

Preliminaries: A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is *additive* if it satisfies the Cauchy equation

$$(CE) \quad f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

Cauchy asked under what conditions an additive function must be linear.

The following lemma is obvious.

Lemma. *If f is additive, then $f(qx) = qf(x)$ for all $q \in \mathbb{Q}$. Thus, if, in addition, f is continuous, then $f(x) = f(1)x$.*

If one is willing to use the axiom of choice, then one can construct a non-linear additive function as follows. By Zorn's Lemma, there exists a maximal $B \subseteq \mathbb{R}$ of numbers which are linearly independent over \mathbb{Q} . Thus, for each x , there exists a unique $\{a_b(x) : b \in B\} \subseteq \mathbb{Q}$ such that $a_b(x) = 0$ for all but a finite number of $b \in B$ and $x = \sum_{b \in B} a_b(x)b$. Clearly, for each $b \in B$, $x \rightsquigarrow a_b(x)$ is additive. On the other hand, because $a_b(x) \in \mathbb{Q}$ for all x , it is obvious that a_b is not linear.

Theorem. *If f is a Lebesgue measurable, additive function, then f is linear.*

Proof. Choose $R > 0$ so that $\{x : |f(x)| \leq R\}$ has positive Lebesgue measure. Then, Vitalli guarantees that there is a $\delta > 0$ such that $[-\delta, \delta] \subseteq \{y - x : |f(x)| \vee |f(y)| \leq R\}$. Hence, $|f(x)| \leq 2R$ if $|x| \leq \delta$. Given any $x \neq 0$, choose $q \in \mathbb{Q}$ so that $\frac{|x|}{\delta} \leq q \leq \frac{2|x|}{\delta}$. Then $x' \equiv \frac{x}{q} \in [-\delta, \delta]$ and so $|f(x)| = q|f(x')| \leq \frac{4R|x|}{\delta}$. More generally, $|f(y) - f(x)| = |f(y - x)| \leq \frac{4R|y-x|}{\delta}$, and so f is continuous. \square

Another Approach

Lemma. *If f is a Lebesgue measurable, additive function that is locally integrable, then $f(x) = xf(1)$.*

Proof. It suffices to show that f is continuous. To this end, let $\rho : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function with compact support and integral 1. Then

$$\rho * f(x) = \int f(t)\rho(x-t) dt = f(x) + \rho * f(0),$$

and so $f = \rho * f - \rho * (0)$ is smooth. \square

Given a Borel probability measure μ on \mathbb{R} , define μ_α for $\alpha \in \mathbb{R}$ to be the distribution of $x \in \mathbb{R} \mapsto \alpha x$ under μ . That is, $\int f d\mu_\alpha = \int f(\alpha x) \mu(dx)$.

Say that $(\alpha, \beta) \in (0, 1)$ is a *Pythagorean pair* if $\alpha^2 + \beta^2 = 1$, and let γ be the standard Gauss measure on \mathbb{R} . That is,

$$\gamma(dx) = (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} dx \text{ and } \hat{\gamma}(\xi) = e^{-\frac{\xi^2}{2}}.$$

Lemma. *If (α, β) is a Pythagorean pair, then $\mu = \mu_\alpha * \mu_\beta$ if and only if $\mu = \gamma_\sigma$ for some $\sigma \geq 0$, in which case $\int x^2 \mu(dx) = \sigma^2$.*

Proof. Since $\hat{\gamma}_\sigma(\xi) = e^{-\frac{\sigma^2 \xi^2}{2}}$, the “if” assertion is trivial. Thus, assume that $\mu = \mu_\alpha * \mu_\beta$.

I begin by proving the last statement under the assumption that μ is symmetric. By assumption, $\hat{\mu}(\xi) = \hat{\mu}(\alpha\xi)\hat{\mu}(\beta\xi)$. Thus, by induction on $n \geq 1$,

$$\hat{\mu}(\xi) = \prod_{m=0}^{n-1} \hat{\mu}(\alpha^m \beta^{n-m} \xi)^{\binom{n}{m}}.$$

Because μ is symmetric, $\hat{\mu}(\xi) = \int \cos(\xi x) \mu(dx)$, and therefore

$$\begin{aligned} & -\log(\hat{\mu}(1)) \\ &= -\sum_{m=0}^{n-1} \binom{n}{m} \log \left(\int \cos(\alpha^m \beta^{n-m} x) \mu(dx) \right). \end{aligned}$$

Since $-\log t \geq 1 - t$ for $t \in [0, 1]$, this means that

$$\begin{aligned} & -\log(\hat{\mu}(1)) \\ & \geq \int \left(\sum_{m=0}^n \binom{n}{m} (1 - \cos(\alpha^m \beta^{n-m} x)) \right) \mu(dx), \end{aligned}$$

and, because

$$\sum_{m=0}^n \binom{n}{m} (1 - \cos(\alpha^m \beta^{n-m} x)) \longrightarrow \frac{x^2}{2},$$

Fatou's Lemma guarantees that

$$\int x^2 \mu(dx) \leq -2 \log(\hat{\mu}(1)).$$

To remove the symmetry assumption, set $\nu = \mu * \mu_{-1}$. Then ν is symmetric and $\nu = \nu_\alpha * \nu_\beta$. Thus, $\int x^2 \nu(dx) < \infty$. Now choose a median a of μ . That is,

$$\mu([a, \infty)) \wedge \mu((-\infty, a]) \geq \frac{1}{2}.$$

Then

$$\mu(\{x : |x - a| \geq t\}) \leq 2\nu(\{x : |x| \geq t\}),$$

and so

$$\begin{aligned} \int x^2 \mu(dx) &\leq 2a^2 + 2 \int |x - a|^2 \mu(dx) \\ &\leq 2a^2 + 4 \int x^2 \nu(dx) < \infty. \end{aligned}$$

Given that μ has a finite second moment σ^2 , note that

$$\int x \mu(x) = (\alpha + \beta) \int x \mu(dx),$$

and therefore, since $\alpha + \beta > 1$, $\int x \mu(dx) = 0$. Thus,

$$\hat{\mu}(\eta) = 1 - \frac{\sigma^2 \eta^2}{2} (1 + o(1)) \quad \text{as } \eta \rightarrow 0,$$

and so, since

$$\hat{\mu}(\xi) = \prod_{m=0}^n \hat{\mu}(\alpha^m \beta^{n-m} \xi)^{\binom{n}{m}},$$

one has that

$$\log(\hat{\mu}(\xi)) = -\frac{\sigma^2 \xi^2}{2}. \quad \square$$

Now suppose that f is a Lebesgue measurable, additive function, and let μ be the distribution of f under γ :

$$\int \varphi \circ f d\gamma = \int \varphi d\mu.$$

Take $\alpha = \frac{3}{5}$ and $\beta = \frac{4}{5}$. Then

$$\begin{aligned} & \hat{\mu}(\alpha\xi)\hat{\mu}(\beta\xi) \\ &= \iint \exp\left(i\xi(f(\alpha x) + f(\beta y))\right) \gamma(dx)\gamma(dy) \\ &= \iint e^{i\xi f(\alpha x + \beta y)} \gamma(dx)\gamma(dy) = \hat{\mu}(\xi). \end{aligned}$$

Hence

$$\int f(x)^2 \gamma(dx) = \int x^2 \mu(dx) < \infty,$$

and so f is locally integrable and therefore linear.

This approach has several advantages. For example, with hardly any change in the argument, one can use it to show that if f is a Lebesgue measurable function with the property that

$$f(x + y) = f(x) + f(y) \quad \text{for a.e. } (x, y) \in \mathbb{R}^2,$$

then there is an $a \in \mathbb{R}$ such that $f(x) = ax$ for a.e. $x \in \mathbb{R}$. Indeed, using Fubini's theorem and the translation invariance of Lebesgue measure, one can show first that $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for a.e. $(x, y) \in \mathbb{R}^2$ when α and β are rational. One then proceeds as before to show that f is locally integrable and is therefore equal to $\tilde{f} \equiv \rho * f - c$ a.e. for some $c \in \mathbb{R}$.

Finally, because \tilde{f} is a continuous and equal a.e. to f , it must be a continuous, additive function and therefore satisfy $\tilde{f}(x) = x\tilde{f}(1)$.

Infinite Dimensions

Suppose that E and F are a pair of Banach spaces over \mathbb{R} and that $\Phi : E \longrightarrow F$ is an additive, Borel measurable function. Then, for each $x \in E$ and $y^* \in F^*$, $t \mapsto \langle \Phi(tx), y^* \rangle$ is an \mathbb{R} -valued, Borel measurable, additive function on \mathbb{R} and is therefore linear. Hence, Φ is a Borel measurable, linear function on E , and one can ask whether it is continuous. As we will now show, the answer is yes.

When E and F are separable, one can use Laurent Schwartz's Borel graph theorem to prove this. Namely, his theorem says that if E and F are separable Banach spaces and $\Phi : E \longrightarrow F$ is a linear map whose graph is Borel measurable, then Φ is continuous. His proof is a *tour de force* based on deep properties of Polish spaces, and using those properties it is easy to show

that Φ is Borel measurable if and only if its graph is. Indeed, a Borel measurable, one-to-one map from one Polish space to another takes Borel sets to Borel sets. Applying this fact to the map $x \in E \mapsto (x, \Phi(x)) \in E \times F$, one sees that the graph $G(\Phi)$ of Γ is Borel measurable if Φ is. Conversely, if $G(\Phi)$ is Borel measurable and π_E and π_F are the natural projection maps of $E \times F$ onto E and F , then $\pi_E \upharpoonright G(\Phi)$ is a one-to-one, Borel measurable map and, as such, its inverse is Borel measurable. Since $\Phi = \pi_F \circ (\pi_E \upharpoonright G(\Phi))^{-1}$, this shows that Φ is Borel measurable.

To prove this result by the technique used earlier, we need to introduce a Gaussian measure on E . For this purpose, take $\mathbb{P} = \gamma^{\mathbb{Z}^+}$ on $\Omega \equiv \mathbb{R}^{\mathbb{Z}^+}$. Given a sequence $\{x_n : n \geq 1\} \subseteq E$ set

$$S_n(\omega) = \sum_{m=1}^n \omega_m x_m \quad \text{for } n \in \mathbb{Z}^+ \text{ and } \omega \in \Omega,$$

$$A \equiv \left\{ \omega : \lim_{n \rightarrow \infty} S_n(\omega) \text{ exists in } E \right\}$$

and

$$S(\omega) = \begin{cases} \lim_{n \rightarrow \infty} S_n(\omega) & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

Since

$$\mathbb{E}^{\mathbb{P}} \left[\sum_{m=1}^{\infty} |\omega_m| \|x_m\|_E \right] = \sqrt{\frac{2}{\pi}} \sum_{m=1}^{\infty} \|x_m\|_E,$$

$\mathbb{P}(A) = 1$ if $\sum_{m=1}^{\infty} \|x_m\|_E < \infty$.

Lemma. *If $f : E \rightarrow \mathbb{R}$ is a Borel measurable, linear map, then*

$$f(x_m)^2 \leq \mathbb{E}^{\mathbb{P}} [(f \circ S)^2] < \infty \text{ for all } m \geq 1.$$

Proof. Since, for every $\omega \in \Omega$, $S(\omega)$ is an element of the closed linear span of $\{x_n : n \geq 1\}$, I will, without loss in generality, assume that E

is separable and therefore that $\mathcal{B}_{E^2} = \mathcal{B}_E \times \mathcal{B}_E$. In particular, this means that the map $(x, y) \in E^2 \mapsto \frac{x+y}{\sqrt{2}} \in E$ is $\mathcal{B}_E \times \mathcal{B}_E$ -measurable.

Next note that

$$\frac{f \circ S(\omega^1) + f \circ S(\omega^2)}{\sqrt{2}} = f \left(\frac{S(\omega^1) + S(\omega^2)}{\sqrt{2}} \right)$$

for $(\omega^1, \omega^2) \in A^2$, and therefore the distribution μ of $f \circ S$ under \mathbb{P} is γ_σ for some $\sigma \geq 0$. Hence, $\mathbb{E}^\mathbb{P} [(f \circ S)^2] = \sigma^2 < \infty$.

To complete the proof, let $m \in \mathbb{Z}^+$ be given, and define $\omega \rightsquigarrow S^{(m)}(\omega)$ relative to the sequence $\{(1 - \delta_{m,n})x_n : n \geq 1\}$. Then $S^{(m)}(\omega)$ is \mathbb{P} -independent of ω_m , and

$$S(\omega) = \omega_m x_m + S^{(m)}(\omega) \text{ for } \omega \in A.$$

Hence

$$\mathbb{E}^\mathbb{P} [(f \circ S)^2] = f(x_m)^2 + \mathbb{E}^\mathbb{P} [(f \circ S^{(m)})^2] \geq f(x_m)^2. \quad \square$$

Theorem. *If $f : E \longrightarrow \mathbb{R}$ is a Borel measurable, additive map, then there exists an $x^* \in E^*$ such that $f(x) = \langle x, x^* \rangle$.*

Proof. We know that f is linear. Suppose it were not continuous. We could then find $\{x_n : n \geq 1\}$ such that $\|x_n\|_E \leq \frac{1}{n^2}$ and $|f(x_n)| \geq n$. Now define $\omega \rightsquigarrow S(\omega)$ accordingly, and thereby get the contradiction that

$$m^2 \leq |f(x_m)|^2 \leq \mathbb{E}^{\mathbb{P}} [(f \circ S)^2] < \infty \text{ for all } m \geq 1. \quad \square$$

Corollary. *Suppose that $\Phi : E \longrightarrow F$ is a additive map with the property that $x \rightsquigarrow \langle \Phi(x), y^* \rangle$ is Borel measurable for each $y^* \in F^*$. Then Φ is continuous.*

Proof. By the closed graph theorem, it suffices to show that $G(\Phi)$ is closed.

By the preceding, we know that $x \rightsquigarrow \langle \Phi(x), y^* \rangle$ is continuous for each $y^* \in F^*$. Now suppose

that $\{x_n : n \geq 1\} \subseteq E$ and that $(x_n, \Phi(x_n)) \longrightarrow (x, y) \in E \times F$. Then, for each $y^* \in F^*$,

$$\langle y, y^* \rangle = \lim_{n \rightarrow \infty} \langle \Phi(x_n), y^* \rangle = \langle \Phi(x), y^* \rangle,$$

and so $y = \Phi(x)$.

A Concluding Comment:

It may be of some interest that, without using the closed graph theorem, one can use the same technique to prove that Borel measurable, linear maps between Banach spaces are continuous. However, in order to do so, one needs a beautiful theorem of X. Fernique which says that if μ is a Borel probability measure on a separable Banach space E and the distribution of

$$(x_1, x_2) \rightsquigarrow \left(\frac{x_1 + x_2}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}} \right)$$

under μ^2 is μ^2 , then

$$\mathbb{E}^\mu \left[e^{\alpha \|x\|_E^2} \right] < \infty$$

for some $\alpha > 0$. In particular, $E^\mu [\|x\|_E^2] < \infty$. Fernique's proof is a remarkably elementary but diabolically clever application of nothing more than the triangle inequality. Given his result, the proof that a Borel measurable, linear map from one Banach space to another must be continuous differs in no substantive way from the one given above when the image space is \mathbb{R} . In the case when the image Banach space is separable, this provides another proof of the Schwartz's Borel graph theorem and therefore the closed graph theorem.