Spectra of C* algebras, Extensions and $\mathbb{R}$-actions

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• Spectra of amenable C*-algebras.

• NC-Selection and semi-split Extensions.

• Study of coherent locally q-compact spaces.

• Application: Exotic line-action on Cuntz algebras.
Conventions and Notations

• Spaces $P, X, Y, \ldots$ are $T_0$ and second countable, algebras $A, B, \ldots$ are separable, ...  

• ... except corona spaces $\beta(P) \setminus P$, multiplier algebras $\mathcal{M}(B)$, and ideals of corona algebras $Q(B) := \mathcal{M}(B)/B$, the space $\text{Prim}(\mathcal{M}(B))$, ...  

• The isomorphisms $\mathcal{I}(A) \cong \mathcal{O}(\text{Prim}(A)) \cong \mathcal{F}(\text{Prim}(A))^{op}$ will be used frequently.  

• $\mathbb{Q} := [0, 1]^\infty$ denotes the Hilbert cube (with its coordinate-wise order).

• A $T_0$ space $X$ is sober (or “point-complete”) if each prime closed subset $F$ of $X$ is a the closure $\overline{\{x\}} = F$ of a singleton $\{x\}$. (Locally) “compact” means (locally) “quasi-compact” in case of $T_0$ spaces.
Spectra of amenable algebras (1)

Characterization of Prim(\(A\)) for amenable \(A\) (H.Harnisch, E.K., M.Rørdam):

**Theorem 1.** A sober space \(X\) is homeomorphic to a primitive ideal space of an amenable \(C^*\)-algebra \(A\), if and only if, there is a Polish l.c. space \(P\) and a continuous map \(\pi: P \to X\) such that
\[
\pi^{-1}: \mathcal{O}(X) \to \mathcal{O}(P) \text{ is injective (=: \(\pi\) is pseudo-epimorphic),}
\]
and
\[
(\bigcap_n \pi^{-1}(U_n))^\circ = \pi^{-1}((\bigcap_n U_n)^\circ) \text{ for each sequence } U_1, U_2, \ldots \in \mathcal{O}(X) \text{ (=: \(\pi\) is pseudo-open).}
\]

The algebra \(A \otimes \mathcal{O}_2 \otimes \mathbb{K}\) is uniquely determined by \(X\) up to (unitarily homotopic) isomorphisms.
Notice: A continuous epimorphism $\pi: P \to X$ is not necessarily pseudo-open, e.g. $\sum_n \alpha_n 3^{-n} \mapsto \sum_n \alpha_n 2^{-n}$ is continuous epimorphism from the Cantor space $\{0,1\}^\infty$ onto $[0,1]$, but no pseudo-open continuous epimorphism from $\{0,1\}^\infty$ onto $[0,1]$ exists.

A map $\Psi: \mathcal{O}(X) \to \mathcal{O}(Y)$ is lower semi-continuous if $(\bigcap_n \Psi(U_n))^\circ = \Psi((\bigcap_n U_n)^\circ)$ for each sequence $U_1, U_2, \ldots \in \mathcal{O}(X)$.

(Thus, $\pi$ is pseudo-open, if and only if, $\Psi := \pi^{-1}$ is lower semi-continuous.)

If one works with closed sets, then one has to replace intersections by unions and interiors by closures.
Proposition 2. If $\Psi : \mathcal{I}(B) \to \mathcal{I}(A)$ is a lower semi-continuous action of $\text{Prim}(B)$ on $A$ and $B$ is stable, then there exists a lower s.c. action $\mathcal{M}(\Psi) : \mathcal{I}(\mathcal{M}(B)) \to \mathcal{I}(A)$ of $\text{Prim}(\mathcal{M}(B))$ on $A$, that has the following properties (i)–(iii):

(i) $\mathcal{M}(\Psi)$ is monotone upper semi-continuous
$(:= \text{sup’s of upward directed families of ideals will be respected}).$

(ii) $\mathcal{M}(\Psi)(J_1) = \mathcal{M}(\Psi)(J_1)$
if $J_1 \cap \delta_{\infty}(\mathcal{M}(B)) = J_2 \cap \delta_{\infty}(\mathcal{M}(B))$.

(iii) $\mathcal{M}(\Psi)(\mathcal{M}(B, I)) = \Psi(I)$ for all $I \in \mathcal{I}(B)$.

The “extension” $\mathcal{M}(\Psi)$ of $\Psi$ with (i)–(iii) is unique.
For strongly p.i. (not necessarily separable) \( B \) and exact \( A \), there is a nuclear *-morphism \( h: A \to B \) with \( \Psi(J) = h^{-1}(h(A) \cap J) \), if and only if, \( \Psi \) is lower s.c. and monotone upper s.c. It yields the following theorem.

**Theorem 3. [NC-selection]** Suppose that \( B \) is stable, \( A \otimes \mathcal{O}_2 \) contains a regular exact \( C^* \)-algebra \( C \subset A \otimes \mathcal{O}_2 \), and that \( \Psi: \mathcal{I}(B) \to \mathcal{I}(A) \) is a lower s.c. action of \( \text{Prim}(B) \) on \( A \).

Then there is a *-morphism \( h: A \to \mathcal{M}(B) \) such that \( \delta_\infty \circ h \) is unitarily equivalent to \( h \), \( \Psi(J) = h^{-1}(h(A) \cap \mathcal{M}(B, J)) \) and that

\[
[h]_J: A/\Psi(J) \to \mathcal{M}(B/J) \cong \mathcal{M}(B)/\mathcal{M}(B, J)
\]

is weakly nuclear for all \( J \in \mathcal{I}(B) \).
Here, a subalgebra $C \subset D$ is regular if $C$ separates the ideals of $D$ and $C \cap (I + J) = (C \cap I) + (C \cap J)$ for all $I, J \in \mathcal{I}(D)$.

Theorem 3 applies to necessary and sufficient criteria for (ideal-system-) equivariant semi-splitness of extensions.

Let $\epsilon: B \to E$ a *-monomorphism onto a closed ideal of $E$ and $\pi: E \to A$ an epimorphism such that $\epsilon(B)$ is the kernel of $\pi$. We denote by $\gamma: A \to Q(B) = \mathcal{M}(B)/B$ the Busby invariant of the extension

$$0 \to B \xrightarrow{\epsilon} E \xrightarrow{\pi} A \to 0.$$
Consider now general “actions” $\psi_B: S \to \mathcal{I}(B)$, $\psi_E: S \to \mathcal{I}(E)$, and $\psi_A: S \to \mathcal{I}(A)$, of a set $S$ on $B$, $E$ and $A$. We require that the extension $E$ is $\psi$-equivariant:

(a) $\epsilon(\psi_B(s)) = \epsilon(B) \cap \psi_E(s) = \epsilon(B)\psi_E(s)$, and

(b) $\psi_A(s) = \pi(\psi_E(s))$ for all $s \in S$.

i.e., $0 \to \psi_B(s) \to \psi_E(s) \to \psi_A(s) \to 0$ is exact for each $s \in S$.

An action $\Phi: \mathcal{I}(A) \to \mathcal{I}(B)$ of $\text{Prim}(A)$ on $B$ is **upper semi-continuous** if $\Psi$ preserves sup of families in $\mathcal{I}(A)$, i.e., $\Psi(I + J) = \Psi(I) + \Psi(J)$ and $\Psi$ is monotone upper semi-continuous.
NC-Selection and Extensions (5)

Lemma 4. There is a unique maximal upper semi-continuous map $\Phi: \mathcal{I}(A) \to \mathcal{I}(B)$ with the property that $\Phi(\psi_A(s)) \subset \psi_B(s)$ for all $s \in S$.

Upper semi-continuous actions $\Phi$ have lower semi-continuous (= inf preserving) adjoint maps $\Psi: \mathcal{I}(B) \to \mathcal{I}(A)$ such that $(\Psi, \Phi)$ build a Galois connection, i.e., $\Psi(J) \supset I$ iff $J \supset \Phi(I)$. The rule is: The upper adjoint is lower semi-continuous.

Applications of Theorem 3 to the adjoint $\Psi$ of $\Phi$ in Lemma 4 implies the following necessary and sufficient criterion (ii):
Theorem 5. Let $B$, $E$, $A$, $\epsilon$, $\pi$, $\gamma$, $\psi_Y : S \to \mathcal{I}(Y)$ (for $Y \in \{B, E, A\}$) be as above, and let $\Phi : \mathcal{I}(A) \to \mathcal{I}(B)$ the map given in Lemma 4.

Suppose, in addition, that $A$ is exact and that $B$ is weakly injective (i.e., has the WEP of Lance).

Then the following properties (i) and (ii) of the extension are equivalent:

(i) The extension has an $S$-equivariant c.p. splitting map, i.e., there is a c.p. map $V : A \to E$ with $\pi \circ V = \text{id}_A$ and $V(\psi_A(s)) \subset \psi_E(s)$ for all $s \in S$.

(ii) The Busby invariant $\gamma : A \to Q(B)$ is nuclear, and,

$$\pi_B(\mathcal{M}(B, \Phi(J))) \supseteq \gamma(J) \quad \forall \ J \in \mathcal{I}(A)$$
Definition 6. A map \( f: X \to [0, \infty) \) is a Dini function if it is lower semi-continuous and 
\[ \sup f(\bigcap_n F_n) = \inf_n \{ \sup f(F_n) \} \]
for every decreasing sequence \( F_1 \supset F_2 \supset \cdots \) of closed subsets of \( X \).

A sober \( T_0 \) space \( X \) is a Dini space if the supports of the Dini functions build a base of the topology of \( X \).

The Dini functions \( f \) are exactly the functions that satisfy the (generalized) Dini Lemma: Every upward directed net of l.s.c. functions converges uniformly to \( f \) if it converges point-wise to \( f \). If a \( T_0 \) space \( X \) is sober, then a function \( f: X \to [0, 1] \) is Dini, if and only if, \( f \) is lower semi-continuous and the restriction \( f: X \setminus f^{-1}(0) \to (0, 1]_{lsc} \) is proper.
Coherent Dini spaces (2)

The class of Dini spaces $X$ coincides with the class of sober locally compact $T_0$ spaces with a countable base of its topology.

A subset $C$ of $X$ is **saturated** if $C = \text{Sat}(C)$, where $\text{Sat}(C)$ means the intersection of all $U \in \mathcal{O}(X)$ with $U \supset C$.

**Definition 7.** A sober $T_0$ space $X$ is **coherent** if the intersection $C_1 \cap C_2$ of two saturated quasi-compact subsets $C_1, C_2 \subset X$ is again quasi-compact.

Below, we consider some partial results concerning the open **Question:**
Is every (second-countable) coherent Dini space $X$ homeomorphic to the primitive ideal spaces $\text{Prim}(A)$ of some amenable $C^*$-algebra $A$?

Let $\mathcal{F}(X)$ denote the lattice of closed subsets $F \subset X$. 
Definition 8. The topological space $\mathcal{F}(X)_{lsc}$ is the set $\mathcal{F}(X)$ with the $T_0$ order topology that is generated by the complements

$$\mathcal{F}(X) \setminus [\emptyset, F] = \{ G \in \mathcal{F}(X) ; \ G \cap U \neq \emptyset \} =: \mu_U$$

of the intervals $[\emptyset, F]$ for all $F \in \mathcal{F}(X)$ (where $U = X \setminus F$).

The Fell-Vietoris topology on $\mathcal{F}(X)$ is the topology, that is generated by the sets $\mu_U$ ($U \in \mathcal{O}(X)$) and the sets $\mu_C := \{ G \in \mathcal{F}(X) ; \ G \cap C = \emptyset \}$ for all quasi-compact $C \subset X$.

$\mathcal{O}(X) \cong \mathcal{F}(X)^{\text{op}}$ defines the Larson topology on $\mathcal{O}(X)$. We denote by $\mathcal{F}(X)_H$ Fell-Vietoris topology.

The space $\mathcal{F}(X)_{lsc}$ is a coherent Dini space, and the space $\mathcal{F}(X)_H$ is a compact Polish space.
Coherent Dini spaces (4)

The ordered Hilbert cube $\mathbb{Q}$ is nothing else $\mathcal{F}(Y)$ for $Y := X_0 \uplus X_0 \uplus \cdots$ where $X_0 := (0, 1]_{lsc}$. The Fell-Vietoris topology becomes the usual Hausdorff topology on $\mathbb{Q}$.

If $X$ is locally quasi-compact sober $T_0$ space, then a dense sequence $g_1, g_2, \ldots$ in the Dini functions $g$ on $X$ with $\sup g(X) = 1$ defines an order isomorphism $\iota: \mathcal{F} \to \mathbb{Q}$ onto a max-closed subset $\iota(\mathcal{F})$ of $\mathbb{Q}$ with $\iota(\emptyset) = 0$, $\iota(X) = 1$ by

$$\iota(F) := (\sup g_1(F), \sup g_2(F), \ldots) \in \mathbb{Q}.$$  

The image $\iota(\mathcal{F}(X))$ is closed in $\mathbb{Q}$ (with Hausdorff topology) and $\iota$ defines a homeomorphism from $\mathcal{F}(X)$ onto $\iota(\mathcal{F}(X))$ with respect to both topologies on $\mathcal{F}(X)$ and $\mathbb{Q}$.
Coherent Dini spaces (5)

In this way, \( X \cong \eta(X) \subset \overline{\eta(X)}^H \setminus \{0\} \subset \mathcal{F}(X) \subset \mathbb{Q} \), considered as Polish spaces, with \( X \ni x \mapsto \eta(x) := \{x\} \in \mathcal{F}(X) \).

**Theorem 9.** Let \( X \) a second countable locally (quasi-)compact sober \( T_0 \) space. Following properties (i)-(iv) of \( X \) are equivalent:

(i) \( X \) is coherent.

(ii) The set \( \mathcal{D}(X) \) of Dini functions on \( X \) is convex.

(iii) \( \mathcal{D}(X) \) is min-closed.

(iv) \( \mathcal{D}(X) \) is multiplicatively closed.
Coherent Dini spaces (6)

It is known that, $X$ is coherent, if and only if, the image $\eta(X) \cong X$ in $\mathcal{F}(X) \setminus \{\emptyset\}$ is closed in $\mathcal{F}(X) \setminus \{\emptyset\}$ with respect to the Fell-Vietoris topology on $\mathcal{F}(X)$.

Lemma 10. (I) Each closed subset $F \subset \mathcal{Q}_H$ is a coherent locally compact sober subspace $F_{lsc}$ of $\mathcal{Q}_{lsc}$, and is the intersection of an decreasing sequence $F_k$ of closed subspaces of $\mathcal{Q}_H$ that are continuously order-isomorphic to spaces $G_k \times \mathbb{Q}$ with $G_k \subset [0, 1]^{n_k}$ a finite union of $n_k$-dimensional (small) cubes.

(II) If $F = \bigcap_k F_k$ for a sequence $F_1 \supset F_2 \supset \cdots$ of closed subsets in $\mathcal{Q}_H$, and if each $(F_k)_{lsc} \subset \mathcal{Q}_{lsc}$ is the primitive ideal space of an amenable $C^*$-algebra, then $F_{lsc}$ is the primitive ideal space of an amenable $C^*$-algebra.
Coherent Dini spaces (7)

Lemma 10 applies to $F := \eta(\mathcal{F}(X))$ for all Dini spaces $X$, and to $F := \{0\} \cup \eta(X)$ for all coherent Dini spaces $X$.

**Corollary 11.** If there is a coherent sober l.c. space $X$ that is not homeomorphic to the primitive ideal space of an amenable $C^*$-algebra, then there is $n \in \mathbb{N}$ and a finite union $Y$ of (Hausdorff-closed and small) cubes in $[0, 1]^n$ such that $Y$ with induced order-topology is not the primitive ideal space of any amenable $C^*$-algebra.

**Theorem 12.** [O.B. Ioffe, E.K.] If $G \subset [0, 1]^n$ is a finite union of (small) cubes, then the space $G_{lsc}$ has a decomposition series $U_1 \subset U_2 \subset \cdots \subset U_k$, by open subsets $U_\ell \subset G_{lsc}$ such that $U_{\ell+1} \setminus U_\ell$ is the primitive ideal space of an amenable $C^*$-algebra.

Now combine above results with the following conjecture.
Coherent Dini spaces (8)

Let $X$ a Dini space and $U \subset X$ open.

**Conjecture 13.** The space $X$ is homeomorphic to the primitive ideal space of an amenable $C^*$-algebra if $U$ and $X \setminus U$ are homeomorphic to primitive ideal spaces of amenable $C^*$-algebras.

This Conjecture implies that Dini spaces are primitive ideal spaces of amenable $C^*$-algebras — if they have decomposition series by open subsets $\{U_\alpha\}$ with coherent spaces $U_{\alpha+1} \setminus U_\alpha$.

A Dini space $X$ is the primitive ideal space of an AF-algebra if $U$ and $X \setminus U$ are primitive ideal spaces of AF-algebras.

**Proposition 14.** Conjecture 13 reduces, in the case where $X$ is coherent, to the case, where $X \setminus U = \{p\}$ is a singleton and $U \cong \text{Prim}(B)$, and where $B$ is an inductive limit of algebras $B_n \cong C_0(\Gamma_n \setminus \{g_n\}) \otimes M_{k_n}$ for connected pointed graphs $(\Gamma_n, g_n)$. 
Theorem 15. [N.Ch. Phillips, E.K.] Suppose that $A$ is an amenable $C^*$-algebra, $G$ an amenable l.c. group, and that $G$ acts minimally by $\alpha : G \to \text{Homeo}(\text{Prim}(A))$ on $\text{Prim}(A)$. Then there exists a continuous group-action $\beta : G \to \text{Aut}(B)$ on the $C^*$-algebra $B := A \otimes \mathcal{O}_2 \otimes \mathbb{K}$ that implements $\alpha$, and has crossed product $B \rtimes_\beta G \cong \mathcal{O}_2 \otimes \mathbb{K}$.

A part of the proof is an $G$-equivariant improvement of Theorem 1. Then the spectra of the actions will be enriched by tensoring (infinitely often if necessary) with the natural action of $G$ on $\mathcal{O}_\infty \cong \mathcal{O}(L_2(G))$.

Definition 16. [N.C. Phillips compactification]

Let $\Xi(P)$ denote the prime $T_0$ space $P \cup \{\infty\}$ with topology given by the system of open subsets

\[ \mathcal{O}(\Xi(P)) = \{\emptyset, \Xi(P) \setminus C ; C \subset P, \text{ compact in } P \} \, . \]
Exotic $\mathbb{R}$-actions (2)

Theorem 17. [N.Ch. Phillips, E.K.] There exists an amenable $C^*$-algebra $A$ with $\text{Prim}(A) \cong \Xi(P)$.

If we apply the above theorems to $\Xi(G)$, we get:

Corollary 18. Every non-compact amenable l.c. group $G$ has a co-action $\hat{\beta}$ on $\mathcal{O}_2 \otimes K$ such that $B := (\mathcal{O}_2 \otimes K) \rtimes \hat{G}$ is prime and the (dual) action $\beta$ of $G$ on $B$ is minimal and topologically free.

If $G := \mathbb{R} = \hat{G}$, there is also an action $\hat{\beta}$ of $\mathbb{R} = \hat{\mathbb{R}}$ on $\mathcal{O}_2$ itself with this property.
The existence problem for extensions reduces in case of non-coherent $X$ to the case where $U \cong \text{Prim}(B)$ with $B \cong B \otimes \mathcal{O}_2 \otimes K$ is an inductive limit of algebras $B_n \cong C_0(\Gamma_n \setminus \{g_n\}) \otimes \mathcal{M}_{k_n}$ for connected pointed graphs $(\Gamma_n, g_n)$, and where $F := X \setminus U$ is homeomorphic to $(0, 1]_{\text{lsc}}$.

This is equivalent to the below formulated question:

Given sequences of positive contractions $T_1, T_2, \ldots \in \mathcal{M}(B)_+$ and isometries $V_n \in \mathcal{M}(B)$ with $T_{n+1} = V_n^* T_n V_n$.

Let $\gamma(J) := \lim_n \|T_n + \mathcal{M}(B, J)\|$, and suppose that, for each $J \in \mathcal{I}(B)$ and $n \in \mathbb{N}$, there is $b := b_{n,J} \in B$ such that

$$(\delta_\infty(T_n) - \gamma(J))_+ - \delta_\infty(b) \in \mathcal{M}(B, J),$$

i.e., $\delta_\infty(\mathcal{M}(\pi_J)(T_n) - \gamma(J)_+) \in \delta_\infty(B/J)$. 

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General extensions (1)
Question 19. Does there exist a contraction $S \in \mathcal{M}(B)_+$ such that

$$\|\mathcal{M}(\pi_J)(S)\| = \|S + \mathcal{M}(B, J) + B\| = \gamma(J)$$

for each $J \in \mathcal{I}(B)$.

If the answer is positive, then the element $\pi_B(S) \in Q(B)$ defines the desired Busby invariant of the desired extension.