QUANTUM SPACETIME and PLANCK SCALES

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Introduction

QST, Quantum Minkowski Space, QFT

QST and Cosmology

Introduction

QM finitely many d. o. f.

$$\Delta q \Delta p \gtrsim \hbar$$

positions = observables, dual to momentum;

exactly implemented by the Born - Heisenberg relations

$$pq - qp = -i\hbar I.$$

NONCOMMUTATIVITY is the root of QM. In **QFT**, **local observables**:

$$A \in \mathfrak{A}(\mathcal{O});$$

$\mathcal{O}$ (*double cones*) - spacetime specifications, in terms of coordinates - accessible through measurements of local observables. **Allows to formulate LOCALITY**:

$$AB = BA$$

whenever

$$A \in \mathfrak{A}(\mathcal{O}_1), B \in \mathfrak{A}(\mathcal{O}_2,),$$

and

$$\mathcal{O}_1, \mathcal{O}_2$$

are spacelike separated.

Born - Heisenberg relations are governed by the C* Algebra of all compact operators on the separable infinite dim Hilbert Space.

Though being a commutativity condition, Locality makes the overall C* of local observables, which has to be **simple**, much more **radically noncommutative**.

Dick has been a pioneer and a master of Noncommutative C* algebras since the beginning of this field; at the same time he devoted deep love and thoughts to Mathematical Physics; a tribute to the Leader and to the unforgettable Friend based on this subject is maybe not out of place.

Locality is OK with **experiments** at all accessible scales; **theory**: in QFT it is OK at all scales, if we neglect GRAVITATIONAL FORCES BETWEEN ELEMENTARY PARTICLES.

If we DON'T we are led to a further step in **noncommutativity**:

Heisenberg: localizing an event in a small region costs energy **(QM)**;

Einstein: energy generates a gravitational field **(CGR)**.

QM + CGR:

PRINCIPLE OF *Gravitational Stability against localization of events* [DFR, 1994, 95]:

The gravitational field generated by the concentration of energy required by the Heisenberg Uncertainty Principle to localize an event in spacetime

should not be so strong to hide the event itself to any distant observer - distant compared to the Planck scale.

Spherically symmetric localization in space with accuracy a: an uncontrollable energy E of order $1/a$ has to be transferred (use universal units where $\hbar = c = G = 1$)

Schwarzschild radius $R \simeq E + U$

if U is the energy already present at the observed spot, in a background spherically symmetric quantum state,

Hence we *must* have that

$$a \gtrsim R \simeq 1/a + U;$$

so that if U is much smaller than 1

$$a \gtrsim 1,$$

i.e. in CGS units

$$a \gtrsim \lambda_P \simeq 1.6 \cdot 10^{-33} cm. \tag{1}$$

if U is much larger than 1,

$$a \gtrsim U,$$

and the "minimal distance" is **dynamical**, **the Effective Planck Length**, which **might diverge**.

Quantum Spacetime can $solve$ the Horizon Problem: divergent Effective Planck Length means instant long range (a causal) correlations, allowing establishment of thermal equilibrium [DMP 2013].

But at $t = 0$ all points instantly connected to one another: a single point. Degrees of freedom collapsing to zero.

An indication in this direction:

fields at a (quantum) point and **interactions vanish at** $t \to 0$ i.e. as $\lambda_{eff} \to \infty$.

(Morsella, Pinamonti, - ; in preparation; Comments at the end).

Neglecting U but no spherical symmetry:

if we measure **one or at most two** space coordinates with great precision a,

but the uncertainty L in another coordinates is **large**, the energy $1/a$ may spread over a region of size L, and

generate a gravitational potential that **vanishes everywhere** as $L \to \infty$

(provided a, as small as we like but non zero, remains constant).

This indicates that the Δq^μ must satisfy **UNCERTAINTY RELATIONS.**

Should be implemented by **commutation relations**.

QUANTUM SPACETIME.

Dependence of Uncertainty Relations, hence of Commutators between coordinates, upon background quantum state i.e. **upon metric tensor.**

CGR: **Geometry** $\sim$ **Dynamics**

QG: **Algebra** $\sim$ **Dynamics**

QST, Quantum Minkowski Space, QFT

Remember: if A, B are self adjoint elements of a C* algbra and ω a state, setting

$$AB - BA = iC$$

we have

$$\Delta_\omega A \cdot \Delta_\omega B \gtrsim (1/2)|\omega(C)|.$$

The Principle of *Gravitational Stability against localization of events* implies :

$$\Delta q_0 \cdot \sum_{j=1}^{3} \Delta q_j \gtrsim 1; \sum_{1 \leq j < k \leq 3} \Delta q_j \Delta q_k \gtrsim 1. \qquad (2)$$

[DFR 1994 - 95]. STUR must be implemented by SPACETIME commutation relations

$$[q_\mu, q_\nu] = i\lambda_P^2 Q_{\mu\nu} \tag{3}$$

imposing **Quantum Conditions** on the $Q_{\mu\nu}$.

SIMPLEST solution:

$$[q^\mu, Q^{\nu,\lambda}] = 0; \tag{4}$$
$$Q_{\mu\nu} Q^{\mu\nu} = 0; \tag{5}$$
$$((1/2)\,[q_0, \ldots, q_3])^2 = I, \tag{6}$$

where $Q_{\mu\nu}Q^{\mu\nu}$ is a scalar, and

$$
\begin{aligned}
[q_0, \ldots, q_3] &\equiv \det \begin{pmatrix} q_0 & \cdots & q_3 \\ \vdots & \ddots & \vdots \\ q_0 & \cdots & q_3 \end{pmatrix} \\
&\equiv \varepsilon^{\mu\nu\lambda\rho} q_\mu q_\nu q_\lambda q_\rho = \\
&= -(1/2) Q_{\mu\nu} (*Q)^{\mu\nu}
\end{aligned} \tag{7}
$$

is a pseudoscalar, hence we use the square in the Quantum Conditions.

Basic model of Quantum Spacetime; **implements exactly Space Time Uncertainty Relations** and is fully **Poincaré covariant**.

The *classical* **Poincaré group acts as symmetries**; translations, in particular, act adding to each q_μ a real multiple of the identity.

The *noncommutative* C* algebra of Quantum Space-time can be associated to the above relations. The procedure [DFR] applies to more general cases.

Assuming that the $q_\lambda, Q_{\mu\nu}$ are selfadjoint operators and that the $Q_{\mu\nu}$ commute *strongly* with one another and with the q_λ, the relations above can be seen as a bundle of Lie Algebra relations based on the joint spectrum of the $Q_{\mu\nu}$.

Regular representations are described by representations of the C* group algebra of the unique simply connected Lie group associated to the corresponding Lie algebra.

The C* algebra of Quantum Spacetime is the C* algebra of a continuos field of group C* algebras based on the spectrum of a commutative C* algebra.

In our case, that spectrum - the joint spectrum of the $Q_{\mu\nu}$ - is the manifold Σ of the real valued antisymmetric 2 - tensors fulfilling the same relations as the $Q_{\mu\nu}$ do: a homogeneous space of the proper orthocronous Lorentz group, identified with the coset space of $SL(2, C)$ mod the subgroup of diagonal matrices. Each of those tensors, can be taken to its rest frame, where the electric and magnetic parts $\mathbf{e}$, $\mathbf{m}$ are **parallel unit vectors**, by a boost, and go back with the inverse boost, specified by **a third vector, orthogonal to those unit vectors**; thus Σ can be viewed as the tangent bundle to two copies of the unit sphere in 3 space - its **base** Σ_1.

Irreducible representations at a point of Σ_1: **Shroedinger p, q in 2 d. o. f.**.

The fibers, with the condition that I is not an independent generator but is represented by I, are the C* algebras of the Heisenberg relations in 2 degrees of freedom - the algebra of all compact operators on a fixed infinite dimensional separable Hilbert space.

The continuos field can be shown to be trivial. Thus the C* algebra $\mathcal{E}$ of Quantum Spacetime is identified with the tensor product of the continuous functions vanishing at infinity on Σ an the algebra of compact operators.

The mathematical generalization of points are pure states.

Optimally localized states: those minimizing

$$\Sigma_\mu (\Delta_\omega q_\mu)^2;$$

minimum $= 2$, reached by states concentrated on Σ_1, at each point **ground state of harmonic oscillator**.

(Given by an **optimal localization map** composed with a probability measure on Σ_1).

But to explore more thoroughly the Quantum Geometry of Quantum Spacetime we must consider $independent$ $events$.

Quantum mechanically n independent events ought to be described by the $n - fold$ tensor product of $\mathcal{E}$ with itself; considering arbitrary values on n we are led to use the direct sum over all n.

If A is the C* algebra with unit over $\mathbb{C}$, obtained adding the unit to $\mathcal{E}$, we will view the n-fold tensor power $\Lambda_n(A)$

of A over $\mathbb{C}$ as an A-bimodule with the product in A,

$$a(a_1 \otimes a_2 \otimes ... \otimes a_n) = (aa_1) \otimes a_2 \otimes ... \otimes a_n;$$
$$(a_1 \otimes a_2 \otimes ... \otimes a_n)a = a_1 \otimes a_2 \otimes ... \otimes (a_n a);$$

and the direct sum

$$\Lambda(A) = \bigoplus_{n=0}^{\infty} \Lambda_n(A)$$

as the A-bimodule tensor algebra,

$$(a_1 \otimes a_2 \otimes ... \otimes a_n)(b_1 \otimes b_2 \otimes ... \otimes b_m) = a_1 \otimes a_2 \otimes ... \otimes (a_n b_1) \otimes b_2 \otimes ... \otimes b_m.$$

This is the natural ambient for the *universal differential calculus*, where the differential is given by

$$d(a_0 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n} (-1)^k a_0 \otimes \cdots \otimes a_{k-1} \otimes I \otimes a_\otimes \cdots \otimes a_n.$$

As usual d is a **graded differential**, i.e., if $\phi \in \Lambda(A), \psi \in \Lambda_n(A)$, we have

$$d^2 = 0;$$
$$d(\phi \cdot \psi) = (d\phi) \cdot \psi + (-1)^n \phi \cdot d\psi.$$

Note that $A = \Lambda_1(A) \subset \Lambda(A)$, and the d-stable subalgebra $\Omega(A)$ of $\Lambda(A)$ generated by A is the *universal differential algebra*. In other words, it is the subalgebra generated by A and

$$da = I \otimes a - a \otimes I$$

as a varies in A.

A curiosity: If τ is a faithful trace on A defined on a two sided ideal J, relative to the the pairing

$$< a_1 \otimes a_2 \otimes ... \otimes a_n, b_1 \otimes b_2 \otimes ... \otimes b_m >= \delta_{n,m} \tau(a_1 b_1 ... a_n b_n)$$

(where at least one of the factors belongs to J) **the Hodge dual of d is the Hochshild boundary**.

In the case of n independent events one is led to describe the spacetime coordinates of the $j-th$ event by $q_j = I \otimes ... \otimes I \otimes q \otimes I ... \otimes I$ (q in the j - th place); in this way, the commutator between the different spacetime components of the q_j would depend on j.

A better choice is to require that it does not; this is achieved as follows.

The centre Z of the multiplier algebra of $\mathcal{E}$ is the algebra of all bounded continuos complex functions on Σ; so that $\mathcal{E}$, and hence A, is in an obvious way a $Z-bimodule$.

We therefore can, and will, replace, in the definition of $\Lambda(A)$, the $\mathbb{C}$ - tensor product by the $Z-bimodule-tensor$ $product$ so that

$$dQ = 0.$$

As a consequence, the q_j and the $2^{-1/2}(q_j - q_k)$, j different from k, and $2^{-1/2}dq$, obey **the same space-time commutation relations,** as does the normalized barycenter coordinates, $n^{-1/2}(q_1+q_2+...+q_n)$; and the latter commutes with the difference coordinates.

These facts allow us to define a *quantum diagonal map* from $\Lambda_n(\mathcal{E})$ to $\mathcal{E}_1$ (the restriction to Σ_1 of $\mathcal{E}$),

$$E^{(n)} : \mathcal{E} \otimes_Z \cdots \otimes_Z \mathcal{E} \longrightarrow \mathcal{E}_1$$

which factors to that restriction map and a conditional expectation which leaves the functions of the barycenter coordinates alone, and evaluates on functions of the difference variables the *universal optimally localized map* (which, when composed with a probability measure on Σ_1, would give the generic optimally localized state).

Replacing the classical diagonal evaluation of a function of n arguments on Minkowski space by the *quantum diagonal map* allows us to define the *Quantum Wick Product*.

But working in $\Omega(A)$ as a subspace of $\Lambda(A)$ allows us to use two structures:

- the tensor algebra structure described above, where both the A bimodule and the Z bimodule structures enter, essential for our reduced universal differential calculus;

- the pre - C* algebra structure of $\Lambda(A)$, which allows us to consider, for each element a of $\Lambda_n(A)$, its modulus $(a^*a)^{1/2}$, its spectrum, and so on.

In particular we can study the geometric operators: separation between two independent events, area, 3 - volume, 4 - volume, given by

$$dq;$$

$$dq \wedge dq;$$

$$dq \wedge dq \wedge dq;$$

$$dq \wedge dq \wedge dq \wedge dq,$$

where, for instance, the latter is given by

$$V = dq \wedge dq \wedge dq \wedge dq =$$

$$\epsilon_{\mu\nu\rho\sigma} dq^{\mu} dq^{\nu} dq^{\rho} dq^{\sigma}.$$

Each of these forms has a number of spacetime components:

e.g. 4 the first one (a vector), 1 the last one (a pseudoscalar).

Each component is a normal operator;

$$THEOREM$$

For each of these forms, **the sum of the square moduli of all spacetime components is bounded below by a multiple of the identity of unit order of magnitude**.

Although that sum is (except for the 4 - volume!) NOT Lorentz invariant, the bound holds in any Lorentz frame.

In particular,

- the four volume operator has pure point spectrum, at distance $5^{1/2} - 2$ from 0;

- the *Euclidean* distance between two independent events has a lower bound of order one in Planck units.

Two distinct points *can never merge to a point*.

However, of course, the state where the minimum is achieved will depend upon the reference frame where the requirement is formulated.

(The structure of length, area and volume operators on QST has been studied in full detail [BDFP 2011]).

Thus the existence of a minimal length is not at all in contradiction with the Lorentz covariance of the model.

In the C* algebra $\mathcal{E}$ of Quantum Spacetime, define [DFR 1995]:

- the **von Neumann functional calculus**: for each $f \in \mathcal{F}L^1(\mathbb{R}^4)$ the **function $f(q)$ of the quantum coordinates q_μ** is given by

$$f(q) \equiv \int \check{f}(\alpha) e^{iq_\mu \alpha^\mu} d^4\alpha \ ,$$

- the **integral over the whole space** and **over 3 -**

space at $q_0 = t$ by

$$\int d^4q f(q) \equiv \int f(x) d^4x = \check{f}(0) = Trf(q),$$
$$\int_{q_0=t} f(q) d^3q \equiv \int e^{ik_0t} \check{f}(k_0, \vec{0}) dk_0 =$$
$$= \lim_m Tr(f_m(q)^* f(q) f_m(q))),$$

where the trace is the ordinary trace at each point of the joint spectrum Σ of the commutators, i.e. a $\mathcal{Z}$ valued trace.

But on more general elements of our algebra both maps give Q - dependent results.

Important to define the interaction Hamiltonian to be used in the Gell'Mann Low formula for the S - Matrix.

QST and QFT

The geometry of Quantum Spacetime and the **free field theories** on it are *fully Poincaré covariant*.

One can introduce interactions in different ways, all preserving spacetime translation and space rotation covariance, all equivalent on ordinary Minkowski space, providing inequivalent approaches on QST; but all of them, sooner or later, meet **problems with Lorentz covariance**, apparently due to the nontrivial action of the Lorentz group on the *centre* of the algebra of Quantum Spacetime.

On this points in our opinion a deeper understanding is needed.

For instance, the interaction Hamiltonian on quantum spacetime

$$\mathcal{H}_I(t) = \lambda \int_{q^0=t} d^3q \ \ :\phi(q)^n:$$

would be Q - dependent; **no invariant probability measure or mean** on Σ; integrating on Σ_1 [DFR 1995] breaks Lorentz invariance.

Covariance is preserved by **Yang Feldmann equations** but missed again at the level of scattering theory.

The **Quantum Wick product** selects a special frame from the start. The interaction Hamiltonian on the quantum spacetime is then given by

$$\mathcal{H}_I(t) = \lambda \int_{q^0=t} d^3q \ \ : \phi(q)^n :_Q$$

where

$$: \phi^n(q) :_Q = E^{(n)}(: \phi(q_1) \cdots \phi(q_n) :)$$

which does not depend on Q any longer, but brakes Lorentz invariance at an earlier stage

The last mentioned approach takes into account, in the very definition of Wick products, the fact that in our Quantum Spacetime n (larger or equal to two) distinct points can never merge to a point. But we can use the canonical *quantum diagonal map* which associates to functions of n independent points a function of a single point, evaluating a conditional expectation which on functions of the differences takes a numerical value, associated with the minimum of the Euclidean distance (**in a given Lorentz frame!**).

The **"Quantum Wick Product"** obtained by this procedure leads to an interaction Hamiltonian on the quantum spacetime given by as a constant operator–valued function of Σ_1 (i.e. $\mathcal{H}_I(t)$ is formally in the tensor product of $\mathcal{C}(\Sigma_1)$ with the algebra of field operators).

The interaction Hamiltonian on the quantum spacetime is then given by

$$\mathcal{H}_I(t) = \lambda \int_{q^0=t} d^3q \ \ :\phi(q)^n :_Q$$

This leads to a unique prescription for the interaction Hamiltonian on quantum spacetime. When used in the Gell'Mann Low perturbative expansion for the S - Matrix, this gives the same result as the **effective non local Hamiltonian** determined by the kernel

$$\exp\left\{-\frac{1}{2}\sum_{j,\mu} a_j^{\mu 2}\right\}\delta^{(4)}\left(\frac{1}{n}\sum_{j=1}^{n} a_j\right).$$

The corresponding perturbative Gell'Mann Low formula is then **free of ultraviolet divergences** at each term of the perturbation expansion [BDFP 2003] .

However, those terms have a meaning only after a sort of adiabatic cutoff: the coupling constant should be changed to a function of time, rapidly vanishing at infinity, say depending upon a cutoff time T.

But the limit $T \to \infty$ is difficult problem, and there are indications it does not exist.

HOWEVER the minimal distance can be taken into account in another way:

replacing, in the Hamiltonian density on Minkowski space, the field at a point by the field at a "quantum point" in Quantum Spacetime

$$< \iota \otimes \omega_x, \phi(q) >$$

where $\phi(q)$ is affiliated to $\mathcal{F} \otimes \mathcal{E}$, $\mathcal{F}$ is the algebra of fields, and ω_x the state of $\mathcal{E}$ optimally localized at x.

The effective non local Hamiltonian density obtained in this way and the previous one have the same spacetime integrals (in the exponent in the Gell'mann - Low formula); but the S - matrix may still differ due to the **Time Ordering**.

This difference is of great importance: the techniques of Perturbative Algebraic Quantum Field Theory developed by Fredenhagen, Reizner, Brunetti, Dütsch may now be applied, to show that the perturbative expansion of the field operators is not only ultraviolet finite, but also term by term convergent in **the Adiabatic Limit** [DMP, 2019].

Unfortunately the same is not yet proved for S - matrix itself. But this result allows one to tackle the problems:

1. Can we estimate the decay of field commutators at spacelike separations, even without results on the convergence of the perturbative expansion?

2. if so, and the commutators do decay rapidly as expected beyond planckian distances, this will be enough to study the S - matrix following Haag, Ruelle, Araki and Hepp.

In order that the so obtained S - matrix has any physical meaning, however, one should apply this procedure to **the renormalized interaction Lagrangean density** of a theory which is renormalizable on the ordinary Minkowski space, with the counter terms defined by that ordinary theory, and **with finite renormalization**

constants depending upon both the Planck length λ_P and the cutoff time T, chosen so that in the limit $\lambda_P \to 0$ and $T \to \infty$ we get back the ordinary renormalized Gell-Mann Low expansion on Minkowski space.

In that case, that result could be taken as source of **predictions to be compared with observations**.

QST and cosmology

Heuristic argument we started with: commutators between coordinates ought to depend on $g_{\mu,\nu}$; scenario:

$$R_{\mu,\nu} - (1/2)Rg_{\mu,\nu} = 8\pi T_{\mu,\nu}(\psi);$$

$$F_g(\psi) = 0;$$

$$[q^\mu, q^\nu] = iQ^{\mu,\nu}(g);$$

Algebra is Dynamics.

Expect: dynamical minimal length.

In particular, divergent near singularities. Would solve Horizon Problem, without inflationary hypothesis.

How solid are these heuristic arguments?

Exact solution of the semiclassical EE, spherically symmetric case confirms:

massless scalar field semiclassical coupling with gravity;

use Quantum Wick product to define Energy - Momentum Tensor $T_Q^{\mu,\nu}(q)$;

Exact EE with source $\omega \otimes \eta_x(T_Q^{\mu,\nu}(q))$, where ω is a KMS state and η_x is a state on $\mathcal{E}$ optimally localized at x;

these simplifying *ansaetze* imply a solution describing spacetime without the horizon problem [DMP 2013]. Near the Big Bang every pair of points were in causal contact, as indicated by the heuristic argument that the range of a-causal effects should diverge.

For the Planck length λ_P is replaced by the **effective Planck length** $\lambda_P/a(t)$, where $a(t)$ is the coefficient in the FRW metric

$$ds^2 = -dt^2 + a(t)^2(dx_1^2 + dx_2^2 + dx_3^2)$$

What happens at the singularity, $t = 0$? Current research (G.Morsella, N.Pinamonti, S.D.):

Two attitudes:

- no limit; asymptotic approaches replace initial conditions?

- state at $t = 0$, given the divergence of the effective Planck length?

We wish to comment on the last, in a simplified picture: Minkowski QST with varying effective Planck length,

$$\lambda_{\mathit{eff}} \rightarrow \infty$$

Repacing λ_P by $\lambda_{\mathit{eff}} \rightarrow \infty$ in the above formulas we get

- **The Quantum Diagonal Map** $E^{(n)} \to 0$;

- **The Fields at a quantum point** $< \iota \otimes \omega_x, \phi(q) > \to 0$;

- The same happens for the interacting field, at least at the lowest perturbative order, in the **Yang - Feldmann approach**;

- **The methods of Perturbative Algebraic Quantum Field Theory**, applied to the effective interaction Hamiltonian, obtained replacing in the Hamiltonian density on Minkowski space, the field at a point by the field at a "quantum point" in Quantum Spacetime, show that fields **tend to zero at all orders in perturbation theory**.[DMP, 2019]

This supports the picture:

Since $\lambda_{\text{eff}} \to \infty$ at the singularity all points are in contact to one another, the universe becames a single point, a system with zero degrees of freedom.

Initial condition or unreachable limit?

In the first case: description of the transition to a flat Universe at nonzero times?

In the second case, different asymtotics at $t \to 0$ replace Initial conditions?

Need for a dynamical picture of Quantum Spacetime.

BUT: observable signatures of QST? DFMP Phys. Rev. D 95, 065009 (2017).

Warning: Quantum effects at Planck scale result from extrapolation of EE to that scale.

But: Newton's law is experimentally checked only for distances (Adelberger *et al, 2003, 2004), i.e. we are extrapolating 31 steps down* **in base 10 - log scale***; while the size of the known universe is "only" 28 steps up.*

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