Stationary actions of higher rank lattices on von Neumann algebras

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Richard Kadison and his mathematical legacy

Rigidity of higher rank lattices in operator algebras and topological dynamics

Higher rank lattices

Let G be a **connected semisimple Lie group** with trivial center, no compact factor, all of whose simple factors have real rank ≥ 2.

Examples

- $G = \mathrm{PSL}_n(\mathbf{R})$ for $n \geq 3$
- $G = \mathrm{PSL}_n(\mathbf{R}) \times \mathrm{PSL}_n(\mathbf{R})$ for $n \geq 3$

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Let $\Gamma < G$ be an **irreducible lattice**, meaning that $\Gamma < G$ is a discrete subgroup with finite covolume such that $\Gamma N < G$ is dense for every nontrivial closed normal subgroup $\mathbf{1} \neq N < G$.

Examples

- If $G = \mathrm{PSL}_n(\mathbf{R})$ for $n \geq 3$, take $\Gamma = \mathrm{PSL}_n(\mathbf{Z})$
- If $G = \mathrm{PSL}_n(\mathbf{R}) \times \mathrm{PSL}_n(\mathbf{R})$ for $n \geq 3$, take $\Gamma = \mathrm{PSL}_n(\mathbf{Z}[\sqrt{2}])$

In this talk, we simply say that $\Gamma < G$ is a **higher rank lattice**.

Representation rigidity

Denote by $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ the **left regular representation** and by τ_Γ the canonical tracial state on the reduced C^*-algebra $C^*_\lambda(\Gamma)$.

A unitary representation $\pi : \Gamma \to \mathcal{U}(H_\pi)$ is **weakly mixing** if it does not contain any nonzero finite dimensional subrepresentation.

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Theorem (BH 2019)

Let $\Gamma < G$ be any higher rank lattice. Let $\pi : \Gamma \to \mathcal{U}(H_\pi)$ be any weakly mixing unitary representation.

Then there is a unique $*$-homomorphism $\Theta : \mathrm{C}^*_\pi(\Gamma) \to \mathrm{C}^*_\lambda(\Gamma)$ such that $\Theta(\pi(\gamma)) = \lambda(\gamma)$ for every $\gamma \in \Gamma$. Moreover,

- $\tau_\Gamma \circ \Theta$ *is the unique tracial state on* $\mathrm{C}^*_\pi(\Gamma)$.
- $\ker(\Theta)$ *is the unique maximal proper ideal of* $\mathrm{C}^*_\pi(\Gamma)$.

Normal subgroup theorem and stabilizer rigidity

Our theorem strengthens Margulis' **normal subgroup theorem** and Stuck-Zimmer's **stabilizer rigidity** result.

Theorem (Margulis 1978)

Let $\Gamma < G$ be any higher rank lattice. Then Γ is **just infinite**, *that is, any normal subgroup $N < \Gamma$ is either trivial or has finite index.*

Proof.

Let $N < \Gamma$ be any infinite index normal subgroup. Then the quasi-regular representation $\lambda_{\Gamma/N}$ is weakly mixing. For every $\gamma \in N$, since $\lambda(\gamma) = \Theta(\lambda_{\Gamma/N}(\gamma)) = \Theta(1) = 1$, we have $N = \mathbf{1}$. □

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Theorem (Stuck-Zimmer 1992)

Let $\Gamma < G$ be any higher rank lattice and $\Gamma \curvearrowright (X, \mu)$ any pmp ergodic action. Then either (X, μ) is finite or the action $\Gamma \curvearrowright (X, \mu)$ is essentially free.

Operator algebraic superrigidity

Our theorem also strengthens **operator algebraic superrigidity** results by Bekka ($\Gamma = \mathrm{PSL}_n(\mathbf{Z})$) and Peterson ($\Gamma$ arbitrary).

Theorem (Bekka 2006, Peterson 2014)

Let $\Gamma < G$ be any higher rank lattice. Let M be any finite factor and $\pi : \Gamma \to \mathcal{U}(M)$ any representation such that $\pi(\Gamma)'' = M$.

Then either M is finite dimensional or π extends to a normal unital $*$-isomorphism $\widetilde{\pi} : \mathrm{L}(\Gamma) \to M$.

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A **character** $\varphi : \Gamma \to \mathbf{C}$ is a normalized positive definite function such that the GNS representation $(\pi_\varphi, H_\varphi, \xi_\varphi)$ generates a finite von Neumann algebra $M = \pi_\varphi(\Gamma)''$.

Theorem (Bekka 2006, Peterson 2014)

Let $\Gamma < G$ be any higher rank lattice. Then for any extreme point $\varphi \in \mathrm{Char}(\Gamma)$, either $\pi_\varphi(\Gamma)''$ is a finite dimensional factor or $\varphi = \delta_e$.

Rigidity in topological dynamics

We obtain a topological analogue of Stuck-Zimmer's result.

Theorem (BH 2019)

Let $\Gamma < G$ be any higher rank lattice. Let $\Gamma \curvearrowright X$ be any minimal action on a compact metrizable space. Then either X is finite or the action $\Gamma \curvearrowright X$ is topologically free.

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Denote by $\mathrm{Sub}(\Gamma)$ the compact metrizable space of all subgroups of Γ endowed with the conjugation action $\Gamma \curvearrowright \mathrm{Sub}(\Gamma)$.

A **Uniformly Recurrent Subgroup (URS)** is a closed Γ-invariant minimal subset of $\mathrm{Sub}(\Gamma)$. The next result answers positively a question of Glasner-Weiss (2014).

Corollary (BH 2019)

Let $\Gamma < G$ be any higher rank lattice. Then any **URS** *of Γ is finite.*

Stationary actions of higher rank lattices on Neumann algebras

What is... a stationary state?

Let H be any lcsc group and $\mu \in \mathrm{Prob}(H)$ any **admissible** Borel probability measure, that is, $\mu \sim \mathrm{Haar}$.

Let $\mathcal{A}$ be any unital C^*-algebra, $\psi \in \mathcal{S}(\mathcal{A})$ any state and $\sigma : H \curvearrowright \mathcal{A}$ any continuous action. Define $\mu * \psi \in \mathcal{S}(\mathcal{A})$ by

$$\mu * \psi = \int_H \psi \circ \sigma_h^{-1} \, \mathrm{d}\mu(h)$$

Following Furstenberg and Hartman-Kalantar, we say that $\phi \in \mathcal{S}(\mathcal{A})$ is μ-**stationary** if $\mu * \phi = \phi$.

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Lemma (Furstenberg)

There always exists a μ-stationary state $\phi \in \mathcal{S}(\mathcal{A})$.

Indeed, choose a nonprincipal ultrafilter $\mathcal{U} \in \beta(\mathbf{N}) \setminus \mathbf{N}$ and define

$$\phi = \lim_{n \to \mathcal{U}} \frac{1}{n+1} \sum_{k=0}^{n} \mu^{*k} * \psi$$

The Poisson boundary

The (H, μ)-**Poisson boundary** is the (unique) ergodic action $H \curvearrowright (B, \nu_B)$ such that $\mu * \nu_B = \nu_B$ and the **Poisson map**

$$\mathrm{L}^\infty(B, \nu_B) \to \mathrm{Har}^\infty(H, \mu) : f \mapsto \widehat{f} = \left(h \mapsto \int_B f(hb)\, \mathrm{d}\nu_B(b)\right)$$

is a H-equivariant isometric isomorphism.

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Theorem (Furstenberg)

Let $\mathcal{A}$ be any separable unital C^*-algebra, $\sigma : H \curvearrowright \mathcal{A}$ any continuous action and $\phi \in \mathcal{S}(\mathcal{A})$ any μ-stationary state.

Then there exists an essentially unique H-equivariant measurable **boundary map** $\beta_\phi : B \to \mathcal{S}(\mathcal{A}) : b \mapsto \phi_b$ *such that*

$$\phi = \int_B \phi_b \, \mathrm{d}\nu_B(b)$$

Stationary ergodic actions on von Neumann algebras

Let $\mathcal{M}$ be any von Neumann algebra, $\varphi \in \mathcal{M}_*$ any normal state and $\sigma : H \curvearrowright \mathcal{M}$ any continuous action. Since $H \curvearrowright \mathcal{M}_*$ is norm continuous, we have $\mu * \varphi \in \mathcal{M}_*$.

Definition (Stationary ergodic action)

We say that $(H, \mu) \curvearrowright (\mathcal{M}, \varphi)$ is a **stationary ergodic action** if $\mu * \varphi = \varphi$ and $\mathcal{M}^H = \mathbf{C}1$.

Examples

- Any state-preserving action $H \curvearrowright (\mathcal{M}, \varphi)$ is stationary.
- The Poisson boundary $(H, \mu) \curvearrowright \mathrm{L}^\infty(B, \nu_B)$ is a stationary ergodic action.

There is no analogue of Furstenberg's lemma in the W^*-setting. Given a continuous action $H \curvearrowright \mathcal{M}$, there need not exist a **normal** μ-stationary state on $\mathcal{M}$.

Structure theory of G/P

Let G be a connected semisimple Lie group with finite center and no compact factor. Choose $K < G$ a maximal compact subgroup and $P < G$ a minimal parabolic subgroup so that $G = KP$.

Example

If $G = \mathrm{SL}_n(\mathbf{R})$, take $K = \mathrm{SO}_n(\mathbf{R})$ and $P < G$ the subgroup of upper triangular matrices.

Denote by $\nu_P \in \mathrm{Prob}(G/P)$ the unique K-invariant Borel probability measure. Then $\nu_P \in \mathrm{Prob}(G/P)$ is G-quasi-invariant.

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Denote by $\nu_P \in \mathrm{Prob}(G/P)$ the unique K-invariant Borel probability measure. Then $\nu_P \in \mathrm{Prob}(G/P)$ is G-quasi-invariant.

More generally, for every parabolic subgroup $P \subset Q \subset G$, denote by $\nu_Q \in \mathrm{Prob}(G/Q)$ the unique K-invariant Borel probability measure, which is also G-quasi-invariant.

The Furstenberg measure

Theorem (Furstenberg 1962-1967)

Let $\Gamma < G$ be any higher rank lattice. There exists a probability measure $\mu_0 \in \mathrm{Prob}(\Gamma)$ for which the following assertions hold:

1. $\mathrm{supp}(\mu_0) = \Gamma$
2. $\mu_0 * \nu_P = \nu_P$, *that is,* ν_P *is* μ_0*-stationary*
3. $(G/P, \nu_P)$ *is the* (Γ, μ_0)*-Poisson boundary*

We will say that $\mu_0 \in \mathrm{Prob}(\Gamma)$ is a **Furstenberg measure**.

Structure of stationary ergodic actions

The **key novelty** is a structure theorem for stationary ergodic actions of higher rank lattices on arbitrary von Neumann algebras.

Theorem (BH 2019)

Let $\Gamma < G$ be any higher rank lattice and $\mu_0 \in \mathrm{Prob}(\Gamma)$ any Furstenberg measure.

Let $(\Gamma, \mu_0) \curvearrowright (M, \phi)$ be any stationary ergodic action. Then the following dichotomy holds:

- *Either ϕ is Γ-invariant.*
- *Or there are a parabolic subgroup $P \subset Q \subsetneq G$ and a Γ-equivariant normal embedding $\mathrm{L}^\infty(G/Q, \nu_Q) \hookrightarrow (M, \phi)$.*

The above result is even new for stationary ergodic actions on **abelian** von Neumann algebras!

Application 1: Rigidity in topological dynamics

Theorem (BH 2019)

Let $\Gamma < G$ be any higher rank lattice. Let $\Gamma \curvearrowright X$ be any minimal action on a compact metrizable space. Then either X is finite or the action $\Gamma \curvearrowright X$ is topologically free.

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Proof.

Assume that X is not finite. We may choose a μ_0-stationary measure $\nu \in \mathrm{Prob}(X)$ so that $(\Gamma, \mu_0) \curvearrowright (X, \nu)$ is ergodic.

By minimality, we have $\mathrm{supp}(\nu) = X$. By stationarity and ergodicity, (X, ν) is a diffuse probability space.

In order to prove that $\Gamma \curvearrowright X$ is topologically free, it suffices to show that $\Gamma \curvearrowright (X, \nu)$ is essentially free.

- If ν is Γ-invariant, this follows from Stuck-Zimmer's result.
- If $(X, \nu) \to (G/Q, \nu_Q)$, this follows from the fact that the projective action $\Gamma \curvearrowright (G/Q, \nu_Q)$ is essentially free. □

Application 2: Classification of stationary characters

We say that a normalized positive definite function $\varphi : \Gamma \to \mathbf{C}$ is a μ_0-**character** if $\mu_0 * \varphi = \sum_{\gamma \in \Gamma} \mu_0(\gamma) \, \varphi \circ \mathrm{Ad}(\gamma)^{-1} = \varphi$.

Theorem (BH 2019)

Let $\Gamma < G$ be any higher rank lattice and $\mu_0 \in \mathrm{Prob}(\Gamma)$ any Furstenberg measure. Then any μ_0-character φ is conjugation invariant, that is, φ is a genuine character.

Moreover, for any extreme point $\varphi \in \mathrm{Char}(\Gamma)$, either $\pi_\varphi(\Gamma)''$ is a finite dimensional factor or $\varphi = \delta_e$.

Our result is reminiscent of Benoist-Quint's classification results of stationary measures on homogeneous spaces (2009).

Using our structure theorem, we obtain a new proof of Peterson's character rigidity result (2014).

Application 3: Representation rigidity

Theorem (BH 2019)

Let $\Gamma < G$ *be any higher rank lattice. Let* $\pi : \Gamma \to \mathcal{U}(H_\pi)$ *be any weakly mixing unitary representation.*

Then there is a unique $*$*-homomorphism* $\Theta : \mathrm{C}^*_\pi(\Gamma) \to \mathrm{C}^*_\lambda(\Gamma)$ *such that* $\Theta(\pi(\gamma)) = \lambda(\gamma)$ *for every* $\gamma \in \Gamma$*. Moreover,*

- $\tau_\Gamma \circ \Theta$ *is the unique tracial state on* $\mathrm{C}^*_\pi(\Gamma)$*.*
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- $\tau_\Gamma \circ \Theta$ *is the unique tracial state on* $\mathrm{C}^*_\pi(\Gamma)$*.*
- $\ker(\Theta)$ *is the unique maximal proper ideal of* $\mathrm{C}^*_\pi(\Gamma)$*.*

Proof.

Set $A = \mathrm{C}^*_\pi(\Gamma)$. By Furstenberg's lemma, there is a μ_0-stationary state $\phi \in \mathcal{S}(A)$. Then $\phi \circ \pi$ is a μ_0-character on Γ.

The classification of μ_0-characters and Kazhdan property (T) imply that $\phi \circ \pi = \delta_e$. This further implies that $\lambda \prec \pi$. □

Proof of the structure theorem

How do we prove the structure theorem?

Theorem (BH 2019)

Let $\Gamma < G$ be any higher rank lattice and $\mu_0 \in \mathrm{Prob}(\Gamma)$ any Furstenberg measure.

Let $(\Gamma, \mu_0) \curvearrowright (M, \phi)$ be any stationary ergodic action. Then the following dichotomy holds:

- *Either ϕ is Γ-invariant.*
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Step 1: The **stationary induction**

Step 2: The **noncommutative Nevo-Zimmer theorem**

Step 3: The **disintegration argument**

Step 1: The stationary induction

When $\Gamma \curvearrowright M$, denote by $\mathrm{Ind}_\Gamma^G(M) = \mathrm{L}^\infty(G/\Gamma) \mathbin{\overline{\otimes}} M$ the induced von Neumann algebra and by $G \curvearrowright \mathrm{Ind}_\Gamma^G(M)$ the induced action.

Theorem (BH 2019)

Let $\Gamma < G$ be a higher rank lattice, $\mu_0 \in \mathrm{Prob}(\Gamma)$ a Furstenberg measure and $\mu \in \mathrm{Prob}(G)$ a K-invariant admissible measure.

Let $(\Gamma, \mu_0) \curvearrowright (M, \phi)$ be any stationary ergodic action. Then there is a **normal** *state $\varphi \in \mathrm{Ind}_\Gamma^G(M)_*$ so that $\mu * \varphi = \varphi$. Moreover,*

$$\phi \text{ is } \Gamma\text{-invariant} \quad \text{if and only if} \quad \varphi \text{ is } G\text{-invariant.}$$

The proof exploits the fact that $(G/P, \nu_P)$ is **simultaneously** the (Γ, μ_0)-Poisson boundary and the (G, μ)-Poisson boundary.

Step 2: The noncommutative Nevo-Zimmer theorem

Our most technical result is the following noncommutative analogue of Nevo-Zimmer's structure theorem.

Theorem (BH 2019)

Let G be any connected semisimple Lie group as before.
Let $\mu \in \mathrm{Prob}(G)$ be any K-invariant admissible measure.
Let $(G, \mu) \curvearrowright (\mathcal{M}, \varphi)$ be any stationary ergodic action. Then the following dichotomy holds:

- *Either φ is G-invariant.*
- *Or there are a parabolic subgroup $P \subset Q \subsetneq G$ and a G-equivariant normal embedding $\mathrm{L}^\infty(G/Q, \nu_Q) \hookrightarrow (\mathcal{M}, \varphi)$.*

Strategy of the proof of noncommutative Nevo-Zimmer

Assuming that φ is not G-invariant, we construct a G-invariant **abelian** von Neumann subalgebra $\mathcal{Z}_0 \subset \mathcal{M}$ for which $\varphi|_{\mathcal{Z}_0}$ is not G-invariant.

Then we apply Nevo-Zimmer's result to obtain a parabolic subgroup $P \subset Q \subsetneq G$ and a G-equivariant normal embedding $\mathrm{L}^\infty(G/Q, \nu_Q) \hookrightarrow (\mathcal{Z}_0, \varphi) \subset (\mathcal{M}, \varphi)$.

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In case $(\mathcal{M}, \varphi) = \mathrm{L}^\infty(X, \nu)$, Nevo-Zimmer construct a projective factor $(X, \nu) \to (G/Q, \nu_Q)$ by using dynamical properties of the measurable **Gauss map** $X \to \mathrm{Gr}(\mathrm{Lie}(G)) : x \mapsto \mathrm{Lie}(G_x)$.

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Conceptual difficulty: There is no such Gauss map when $\mathcal{M}$ is noncommutative!

When Ge-Kadison meet noncommutative dynamics

In case $(\mathcal{M}, \varphi)$ is arbitrary, we start the proof in a similar fashion as Nevo-Zimmer until we reach the critical point where we would need to use the Gauss map.

From that point on, we develop a new strategy that relies on **Margulis' operation** and **Ge-Kadison's splitting theorem** for tensor product von Neumann algebras.

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Theorem (Ge-Kadison 1995)

Let N be any factor, B any von Neumann algebra and

$$N \mathbin{\overline{\otimes}} \mathbf{C}1_B \subset M \subset N \mathbin{\overline{\otimes}} B$$

any intermediate von Neumann subalgebra. Then there exists a von Neumann subalgebra $C \subset B$ such that $M = N \mathbin{\overline{\otimes}} C$.