Endomorphisms - old and not so old

Joachim Cuntz

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SIMILARITY OF OPERATOR ALGEBRAS

BY

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Dedicated to M. H. Stone on his seventy fifth birthday

1. Introduction

When viewed in a certain light, Tomita's theorem (the main result of the Tomita–Takesaki theory—see [3, 14, 15, 16, 17]) appears as the combination of a result on "unbounded" similarity between self-adjoint operator algebras and the special structure of a von Neumann algebra and its commutant relative to a joint separating vector. The main purpose of this article is to introduce and develop the theory of such similarities. (See section 3.) Our secondary purpose is to present a full proof of Tomita's theorem in the style mentioned. (See section 4.) In connection with this argument, we develop a new density result (Theorem 4.10). In section 2 we prove a bounded similarity result.

The author is indebted to the Centre Universitaire de Marseille-Luminy, the University of Newcastle and the Zentrum für interdiziplinaire Forschung Universität Bielefeld for their hospitality during various stages of this work and to J. Ringrose, M. Takesaki & A. Van Daele, for important insights into the Tomita–Takesaki theory. Thanks are due to the NSF (USA) and SRC (UK) for partial support.

2. Bounded similarity

If $\mathcal{H}$ is a complex Hilbert space and H is an operator on $\mathcal{H}$ such that $0 < aI \leqslant H \leqslant bI$, then H is bounded and sp (H), the spectrum of H, lies in $[a, b]$. In addition, H has an inverse with spectrum in $[b^{-1}, a^{-1}]$. If $\varphi(T) = HTH^{-1}$ for T in $\mathcal{B}(\mathcal{H})$, then φ is a bounded operator on $\mathcal{B}(\mathcal{H})$ and sp (φ) (relative to $\mathcal{B}(\mathcal{B}(\mathcal{H}))$) is contained in $[ab^{-1}, a^{-1}b]$. To see this, note that left multiplication by H on $\mathcal{B}(\mathcal{H})$ has the same spectrum as H, that right multiplication by H^{-1} has the same spectrum as H^{-1}, and that these two multiplications commute.

We employ the Banach-algebra-valued, holomorphic function calculus (see, for example, [1; Chapter VII]) to discuss holomorphic functions f of an element A of a Banach

A unique, even bizarre, institution in West Philadelphia since 1950, the Divine Tracy is up for sale. The asking price for the 140-room hotel is $10 million.

In a city where hotels boast what makes them luxurious – from the thread count in the bed sheets to spa services for pets – the Divine Tracy stands out for the niche it serves. For a flat $50 a night rate, guests can enjoy austere but consistent accommodations in which men and women are housed on separate floors.

Guests must adhere to a so-called International Modest Code, which was developed by Father Divine, the spiritual leader of a Christian-based ministry called the Palace Mission.

The code, which sets high standards for behavior, provides that guest do not smoke, drink alcohol, use obscenities, vulgarity, profanity, receive gifts, presents, tips or bribes. There is no eating food in the rooms and the dress code bespeaks of modesty. Women are not permitted to wear pants, shorts or miniskirts; men must not don sleeveless shirts, have their shirts untucked or wear shoes sans socks. In addition, there must be no "undue mixing of the sexes" but men and women may converse in the lobby. Rooms do not have televisions.

(Note: At the time the rate was $50 for an entire week.)

TECHNISCHE UNIVERSITÄT BERLIN

SIMPLE C^*-ALGEBRAS GENERATED

BY ISOMETRIES

Joachim Cuntz

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FACHBEREICH 3

The C*-algebra $\mathcal{A}[\alpha]$

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Description of $\mathcal{A}[\alpha]$ in the Fourier transform picture

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This dual construction of $\mathcal{A}[\varphi]$ can be generalized to an endomorphism φ of a not necessarily abelian discrete group G satisfying

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We will however consider only the case where G is abelian (and φ satisfies the three conditions above).

Under these conditions the algebra $\mathcal{A}[\alpha] = \mathcal{A}[\varphi]$ always contains a canonical commutative subalgebra D which is a Cartan subalgebra. It is generated by the translates $\lambda_g s_\varphi^n s_\varphi^{\star n} \lambda_g^\star$ of the range projections $s_\varphi^n s_\varphi^{\star n}$. The spectrum of D is the φ-adic completion $\bar{G} = \varprojlim G/\varphi^n G$ of G. This completion is actually a compact abelian group.

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Theorem (Cuntz-Vershik) $\mathcal{A}[\alpha]$ is simple and purely infinite. It can be described as a universal C*-algebra with a natural set of generators and relations (as a consequence $\mathcal{A}[\alpha]$ is also isomorphic to a crossed product $B \rtimes \mathbb{N}$).

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The K-theory of $\mathcal{A}(\alpha) = \mathcal{A}(\varphi)$ fits into an exact sequence of the form

$$K_* C^*(G) \xrightarrow{1-b(\varphi)} K_* C^*(G) \longrightarrow K_* \mathcal{A}[\varphi] \longrightarrow K_* C^*(G)$$

where the map $b(\varphi)$ satisfies the equation $b(\varphi)\varphi_* = N(\varphi)$ with $N(\varphi) = |G/\varphi G| = |\mathrm{Ker}\,\alpha|$.

Proof of the K-theory formula

We have seen that $D = \varinjlim C(G/\varphi^n G)$ and $B = D \rtimes G$. This gives a representation of B as an inductive limit of the algebras $C(G/\varphi^n G) \rtimes G$ and a corresponding inductive limit description of $K_* B$ (with connecting map $b(\varphi)$ at each level). The semigroup $\mathbb{N}$ acts on B by the endomorphism φ and $\mathcal{A}[\varphi] \cong B \rtimes \mathbb{N}$. Applying the Pimsner-Voiculescu sequence and using the fact that φ acts as the shift in the inductive limit representation of $K_* B$ gives the result.

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Example Let $K = (\mathbb{Z}/n)^{\mathbb{N}}$ and α the one-sided shift. Then $\mathcal{A}[\alpha] \cong \mathcal{O}_n$ and the formula for the K-theory gives the expected result.

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Remark Let G be a discrete amenable group and φ an injective endomorphism such that $\bigcap \varphi^n(G) = \{e\}$, but for which $G/\varphi G$ is infinite. Then $\mathcal{A}[\varphi]$ "looks like" $\mathcal{O}_\infty \otimes C^*G$ but is simple and purely infinite. Moreover in this case $K_*(\mathcal{A}[\varphi]) \cong K_*(C^*G)$ (Felipe Vieira).

Example Let α be an endomorphism of $K = \mathbb{T}^n$ and $\varphi = \hat{\alpha}$ the dual endomorphism of $G = \mathbb{Z}^n$. We assume that $\det \varphi \neq 0$.
We know that there is an isomorphism of $K_*(C(\mathbb{T}^n))$ with the exterior algebra $\Lambda^*\mathbb{Z}^n = \bigoplus_{p=0}^n \Lambda^p\mathbb{Z}$, preserving the grading (and the exterior product). The endomorphism φ_* of $K_*(C(\mathbb{T}^n))$ induced by φ corresponds to the endomorphism $\Lambda\varphi$ of $\Lambda^*\mathbb{Z}^n$.
The associated endomorphism $b(\varphi)$ of $\Lambda^*\mathbb{Z}^n$ is determined by the formula $b(\varphi)\varphi_* = N(\varphi)\,\mathrm{id}$. In the present case we have $N(\varphi) = |\det\varphi|$. Now, the unique solution b (in endomorphisms of $\Lambda\mathbb{Z}^n$) for the equation $b\,\Lambda\varphi = |\det\varphi|\,\mathrm{id}$ corresponds under the Poincaré isomorphism $D : \Lambda G \cong \Lambda G'$ to $\mathrm{sgn}(\det\varphi)\Lambda\varphi'$ (here we write G' for the algebraic dual $\mathrm{Hom}\,(G, \mathbb{Z})$ and denote by φ' the endomorphism of G' which is dual to φ. The restriction of b to $\Lambda^1\mathbb{Z}^n \cong \mathbb{Z}^n$ for instance is the complementary matrix to φ determined by Cramer's rule. Thus we obtain

$$K_*\mathcal{A}[\alpha] \cong \Lambda G/(1 - D\Lambda\varphi' D^{-1})\Lambda G \;\oplus\; \mathrm{Ker}\,(1 - D\Lambda\varphi' D^{-1})$$

where the first term has the natural even/odd grading. The second term $\mathrm{Ker}\,(1 - D\Lambda\varphi' D^{-1})$ is $\Lambda^n\mathbb{Z}^n \cong \mathbb{Z}$ if $\det\varphi > 0$ and $\{0\}$ if $\det\varphi < 0$. It contributes to K_0 if n is odd and to K_1 if n is even.

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This result has been obtained independently by Exel-an Huef-Raeburn using a rather different approach.

Example Consider the solenoid group

$$K = \varprojlim_{p} \mathbb{T} \qquad G = \mathbb{Z}[\frac{1}{p}]$$

with the endomorphism φ_q determined on G by $\varphi_q(x) = qx$ (q prime to p). The description of G as an inductive limit of groups of the form $\mathbb{Z}$ immediately leads to the formulas

$$K_1(C^*G) = \mathbb{Z}[\frac{1}{p}] \qquad K_0(C^*G) = \mathbb{Z}$$

Now φ_q acts as id on $K_0(C^*G)$ and by multiplication by q on $K_1(C^*G)$. Since $N(\varphi_q) = q$, we get that $b(\varphi) = q\,\mathrm{id}$ on $K_0(C^*G)$ and $b(\varphi) = \mathrm{id}$ on $K_1(C^*G)$. Thus the exact sequence shows that

$$K_0(\mathcal{A}[\varphi]) = \mathbb{Z}/(q-1) + \mathbb{Z}[\frac{1}{p}] \qquad K_1(\mathcal{A}[\varphi]) = \mathbb{Z}[\frac{1}{p}]$$

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Example Another interesting generalization of $\mathcal{O}_p$, p prime, arises as follows: Consider $G = (\mathbb{Z}/p)[t]$. This is actually not only an abelian group but a ring. Multiplication by a non-zero element x gives an injective endomorphism of G. For $x = t$ we obtain $\mathcal{A}[\varphi] \cong \mathcal{O}_p$. Partial computations concerning the K-theory of $\mathcal{A}[\varphi]$ for more general X have been made by Cuntz-Li.