepsilon \equiv \sqrt{\gamma^2 + \frac{2r}{\sigma^2}} \tag{3}$$

Thus, the two stationary solutions of (1) are $S^{\gamma+\varepsilon}$ and $S^{\gamma-\varepsilon}$, which we term the value of a pseudo-share and the value of a pseudo-bond, respectively. The terminology reflects the observation that if $r = \delta = 0$, then $\gamma = \varepsilon = 1/2$, so the pseudo-share is a share, while the pseudo-bond is a bond.

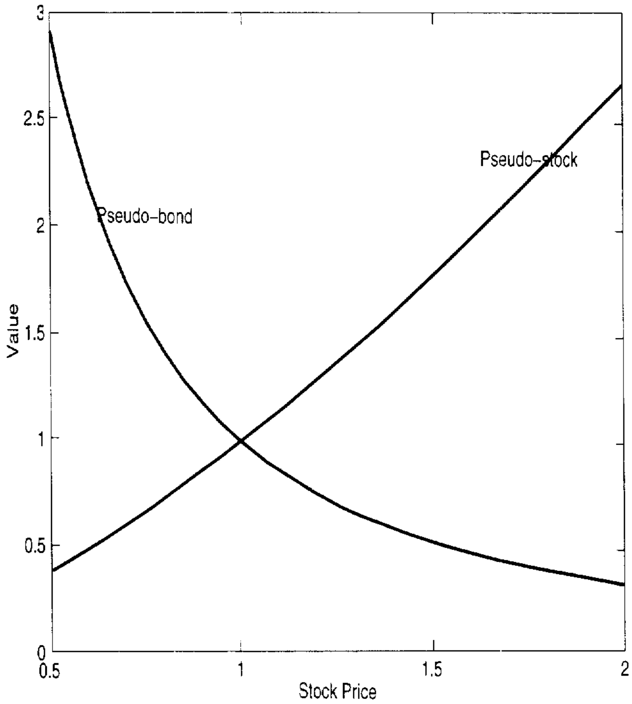
For arbitrary r and δ, these pseudo-securities are both non-negative and convex functions of the stock price, as shown in Exhibits 1 and 2. Since $\gamma + \varepsilon \geq 1$, the pseudo-share rises with the stock price, while since $\gamma - \varepsilon \leq 0$, the pseudo-bond falls with the stock price.

We have thus identified a pair of payoffs with the stationarity property. It can be shown that these payoffs can be spanned by the payoffs of a continuum of European options of all strikes and the same maturity. In fact, the pseudo-share can be created out of a static position in calls alone:

$$S_T^{\gamma+\varepsilon} = \int_0^\infty (\gamma+\varepsilon)(\gamma+\varepsilon-1)K^{\gamma+\varepsilon-2}(S_T-K)^+ dK \tag{4}$$

EXHIBIT 1

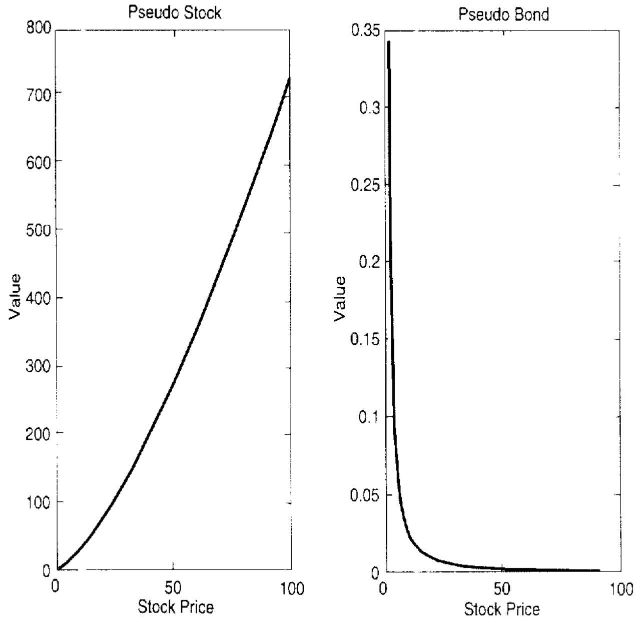
Value of a Pseudo-Stock and a Pseudo-Bond Graphed Against Stock Prices



$(r = 0.10,\ d = 0.0,\ \sigma = 0.3)$

EXHIBIT 2

Values of a Pseudo-Stock and a Pseudo-Bond Graphed Against Stock Prices



$(r = 0.10,\ d = 0.0,\ \sigma = 0.3)$

The value of this portfolio of calls at any prior time $t \le T$ is simply $S_t^{\gamma+\varepsilon}$. Similarly, a static portfolio of vanilla puts can be used to create the pseudo-bond:

$$S_T^{\gamma-\varepsilon} = \int_0^\infty (\gamma - \varepsilon)(\gamma - \varepsilon - 1)K^{\gamma-\varepsilon-2}(K - S_T)^+ \, dK \qquad (5)$$

Once again, the prior value of the put portfolio is simply $S_t^{\gamma-\varepsilon}$.

II. HEDGING AMERICAN BINARY CALLS

We focus on American binary calls, leaving the corresponding results for puts as an exercise for the reader. Thus, we assume that the initial stock price S_0 is less than the barrier H. To hedge American binary calls (henceforth ABCs), we will use static positions in both vanilla and barrier options with the same maturity T. The barrier options are ultimately hedged with static positions in vanilla options, so that the final ABC hedge involves only standard options.

Exotic Options

Exhibit 3 describes all the exotic options we discuss. For compactness, we adopt this notation for option payoffs in Exhibit 3 and elsewhere:

$$1(S > H) \equiv \begin{Bmatrix} 1, & \text{if } S > H \\ 0, & \text{if } S \le H \end{Bmatrix}$$

We first establish the parity relation:

$$ABC_t = \frac{1}{H^{\gamma+\varepsilon}} [PS \vee NC_t + UIPS \vee NP_t] \quad \text{for } \tau_h \ge t \qquad (6)$$

The right-hand side represents the initial cost of a portfolio consisting of $1/(H^{\gamma+\varepsilon})$ pseudo-share-or-nothing calls and $1/(H^{\gamma+\varepsilon})$ up-and-in pseudo-share-or-nothing puts. Along paths that avoid the barrier by maturity, both the ABC and this portfolio

EXHIBIT 3

Exotic Options

Abbreviation	Name	Payoff
ABC	American binary call	Pays \$1 at the hitting time if this occurs before maturity
PS	Pseudo-share	Pays $S^{\gamma+\varepsilon}$ at maturity
PB	Pseudo-bond	Pays $S^{\gamma-\varepsilon}$ at maturity
PS∨NC	Pseudo-share-or-nothing call	Pays $S^{\gamma+\varepsilon}1(S > H)$ at maturity
PB∨NC	Pseudo-bond-or-nothing call	Pays $S^{\gamma-\varepsilon}1(S > H)$ at maturity
UIPS∨NC	Up-and-in pseudo-share-or-nothing call	Pays $S^{\gamma+\varepsilon}1(S > H)$ at maturity if the barrier has been hit
UIPS∨NP	Up-and-in pseudo-share-or-nothing put	Pays $S^{\gamma+\varepsilon}1(S < H)$ at maturity if the barrier has been hit

have zero value. Along paths that hit the barrier before maturity, the ABC pays one dollar at the hitting time, while the securities in the portfolio combine to pay the value of $1/(H^{\gamma-\varepsilon})$ pseudo-shares at the hitting time.

Since S = H at the hitting time, and the pseudo-shares are stationary, the portfolio is also worth one dollar at the hitting time. Thus, the portfolio replicates the payoffs of the ABC along all paths, and must have the same value to avoid arbitrage.

Replicating the Value of an UIPS∨NP

To replicate the value of the up-and-in pseudo-share-or-nothing put in (6), we form a portfolio of options whose payoff lies strictly above the barrier. Consequently, along paths that avoid the barrier, the terminal value of both the up-and-in option and this portfolio are zero. Along a path that hits the barrier, we need the payoff on the portfolio to give rise to a value that matches that of the up-and-in option at the first hitting time of the barrier. At this hitting time, the up-and-in option knocks into a pseudo-share-or-nothing put. To find the payoff above the barrier whose value at the barrier is that of a pseudo-share-or-nothing put, we use the *symmetry principle*, developed by Carr and Chou [1996]:

Consider a payoff with support below the barrier:

$$f(S_T) = \begin{cases} \phi(S_T) & \text{if } S_T \in (0, H) \\ 0 & \text{otherwise} \end{cases}$$

Then in the Black-Scholes model, there exists a drift-adjusted reflected payoff $f^r(S_t)$ with support above H whose value matches that of f at any time t when the spot price is at H. This payoff is given by:

$$f^r(S_T) = \begin{cases} (S_T/H)^{2\gamma}\phi(H^2/S_T) & \text{if } S_T \in (H, \infty) \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

A slightly more general version of this symmetry principle is proved in Appendix A. Letting $\phi(S_T) = S_T^{\gamma+\varepsilon}$ implies that the drift-adjusted reflected payoff of the pseudo-share-or-nothing put is:

$$f^r(S_T) = \begin{cases} (S_T/H)^{2\gamma}(H^2/S_T)^{\gamma+\varepsilon} & \text{if } S_T \in (H, \infty) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} H^{2\varepsilon}S_T^{\gamma-\varepsilon} & \text{if } S_T \in (H, \infty) \\ 0 & \text{otherwise} \end{cases}$$

This payoff is that of $H^{2\varepsilon}$ units of a pseudo-bond-or-nothing call. Since the payoffs of the up-and-in pseudo-stock-or-nothing put have been matched along all possible paths, absence of arbitrage requires:

$$UIPS\vee NP_t = H^{2\varepsilon}PB\vee NC_t \text{ for } \tau_h \geq t \quad (8)$$

Replicating an ABC with Path-Independent Options

Substituting (8) in (6) implies that an American binary call has the same value as a portfolio of path-independent securities:

$$ABC_t = \frac{1}{H^{\gamma+\varepsilon}}PS\vee NC_t + \frac{1}{H^{\gamma-\varepsilon}}PB\vee NC_t \quad \text{for } \tau_h \geq t \quad (9)$$

Thus, the payoffs of an American binary call value can be created with a static portfolio consisting of $1/(H^{\gamma+\varepsilon})$ pseudo-share-or-nothing calls and $1/(H^{\gamma-\varepsilon})$ pseudo-bond-or-nothing calls, with the latter two calls struck at H. Once again, if the stock price avoids the barrier before T, then all options expire worthless. If the stock price touches the barrier before T, then the two path-independent options can again be sold at the hitting time for a total of one dollar (see Exhibit 4).

Replicating Path-Independent Options with Vanilla Options

Note that prior to hitting the barrier, the payoff from each of the two path-independent calls can be statically replicated with a portfolio of vanilla options. From Exhibit 3, these calls have a payoff at T of $S_T^{\gamma\pm\varepsilon}1(S_T > H)$, with the plus sign holding when the underlying is the pseudo-share and the minus sign holding when the underlying is the pseudo-bond.

Assuming only that markets are frictionless,

Exhibit 4
Values of an American Binary Call Graphed Against Stock Price and Time

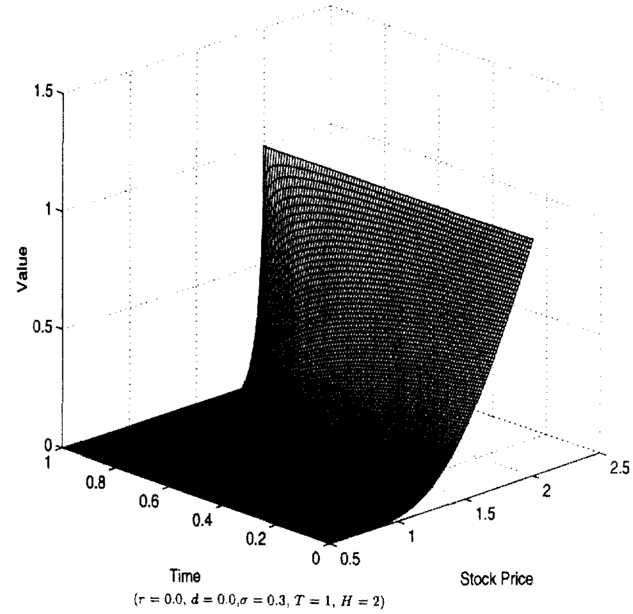
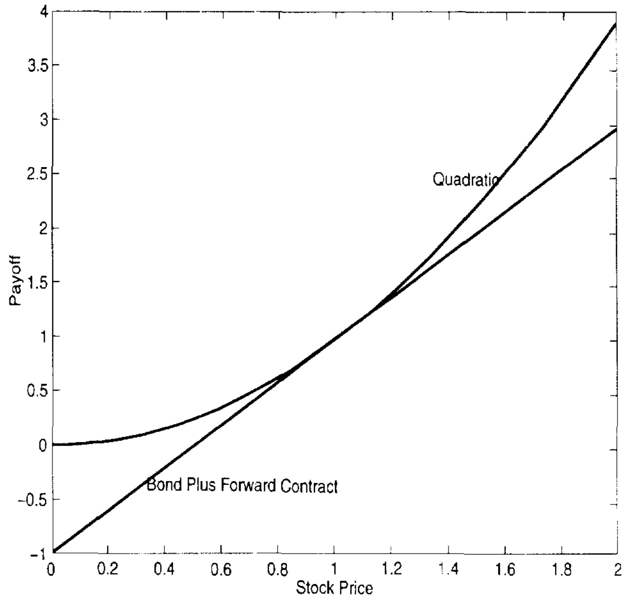


Exhibit 5
Quadratic Payoff and Static Hedge



Quadratic payoff f(S) = S2 created by buying one bond, two forward contracts with delivery price 1, and two dK puts at all strikes $K \leq 1$ and two dK calls of all strikes $K \geq 1$.

Breeden and Litzenberger [1978] show that any path-independent payoff occurring at T can be achieved by a portfolio of European calls and puts maturing at T. In particular, using a proof given in Carr and Madan [1998], Appendix B shows that the value V_t^f of any twice differentiable payoff f(S) can be expanded about an arbitrary point κ as:[1]

$$f(S_T) = f(\kappa) + f'(\kappa)(S_T - \kappa) +$$

$$\int_0^\kappa f''(K)(K - S_T)^+ dK + \int_\kappa^\infty f''(K)(S_T - K)^+ dK \qquad (10)$$

The first two terms in (10) combine to give the tangent to the curve f at the point κ. Exhibit 5 illustrates a tangency at $\kappa = 1$ for the quadratic payoff $f(S) = S^2$. Since $f''(K) = 2$, the quadratic payoff is achieved by adding 2dK puts for all strikes K below 1 and 2dK calls for all strikes K above 1.

Absence of arbitrage and (10) imply that any prior value V_t of the payoff f decomposes as:

$$V_t = f(\kappa)B_t + f'(\kappa)I_t(\kappa) + \int_0^\kappa f''(K)P_t(K)dK +$$

$$\int_\kappa^\infty f''(K)C_t(K)dK, \ t \in [0, T] \qquad (11)$$

where B_t is the time t value of the unit bond, $I_t(\kappa)$ is the time t value of the forward contract with delivery price κ, and $P_t(K)$, $K \leq \kappa$ and $C_t(K)$, $K \geq \kappa$ are the time t values of puts and calls struck at K, respectively.

Strictly speaking, (11) cannot be used to value the payoffs $S_T^{\gamma+\varepsilon}1(S_T > H)$ or $S_T^{\gamma-\varepsilon}1(S_T > H)$ since neither payoff is twice differentiable. Fortunately, (11) still holds in this case if $f(\cdot)$ is interpreted as a generalized function.[2] Thus, from (11), the payoff $S_T^{\gamma+\varepsilon}1(S_T > H)$ can be replicated by a static portfolio formed at t consisting of $H^{\gamma+\varepsilon}$ bond-or-nothing calls struck at H, $(\gamma + \varepsilon)H^{\gamma+\varepsilon-1}$ vanilla calls struck at H, and the infinitesimal position $(\gamma + \varepsilon)(\gamma + \varepsilon - 1)dK$ in all vanilla calls struck above H:

$$S_T^{\gamma+\varepsilon}1(S_T > H) = H^{\gamma+\varepsilon}1(S_T > H) +$$

$$(\gamma + \varepsilon)H^{\gamma+\varepsilon-1}(S_T - H)^+ +$$

$$\int_H^\infty (\gamma+\varepsilon)(\gamma+\varepsilon-1)K^{\gamma+\varepsilon-2}(S_T-K)^+ dK$$

Similarly, the payoff $S_T^{\gamma-\varepsilon}1(S_T > H)$ can be replicated by a static portfolio formed at t consisting of $H^{\gamma-\varepsilon}$ bond-or-nothing calls struck at H, $(\gamma-\varepsilon)H^{\gamma-\varepsilon-1}$ vanilla calls struck at H, and the infinitesimal position $(\gamma-\varepsilon)(\gamma-\varepsilon-1)dK$ in all vanilla calls struck above H:

$$S_T^{\gamma-\varepsilon}1(S_T > H) = H^{\gamma-\varepsilon}1(S_T > H) +$$

$$(\gamma-\varepsilon)H^{\gamma-\varepsilon-1}(S_T - H)^+ +$$

$$\int_H^\infty (\gamma-\varepsilon)(\gamma-\varepsilon-1)K^{\gamma-\varepsilon-2}(S_T-K)^+ dK$$

The bond-or-nothing call (B∨NC) is a vertical spread of vanilla calls:

$$1(S_T > H) = \lim_{\Delta H \downarrow 0} \frac{(S_T - H + \Delta H)^+ - (S_T - H)^+}{\Delta H}$$

Replicating American Binary Options with Vanilla Options

It follows from (9) that the value at t of an American binary call maturing at T can be expressed in terms of the contemporaneous prices of vanilla calls maturing at T:

$$ABC_t(T) = 2B \vee NC_t(T) + \frac{2\gamma}{H}C_t(H, T) +$$

$$\int_H^\infty n_c(K)C_t(K, T)dK \qquad (12)$$

where:

$$n_c(K) \equiv \left[(\gamma+\varepsilon)(\gamma+\varepsilon-1)\left(\frac{K}{H}\right)^{\gamma+\varepsilon-2} + \right.$$

$$\left.(\gamma-\varepsilon)(\gamma-\varepsilon-1)\left(\frac{K}{H}\right)^{\gamma-\varepsilon-2}\right]\frac{1}{H^2}$$

We note that if $r = \delta$, then, from (2), n_c simplifies to:

$$n_c(K) = \frac{2r}{\sigma^2 H^2}\left[\left(\frac{K}{H}\right)^{\gamma+\varepsilon-2} + \left(\frac{K}{H}\right)^{\gamma-\varepsilon-2}\right]$$

Thus, if $r = \delta = 0$, then $n_c = 0$, and the American binary call can be spanned by positions in options struck only at H:

$$ABC_t(T) = 2B \vee NC_t(T) + \frac{1}{H}C_t(H, T)$$

One can repeat this analysis to develop a static hedge for an American binary *put*, paying a dollar at the first passage time to a lower barrier L. The payoff to be created is:

$$\left[\left(\frac{S}{L}\right)^{\gamma+\varepsilon} + \left(\frac{S}{L}\right)^{\gamma-\varepsilon}\right]1(S < L)$$

From (11), this path-independent payoff can be created out of vanilla puts maturing at T and struck at and below L.

III. STATIC HEDGING OF BARRIER OPTIONS

State Pricing Densities

Breeden and Litzenberger [1978] show that static positions in vanilla options can be used to uncover the state pricing density, where states are defined by terminal stock price levels. In fact, using a proof given in Carr and Madan [1998], Appendix C shows that this state pricing density g(Z) is given by the cost of a position in $1/[(\Delta K)^2]$ butterfly spreads centered at Z, as $\Delta K \downarrow 0$:

$$g_t(Z) =$$

$$\begin{cases} \dfrac{\partial^2 P_t(Z)}{\partial K^2} \equiv \lim_{\Delta K \downarrow 0} \dfrac{P_t(Z-\Delta K) - 2P_t(Z) + P_t(Z+\Delta K)}{(\Delta K)^2} & \text{for } Z \le \kappa \\ \dfrac{\partial^2 C_t(Z)}{\partial K^2} \equiv \lim_{\Delta K \downarrow 0} \dfrac{C_t(Z-\Delta K) - 2C_t(Z) + C_t(Z+\Delta K)}{(\Delta K)^2} & \text{for } Z > \kappa \end{cases}$$

(13)

where κ is an arbitrary constant.

By analogy, we represent a state pricing density in terms of option prices, where states are now given by first passage times to a constant barrier. In contrast to (13), the representation given is valid only in the Black-Scholes model. In common with (13), the representation allows us to uncover the static hedge associated with an arbitrary payoff defined over the relevant states. In our case, this payoff occurs at the first passage time to the barrier.

Recall that an American binary call maturing at T pays one dollar at the first passage time as long as this hitting time occurs before T. Consequently, a calendar spread of an American binary call maturing at T + ΔT over one maturing at T delivers a dollar if and only if the first hitting time is between T and T + ΔT. If an investor purchases 1/(ΔT) such spreads, then as ΔT approaches zero, the payoff from the position at T approaches that of a delta function centered at T:

$$\lim_{\Delta T \downarrow 0} \frac{ABC_T(T+\Delta T) - ABC_t(T)}{\Delta T} = \delta(\tau_h - T)$$

Recall from (9) that, in the absence of arbitrage, the cost at t of purchasing an American binary call maturing at T is:

$$ABC_t(T) = \frac{1}{H^{\gamma+\varepsilon}} PS \vee NC_t(T) +$$

$$\frac{1}{H^{\gamma-\varepsilon}} PB \vee NC_t(T) \quad \text{for } S_t < H, \tau \geq t$$

It follows that the cost of obtaining the delta function payoff is:

$$ABC_t'(T) = \frac{1}{H^{\gamma+\varepsilon}} PS \vee NC_t'(T) +$$

$$\frac{1}{H^{\gamma-\varepsilon}} PB \vee NC_t'(T) \quad \text{for } S_t < H, \tau \geq t$$

where $PS \vee NC'_t(T)$ is the time t cost as $\Delta T \downarrow 0$ of a vanilla call portfolio with payoffs:

$$\lim_{\Delta T \downarrow 0} \frac{(S_{T+\Delta T}/H)^{\gamma+\varepsilon} 1(S_{T+\Delta T} > H)}{\Delta T} \text{ at } T + \Delta T$$

and

$$\lim_{\Delta T \downarrow 0} -\frac{(S_T/H)^{\gamma+\varepsilon} 1(S_T > H)}{\Delta T} \text{ at } T$$

Similarly, $PB \vee NC'_t(T)$ is the time t cost as $\Delta T \downarrow 0$ of a vanilla call portfolio with payoffs:

$$\lim_{\Delta T \downarrow 0} \frac{(S_{T+\Delta T}/H)^{\gamma-\varepsilon} 1(S_{T+\Delta T} > H)}{\Delta T} \text{ at } T + \Delta T$$

and

$$\lim_{\Delta T \downarrow 0} -\frac{(S_T/H)^{\gamma-\varepsilon} 1(S_T > H)}{\Delta T} \text{ at } T$$

In analogy with the representation in (13), we define $\phi_t(T) \equiv ABC'_t(T)$ as the state pricing density, where states are henceforth defined by first passage times to the barrier H. This density can be observed from the market prices of calls struck above H and maturing before T. The appropriate static position in these calls allows an investor to receive an infinite payoff if the first passage time is T, and no payoff otherwise.

Up-and-Out Put Options

To show how to statically replicate the payoffs of any (single) barrier option using a portfolio of vanilla options, for concreteness, we focus on an up-and-out put struck below the initial stock price. By in-out parity, the out put can be statically replicated if the in put can be:

$$UOP_t = P_t - UIP_t \text{ for } t \leq \tau_h$$

To value the up-and-in put, let P(S, t) be the well-known Black-Scholes European put formula:

$$P(S, t) = Ke^{-r(T-t)}N[-d_2(S, t)] -$$

$$Se^{-\delta(T-t)}N[-d_1(S, t)] \text{ for } t \in [0, T] \qquad (14)$$

where:

$$-d_2(S, t) \equiv \frac{\ln(K/S)}{\sigma\sqrt{T-t}} + \gamma\sigma\sqrt{T-t}$$

$$-d_1(S, t) \equiv -d_2(S, t) - \sigma\sqrt{T-t}$$

and where

$$N(d) \equiv \int_{-\infty}^{d} \frac{e^{-x^2/2}}{\sqrt{2\pi}}$$

is the standard normal distribution function.

Then, the up-and-in put value at t is given by:

$$UIP_t = \int_t^T \phi_t(u)p(u)du \tag{15}$$

where:

$$p(u) \equiv P(H, u) \tag{16}$$

is the vanilla put value at the barrier. Thus, the payoffs of the up-and-in put can be statically replicated by a portfolio of vanilla calls struck above H and maturing before T. To determine the position in each call, integrate (15) by parts:

$$UIP_t = -\int_t^T ABC_t(u)p'(u)du \tag{17}$$

where from (14) and (16), the put's time derivative is given by:

$$p'(u) = rKe^{-r(T-u)}N[-d_2(H, u)] -$$

$$\delta He^{-\delta(T-u)}N[-d_1(H, u)] -$$

$$\frac{\sigma Ke^{-r(T-u)}}{2\sqrt{T-u}}N'[-d_2(H, u)] \quad \text{for } u \in [t, T]$$

Combining (17) with (12) gives the position in calls:

$$UIP_t = \int_t^T [-p'(u)] \times$$

$$[2B \vee NC_t(u) + \frac{2\gamma}{H}C_t(H, u) +$$

$$\int_H^\infty n_c(K)C_t(K, u)dK]du \tag{18}$$

Thus, this static hedge uses calls maturing up to T and struck at or beyond the barrier. However, if $r = \delta = 0$, then $n_c(K) = 0$, so only calls struck at or near the barrier are used. Of course, when the integral in (18) does not vanish, it must be discretized in practice.

Up-and-Out Claims

More generally, we may consider an up-and-out claim with a payoff of $f(S_T)$ paid at maturity if the barrier is avoided and a time-dependent rebate of $g(\tau_h)$ paid at the hitting time otherwise. Letting V(S, t) denote the Black-Scholes value at t of the payoff $f(S_T)1(S_T < H)$, the value of this up-and-out claim is given by:[4]

$$UOV_t = V(S, t) + \int_t^T \phi_t(u)[g(u) - V(H, u)]du \tag{19}$$

Integrating by parts gives:

$$UOV_t = V(S, t) + ABC_t(T)[g(T) - V(H, T)] -$$

$$\int_t^T ABC_t(u)[g'(u) - \frac{\partial}{\partial u}V(H, u)]du \tag{20}$$

since $ABC_t(t) = 0$ for $S_t < H$. The static hedge for the first term is obtained from (11), while the static hedge for the second and third terms can be obtained from the appropriate weighting of the static hedge for an American binary call given in (12).

Once again, calls of all strikes $\geq$ H and all maturities $\leq$ T are used in the hedge. For down-and-out claims, one would instead use the static hedge for an American binary put.

IV. SIMULATING DYNAMIC AND STATIC HEDGES OF AMERICAN BINARY CALL

If one can trade continuously, and if a continuum of strikes is available, then static and dynamic hedges both work perfectly under zero transaction costs. To compare the effectiveness of static and dynamic hedging with discrete trading, discrete strikes, and transaction costs, we simulate static and dynamic hedges for the sale of a six-month American binary call with barrier \$110.

We assume an initial stock price of \$100, a constant interest rate of 5%, no dividends, and a constant volatility of 20%. We vary the frequency of the dynamic hedge from a low of four times over six months to a high of 1,024 times. We also assume proportional transaction costs of 0.1%.

Recall from (12) that the static hedge decomposes into the sum of three positions:

1. Two European binary calls struck at H.
2. A small position in a standard call struck at H.
3. An infinitesimally small position in calls struck above H.

We initially assume that strikes are available in \$1 increments with call prices given by the Black-Scholes formula. To create the first position, we use a vertical spread centered at the barrier of \$110. To create the second position, we assume that a vanilla call is available with a strike of \$110. To create the last term, we assume that the highest strike available is \$15 above the barrier of \$110.

Our results show that the cost of the static hedge is dominated by the first term. Accordingly, we vary the distance between strikes in the vertical spread from a low of \$1 to a high of \$10. We consider proportional transaction costs of both 0.1% and 0.5%.

Mean-Variance Analysis

Exhibit 6 plots the mean and standard deviation of the profit and loss from three classes of strategies:

1. Dynamic hedging with proportional transaction costs of 0.1%.
2. Static hedging with proportional transaction costs of 0.1%.
3. Static hedging with proportional transaction costs of 0.5%.

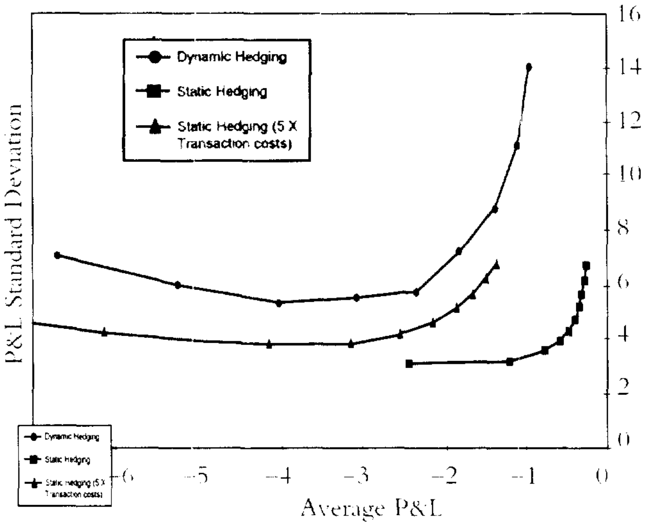
The results show that all strategies result in negative P&L. This is due to the assumption that the American binary call is sold for the (zero-transaction cost) Black-Scholes value. For the dynamic hedge, the shorter the time interval between trades, the lower the standard deviation of P&L, but the lower also the mean P&L. Similarly, for both classes of static hedges, the smaller the spatial interval between strikes, the lower the standard deviation of P&L, but also the lower the mean P&L. This lowering of the mean P&L results from larger long and short positions taken in the two legs of the vertical spread.

Stochastic Dominance

Comparing dynamic and static hedges, we conclude that each possible dynamic hedging strategy is mean-variance dominated by a static hedging strategy with five times higher transaction costs. However, it is well known that mean-variance analysis is inappropriate when outcomes are unbounded above or when P&L distributions are not symmetric. To examine this issue, we pick the best strategy from each of the three classes and compare their distribution functions.[5] The strategies compared are:

1. Dynamic hedging sixty-four times over six months with proportional transaction costs of 0.1%.
2. Static hedging with strikes \$4 apart with proportional transaction costs of 0.1%.

EXHIBIT 6
Static versus Dynamic Hedging — 20% Volatility



3. Static hedging with strikes $6 apart with proportional transaction costs of 0.5%.

Exhibit 7 shows that all three distribution functions graph against P&L in the same manner that binary call values graph against stock price. In other words, all three strategies are effective in reducing risk.

Since the static hedge with 0.1% transaction costs has a lower distribution function than the static hedge with 0.5% transaction costs, one can draw the obvious conclusion that any investor preferring more to less prefers the static hedge with the lower transaction costs, since there is greater probability of exceeding any fixed P&L level.

Since the distribution functions for the static hedges each cross the distribution function for the dynamic hedge, however, one cannot conclude that either static hedge first-order stochastically dominates the dynamic hedge or vice versa. Yet the graph of areas under distribution functions shown in Exhibit 8 shows that both static hedges second-order stochastically dominate the dynamic hedge.[6]

We can conclude that when hedging the sale of an American binary call, any risk-averse investor preferring more to less will prefer the static hedge we have described over standard dynamic hedging. Several caveats are in order, however. First, one may not be able to buy and sell calls at Black-Scholes prices. Second, it may well be that under discrete rebalancing and transaction costs, dynamic hedges other than standard delta-hedging will prevail. Similarly, with discrete strikes and and transaction costs, there may exist better static hedges than the one we consider. Finally, whether or not one considers discreteness or transaction costs, it is likely that the best hedging strategy will allow for dynamic positioning in both standard options and the underlying.

V. SUMMARY

By combining stationarity and symmetry principles, we have shown how to establish static hedges of American binary options. We show how the ability to statically hedge these fundamental path-dependent securities implies that all other single-barrier options can be statically hedged. Our simulation of dynamic and static hedges of American binary calls shows that static hedging dominates dynamic hedging in both a mean-variance framework and in a more general utility maximization setting.

It would be relatively straightforward to extend the analysis to multiple-barrier options and to lookback options. A much more challenging problem is to extend the analysis to time- or spatially dependent vari-

EXHIBIT 7
Static versus Dynamic P&L Distributions

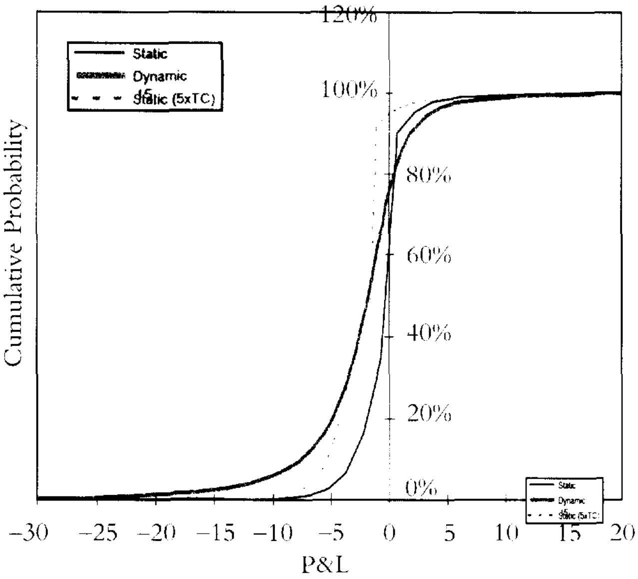
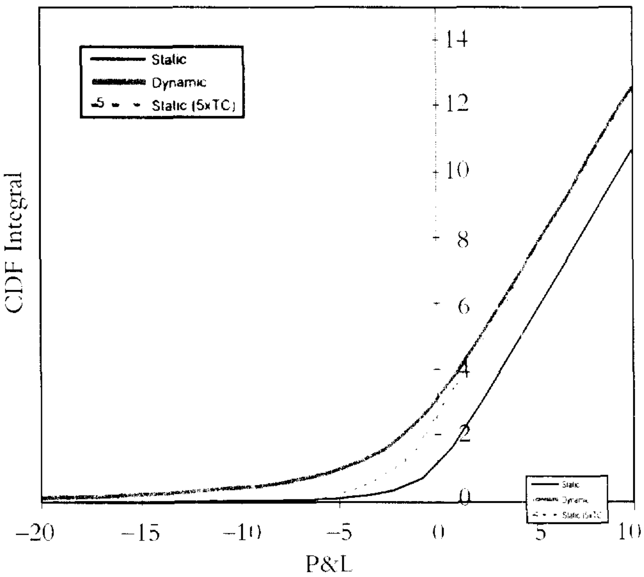


EXHIBIT 8
Static versus Dynamic: Integral of the CDF



ance rates, interest rates, and dividend yields, while retaining analytical solutions for values and hedges.

APPENDIX A

Proof of (7)

To statically hedge down-options, we prove a slightly more general version of the symmetry principle.

Consider a payoff with support in the interval (A, B):

$$f(S_T) = \begin{cases} \phi(S_T) & \text{if } S_T \in (A, B) \\ 0 & \text{otherwise} \end{cases}$$

Then, in the Black-Scholes model, there exists a drift-adjusted reflected payoff $f^r(S_t)$ whose value matches that of f at any time t when the spot price is at H.

This payoff is given by:

$$f^r(S_T) = \begin{cases} (S_T/H)^{2\gamma}\phi(H^2/S_T) & \text{if } S_T \in \left(\frac{H^2}{B}, \frac{H^2}{A}\right) \\ 0 & \text{otherwise} \end{cases} \qquad \text{(A-1)}$$

Proof. From (14) and (13) as $\kappa \uparrow \infty$, the state pricing density in the Black-Scholes model is:

$$g_t(S_t, S_T) = \frac{e^{-r\tau}}{S_T\sigma\sqrt{\tau}} N'\left(\frac{\ln(S_T/S_t)}{\sigma\sqrt{\tau}} + \gamma\sigma\sqrt{\tau}\right) \qquad \text{(A-2)}$$

where $\tau \equiv T - t$. Thus, the value of f when the spot price is at H at time t is:

$$V_f(H, t) = \int_A^B \phi(S_T) g_t(H, S_T) dS_T$$

$$= \int_A^B \phi(S_T) \frac{e^{-r\tau}}{S_T\sigma\sqrt{\tau}} N'\left(\frac{\ln(S_T/H)}{\sigma\sqrt{\tau}} + \gamma\sigma\sqrt{\tau}\right) dS_T$$

Let $\hat{S}_T = (H^2/S_T)$. Then, $dS_T = -(H^2/\hat{S}_T^2)d\hat{S}_T$ and:

$$V_f(H, t) = -\int_{H^2/A}^{H^2/B} \phi\left(\frac{H^2}{\hat{S}_T}\right) \frac{e^{-r\tau}}{\hat{S}_T\sigma\sqrt{\tau}} \times N'\left(\frac{\ln(H/\hat{S}_T)}{\sigma\sqrt{\tau}} + \gamma\sigma\sqrt{\tau}\right) d\hat{S}_T$$

$$= \int_{H^2/B}^{H^2/A} \left(\frac{\hat{S}_T}{H}\right)^{2\gamma} \phi\left(\frac{H^2}{\hat{S}_T}\right) \frac{e^{-r\tau}}{\hat{S}_T\sigma\sqrt{\tau}} \times N'\left(\frac{\ln(\hat{S}_T/H)}{\sigma\sqrt{\tau}} + \gamma\sigma\sqrt{\tau}\right) d\hat{S}_T$$

$$= \int_{H^2/B}^{H^2/A} \left[\left(\frac{\hat{S}_T}{H}\right)^{2\gamma} \phi\left(\frac{H^2}{\hat{S}_T}\right)\right] g_t(H, \hat{S}_T) d\hat{S}_T$$

$$= V_{f^r}(H, t)$$

■

Equation (7) arises from setting A = 0 and B = H in (A-1).

APPENDIX B

Proof of (11)

The Fundamental Theorem of Calculus implies that for any fixed κ:

$$f(S) = f(\kappa) - 1(S < \kappa)\int_S^\kappa f'(u)du + 1(S > \kappa)\int_\kappa^S f'(u)du$$

$$= f(\kappa) - 1(S < \kappa)\int_S^\kappa [f'(\kappa) - \int_u^\kappa f''(v)dv]du +$$

$$1(S > \kappa)\int_\kappa^S [f'(\kappa) - \int_\kappa^u f''(v)dv]du$$

Noting that $f'(\kappa)$ does not depend on u, and reversing the order of integration yields:

$$f(S) = f(\kappa) + f'(\kappa)(S - \kappa) + 1(S < \kappa)\int_S^\kappa\int_S^v f''(v)dudv +$$

$$1(S > \kappa)\int_\kappa^S\int_v^S f''(v)dudv$$

Performing the integral over u yields:

$$f(S) = f(\kappa) + f'(\kappa)(S - \kappa) + 1(S < \kappa)\int_S^\kappa f''(v)(v - S)dv +$$

$$1(S > \kappa)\int_\kappa^S f''(v)(S - v)dv$$

and this may be equivalently rewritten as:

$$f(S) = f(\kappa) + f'(\kappa)(S-\kappa) +$$

$$\int_0^{\kappa} f''(v)(v-S)^+ dv + \int_{\kappa}^{\infty} f''(v)(S-v)^+ dv$$

The payoff may be interpreted as the payoff arising from a static position in bonds, forward contracts, and vanilla options. Letting V_t^f denote the time t value of the payoff F, absence of arbitrage requires that:

$$V_t^f = f(\kappa)B_t + f'(\kappa)I_t(\kappa) + \int_0^{\kappa} f''(K)P_t(K)dK +$$

$$\int_{\kappa}^{\infty} f''(K)C_t(K)dK \qquad \text{for } t \in [0, T]$$

where B_t is the time t value of the unit bond, $I_t(\kappa)$ is the time t value of the forward contract with delivery price κ, and $P_t(K)$, $K \le \kappa$ and $C_t(K)$, $K \ge \kappa$ are the time t values of puts and calls struck at K, respectively.

APPENDIX C
Proof of (13)

Integrating Equation (11) by parts gives:

$$V_t = f(\kappa)B_t + f'(\kappa)I_t(\kappa) + f'(K)P_t(K)\Big|_0^{\kappa} -$$

$$\int_0^{\kappa} f'(K)P_t'(K)dK + f'(K)C_t(K)\Big|_{\kappa}^{\infty} -$$

$$\int_{\kappa}^{\infty} f'(K)C_t'(K)dK$$

Since $P(0) = 0$ and $C(\infty) = 0$, and $I(\kappa) = C(\kappa) - P(\kappa)$, the second, third, and fifth terms cancel. Integrating by parts once again:

$$V_t = f(\kappa)B_t - f(K)P_t'(K)\Big|_0^{\kappa} +$$

$$\int_0^{\kappa} f(K)P_t''(K)dK - f(K)C_t'(K)\Big|_{\kappa}^{\infty} +$$

$$\int_{\kappa}^{\infty} f(K)C_t''(K)dK$$

Noting that $C_t'(\infty) = P_t'(0) = 0$ and $P_t'(\kappa) - C_t'(\kappa) = B_t$, by differentiating put-call parity, we observe that:

$$V_t = \int_0^{\infty} f(K)g_t(K)dK$$

where $g_t(K)$ is the second derivative with respect to strike of an option price:

$$g_t(K) \equiv \begin{cases} \dfrac{\partial^2 P_t(K)}{\partial K^2} & \text{for } K \le \kappa \\[2ex] \dfrac{\partial^2 C_t(K)}{\partial K^2} & \text{for } K > \kappa \end{cases}$$

ENDNOTES

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[1]This result is used to represent the stationary security payoffs in (4) and (5) in terms of options, by taking limits as κ approaches zero and infinity, respectively.

[2]Generalized functions always have derivatives of all orders, which are themselves generalized functions. For an introduction to generalized functions, see Richards and Youn [1990].

[3]To derive the result, consider the left and right limits as κ approaches H.

[4]With states defined by stock price levels, the state pricing density g can be used to determine this value (see A-2 in Appendix A). Equivalently, the expansion (11) can be used with $\kappa = H$ and the Black-Scholes formula (14) used to value the vanilla puts at the barrier.

[5]For the purpose of selecting the best strategy from each class, we assume constant absolute risk aversion of 0.1 and pick the strategy that maximizes expected utility.

[6]See Huang and Litzenberger [1998, Chapter 2] for a discussion of stochastic dominance.

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