Close Operator Algebras

Stuart White

University of Oxford

Richard Kadison: A Mathematical Legacy

With thanks to my collaborators:

Cameron, Christensen, Hirshberg, Kirchberg, Perera, Sinclair, Smith, Toms, Wiggins, Winter.

Close Operator Algebras

Perturbation problems

- When can we perturb approximate behaviour to exact behaviour?
- e.g. approximate projections are near to projections.
- i.e., if $T \in \mathcal{B}(\mathcal{H})$ has $\|T - T^*\|, \|T - T^2\|$ small, then there exists a projection $P \in \mathcal{B}(\mathcal{H})$ with $\|T - P\|$ small.
- Questions of this nature are ubiquitous in operator algebras.

Close Operator Algebras

Perturbation problems

- When can we perturb approximate behaviour to exact behaviour?
- e.g. approximate projections are near to projections.
- i.e., if $T \in \mathcal{B}(\mathcal{H})$ has $\|T - T^*\|, \|T - T^2\|$ small, then there exists a projection $P \in \mathcal{B}(\mathcal{H})$ with $\|T - P\|$ small.
- Questions of this nature are ubiquitous in operator algebras.

Kadison and Kastler (72)

- Consider all operator algebras on a fixed Hilbert space $\mathcal{H}$.
- If two algebras are close, can one be perturbed into the other?

Close Operator Algebras

Kadison and Kastler (72)

- Consider all operator algebras on a fixed Hilbert space $\mathcal{H}$.
- If two algebras are close, can one be perturbed into the other?

Close operator algebras

- Unit balls close in Hausdorff metric arising from operator norm
- $d(\mathcal{M},\mathcal{N}) < \gamma$ iff every operator x in the unit ball of $\mathcal{M}$ is within γ of an operator in the unit ball of $\mathcal{N}$, and vice versa.

Close Operator Algebras

Kadison and Kastler (72)

- Consider all operator algebras on a fixed Hilbert space $\mathcal{H}$.
- If two algebras are close, can one be perturbed into the other?

Close operator algebras

- Unit balls close in Hausdorff metric arising from operator norm
- $d(\mathcal{M},\mathcal{N}) < \gamma$ iff every operator x in the unit ball of $\mathcal{M}$ is within γ of an operator in the unit ball of $\mathcal{N}$, and vice versa.

In what sense can one speak of "perturbation" of a von Neumann algebra? We have not "moved" it by some process — "adding a term," for instance. There is such a a process available, however. If a von Neumann algebra is transformed by a unitary operator close to the identity operator, the result is a "slight perturbation" of the original algebra.

CLOSE OPERATOR ALGEBRAS

CLOSE OPERATOR ALGEBRAS

- Unit balls close in Hausdorff metric arising from operator norm
- $d(\mathcal{M}, \mathcal{N}) < \gamma$ iff every operator x in the unit ball of $\mathcal{M}$ is within γ of an operator in the unit ball of $\mathcal{N}$, and vice versa.

In what sense can one speak of "perturbation" of a von Neumann algebra? We have not "moved" it by some process — "adding a term," for instance. There is such a a process available, however. If a von Neumann algebra is transformed by a unitary operator close to the identity operator, the result is a "slight perturbation" of the original algebra.

EXAMPLE: SMALL UNITARY PERTURBATION

- $d(u\mathcal{M}u^*, \mathcal{M}) \leqslant 2\|u - 1\|$.
- Natural question: Do close operator algebras necessarily arise from small unitary perturbations?

Close Operator Algebras

Close operator algebras

- Unit balls close in Hausdorff metric arising from operator norm
- $d(\mathcal{M}, \mathcal{N}) < \gamma$ iff every operator x in the unit ball of $\mathcal{M}$ is within γ of an operator in the unit ball of $\mathcal{N}$, and vice versa.

Example: small unitary perturbation

- $d(u\mathcal{M}u^*, \mathcal{M}) \leqslant 2\|u - 1\|$.
- Natural question: Do close operator algebras necessarily arise from small unitary perturbations?

We believe that this is the only possible method of perturbing an operator algebra–that suitably close operator algebras are unitarily equivalent.

Close Operator Algebras

We believe that this is the only possible method of perturbing an operator algebra–that suitably close operator algebras are unitarily equivalent.

Questions

1. Do close operator algebras share the same properties?
2. Must they be isomorphic? via an isomorphism close to the inclusion into the underlying $\mathcal{B}(\mathcal{H})$?
3. Must they be (small) unitary perturbations?

Close Operator Algebras

We believe that this is the only possible method of perturbing an operator algebra–that suitably close operator algebras are unitarily equivalent.

Questions

1. Do close operator algebras share the same properties?
2. Must they be isomorphic? via an isomorphism close to the inclusion into the underlying $\mathcal{B}(\mathcal{H})$?
3. Must they be (small) unitary perturbations?

One sided versions: if $A \subset_\gamma B$, for γ small,

- must $A \hookrightarrow B$?
- can such an embedding be taken close to $A \hookrightarrow \mathcal{B}(\mathcal{H})$?
- can such an embedding be implemented by a unitary (close to 1)?

Close Operator Algebras

Questions

1. Do close operator algebras share the same properties?
2. Must they be isomorphic? via an isomorphism close to the inclusion into the underlying $\mathcal{B}(\mathcal{H})$?
3. Must they be (small) unitary perturbations?

One sided versions: if $A \subset_\gamma B$, for γ small,

- must $A \hookrightarrow B$?
- can such an embedding be taken close to $A \hookrightarrow \mathcal{B}(\mathcal{H})$?
- can such an embedding be implemented by a unitary (close to 1)?

Many other variants: e.g. replace $\mathcal{B}(\mathcal{H})$ by a finite factor

- Work both with operator norm, and 2-norm.
- Ideas play role in Popa's intertwining by bimodules.

Stability of type

Theorem (Kadison and Kastler)

Sufficiently close von Neumann algebras have close type decompositions.

Stability of type

Theorem (Kadison and Kastler)

Sufficiently close von Neumann algebras have close type decompositions.

- $\mathcal{M}$ and $\mathcal{N}$ have type decomposition $\bigoplus_i \mathcal{M}p_i$ and $\bigoplus_i \mathcal{N}q_i$ for $i = I_1, I_2, \ldots, I_\infty, II_1, II_\infty, III$.
- Then $\forall \epsilon > 0$, $\exists \delta > 0$ s.t.

$$d(\mathcal{M}, \mathcal{N}) < \delta \implies \|p_i - q_i\| < \epsilon, \text{ for all } i.$$

- Close factors are of the same type.

K-THEORY

KHOSHKAM, RAEBURN-PHILIPS

- When A and B are close, projections in A are close to those in B.
- Consequence: sufficiently close C^*-algebras have the same dimension range.
- Consequence II: sufficiently close separable AF C^*-algebras are isomorphic.

K-THEORY

KHOSHKAM, RAEBURN-PHILIPS

- When A and B are close, projections in A are close to those in B.
- Consequence: sufficiently close C^*-algebras have the same dimension range.
- Consequence II: sufficiently close separable AF C^*-algebras are isomorphic.

TO HANDLE K-THEORY WE NEED

If A and B are close, must $d_{cb}(A, B) := \sup_n d(M_n(A), M_n(B))$ be small?

K-THEORY

KHOSHKAM, RAEBURN-PHILIPS

- When A and B are close, projections in A are close to those in B.
- Consequence: sufficiently close C^*-algebras have the same dimension range.
- Consequence II: sufficiently close separable AF C^*-algebras are isomorphic.

TO HANDLE K-THEORY WE NEED

If A and B are close, must $d_{cb}(A,B) := \sup_n d(M_n(A), M_n(B))$ be small?

PROPOSITION (KHOSHKAM)

Suppose $d_{cb}(A,B)$ sufficiently small. Then $K_*(A) \cong K_*(B)$.

K-THEORY

KHOSHKAM, RAEBURN-PHILIPS

- When A and B are close, projections in A are close to those in B.
- Consequence: sufficiently close C^*-algebras have the same dimension range.
- Consequence II: sufficiently close separable AF C^*-algebras are isomorphic.

TO HANDLE K-THEORY WE NEED

If A and B are close, must $d_{cb}(A, B) := \sup_n d(M_n(A), M_n(B))$ be small?

PROPOSITION (KHOSHKAM)

Suppose $d_{cb}(A, B)$ sufficiently small. Then $K_*(A) \cong K_*(B)$.

THEOREM

Suppose $d_{cb}(A, B)$ sufficiently small. Then $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$.

Kadison's Similarity problem (55)

Operator algebra version of unitarisability problem for group representations.

Question

Let A be a C^*-algebra. When is it the case that a bounded homomorphism $\theta : A \to \mathcal{B}(\mathcal{H})$ is similar to a *-homomorphism?

Kadison's Similarity problem (55)

Operator algebra version of unitarisability problem for group representations.

Question

Let A be a C^*-algebra. When is it the case that a bounded homomorphism $\theta : A \to \mathcal{B}(\mathcal{H})$ is similar to a $*$-homomorphism?

- Yes, if $\theta(A)$ has a cyclic vector (Haagerup).
- Yes for properly infinite von Neumann algebras, and hence stable C^*-algebras.
- Yes for II_1 factors with property Γ.
- Profound reformulations (Kirchberg, Pisier,...)

Similarities and close operator algebras

Theorem

Let A be a C^*-algebra with the similarity property. Then any C^*-algebra sufficiently close to A also has the similarity property.

Similarities and close operator algebras

Theorem

Let A be a C^*-algebra with the similarity property. Then any C^*-algebra sufficiently close to A also has the similarity property.

- If A has similarity property, get $d_{cb}(A, B) \leqslant C_A d(A, B)$ for all B.
- So if A is nuclear, or $\mathcal{Z}$-stable, or stable, and $d(A, B)$ sufficiently small, then $K_*(A) \cong K_*(B)$ and $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$.

Similarities and close operator algebras

Theorem

Let A be a C^*-algebra with the similarity property. Then any C^*-algebra sufficiently close to A also has the similarity property.

- If A has similarity property, get $d_{cb}(A, B) \leqslant C_A d(A, B)$ for all B.
- So if A is nuclear, or $\mathcal{Z}$-stable, or stable, and $d(A, B)$ sufficiently small, then $K_*(A) \cong K_*(B)$ and $\mathrm{Cu}(A) \cong \mathrm{Cu}(B)$.

Theorem

d and d_{cb} are equivalent metrics if and only if the similarity problem has a positive answer.

Christensen's breakthroughs 77-80

Injective vNAs are perturbation rigid: V1

Close injective von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ arise from small unitary perturbations.

- Can in fact take $u \in (\mathcal{M} \cup \mathcal{N})''$.

Christensen's breakthroughs 77-80

Injective vNAs are perturbation rigid: V1

Close injective von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ arise from small unitary perturbations.

- Can in fact take $u \in (\mathcal{M} \cup \mathcal{N})''$.

Idea

- Injectivity of $\mathcal{N}$, gives ucp map $\Phi : \mathcal{M} \to \mathcal{N}$ which is close to the inclusion $\mathcal{M} \hookrightarrow \mathcal{B}(\mathcal{H})$.
- Φ is almost multiplicative, so writing $\Phi(\cdot) = p\pi(\cdot)p$ the Steinespring projection almost commutes with unitaries in $\mathcal{M}$.
- Use injectivity (property P) of $\mathcal{M}$, to find a projection near to p in $\mathcal{M}'$. In this way obtain $*$-homomorphism $\Psi : \mathcal{M} \to \mathcal{N}$ near Φ.

Christensen's breakthroughs 77-80

Injective vNAs are perturbation rigid: V1

Close injective von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ arise from small unitary perturbations.

- Can in fact take $u \in (\mathcal{M} \cup \mathcal{N})''$.

Idea

- Injectivity of $\mathcal{N}$, gives ucp map $\Phi : \mathcal{M} \to \mathcal{N}$ which is close to the inclusion $\mathcal{M} \hookrightarrow \mathcal{B}(\mathcal{H})$.
- Φ is almost multiplicative, so writing $\Phi(\cdot) = p\pi(\cdot)p$ the Steinespring projection almost commutes with unitaries in $\mathcal{M}$.
- Use injectivity (property P) of $\mathcal{M}$, to find a projection near to p in $\mathcal{M}'$. In this way obtain $*$-homomorphism $\Psi : \mathcal{M} \to \mathcal{N}$ near Φ.

But:

We don't want to assume conditions on $\mathcal{M}$ and $\mathcal{N}$.

CHRISTENSEN'S BREAKTHROUGHS 77-80

INJECTIVE VNAS ARE PERTURBATION RIGID: V1

Close injective von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ arise from small unitary perturbations.

- Can in fact take $u \in (\mathcal{M} \cup \mathcal{N})''$.

INJECTIVE VNAS PERTURBATION RIGID: V2

- If $\mathcal{M}$ is injective, and nearly contained in $\mathcal{N}$, then there is a unitary $u \in (\mathcal{M} \cup \mathcal{N})''$ close to 1, with $u\mathcal{M}u^* \subseteq \mathcal{N}$.
- In particular if $\mathcal{M}$ and $\mathcal{N}$ are close and $\mathcal{M}$ is injective, then $\mathcal{M}$ and $\mathcal{N}$ are small unitary perturbations.

Christensen's breakthroughs 77-80

Injective vNAs are perturbation rigid: V1

Close injective von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ arise from small unitary perturbations.

- Can in fact take $u \in (\mathcal{M} \cup \mathcal{N})''$.

Injective vNAs perturbation rigid: V2

- If $\mathcal{M}$ is injective, and nearly contained in $\mathcal{N}$, then there is a unitary $u \in (\mathcal{M} \cup \mathcal{N})''$ close to 1, with $u\mathcal{M}u^* \subseteq \mathcal{N}$.
- In particular if $\mathcal{M}$ and $\mathcal{N}$ are close and $\mathcal{M}$ is injective, then $\mathcal{M}$ and $\mathcal{N}$ are small unitary perturbations.

AF Algebras

If A and B are close with A separable and AF, then B is AF. Hence $A \cong B$.

Christensen's breakthroughs 77-80

Injective vNAs are perturbation rigid: V1

Close injective von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ arise from small unitary perturbations.

- Can in fact take $u \in (\mathcal{M} \cup \mathcal{N})''$.

Injective vNAs perturbation rigid: V2

- If $\mathcal{M}$ is injective, and nearly contained in $\mathcal{N}$, then there is a unitary $u \in (\mathcal{M} \cup \mathcal{N})''$ close to 1, with $u\mathcal{M}u^* \subseteq \mathcal{N}$.
- In particular if $\mathcal{M}$ and $\mathcal{N}$ are close and $\mathcal{M}$ is injective, then $\mathcal{M}$ and $\mathcal{N}$ are small unitary perturbations.

AF Algebras

If A and B are close with A separable and AF, then B is AF. Hence $A \cong B$.

- In fact there is a unitary $u \in (A \cup B)''$ with $uAu^* = B$.

Two counter examples from the 80's

Choi-Christensen

There exist non-separable non-isomorphic C^*-algebras A and B, which can be represented arbitrarily closely on a Hilbert space.

- These can be constructed $(1+\epsilon)$-completely order isomorphic for any $\epsilon > 0$.

Two counter examples from the 80's

Choi-Christensen

There exist non-separable non-isomorphic C^*-algebras A and B, which can be represented arbitrarily closely on a Hilbert space.

- These can be constructed $(1 + \epsilon)$-completely order isomorphic for any $\epsilon > 0$.

Johnson

For any $\epsilon > 0$, can find two reps $\pi_i : A := C[0, 1] \otimes \mathcal{K} \to \mathcal{B}(\mathcal{H})$, with $d(\pi_1(A), \pi_2(A)) < \epsilon$, but no isomorphism $\theta : \pi_1(A) \to \pi_2(A)$ can be uniformly close to $\pi_1(A) \hookrightarrow \mathcal{B}(\mathcal{H})$.

Two counter examples from the 80's

Choi-Christensen

There exist non-separable non-isomorphic C^*-algebras A and B, which can be represented arbitrarily closely on a Hilbert space.

- These can be constructed $(1+\epsilon)$-completely order isomorphic for any $\epsilon > 0$.

Johnson

For any $\epsilon > 0$, can find two reps $\pi_i : A := C[0,1] \otimes \mathcal{K} \to \mathcal{B}(\mathcal{H})$, with $d(\pi_1(A), \pi_2(A)) < \epsilon$, but no isomorphism $\theta : \pi_1(A) \to \pi_2(A)$ can be uniformly close to $\pi_1(A) \hookrightarrow \mathcal{B}(\mathcal{H})$.

Conclusion

- Measure distance between C^*-algebras in operator norm
- Measure small uniform perturbations in point norm

SOME POSITIVE C^*-ALGEBRA RESULTS

THEOREM

Let A be separable and nuclear and $d(A, B)$ small on a separable Hilbert space.[a] Then there is a unitary $u \in (A \cup B)''$ with $uAu^* = B$.

[a] $d(A, B) < 10^{-11}$ will do!

Some positive C^*-algebra results

Theorem

Let A be separable and nuclear and $d(A, B)$ small on a separable Hilbert space.[a] Then there is a unitary $u \in (A \cup B)''$ with $uAu^* = B$.

[a] $d(A, B) < 10^{-11}$ will do!

- For $\epsilon > 0$, we can choose $\delta > 0$ such that if $d(A, B) < \delta$, then for all finite subsets $\mathcal{F}$ of the unit ball of A, can choose a unitary $u \in (A \cup B)''$ with $uAu^* = B$ and

$$\|uxu^* - x\| < \epsilon, \quad x \in \mathcal{F}.$$

- This is what I mean by 'measure small uniform perturbations in point norm.'
- Proof follows Erik's strategy for injective von Neumann algebras, in a point norm way.
- Separability crucial to use an Elliott intertwining argument.

SOME POSITIVE C^*-ALGEBRA RESULTS

THEOREM

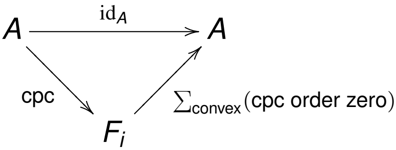
Let A be separable and nuclear and $d(A, B)$ small on a separable Hilbert space.[a] Then there is a unitary $u \in (A \cup B)''$ with $uAu^* = B$.

[a] $d(A, B) < 10^{-11}$ will do!

THEOREM

Let A be separable and nuclear, A nearly contained in B. Then A embeds into B.

- Again, can produce embeddings with point norm control.
- Uses an improved completely positive approximation property:



A non-amenable result

Theorem

There exist non-amenable II_1 factors $\mathcal{M}$, for which sufficiently close algebras are small unitary perturbations.

A non-amenable result

Theorem

There exist non-amenable II_1 factors $\mathcal{M}$, for which sufficiently close algebras are small unitary perturbations.

- eg. $\mathcal{M} = (L^\infty(X,\mu) \rtimes SL_n(\mathbb{Z})) \otimes \mathcal{R}$ for $n \geqslant 3$ and a free ergodic pmp action.

A non-amenable result

Theorem

There exist non-amenable II_1 factors $\mathcal{M}$, for which sufficiently close algebras are small unitary perturbations.

- eg. $\mathcal{M} = (L^\infty(X,\mu) \rtimes SL_n(\mathbb{Z})) \otimes \mathcal{R}$ for $n \geqslant 3$ and a free ergodic pmp action.

- The '$\otimes\mathcal{R}$' ensures $\mathcal{M} = \mathcal{M}_0 \otimes \mathcal{R}$ has the similarity property.
- $\otimes\mathcal{R}$ needed, but first step is to remove it!
- Show that any close $\mathcal{N}$ can be perturbed to $\mathcal{N}_0 \otimes \mathcal{R}$ for same copy of $\mathcal{R}$, and $d_{cb}(\mathcal{M}_0, \mathcal{N}_0)$ small.

A non-amenable result

Theorem

There exist non-amenable II_1 factors $\mathcal{M}$, for which sufficiently close algebras are small unitary perturbations.

- eg. $\mathcal{M} = (L^\infty(X,\mu) \rtimes SL_n(\mathbb{Z})) \otimes \mathcal{R}$ for $n \geqslant 3$ and a free ergodic pmp action.

Reduced to:

- $\mathcal{M}_0 = (L^\infty(X,\mu) \rtimes SL_n(\mathbb{Z})$, and $d_{cb}(\mathcal{M}_0, \mathcal{N}_0)$ small.

- Can use Christensen's results to perturb the copy of $L^\infty(X,\mu)$ as a masa in $\mathcal{N}_0$, and produce normalisers of this.
- Show that $L^\infty(X,\mu)$ is Cartan in N_0, inducing same equivalence relation as $SL_n(\mathbb{Z}) \curvearrowright (X,\mu)$.
- uses a lot of ideas of Popa. e.g. if we only assume $d(\mathcal{M}_0, \mathcal{N}_0)$ small at this point, we need his answer to a Baton-Rouge question of Kadison: existence of masas for II_1 factors inside specifed irreducible subfactors.

A NON-AMENABLE RESULT

THEOREM

There exist non-amenable II_1 factors $\mathcal{M}$, for which sufficiently close algebras are small unitary perturbations.

- eg. $\mathcal{M} = (L^\infty(X,\mu) \rtimes SL_n(\mathbb{Z})) \otimes \mathcal{R}$ for $n \geqslant 3$ and a free ergodic pmp action.

REDUCED TO:

- $\mathcal{M}_0 = (L^\infty(X,\mu) \rtimes SL_n(\mathbb{Z})$, and $d_{cb}(\mathcal{M}_0, \mathcal{N}_0)$ small.

GET

- $\mathcal{N}_0$ a twisted crossed product $L^\infty(X,\mu) \rtimes_\omega SL_m(\mathbb{Z})$
- ω a 2-cocycle, uniformly close to 1.

- $SL_n(\mathbb{Z})$ for $n \geqslant 3$, gives vanishing of a bounded cohomology group $H^2_b(SL_n(\mathbb{Z}), L^\infty_{\mathbb{R}}(X,\mu))$ (using work of Monod, Burger-Monod, Monod-Shalom).

Some questions[1]

1. Suppose $A \subset_\delta B$ where B is nuclear and A has similarity property. Must $A \hookrightarrow B$?
    - Christensen: $\mathcal{M} \subset_\delta \mathcal{N}$, with $\mathcal{M}$ having similarity property and $\mathcal{N}$ injective gives embedding $\mathcal{M} \hookrightarrow \mathcal{N}$.
    - Partial result: Yes for B a type I C^*-algebra.

[1] Caveat: Appropriate separability assumptions in all questions.

Some questions[1]

1. Suppose $A \subset_\delta B$ where B is nuclear and A has similarity property. Must $A \hookrightarrow B$?
   - Christensen: $\mathcal{M} \subset_\delta \mathcal{N}$, with $\mathcal{M}$ having similarity property and $\mathcal{N}$ injective gives embedding $\mathcal{M} \hookrightarrow \mathcal{N}$.
   - Partial result: Yes for B a type I C^*-algebra.
2. Can one produce close isomorphic copies of a II_1 factor $\mathcal{M}$ (perhaps $L^\infty(X,\mu) \rtimes \mathbb{F}_n$) on $\mathcal{H}$, for which no isomorphism can be close to the inclusion into $\mathcal{B}(\mathcal{H})$?
   - We know that for any free ergodic pmp action $\mathbb{F}_n \curvearrowright (X,\mu)$, the crossed product $\mathcal{M} = L^\infty(X,\mu) \rtimes \mathbb{F}_n$ is perturbation rigid: i.e., isomorphic to any von Neumann algebra it is close to.
   - $\mathbb{F}_k$ chosen as the bounded group cohomology $H^2(\mathbb{F}_k, L^\infty(X,\mu))$ is non-trivial.

[1] Caveat: Appropriate separability assumptions in all questions.

Some questions[1]

2. Can one produce close isomorphic copies of a II_1 factor $\mathcal{M}$ (perhaps $L^\infty(X,\mu) \rtimes \mathbb{F}_n$) on $\mathcal{H}$, for which no isomorphism can be close to the inclusion into $\mathcal{B}(\mathcal{H})$?
   - We know that for any free ergodic pmp action $\mathbb{F}_n \curvearrowright (X,\mu)$, the crossed product $\mathcal{M} = L^\infty(X,\mu) \rtimes \mathbb{F}_n$ is perturbation rigid: i.e., isomorphic to any von Neumann algebra it is close to.
   - $\mathbb{F}_k$ chosen as the bounded group cohomology $H^2(\mathbb{F}_k, L^\infty(X,\mu))$ is non-trivial.
3. If A and B are close and A is $\mathcal{Z}$-stable, must B be $\mathcal{Z}$-stable?
   - Close II_1 factors are simultaneously McDuff.
   - Moreover McDuffness perturbs: if $\mathcal{M}$ McDuff, and $\mathcal{N}$ close to $\mathcal{M}$ can make a small unitary perturbation so that $\mathcal{M} = \mathcal{M}_0 \otimes \mathcal{R}$ and $\mathcal{N} = \mathcal{N}_0 \otimes \mathcal{R}$ with the same copy of $\mathcal{R}$ and $\mathcal{M}_0$ and $\mathcal{N}_0$ close.

[1] Caveat: Appropriate separability assumptions in all questions.

Some questions[1]

3. If A and B are close and A is $\mathcal{Z}$-stable, must B be $\mathcal{Z}$-stable?
   - Close II_1 factors are simultaneously McDuff.
   - Moreover McDuffness perturbs: if $\mathcal{M}$ McDuff, and $\mathcal{N}$ close to $\mathcal{M}$ can make a small unitary perturbation so that $\mathcal{M} = \mathcal{M}_0 \otimes \mathcal{R}$ and $\mathcal{N} = \mathcal{N}_0 \otimes \mathcal{R}$ with the same copy of $\mathcal{R}$ and $\mathcal{M}_0$ and $\mathcal{N}_0$ close.
4. If $\mathcal{M}$ and $\mathcal{N}$ are close II_1 factors, what can we say about their subfactors?
   - Cartan decompositions, tensor product decompositions, transfer to close subalgebras.

[1] Caveat: Appropriate separability assumptions in all questions.

Some questions[1]

4. If $\mathcal{M}$ and $\mathcal{N}$ are close II_1 factors, what can we say about their subfactors?
   - Cartan decompositions, tensor product decompositions, transfer to close subalgebras.
5. Suppose $C \subset A$ and $D \subset B$ are inclusions of nuclear C^*-algebras with $d(A, B)$ and $d(C, D)$ small. Is there an isomorphism $\theta : A \to B$ with $\theta(C) = D$?
   - This works for injective von Neumann algebras: apply Christensen's theorem twice.
   - It also works if C (and hence D) is an ideal.
   - But in general the Elliott intertwining arguments needed to construct isomorphisms between C^*-algebras don't work well with inclusions.

[1] Caveat: Appropriate separability assumptions in all questions.