Stationary actions of higher rank lattices on von Neumann algebras

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(joint work with Rémi Boutonnet)

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Richard Kadison and his mathematical legacy
Rigidity of higher rank lattices in operator algebras and topological dynamics
Higher rank lattices

Let $G$ be a connected semisimple Lie group with trivial center, no compact factor, all of whose simple factors have real rank $\geq 2$.

Examples

- $G = \text{PSL}_n(\mathbb{R})$ for $n \geq 3$
- $G = \text{PSL}_n(\mathbb{R}) \times \text{PSL}_n(\mathbb{R})$ for $n \geq 3$
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Let $\Gamma < G$ be an irreducible lattice, meaning that $\Gamma < G$ is a discrete subgroup with finite covolume such that $\Gamma N < G$ is dense for every nontrivial closed normal subgroup $1 \neq N < G$.

**Examples**
- If $G = \text{PSL}_n(\mathbb{R})$ for $n \geq 3$, take $\Gamma = \text{PSL}_n(\mathbb{Z})$
- If $G = \text{PSL}_n(\mathbb{R}) \times \text{PSL}_n(\mathbb{R})$ for $n \geq 3$, take $\Gamma = \text{PSL}_n(\mathbb{Z}[\sqrt{2}])$

In this talk, we simply say that $\Gamma < G$ is a higher rank lattice.
Denote by \( \lambda : \Gamma \to U(\ell^2(\Gamma)) \) the **left regular representation** and by \( \tau_\Gamma \) the canonical tracial state on the reduced \( C^* \)-algebra \( C^*_\lambda(\Gamma) \).

A unitary representation \( \pi : \Gamma \to U(H_\pi) \) is **weakly mixing** if it does not contain any nonzero finite dimensional subrepresentation.
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A unitary representation $\pi : \Gamma \to U(H_\pi)$ is \textbf{weakly mixing} if it does not contain any nonzero finite dimensional subrepresentation.

\textbf{Theorem (BH 2019)}

Let $\Gamma < G$ be any higher rank lattice. Let $\pi : \Gamma \to U(H_\pi)$ be any weakly mixing unitary representation.

Then there is a unique $\ast$-homomorphism $\Theta : C^*_\pi(\Gamma) \to C^*_\lambda(\Gamma)$ such that $\Theta(\pi(\gamma)) = \lambda(\gamma)$ for every $\gamma \in \Gamma$. Moreover,

- $\tau_\Gamma \circ \Theta$ is the unique tracial state on $C^*_\pi(\Gamma)$.
- $\ker(\Theta)$ is the unique maximal proper ideal of $C^*_\pi(\Gamma)$.
Our theorem strengthens Margulis’ **normal subgroup theorem** and Stuck-Zimmer’s **stabilizer rigidity** result.

**Theorem (Margulis 1978)**

*Let $\Gamma < G$ be any higher rank lattice. Then $\Gamma$ is just infinite, that is, any normal subgroup $N < \Gamma$ is either trivial or has finite index.*

**Proof.**

Let $N < \Gamma$ be any infinite index normal subgroup. Then the quasi-regular representation $\lambda_{\Gamma/N}$ is weakly mixing. For every $\gamma \in N$, since $\lambda(\gamma) = \Theta(\lambda_{\Gamma/N}(\gamma)) = \Theta(1) = 1$, we have $N = 1$. \qed
Normal subgroup theorem and stabilizer rigidity

Our theorem strengthens Margulis’ **normal subgroup theorem** and Stuck-Zimmer’s **stabilizer rigidity** result.

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**Theorem (Stuck-Zimmer 1992)**

Let $\Gamma < G$ be any higher rank lattice and $\Gamma \curvearrowright (X, \mu)$ any pmp ergodic action. Then either $(X, \mu)$ is finite or the action $\Gamma \curvearrowright (X, \mu)$ is essentially free.
Our theorem also strengthens **operator algebraic superrigidity** results by Bekka ($\Gamma = \text{PSL}_n(\mathbb{Z})$) and Peterson ($\Gamma$ arbitrary).

**Theorem (Bekka 2006, Peterson 2014)**

Let $\Gamma < G$ be any higher rank lattice. Let $M$ be any finite factor and $\pi : \Gamma \to \mathcal{U}(M)$ any representation such that $\pi(\Gamma)'' = M$.

Then either $M$ is finite dimensional or $\pi$ extends to a normal unital $\ast$-isomorphism $\tilde{\pi} : L(\Gamma) \to M$. 
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A **character** $\varphi : \Gamma \to \mathbb{C}$ is a normalized positive definite function such that the GNS representation $(\pi_\varphi, H_\varphi, \xi_\varphi)$ generates a finite von Neumann algebra $M = \pi_\varphi(\Gamma)''$.

**Theorem (Bekka 2006, Peterson 2014)**

Let $\Gamma < G$ be any higher rank lattice. Then for any extreme point $\varphi \in \text{Char}(\Gamma)$, either $\pi_\varphi(\Gamma)''$ is a finite dimensional factor or $\varphi = \delta_e$. 

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We obtain a topological analogue of Stuck-Zimmer’s result.

**Theorem (BH 2019)**

Let $\Gamma < G$ be any higher rank lattice. Let $\Gamma \act X$ be any minimal action on a compact metrizable space. Then either $X$ is finite or the action $\Gamma \act X$ is topologically free.
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Let $\Gamma < G$ be any higher rank lattice. Let $\Gamma \curvearrowleft X$ be any minimal action on a compact metrizable space. Then either $X$ is finite or the action $\Gamma \curvearrowleft X$ is topologically free.

Denote by $\text{Sub}(\Gamma)$ the compact metrizable space of all subgroups of $\Gamma$ endowed with the conjugation action $\Gamma \curvearrowleft \text{Sub}(\Gamma)$.

A **Uniformly Recurrent Subgroup (URS)** is a closed $\Gamma$-invariant minimal subset of $\text{Sub}(\Gamma)$. The next result answers positively a question of Glasner-Weiss (2014).

**Corollary (BH 2019)**

Let $\Gamma < G$ be any higher rank lattice. Then any URS of $\Gamma$ is finite.
Stationary actions of higher rank lattices on Neumann algebras
What is... a stationary state?

Let $H$ be any lcsc group and $\mu \in \text{Prob}(H)$ any \textbf{admissible} Borel probability measure, that is, $\mu \sim \text{Haar}$.

Let $\mathcal{A}$ be any unital $C^*$-algebra, $\psi \in S(\mathcal{A})$ any state and $\sigma : H \curvearrowright \mathcal{A}$ any continuous action. Define $\mu * \psi \in S(\mathcal{A})$ by

$$
\mu * \psi = \int_H \psi \circ \sigma^{-1}_h \, d\mu(h)
$$

Following Furstenberg and Hartman-Kalantar, we say that $\phi \in S(\mathcal{A})$ is $\mu$-\textbf{stationary} if $\mu * \phi = \phi$. 
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\textbf{Lemma (Furstenberg)}

There always exists a $\mu$-stationary state $\phi \in \mathcal{S}(A)$.

Indeed, choose a nonprincipal ultrafilter $\mathcal{U} \in \beta(\mathbb{N}) \setminus \mathbb{N}$ and define

$$
\phi = \lim_{n \to \mathcal{U}} \frac{1}{n+1} \sum_{k=0}^{n} \mu^k * \psi
$$
The Poisson boundary

The \((H, \mu)\)-Poisson boundary is the (unique) ergodic action \(H \curvearrowright (B, \nu_B)\) such that \(\mu \ast \nu_B = \nu_B\) and the Poisson map

\[
L^\infty(B, \nu_B) \to \text{Har}^\infty(H, \mu) : f \mapsto \hat{f} = \left( h \mapsto \int_B f(hb) \, d\nu_B(b) \right)
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is a \(H\)-equivariant isometric isomorphism.
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**Theorem (Furstenberg)**

Let \(A\) be any separable unital \(C^*\)-algebra, \(\sigma : H \curvearrowright A\) any continuous action and \(\phi \in S(A)\) any \(\mu\)-stationary state.

Then there exists an essentially unique \(H\)-equivariant measurable boundary map \(\beta_\phi : B \to S(A) : b \mapsto \phi_b\) such that

\[
\phi = \int_B \phi_b \, d\nu_B(b)
\]
Let $\mathcal{M}$ be any von Neumann algebra, $\varphi \in \mathcal{M}_*$ any normal state and $\sigma : H \curvearrowright \mathcal{M}$ any continuous action. Since $H \curvearrowright \mathcal{M}_*$ is norm continuous, we have $\mu \ast \varphi \in \mathcal{M}_*$.

**Definition (Stationary ergodic action)**

We say that $(H, \mu) \curvearrowright (\mathcal{M}, \varphi)$ is a **stationary ergodic action** if $\mu \ast \varphi = \varphi$ and $\mathcal{M}^H = \mathbb{C}1$.

**Examples**

- Any state-preserving action $H \curvearrowright (\mathcal{M}, \varphi)$ is stationary.
- The Poisson boundary $(H, \mu) \curvearrowright L^\infty(B, \nu_B)$ is a stationary ergodic action.

There is no analogue of Furstenberg’s lemma in the $W^*$-setting. Given a continuous action $H \curvearrowright \mathcal{M}$, there need not exist a **normal** $\mu$-stationary state on $\mathcal{M}$.
Structure theory of $G/P$

Let $G$ be a connected semisimple Lie group with finite center and no compact factor. Choose $K < G$ a maximal compact subgroup and $P < G$ a minimal parabolic subgroup so that $G = KP$.

**Example**

If $G = \text{SL}_n(\mathbb{R})$, take $K = \text{SO}_n(\mathbb{R})$ and $P < G$ the subgroup of upper triangular matrices.

Denote by $\nu_P \in \text{Prob}(G/P)$ the unique $K$-invariant Borel probability measure. Then $\nu_P \in \text{Prob}(G/P)$ is $G$-quasi-invariant.
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Denote by $\nu_P \in \text{Prob}(G/P)$ the unique $K$-invariant Borel probability measure. Then $\nu_P \in \text{Prob}(G/P)$ is $G$-quasi-invariant.

More generally, for every parabolic subgroup $P \subset Q \subset G$, denote by $\nu_Q \in \text{Prob}(G/Q)$ the unique $K$-invariant Borel probability measure, which is also $G$-quasi-invariant.
The Furstenberg measure

Theorem (Furstenberg 1962-1967)

Let \( \Gamma \subset G \) be any higher rank lattice. There exists a probability measure \( \mu_0 \in \text{Prob}(\Gamma) \) for which the following assertions hold:

1. \( \text{supp}(\mu_0) = \Gamma \)
2. \( \mu_0 \ast \nu_P = \nu_P \), that is, \( \nu_P \) is \( \mu_0 \)-stationary
3. \( (G/P, \nu_P) \) is the \( (\Gamma, \mu_0) \)-Poisson boundary

We will say that \( \mu_0 \in \text{Prob}(\Gamma) \) is a **Furstenberg measure**.
The **key novelty** is a structure theorem for stationary ergodic actions of higher rank lattices on arbitrary von Neumann algebras.

**Theorem (BH 2019)**

Let $\Gamma < G$ be any higher rank lattice and $\mu_0 \in \text{Prob}(\Gamma)$ any Furstenberg measure.

Let $(\Gamma, \mu_0) \curvearrowright (M, \phi)$ be any stationary ergodic action. Then the following dichotomy holds:

- Either $\phi$ is $\Gamma$-invariant.
- Or there are a parabolic subgroup $P \subset Q \subsetneq G$ and a $\Gamma$-equivariant normal embedding $L^\infty(G/Q, \nu_Q) \hookrightarrow (M, \phi)$.

The above result is even new for stationary ergodic actions on **abelian** von Neumann algebras!
Application 1: Rigidity in topological dynamics

Theorem (BH 2019)

Let $\Gamma \vartriangleleft G$ be any higher rank lattice. Let $\Gamma \curvearrowright X$ be any minimal action on a compact metrizable space. Then either $X$ is finite or the action $\Gamma \curvearrowright X$ is topologically free.
Application 1: Rigidity in topological dynamics

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Proof.

Assume that \( X \) is not finite. We may choose a \( \mu_0 \)-stationary measure \( \nu \in \text{Prob}(X) \) so that \( (\Gamma, \mu_0) \curvearrowright (X, \nu) \) is ergodic.

By minimality, we have \( \text{supp}(\nu) = X \). By stationarity and ergodicity, \( (X, \nu) \) is a diffuse probability space.

In order to prove that \( \Gamma \curvearrowright X \) is topologically free, it suffices to show that \( \Gamma \curvearrowright (X, \nu) \) is essentially free.

- If \( \nu \) is \( \Gamma \)-invariant, this follows from Stuck-Zimmer’s result.
- If \( (X, \nu) \to (G/Q, \nu_Q) \), this follows from the fact that the projective action \( \Gamma \curvearrowright (G/Q, \nu_Q) \) is essentially free.
Application 2: Classification of stationary characters

We say that a normalized positive definite function $\varphi : \Gamma \rightarrow \mathbb{C}$ is a $\mu_0$-character if $\mu_0 \ast \varphi = \sum_{\gamma \in \Gamma} \mu_0(\gamma) \varphi \circ \text{Ad}(\gamma)^{-1} = \varphi$.

**Theorem (BH 2019)**

Let $\Gamma < G$ be any higher rank lattice and $\mu_0 \in \text{Prob}(\Gamma)$ any Furstenberg measure. Then any $\mu_0$-character $\varphi$ is conjugation invariant, that is, $\varphi$ is a genuine character.

Moreover, for any extreme point $\varphi \in \text{Char}(\Gamma)$, either $\pi_{\varphi}(\Gamma)''$ is a finite dimensional factor or $\varphi = \delta_e$.

Our result is reminiscent of Benoist-Quint’s classification results of stationary measures on homogeneous spaces (2009).
Using our structure theorem, we obtain a new proof of Peterson’s character rigidity result (2014).
**Theorem (BH 2019)**

Let $\Gamma < G$ be any higher rank lattice. Let $\pi : \Gamma \rightarrow \mathcal{U}(H_{\pi})$ be any weakly mixing unitary representation.

Then there is a unique $*$-homomorphism $\Theta : C^*_\pi(\Gamma) \rightarrow C^*_\lambda(\Gamma)$ such that $\Theta(\pi(\gamma)) = \lambda(\gamma)$ for every $\gamma \in \Gamma$. Moreover,

- $\tau_{\Gamma} \circ \Theta$ is the unique tracial state on $C^*_\pi(\Gamma)$.
- $\ker(\Theta)$ is the unique maximal proper ideal of $C^*_\pi(\Gamma)$. 

**Proof.**

Set $A = C^*_\pi(\Gamma)$. By Furstenberg's lemma, there is a $\mu_0$-stationary state $\phi \in S(A)$. Then $\phi \circ \pi$ is a $\mu_0$-character on $\Gamma$.

The classification of $\mu_0$-characters and Kazhdan property (T) imply that $\phi \circ \pi = \delta e$. This further implies that $\lambda \preceq \pi$.
Application 3: Representation rigidity

**Theorem (BH 2019)**

Let $\Gamma < G$ be any higher rank lattice. Let $\pi : \Gamma \to \mathcal{U}(H_\pi)$ be any weakly mixing unitary representation.

Then there is a unique $\ast$-homomorphism $\Theta : C^*_\pi(\Gamma) \to C^*_\lambda(\Gamma)$ such that $\Theta(\pi(\gamma)) = \lambda(\gamma)$ for every $\gamma \in \Gamma$. Moreover,

- $\tau_\Gamma \circ \Theta$ is the unique tracial state on $C^*_\pi(\Gamma)$.
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**Proof.**

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The classification of $\mu_0$-characters and Kazhdan property (T) imply that $\phi \circ \pi = \delta_e$. This further implies that $\lambda \prec \pi$. \qed
Proof of the structure theorem
How do we prove the structure theorem?

Theorem (BH 2019)

Let $\Gamma < G$ be any higher rank lattice and $\mu_0 \in \text{Prob}(\Gamma)$ any Furstenberg measure.

Let $(\Gamma, \mu_0) \curvearrowright (M, \phi)$ be any stationary ergodic action. Then the following dichotomy holds:

- Either $\phi$ is $\Gamma$-invariant.
- Or there are a parabolic subgroup $P \subset Q \subset G$ and a $\Gamma$-equivariant normal embedding $L^\infty(G/Q, \nu_Q) \hookrightarrow (M, \phi)$.
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**Step 1:** The stationary induction

**Step 2:** The noncommutative Nevo-Zimmer theorem

**Step 3:** The disintegration argument
Step 1: The stationary induction

When $\Gamma \actson M$, denote by $\text{Ind}_\Gamma^G(M) = L^\infty(G/\Gamma) \boxtimes M$ the induced von Neumann algebra and by $G \actson \text{Ind}_\Gamma^G(M)$ the induced action.

**Theorem (BH 2019)**

Let $\Gamma < G$ be a higher rank lattice, $\mu_0 \in \text{Prob}(\Gamma)$ a Furstenberg measure and $\mu \in \text{Prob}(G)$ a $K$-invariant admissible measure.

Let $(\Gamma, \mu_0) \actson (M, \phi)$ be any stationary ergodic action. Then there is a normal state $\varphi \in \text{Ind}_\Gamma^G(M)_*$ so that $\mu \ast \varphi = \varphi$. Moreover,

$$\phi \text{ is } \Gamma\text{-invariant} \quad \text{if and only if} \quad \varphi \text{ is } G\text{-invariant}.$$  

The proof exploits the fact that $(G/P, \nu_P)$ is simultaneously the $(\Gamma, \mu_0)$-Poisson boundary and the $(G, \mu)$-Poisson boundary.
Our most technical result is the following noncommutative analogue of Nevo-Zimmer’s structure theorem.

**Theorem (BH 2019)**

Let $G$ be any connected semisimple Lie group as before. Let $\mu \in \text{Prob}(G)$ be any $K$-invariant admissible measure. Let $(G, \mu) \curvearrowright (\mathcal{M}, \varphi)$ be any stationary ergodic action. Then the following dichotomy holds:

- Either $\varphi$ is $G$-invariant.
- Or there are a parabolic subgroup $P \subset Q \subsetneq G$ and a $G$-equivariant normal embedding $L^\infty(G/Q, \nu_Q) \hookrightarrow (\mathcal{M}, \varphi)$. 
Assuming that $\varphi$ is not $G$-invariant, we construct a $G$-invariant abelian von Neumann subalgebra $\mathcal{Z}_0 \subset \mathcal{M}$ for which $\varphi|_{\mathcal{Z}_0}$ is not $G$-invariant.

Then we apply Nevo-Zimmer’s result to obtain a parabolic subgroup $P \subset Q \subset G$ and a $G$-equivariant normal embedding $L^\infty(G/Q, \nu_Q) \hookrightarrow (\mathcal{Z}_0, \varphi) \subset (\mathcal{M}, \varphi)$.
Assuming that $\varphi$ is not $G$-invariant, we construct a $G$-invariant \textbf{abelian} von Neumann subalgebra $Z_0 \subset M$ for which $\varphi|_{Z_0}$ is not $G$-invariant.

Then we apply Nevo-Zimmer’s result to obtain a parabolic subgroup $P \subset Q \subset G$ and a $G$-equivariant normal embedding $L^\infty(G/Q, \nu_Q) \hookrightarrow (Z_0, \varphi) \subset (M, \varphi)$.

In case $(M, \varphi) = L^\infty(X, \nu)$, Nevo-Zimmer construct a projective factor $(X, \nu) \to (G/Q, \nu_Q)$ by using dynamical properties of the measurable \textbf{Gauss map} $X \to \text{Gr}(\text{Lie}(G)) : x \mapsto \text{Lie}(G_x)$. 
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\textbf{Conceptual difficulty:} There is no such Gauss map when $\mathcal{M}$ is noncommutative!
In case \((\mathcal{M}, \varphi)\) is arbitrary, we start the proof in a similar fashion as Nevo-Zimmer until we reach the critical point where we would need to use the Gauss map.

From that point on, we develop a new strategy that relies on Margulis’ operation and Ge-Kadison’s splitting theorem for tensor product von Neumann algebras.
When Ge-Kadison meet noncommutative dynamics

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**Theorem (Ge-Kadison 1995)**

Let \(N\) be any factor, \(B\) any von Neumann algebra and

\[
N \overline{\otimes} C_1 B \subset M \subset N \overline{\otimes} B
\]

any intermediate von Neumann subalgebra. Then there exists a von Neumann subalgebra \(C \subset B\) such that \(M = N \overline{\otimes} C\).