C*-algebras of stable rank one and their Cuntz semigroups

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Richard Kadison and his mathematical legacy
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Copenhagen



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Application: Non-stable *K*-theory (Rieffel 1983):

- $K_1(A) = \mathcal{U}(A)/\mathcal{U}_0(A)$. (No matrix amplifications needed.)
- projections p and q are equivalent iff $[p] = [q] \in K_0(A)$



Examples among unital, stably finite, simple C*-algebras:

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- → Question: Do all simple group C*-algebras have stable rank one?



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$$\begin{array}{c|cccc} A & V(A) & K_0(A) & \operatorname{Cu}(A) \\ \hline \mathbb{C} \text{ or } M_n(\mathbb{C}) & \mathbb{N} & \mathbb{Z} & \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\} \\ \end{array}$$

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C([0,1])	1	\mathbb{Z}	$Lsc([0,1],\overline{\mathbb{N}})$

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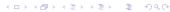
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Theorem (APT 2017)

- Cu admits internal-hom (closed monoidal category)
- starting point to develop UCT for bivariant Cuntz semigroups:

$$\mathsf{Cu}(A,B) \to \llbracket \mathsf{Cu}(A), \mathsf{Cu}(B)
rbracket$$



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- Cu is complete and cocomplete.
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Corollary

Ultraproduct $\prod_{\mathcal{U}} A_k$ is simple iff either:

- almost all A_k are simple, purely infinite; or:
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Characterization of when limit (quasi)traces on $\prod_{\mathcal{U}} A_k$ are dense in $QT(\prod_{\mathcal{U}} A_i)$.

(Generalizing Ozawa's 2013 'no silly traces' result. Uses that Cu(-) encodes simplex of (quasi)traces.)

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If A has stable rank one and strict comparison, then

$$Cu(A) \cong V(A) \sqcup LAff(QT(A))_{++} \cong Cu(\mathcal{Z} \otimes A).$$

Toms-Winter conjecture 2005

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Corollary (T 2018)

Toms-Winter conjecture holds for:

- C*-algebras with stable rank one and locally finite nuclear dimension (in particular, stable rank one ASH-algebras).
- minimal crossed products $C(X) \rtimes \mathbb{Z}$ (using Lutley 2017).



Theorem (Antoine-Perera-Robert-T 2018)

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Proof.

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Given Hilbert modules E and F there is a Hilbert module G with G → E and G → F, and such that any Hilbert module H with H → E and H → F also satisfies H → G.

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- We can apply methods from semilattice theory to study C*-algebras of stable rank one.



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Theorem (APRT 2018)

Blackadar-Handelman conjecture holds for stable rank one.



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Is this the only obstruction?

If A is simple and non-elementary, then (2) holds for every k. (Glimm 1960s)



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Theorem (APRT 2018, Global Glimm Halving)

Let A be unital with stable rank one, and $k \in \mathbb{N}$. TFAE:

- A has no irreducible representation of dimension < k.
- ② there exists $M_k(C_0((0,1])) \rightarrow A$ with full image.
- **3** there exists a Hilbert A-module E such that $E^{\oplus k} \subseteq A \subseteq E^{\oplus n}$, for some n.

The rank of a Hilbert A-module E is

$$\widehat{\mathsf{E}} \colon \operatorname{\mathsf{QT}}(\mathsf{A}) o [0,\infty], \quad \widehat{\mathsf{E}}(au) = \mathsf{d}_{ au}(\mathsf{E}).$$

The rank problem

Describe $\{\widehat{E} : E \text{ Hilbert } A\text{-module}\}.$

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Theorem (APRT 2018)

Let A be unital, stable rank one, no finite-dimensional representations. Then for every $f \in LAff(QT(A))_{++}$ there is Hilbert A-module E with $\widehat{E} = f$.



References

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