Some Applications of Gaussian Measures to Analysis

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Given a probability measure μ on $\mathbb{R}$ and a number $\sigma \in \mathbb{R}$, define μ_σ to be the distribution of $x \rightsquigarrow \sigma x$ under μ. Let γ be the standard Gaussian measure given by

$$\gamma(dx) = (2\pi)^{-\frac{1}{2}}\, e^{-\frac{x^2}{2}}\, dx \qquad \text{on } \mathbb{R}.$$

If α and β are positive numbers that satisfy $\alpha^2 + \beta^2 = 1$, then $\mu = \mu_\alpha * \mu_\beta$ if and only if $\mu = \gamma_\sigma$ for some $\sigma \geq 0$. This elementary fact, for which I will give a proof, has interesting applications to analysis. As a preliminary example, I will show that it allows one to prove that the only Lebesgue measurable solutions f to Cauchy's equation

$$f(x+y) = f(x) + f(y)$$

are linear. I will then apply it to give a proof that if Φ is a Borel measurable, linear map from one Banach space to another, then Φ must be continuous, a result that is equivalent to Laurent Schwartz's Borel graph theorem in the case when the Banach spaces are separable.