Classification of $\mathbb{Z}^N$-actions
on simple $C^*$-algebras

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Classification of amenable $C^*$-algebras
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Goal (too ambitious)

Classify (strongly) outer actions of discrete amenable groups on simple classifiable $C^*$-algebras (Kirchberg algebras, UHF, AF, AT, AH algebras, the Jiang-Su alg. etc.) up to cocycle conjugacy.

Outer actions of finite groups on unital simple $C^*$-algebras do not always have the Rohlin property. In fact, if an action $\alpha : \Gamma \curvearrowright A$ of a finite group $\Gamma$ on a unital simple $C^*$-algebra $A$ has the Rohlin property, then $K_0(A)$ and $K_1(A)$ are completely cohomologically trivial as $\Gamma$-modules.

Goal (realistic)

Classify (strongly) outer actions of $\mathbb{Z}^N$ (or “poly-$\mathbb{Z}$” groups) on simple classifiable $C^*$-algebras up to cocycle conjugacy.
Cocycle conjugacy

**Definition**

Let $\alpha : \Gamma \curvearrowright A$ be an action of a countable discrete group $\Gamma$ on a unital $C^*$-algebra $A$. $(u_g)_{g \in \Gamma} \subset U(A)$ is called an $\alpha$-cocycle if

$$u_g \alpha_g(u_h) = u_{gh} \quad \forall g, h \in \Gamma.$$ 

Two actions $\alpha, \beta : \Gamma \curvearrowright A$ are said to be cocycle conjugate, if

$$\exists \gamma \in \text{Aut}(A), \quad \exists (u_g)_g \alpha \text{-cocycle}$$

$$\text{Ad} u_g \circ \alpha_g = \gamma \circ \beta_g \circ \gamma^{-1} \quad \forall g \in \Gamma.$$ 

**Strong cocycle conjugacy** further requires

$$\exists v_n \in U(A), \quad \| u_g - v_n \alpha_g(v_n^*) \| \to 0 \quad \forall g \in \Gamma.$$
An automorphism of the form $\text{Ad} \, u$ is said to be inner.
An action $\alpha : \Gamma \curvearrowright A$ is said to be outer if $\alpha_g$ is not inner for every $g \in \Gamma \setminus \{e\}$.

Let $T(A)$ denote the set of tracial states and let $\pi_\tau$ be the GNS representation by $\tau \in T(A)$.
$\alpha \in \text{Aut}(A)$ is said to be not weakly inner if the extension $\bar{\alpha}$ on $\pi_\tau(A)''$ is not inner for any $\tau \in T(A)^\alpha$, that is, there does not exist a unitary $U \in \pi_\tau(A)''$ such that $\bar{\alpha} = \text{Ad} \, U$.
An action $\alpha : \Gamma \curvearrowright A$ is said to be strongly outer if $\alpha_g$ is not weakly inner for every $g \in \Gamma \setminus \{e\}$.

If $T(A) = \{\tau\}$, then

$\alpha : \Gamma \curvearrowright A$ is strongly outer $\iff \bar{\alpha} : \Gamma \curvearrowright \pi_\tau(A)''$ is outer.
**Z-actions on Kirchberg algebras**

Complete classification is known for outer actions of $\mathbb{Z}$ on unital Kirchberg algebras.

**Theorem (H. Nakamura 2000)**

Let $A$ be a unital Kirchberg algebra and let $\alpha : \mathbb{Z} \curvearrowright A$ be an outer action. Then $\alpha$ has the Rohlin property.

**Theorem (H. Nakamura 2000)**

Let $A$ be a unital Kirchberg algebra. For two outer actions $\alpha, \beta : \mathbb{Z} \curvearrowright A$, the following are equivalent.

1. $KK(\alpha_1) = KK(\beta_1)$.
2. $\alpha$ and $\beta$ are cocycle conjugate via $\gamma \in \text{Aut}(A)$ satisfying $KK(\gamma) = 1$. 
We say that a group $\Gamma$ is poly-$\mathbb{Z}$ if there exists a normal series

$$\{e\} = \Gamma_0 \triangleleft \Gamma_1 \triangleleft \Gamma_2 \triangleleft \cdots \triangleleft \Gamma_m = \Gamma$$

such that $\Gamma_{i+1}/\Gamma_i \cong \mathbb{Z}$.

**Theorem (M. Izumi and M)**

Let $\Gamma$ be a poly-$\mathbb{Z}$ group and let $A$ be either $O_2$, $O_\infty$ or $O_\infty \otimes B$ with $B$ being a UHF algebra of infinite type. Then there exists a unique strong cocycle conjugacy class of outer $\Gamma$-actions on $A$.

Why unique?

- For $\Gamma$ as above, its classifying space $BG$ has the homotopy type of a finite CW complex.
- For $A$ as above, the homotopy group $\pi_n(\text{Aut}(A))$ is trivial for every $n \geq 0$ (M. Dadarlat 2007).
In particular, for every poly $\mathbb{Z}$-group $\Gamma$ and $A = \mathcal{O}_2, \mathcal{O}_\infty, \mathcal{O}_\infty \otimes B$ as in the previous slide, any outer action $\alpha : \Gamma \curvearrowright A$ is asymptotically representable, i.e. there exist continuous paths of unitaries $(v_g(t))_{g \in \Gamma, t \in [0, \infty)}$ in $A$ such that

$$
\|v_g(t)v_h(t) - v_{gh}(t)\| \to 0 \quad \forall g, h \in \Gamma,
$$

$$
\|\alpha_g(v_h(t)) - v_{ghg^{-1}}(t)\| \to 0 \quad \forall g, h \in \Gamma,
$$

$$
\|v_g(t)av_g(t)^* - \alpha_g(a)\| \to 0 \quad \forall g \in \Gamma, a \in A.
$$

**Theorem (M. Izumi and M)**

Let $\Gamma$ be a poly-$\mathbb{Z}$ group and let $A$ be a unital Kirchberg algebra. Let $\alpha : \Gamma \curvearrowright A$ and $\sigma : \Gamma \curvearrowright \mathcal{O}_\infty$ be outer actions. Then $(A, \alpha)$ is strongly cocycle conjugate to $(A \otimes \mathcal{O}_\infty, \alpha \otimes \sigma)$. In particular, $\alpha$ has the Rohlin property.
Poly-$\mathbb{Z}$ groups of rank two

For a $\Gamma$-action $\alpha$, the first classification invariant is $KK(\alpha_g)$. When two actions $\alpha$ and $\beta$ satisfy $KK(\alpha_g) = KK(\beta_g)$, there exist homotopies $(\sigma_g(t))_{t \in [0,1]}$ in $\text{Aut}(A \otimes \mathbb{K})$ connecting $\alpha_g \otimes \text{id}_\mathbb{K}$ and $\beta_g \otimes \text{id}_\mathbb{K}$. For each $g, h \in \Gamma$, $(\sigma_g(t) \circ \sigma_h(t) \circ \sigma_{gh}(t)^{-1})_{t \in [0,1]}$ is a loop in $\text{Aut}(A \otimes \mathbb{K})_0$, which gives rise to a cohomology class

$$c(\alpha, \beta) \in H^2(\Gamma, \pi_1(\text{Aut}(A \otimes \mathbb{K})_0)) \cong H^2(\Gamma, KK^1(A, A)).$$

Theorem (M. Izumi and M)

Let $\Gamma$ be either $\mathbb{Z}^2$ or $\langle a, b \mid bab^{-1} = a^{-1} \rangle$. For outer actions $\alpha, \beta$ of $\Gamma$ on a unital Kirchberg algebra $A$, the following are equivalent.

1. $\alpha$ and $\beta$ are cocycle conjugate via $\gamma \in \text{Aut}(A)$ satisfying $KK(\gamma) = 1$.

2. $KK(\alpha_g) = KK(\beta_g)$ for all $g \in \Gamma$ and $c(\alpha, \beta) = 0$. 
$\mathbb{Z}$-actions on UHF algebras

**Theorem (A. Kishimoto 1995)**

Let $A$ be a UHF algebra and let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. Then $\alpha$ has the **Rohlin property**, i.e. for any $m \in \mathbb{N}$, there exist sequences of projections $(e_n)_n, (f_n)_n$ in $A$ such that

\[
\sum_{i=0}^{m-1} \alpha^i(e_n) + \sum_{j=0}^{m} \alpha^j(f_n) \to 1,
\]

\[[e_n, a] \to 0 \quad \text{and} \quad [f_n, a] \to 0 \quad \forall a \in A.\]

**Theorem (A. Kishimoto 1995)**

Let $A$ be a UHF algebra. All strongly outer $\mathbb{Z}$-actions on $A$ are strongly cocycle conjugate to each other.
**Theorem (H. Nakamura 1999)**

Let $A$ be a UHF algebra and let $\alpha : \mathbb{Z}^2 \curvearrowright A$ be a strongly outer action. Then $\alpha$ has the Rohlin property.

Since $\text{Aut}(A)$ is path-connected, for $\alpha, \beta : \mathbb{Z}^2 \curvearrowright A$, we can define a loop $(\sigma_g(t) \circ \sigma_h(t) \circ \sigma_{gh}(t)^{-1})_{t \in [0,1]}$ in $\text{Aut}(A)$ for each $g, h \in \mathbb{Z}^2$ as in the case of Kirchberg algebras, and obtain a cohomology class

$$c(\alpha, \beta) \in H^2(\mathbb{Z}^2, \pi_1(\text{Aut}(A))) \cong \pi_1(\text{Aut}(A)).$$

**Theorem (T. Katsura and M 2008)**

Let $A$ be a UHF algebra. Two strongly outer actions $\alpha, \beta : \mathbb{Z}^2 \curvearrowright A$ are strongly cocycle conjugate if and only if $c(\alpha, \beta) = 0$.

The fundamental group $\pi_1(\text{Aut}(A))$ is isomorphic to a (possibly infinite) direct product of finite cyclic groups (K. Thomsen 1987).
A UHF algebra $A$ is said to be of **infinite type** if $A \otimes A \cong A$.

**Theorem (M)**

Let $A$ be a UHF algebra of infinite type and let $\alpha : \mathbb{Z}^N \curvearrowright A$ be a strongly outer action. Then $\alpha$ has the Rohlin property.

**Theorem (M)**

Let $A$ be a UHF algebra of infinite type. Then, all strongly outer actions of $\mathbb{Z}^N$ on $A$ are mutually strongly cocycle conjugate to each other.

When $A$ is a UHF algebra of infinite type, it is known that the homotopy group $\pi_n(\text{Aut}(A))$ is trivial for every $n \geq 0$ (K. Thomsen 1987).
We can generalize Kishimoto's results for UHF algebras to certain AH algebras.

**Theorem (M 2010)**

Let $A$ be a unital simple AH algebra with slow dimension growth, real rank zero and finitely many extremal tracial states. Let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. If $\alpha_k$ is approximately inner for some $k \in \mathbb{N}$, then $\alpha$ has the Rohlin property.

**Theorem (M 2010)**

Let $A$ be a unital simple AH algebra with slow dimension growth and real rank zero. If two actions $\alpha, \beta : \mathbb{Z} \curvearrowright A$ have the Rohlin property and $\alpha_1 \circ \beta_{-1}$ is asymptotically inner, then $\alpha$ and $\beta$ are cocycle conjugate.
Uniqueness of asymptotically representable actions

Let $A$ be a unital simple AH algebra with real rank zero, slow dimension growth and finitely many extremal tracial states.

Theorem (M)

For an approximately representable action $\alpha : \mathbb{Z}^N \curvearrowright A$, the following are equivalent.

1. $\alpha$ is strongly outer.
2. $\alpha$ has the Rohlin property.

Theorem (M)

Let $\alpha, \beta : \mathbb{Z}^N \curvearrowright A$ be strongly outer, asymptotically representable actions. Then they are cocycle conjugate.
We denote the Jiang-Su algebra by $\mathcal{Z}$ and the universal UHF algebra by $Q$.

Let $\mathcal{C}$ be the set of all unital separable simple nuclear $\mathcal{Z}$-stable $C^*$-algebras $A$ such that $A \otimes Q$ has tracial rank zero.

**Theorem (W. Winter, H. Lin and Z. Niu 2008)**

*For $A, B \in \mathcal{C}$ satisfying the UCT, if there exists an ordered isomorphism $\varphi : K_*(A) \to K_*(B)$, then there exists an isomorphism $\Phi : A \to B$ inducing $\varphi$.***

A unital simple ASH algebra $A$ is $\mathcal{Z}$-stable if and only if $A$ has slow dimension growth (A. Toms and W. Winter).

If $A$ is a unital simple ASH algebra whose projections separate traces, then $A \otimes B$ has tracial rank zero for any UHF algebra $B$ (W. Winter 2007).
\( \mathbb{Z} \)-stability of crossed products

**Theorem (Y. Sato and M)**

Suppose that \( A \in \mathcal{C} \) satisfies the UCT and has finitely many extremal tracial states. Let \( \alpha : \Gamma \curvearrowright A \) be a strongly outer action. Assume either of the following.

1. \( \Gamma = \mathbb{Z} \).
2. \( \Gamma = \mathbb{Z}^N \) and \( T(A)^\alpha = T(A) \).
3. \( \Gamma \) is a finite group.

Then \( (A, \alpha) \) is strongly cocycle conjugate to \( (A \otimes \mathbb{Z}, \alpha \otimes \text{id}) \). In particular, \( A \rtimes_{\alpha} \Gamma \) is \( \mathbb{Z} \)-stable.

Note that \( \alpha \) may not have the Rohlin property, because \( A \) may not have rich projections. Instead we use the weak Rohlin property, in which projections are replaced with positive elements. By using this property, one can construct a unital embedding of \( \mathbb{Z} \) into \( (A^\infty \cap A')^\alpha \), which implies the conclusion.
Closedness of $\mathcal{C}$ under taking crossed products

Combining the theorem in the previous slide with the results of N. C. Phillips and A. Kishimoto, we get the following two theorems.

**Theorem (Y. Sato and M)**

Suppose that $A \in \mathcal{C}$ satisfies the UCT and has finitely many extremal tracial states. Let $\Gamma$ be a finite group and let $\alpha : \Gamma \curvearrowright A$ be a strongly outer action. Then $A \rtimes_{\alpha} \Gamma$ belongs to $\mathcal{C}$.

**Theorem (Y. Sato and M)**

Suppose that $A \in \mathcal{C}$ satisfies the UCT and has a unique tracial state. Let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. If $\alpha_{k}$ induces the identity on $K_{*}(A) \otimes \mathbb{Q}$ for some $k \in \mathbb{N}$, then $A \rtimes_{\alpha} \mathbb{Z}$ belongs to $\mathcal{C}$.
Z-actions and $\mathbb{Z}^2$-actions on $\mathcal{Z}$ (1/2)

Theorem (Y. Sato 2010)

All strongly outer $\mathbb{Z}$-actions on $\mathcal{Z}$ are strongly cocycle conjugate to each other.

Theorem (Y. Sato and M)

All strongly outer $\mathbb{Z}^2$-actions on $\mathcal{Z}$ are strongly cocycle conjugate to each other.

We sketch the proof.

Let $\alpha, \beta : \mathbb{Z}^2 \curvearrowright \mathcal{Z}$ be two strongly outer actions.

1. By the theorem mentioned before, we may replace $\alpha, \beta$ with $\alpha \otimes \text{id}, \beta \otimes \text{id} : \mathbb{Z}^2 \curvearrowright \mathcal{Z} \otimes \mathcal{Z}$.

2. We can find an ‘almost’ $\alpha$-cocycle $(u_g)_{g \in \mathbb{Z}^2} \subset \mathcal{Z}$ such that $\text{Ad} \ u_g \circ \alpha_g \approx \beta_g$ on a large finite subset of $\mathcal{Z}$. 
(3) $Z = \{ f : [0, 1] \to M_{2\infty} \otimes M_{3\infty} \mid f(0) \in M_{2\infty}, \ f(1) \in M_{3\infty} \}$ is a unital subalgebra of $\mathcal{Z}$ (M. Rørdam and W. Winter 2010).

(4) For the $\mathbb{Z}^2$-action $\alpha \otimes \text{id}$ on the UHF algebra $\mathcal{Z} \otimes M_{2\infty}$, the cohomology vanishing theorem is known. (T. Katsura and M 2008). Hence there exists a unitary $v_0 \in \mathcal{Z} \otimes M_{2\infty}$ such that $u_g \otimes 1 \approx v_0(\alpha_g \otimes \text{id})(v_0^*)$.

(5) In the same way, we obtain a unitary $v_1 \in \mathcal{Z} \otimes M_{3\infty}$ such that $u_g \otimes 1 \approx v_1(\alpha_g \otimes \text{id})(v_1^*)$.

(6) By modifying $v_0, v_1$ a little bit, we can find a path of unitaries $(v_t)_{t \in [0, 1]} \subset \mathcal{Z} \otimes M_{2\infty} \otimes M_{3\infty}$ connecting $v_0$ and $v_1$, without breaking the condition $u_g \otimes 1 \approx v_t(\alpha_g \otimes \text{id})(v_t^*)$.

(7) $\nu = (v_t)_t$ is regarded as an element of $\mathcal{Z}$. Therefore the ‘almost’ $(\alpha \otimes \text{id})$-cocycle $(u_g \otimes 1)_{g \in \mathbb{Z}^2}$ is approximated by $(\alpha \otimes \text{id})$-coboundaries in $\mathcal{Z} \otimes \mathcal{Z}$.

(8) Evans-Kishimoto intertwining argument completes the proof.
Open problems

- Classify outer actions of $\mathbb{Z}^N$ ($N \geq 3$) or poly-$\mathbb{Z}$ groups on unital Kirchberg algebras.
- Classify strongly outer actions of $\mathbb{Z}^N$ on a general UHF algebra when $N \geq 3$.
- Show the uniqueness of strongly outer actions of poly-$\mathbb{Z}$ groups on a UHF algebra of infinite type.
- Show the uniqueness of strongly outer $\mathbb{Z}^N$-actions on the Jiang-Su algebra $\mathcal{Z}$ when $N \geq 3$.
- Classify strongly outer $\mathbb{Z}^2$-actions on a unital simple AF algebra (as much as possible).
- Classify strongly outer $\mathbb{Z}$-actions on $A \in \mathcal{C}$ satisfying the UCT (as much as possible).
References