CLOSE OPERATOR ALGEBRAS

Stuart White

University of Oxford

Richard Kadison: A Mathematical Legacy

WITH THANKS TO MY COLLABORATORS:
Cameron, Christensen, Hirshberg, Kirchberg, Perera, Sinclair, Smith, Toms, Wiggins, Winter.
When can we perturb approximate behaviour to exact behaviour?

e.g. approximate projections are near to projections.

i.e., if \( T \in \mathcal{B}(\mathcal{H}) \) has \( \| T - T^* \|, \| T - T^2 \| \) small, then there exists a projection \( P \in \mathcal{B}(\mathcal{H}) \) with \( \| T - P \| \) small.

Questions of this nature are ubiquitous in operator algebras.
**Close Operator Algebras**

**Perturbation Problems**

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**Kadison and Kastler (72)**

- Consider all operator algebras on a fixed Hilbert space $\mathcal{H}$.
- If two algebras are close, can one be perturbed into the other?
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Unit balls close in Hausdorff metric arising from operator norm $d$.

$d(\mathcal{M}, \mathcal{N}) < \gamma$ iff every operator $x$ in the unit ball of $\mathcal{M}$ is within $\gamma$ of an operator in the unit ball of $\mathcal{N}$, and vice versa.
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In what sense can one speak of “perturbation” of a von Neumann algebra? We have not “moved” it by some process — “adding a term,” for instance. There is such a a process available, however. If a von Neumann algebra is transformed by a unitary operator close to the identity operator, the result is a “slight perturbation” of the original algebra.
Close Operator Algebras

Unit balls close in Hausdorff metric arising from operator norm

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In what sense can one speak of “perturbation” of a von Neumann algebra? We have not “moved” it by some process — “adding a term,” for instance. There is such a process available, however. If a von Neumann algebra is transformed by a unitary operator close to the identity operator, the result is a “slight perturbation” of the original algebra.

Example: Small Unitary Perturbation

\[ d(u\mathcal{M}u^*, \mathcal{M}) \leq 2\|u - 1\|. \]

Natural question: Do close operator algebras necessarily arise from small unitary perturbations?
**Close Operator Algebras**

**Close operator algebras**
- Unit balls close in Hausdorff metric arising from operator norm
- \( d(\mathcal{M}, \mathcal{N}) < \gamma \) iff every operator \( x \) in the unit ball of \( \mathcal{M} \) is within \( \gamma \) of an operator in the unit ball of \( \mathcal{N} \), and vice versa.

**Example: Small unitary perturbation**
- \( d(u\mathcal{M}u^*, \mathcal{M}) \leq 2\|u - 1\| \).
- Natural question: Do close operator algebras necessarily arise from small unitary perturbations?

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**Questions**

1. Do close operator algebras share the same properties?
2. Must they be isomorphic? via an isomorphism close to the inclusion into the underlying $\mathcal{B}(\mathcal{H})$?
3. Must they be (small) unitary perturbations?
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One sided versions: if $A \subset_\gamma B$, for $\gamma$ small,

- must $A \hookrightarrow B$?
- can such an embedding be taken close to $A \hookrightarrow \mathcal{B}(\mathcal{H})$?
- can such an embedding be implemented by a unitary (close to 1)?
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Many other variants: e.g. replace $\mathcal{B}(\mathcal{H})$ by a finite factor

- Work both with operator norm, and 2-norm.
- Ideas play role in Popa’s intertwining by bimodules.
STABILITY OF TYPE

THEOREM (KADISON AND KASTLER)
Sufficiently close von Neumann algebras have close type decompositions.
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- $\mathcal{M}$ and $\mathcal{N}$ have type decomposition $\bigoplus_i \mathcal{M} p_i$ and $\bigoplus_i \mathcal{N} q_i$ for $i = I_1, I_2, \ldots, I_8, II_1, II_8, III$.
- Then $\forall \epsilon > 0, \exists \delta > 0$ s.t.
  \[ d(\mathcal{M}, \mathcal{N}) < \delta \implies \| p_i - q_i \| < \epsilon, \text{ for all } i. \]
- Close factors are of the same type.
**K-THEORY**

**KHOSHKAM, RAEBURN-PHILIPS**

- When $A$ and $B$ are close, projections in $A$ are close to those in $B$.
- Consequence: sufficiently close $C^*$-algebras have the same dimension range.
- Consequence II: sufficiently close separable AF $C^*$-algebras are isomorphic.
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If $A$ and $B$ are close, must $d_{cb}(A, B) := \sup_n d(M_n(A), M_n(B))$ be small?
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**Proposition (Khoshkam)**

Suppose $d_{cb}(A, B)$ sufficiently small. Then $K_*(A) \cong K_*(B)$.
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**Proposition (Khoshkam)**
Suppose $d_{cb}(A, B)$ sufficiently small. Then $K_*(A) \cong K_*(B)$.

**Theorem**
Suppose $d_{cb}(A, B)$ sufficiently small. Then $\text{Cu}(A) \cong \text{Cu}(B)$.
Kadison’s Similarity Problem (55)

Operator algebra version of unitarisability problem for group representations.

**Question**

Let $A$ be a $C^*$-algebra. When is it the case that a bounded homomorphism $\theta : A \to B(\mathcal{H})$ is similar to a $\ast$-homomorphism?

Yes, if $\theta$ has a cyclic vector (Haagerup).

Yes for properly infinite von Neumann algebras, and hence stable $C^*$-algebras.

Yes for $\text{II}_1$ factors with property $\Gamma$.

Profound reformulations (Kirchberg, Pisier, . . . )
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Let $A$ be a $C^*$-algebra with the similarity property. Then any $C^*$-algebra sufficiently close to $A$ also has the similarity property.
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- If $A$ has similarity property, get $d_{cb}(A, B) \leq C_A d(A, B)$ for all $B$.
- So if $A$ is nuclear, or $\mathcal{Z}$-stable, or stable, and $d(A, B)$ sufficiently small, then $K_*(A) \cong K_*(B)$ and $\text{Cu}(A) \cong \text{Cu}(B)$. 
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THEOREM

$d$ and $d_{cb}$ are equivalent metrics if and only if the similarity problem has a positive answer.
Injective vNas are perturbation rigid: V1

Close injective von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$ arise from small unitary perturbations.

- Can in fact take $u \in (\mathcal{M} \cup \mathcal{N})''$. 
**Christensen’s breakthroughs 77-80**

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**Idea**

- Injectivity of $\mathcal{N}$, gives ucp map $\Phi : \mathcal{M} \to \mathcal{N}$ which is close to the inclusion $\mathcal{M} \hookrightarrow \mathcal{B}(\mathcal{H})$.

- $\Phi$ is almost multiplicative, so writing $\Phi(\cdot) = p\pi(\cdot)p$ the Steinespring projection almost commutes with unitaries in $\mathcal{M}$.

- Use injectivity (property $P$) of $\mathcal{M}$, to find a projection near to $p$ in $\mathcal{M}'$. In this way obtain *-homomorphism $\Psi : \mathcal{M} \to \mathcal{N}$ near $\Phi$. 


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**But:**

We don’t want to assume conditions on $\mathcal{M}$ and $\mathcal{N}$. 

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- If $\mathcal{M}$ is injective, and nearly contained in $\mathcal{N}$, then there is a unitary $u \in (\mathcal{M} \cup \mathcal{N})''$ close to 1, with $u\mathcal{M}u^* \subseteq \mathcal{N}$.
- In particular if $\mathcal{M}$ and $\mathcal{N}$ are close and $\mathcal{M}$ is injective, then $\mathcal{M}$ and $\mathcal{N}$ are small unitary perturbations.
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**AF Algebras**

If $A$ and $B$ are close with $A$ separable and AF, then $B$ is AF. Hence $A \cong B$. 
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AF Algebras

If $A$ and $B$ are close with $A$ separable and AF, then $B$ is AF. Hence $A \cong B$.

- In fact there is a unitary $u \in (A \cup B)''$ with $uAu^* = B$. 
Two counter examples from the 80's

Choi-Christensen

There exist non-separable non-isomorphic $C^*$-algebras $A$ and $B$, which can be represented arbitrarily closely on a Hilbert space.

- These can be constructed $(1 + \epsilon)$-completely order isomorphic for any $\epsilon > 0$. 

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There exist non-separable non-isomorphic C*-algebras $A$ and $B$, which can be represented arbitrarily closely on a Hilbert space.

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**Johnson**

For any $\varepsilon > 0$, can find two reps $\pi_i : A := C[0, 1] \otimes \mathcal{K} \to \mathcal{B}(\mathcal{H})$, with $d(\pi_1(A), \pi_2(A)) < \varepsilon$, but no isomorphism $\theta : \pi_1(A) \to \pi_2(A)$ can be uniformly close to $\pi_1(A) \hookrightarrow \mathcal{B}(\mathcal{H})$. 
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**Conclusion**

- Measure distance between $C^*$-algebras in operator norm
- Measure small uniform perturbations in point norm
THEOREM

Let $A$ be separable and nuclear and $d(A, B)$ small on a separable Hilbert space. Then there is a unitary $u \in (A \cup B)^\prime\prime$ with $uAu^* = B$.

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$^a d(A, B) < 10^{-11}$ will do!
Some positive $C^*$-algebra results

**Theorem**

Let $A$ be separable and nuclear and $d(A, B)$ small on a separable Hilbert space.\(^a\) Then there is a unitary $u \in (A \cup B)^\prime\prime$ with $uAu^* = B$.

\(^a\) $d(A, B) < 10^{-11}$ will do!

- For $\epsilon > 0$, we can choose $\delta > 0$ such that if $d(A, B) < \delta$, then for all finite subsets $\mathcal{F}$ of the unit ball of $A$, can choose a unitary $u \in (A \cup B)^\prime\prime$ with $uAu^* = B$ and

$$
\|uxu^* - x\| < \epsilon, \quad x \in \mathcal{F}.
$$

- This is what I mean by ‘measure small uniform perturbations in point norm.’
- Proof follows Erik’s strategy for injective von Neumann algebras, in a point norm way.
- Separability crucial to use an Elliott intertwining argument.
**Some positive C*-algebra results**

**Theorem**

Let $A$ be separable and nuclear and $d(A, B)$ small on a separable Hilbert space.\(^a\) Then there is a unitary $u \in (A \cup B)^{\prime\prime}$ with $uAu^* = B$.

\(^a\) $d(A, B) < 10^{-11}$ will do!

**Theorem**

Let $A$ be separable and nuclear, $A$ nearly contained in $B$. Then $A$ embeds into $B$.

- Again, can produce embeddings with point norm control.
- Uses an improved completely positive approximation property:

  $\begin{align*}
  A \xrightarrow{id_A} A \\
  \xleftarrow{\text{cpc}} F_i \xrightarrow{\sum_{\text{convex}} (\text{cpc order zero})} A
  \end{align*}$
A non-amenable result

**Theorem**

There exist non-amenable II$_1$ factors $\mathcal{M}$, for which sufficiently close algebras are small unitary perturbations.
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- eg. $\mathcal{M} = (L^\infty(X, \mu) \rtimes SL_n(\mathbb{Z})) \otimes \mathcal{R}$ for $n \geq 3$ and a free ergodic pmp action.
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- The ‘$\otimes \mathcal{R}$’ ensures $\mathcal{M} = \mathcal{M}_0 \otimes \mathcal{R}$ has the similarity property.

- $\otimes \mathcal{R}$ needed, but first step is to remove it!

- Show that any close $\mathcal{N}$ can be perturbed to $\mathcal{N}_0 \otimes \mathcal{R}$ for same copy of $\mathcal{R}$, and $d_{cb}(\mathcal{M}_0, \mathcal{N}_0)$ small.
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**Reduced to:**

- $\mathcal{M}_0 = (L^\infty(X, \mu) \rtimes SL_n(\mathbb{Z}),$ and $d_{cb}(\mathcal{M}_0, \mathcal{N}_0)$ small.

- Can use Christensen’s results to perturb the copy of $L^\infty(X, \mu)$ as a masa in $\mathcal{N}_0$, and produce normalisers of this.

- Show that $L^\infty(X, \mu)$ is Cartan in $\mathcal{N}_0$, inducing same equivalence relation as $SL_n(\mathbb{Z}) \sim (X, \mu)$.

- uses a lot of ideas of Popa. e.g. if we only assume $d(\mathcal{M}_0, \mathcal{N}_0)$ small at this point, we need his answer to a Baton-Rouge question of Kadison: existence of masas for II$_1$ factors inside specified irreducible subfactors.
**A non-amenable result**

**Theorem**

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**Reduced to:**

- $\mathcal{M}_0 = (L^\infty(X, \mu) \rtimes SL_n(\mathbb{Z}))$, and $d_{cb}(\mathcal{M}_0, \mathcal{N}_0)$ small.

**Get**

- $\mathcal{N}_0$ a twisted crossed product $L^\infty(X, \mu) \rtimes_\omega SL_m(\mathbb{Z})$
- $\omega$ a 2-cocycle, uniformly close to 1.

- $SL_n(\mathbb{Z})$ for $n \geq 3$, gives vanishing of a bounded cohomology group $H^2_b(SL_n(\mathbb{Z}), L^\infty(X, \mu))$ (using work of Monod, Burger-Monod, Monod-Shalom).
Some questions

1. Suppose $A \subset_\delta B$ where $B$ is nuclear and $A$ has similarity property. Must $A \hookrightarrow B$?

   - Christensen: $\mathcal{M} \subset_\delta \mathcal{N}$, with $\mathcal{M}$ having similarity property and $\mathcal{N}$ injective gives embedding $\mathcal{M} \hookrightarrow \mathcal{N}$.
   - Partial result: Yes for $B$ a type I C*-algebra.

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**Some questions**\(^1\)

1. Suppose \( A \triangleleft_\delta B \) where \( B \) is nuclear and \( A \) has similarity property. Must \( A \leftrightarrow B? \)
   - Christensen: \( M \triangleleft_\delta N \), with \( M \) having similarity property and \( N \) injective gives embedding \( M \leftrightarrow N \).
   - Partial result: Yes for \( B \) a type I \( C^* \)-algebra.

2. Can one produce close isomorphic copies of a \( II_1 \) factor \( M \) (perhaps \( L^\infty(X, \mu) \rtimes F_n \)) on \( \mathcal{H} \), for which no isomorphism can be close to the inclusion into \( B(\mathcal{H})? \)
   - We know that for any free ergodic pmp action \( F_n \curvearrowright (X, \mu) \), the crossed product \( M = L^\infty(X, \mu) \rtimes F_n \) is perturbation rigid: i.e., isomorphic to any von Neumann algebra it is close to.
   - \( F_k \) chosen as the bounded group cohomology \( H^2(F_k, L^\infty(X, \mu)) \) is non-trivial.

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2 Can one produce close isomorphic copies of a II\(_1\) factor \(\mathcal{M}\) (perhaps \(L^\infty(X, \mu) \rtimes \mathbb{F}_n\)) on \(\mathcal{H}\), for which no isomorphism can be close to the inclusion into \(B(\mathcal{H})\)?

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- \(\mathbb{F}_k\) chosen as the bounded group cohomology \(H^2(\mathbb{F}_k, L^\infty(X, \mu))\) is non-trivial.

3 If \(A\) and \(B\) are close and \(A\) is \(\mathcal{Z}\)-stable, must \(B\) be \(\mathcal{Z}\)-stable?

- Close II\(_1\) factors are simultaneously McDuff.
- Moreover McDuffness perturbs: if \(\mathcal{M}\) McDuff, and \(\mathcal{N}\) close to \(\mathcal{M}\) can make a small unitary perturbation so that \(\mathcal{M} = \mathcal{M}_0 \otimes \mathcal{R}\) and \(\mathcal{N} = \mathcal{N}_0 \otimes \mathcal{R}\) with the same copy of \(\mathcal{R}\) and \(\mathcal{M}_0\) and \(\mathcal{N}_0\) close.

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**Some Questions**

3. If $A$ and $B$ are close and $A$ is $\mathcal{Z}$-stable, must $B$ be $\mathcal{Z}$-stable?
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   - Moreover McDuffness perturbs: if $\mathcal{M}$ McDuff, and $\mathcal{N}$ close to $\mathcal{M}$ can make a small unitary perturbation so that $\mathcal{M} = \mathcal{M}_0 \otimes \mathcal{R}$ and $\mathcal{N} = \mathcal{N}_0 \otimes \mathcal{R}$ with the same copy of $\mathcal{R}$ and $\mathcal{M}_0$ and $\mathcal{N}_0$ close.

4. If $\mathcal{M}$ and $\mathcal{N}$ are close $\text{II}_1$ factors, what can we say about their subfactors?
   - Cartan decompositions, tensor product decompositions, transfer to close subalgebras.

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Some questions

4 If $\mathcal{M}$ and $\mathcal{N}$ are close $II_1$ factors, what can we say about their subfactors?
   - Cartan decompositions, tensor product decompositions, transfer to close subalgebras.

5 Suppose $C \subset A$ and $D \subset B$ are inclusions of nuclear $C^*$-algebras with $d(A, B)$ and $d(C, D)$ small. Is there an isomorphism $\theta : A \to B$ with $\theta(C) = D$?
   - This works for injective von Neumann algebras: apply Christensen’s theorem twice.
   - It also works if $C$ (and hence $D$) is an ideal.
   - But in general the Elliott intertwining arguments needed to construct isomorphisms between $C^*$-algebras don’t work well with inclusions.

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