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Keeping the field alive — reflections on Kadison's pivotal role.

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Talk given at the conference
"Richard Kadison and his mathematical legacy — A memorial conference".
Copenhagen, November 29–30, 2019.

Bernard of Chartres ($\sim$ 1100):

“We Moderns are like dwarves perched on the shoulders of the Ancients (the giants), and thus we are able to see more and farther than the latter.”

“Kadison’s global vision of the field was certainly essential for my own development.”

(Words of Alain Connes when Kadison was awarded the Steele prize in 1999 for Lifetime Achievement.)

University of Pennsylvania (Penn) being a “powerhouse” in operator algebras around 1970.

- R. Kadison
- E. Effros
- S. Sakai
- R. Powers
- M. Fell
- L. Pukansky
- (1968/69: M. Takesaki)

Historical precedent of “keeping the field alive”.

- Legendre (1752–1833) and elliptic functions.
- Abel (1802-1829) and Jacobi (1804-1851).

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Quoting Kadison:

"When somebody sneezes in algebraic geometry,
it's like thunder in any other field."

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“On rings of operators”; I, II, III, IV (1936–1943) by Murray-von Neumann.

Two important results — complicated proofs

(i) E and F finite projections $\implies$ $E \vee F$ finite projection.

(ii) Additivity of the trace in finite factors.

Simplifying the proofs of (i) and (ii)

(i) Key ingredient is Kaplansky's formula (1951):

$$E \vee F - F \sim E - (E \wedge F)$$

(ii) Starting with the dimension function on projections in a II_1 factor M, the problem is to extend this to an *additive* trace on M. Kadison gave an elegant and greatly simplified proof of the additivity in 1955.

Two important results proved by Kadison pertaining to von Neumann algebras.

Theorem A (1956)

A C^*-algebra R is a von Neumann algebra if and only if (i) each bounded increasing net in R_{sa} has a least upper bound in R_{sa} and (ii) R has a separating family of normal states.

Theorem B (1966)

Each von Neumann algebra has only inner derivations.

Theorem (Wiener, 1931)

Suppose

- $f\left(e^{i\theta}\right) = \sum_{-\infty}^{\infty} c_n e^{in\theta}$,
- $\sum_{-\infty}^{\infty} |c_n| < \infty$, *and*
- $f\left(e^{i\theta}\right) \neq 0$ *for every real* θ.

Then $\dfrac{1}{f\left(e^{i\theta}\right)} = \sum_{-\infty}^{\infty} \gamma_n e^{in\theta}$ *with* $\sum_{-\infty}^{\infty} |\gamma_n| < \infty$.

Proved by Gelfand in 1938 using (commutative) Banach algebra theory.
(His doctoral thesis; Advisor: Kolmogorov.)

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“After a decade of mystery the mist finally disappeared, and there was nothing there.”

Gelfand-Naimark (1943)

B^*-algebra: Banach $*$-algebra with $\|A^*A\| = \|A^*\|\,\|A\|$, plus

(i) $A^*A + I$ has an inverse

(ii) $\|A\| = \|A^*\|$

Gelfand and Naimark conjectured that (i) and (ii) were superfluous.

(i) Proved to be superfluous in 1952 by Fukamiya (c.f. also Kaplansky).

(ii) Proved to be superfluous in 1960 by Kadison and Glimm.

Kadison's favourite discovery?

Kadison's Transitivity Theorem (1957)

If a C^*-algebra $\mathfrak{A}$ of operators acting on a Hilbert space has no nontrivial invariant closed subspace, then it has no nontrivial invariant linear vector space.

Corollary 1

Let $\mathfrak{S} = \{A \in \mathfrak{A} \mid \rho(A^*A) = 0\}$, where ρ is a state of $\mathfrak{A}$. If ρ is pure state, then $\mathfrak{A}/\mathfrak{S}$ is complete in the inner product $\langle A + \mathfrak{S}, B + \mathfrak{S} \rangle = \rho(B^*A)$.

Corollary 2

$\mathfrak{S}$ is a maximal left ideal of $\mathfrak{A}$ if and only if ρ is a pure state.

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Kadison's Transitivity Theorem + Kadison/Glimm (1960)

Let A be a (unital) C^*-algebra acting irreducibly on a Hilbert space H, and suppose that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are elements of H and that $x_1, \ldots, x_n$ are linearly independent. Then there exists an operator $u \in A$ such that $u(x_j) = y_j$ for $j = 1, \ldots, n$.

- If there is a s.a. operator v on H such that $v(x_j) = y_j$ for $j = 1, \ldots, n$, then we may choose u to be s.a. also.
- If there is a unitary v on H such that $v(x_j) = y_j$ for $j = 1, \ldots, n$, then we may choose u to be a unitary also — we may even suppose that $u = e^{iw}$ for some element $w \in A_{sa}$.

Corollary 1

Two pure states ρ and σ of a unital C^*-algebra A are unitarily equivalent (i.e. $\rho(a) = \sigma(u^* a u)$ for some unitary $u \in A$ and all $a \in A$) provided $\|\rho - \sigma\| < 2$.

Corollary 2

Two pure states are unitarily equivalent provided the corresponding representations are.

Corollary 3

The $*$-operation is isometric in B^*-algebras.

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Definition

The *enveloping von Neumann algebra* (denoted A'') of the (unital) C^*-algebra A is the strong closure of the universal representation of A.

Theorem (Kadison, 1951)

Let $\mathcal{S} = \{\phi \in A^* \mid \|\phi\| = \phi(1) = 1\}$ be the (convex, w^*-compact) state space of A. There is then an order preserving linear isometry $^\wedge$ of A''_{sa} onto $\mathrm{Aff}(\mathcal{S})$ (the real-valued bounded affine functions on $\mathcal{S}$) such that $(A_{sa})^\wedge = \mathrm{Aff}_c(\mathcal{S})$ $(\subseteq \mathrm{Aff}(\mathcal{S}))$, where c denotes "continuous".

Kadison-Singer Conjecture (1959), recently (2013) proven true by A. Marcus, D. Spielman, N. Srivastava.

Consider the separable Hilbert space $H = l^2(\mathbb{Z})$, and let D denote the masa in $B(H)$ consisting of diagonal operators on H. Let $\rho\colon D \to \mathbb{C}$ be a pure state on D (equivalently, ρ is multiplicative). There exists a unique extension of ρ to a state (necessarily pure) $\widetilde{\rho}$ on $\mathcal{B}(H)$.

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Problem posed by Kadison in 1955, and solved by U. Haagerup in 1983:

Let $\phi\colon A \to \mathcal{B}(H)$ be a bounded non-selfadjoint representation of a C^*-algebra A on a Hilbert space H, and assume there is an $x \in H$ such that $\{\phi(a)x \mid a \in A\}$ is dense in H. Then there exists an invertible T in $\mathcal{B}(H)$ such that $T\phi T^{-1}$ is a $*$-representation of A.

In 1952, Fuglede and Kadison defined their determinant

$$\det_\tau^{\mathrm{FK}} : \begin{cases} Inv(N) \to \mathbb{R}_+^* \\ x \mapsto \exp\left(\tau\left(\log(x^*x)^{\frac{1}{2}}\right)\right) \end{cases}$$

where N is a finite factor with (unique) normalized trace τ. They showed that $\det_\tau^{\mathrm{FK}}$ is a homomorphism of groups.

Much later this determinant, and generalizations of it for appropriate groups of invertible elements in Banach algebras, has found applications in topology. (Ref. Pierre de la Harpe, "Fuglede-Kadison determinant: theme and variations" (2013).)